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Abstract

The use of composite materials has become increasingly famous for manufacturing high performance structures the past few decades, especially within the automotive and aerospace industries. One would ask "Why composite materials?". The answer is simple enough: due to their particular characteristic of combining the best properties of the separate materials which constitute the final product. This enables the design of components that would be more lightweight yet stronger, meaning that they are characterized by superior weight-to-strength ratio, compared to conventional materials (i.e. metallic alloys).

Despite the beneficial characteristics of composite materials, they turn designing into a complex procedure, due to the fact that their distinctive nature generates an augmented number of design parameters and manufacturing constraints. All variables that govern a particular designing problem, affect the mechanical properties of the final product, which may be crucial for the sustainability of a whole engineering structure. Thus, it becomes a substantial necessity to select the proper parameters that, for each case, will lead to the optimal result.

An efficient way to increase the possibility of resulting in the most fruitful combination of the pre-referenced parameters, is the implementation of an optimization technique. Optimization based design guarantees a final product that besides fulfilling all imposed manufacturing targets and constraints, is characterized with lower costs and resources' demands, due to the accurate selection and use of the composite materials.

The immediate scope of the current thesis is the development of an algorithm that deals to find the responses of specified load cases upon two dimensional plates made out of fibrous composite materials. The profound target is to familiarize the reader with the three different theories: *Optimization Philosophy, Finite Element Analysis and Classic Laminate Theory.*

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Author,

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1. Introduction

1.1 The philosophy of Optimization

People tend to optimize the procedures that they undertake, no matter their complexity, in order to save time and money. The same technique is applied in every engineering field, driven by the market antagonism and need of continuous improvement in the produced goods. So, the necessity for developing ameliorated composite parts arises, in order to serve the needs of the increasingly rigorous market. Optimization of a laminated composite product is not a simple case, because except from the numerous deterministic variables (i.e. thickness, type of materials, orientations, environmental circumstances etc.) there are a lot more stochastic parameters that affect the final result (i.e. manufacturer's experience, manufacturability based on the part's design, flaws due to materials/consumables imperfections etc.) Effectively, the optimization process should be specified with a chosen performance measure, based on specific needs, which will be used for describing the desired quality of the final part. The analysis, measurement and validation of the result, works as input for further optimization, so it is considered a never-ending, cyclic process that leads to superior final products.

1.2 Composite materials

A composite material is defined as an enhanced combination of two or more separate materials, with different properties that are joined together in order to give a superior result. The individual materials preserve their identities in the final product, permitting their after-blending identification, fact that distinguishes a composite material from a mixture. A general categorization distinguishes 3 types of composites: natural, early and modern composite materials.

<u>Natural Composites</u>

Natural composite materials exist in a great percentage within both *fauna and flora.* Wood is a characteristic example which is made from long fibres of cellulose, which are polymer-based substances, held together by a matrix material, called lignin. Lignin is considered weaker than cellulose, with the latter not being either especially strong. Although, they result in a combination with much greater properties, that is exploited in human engineering activities from the dawn of modern civilizations.

<u>Early Composites</u>

People are occupied with the search of stronger materials for at least 5,000 to 6,000 years. The first combinations of mud with straw or rice, resulted in the construction of mud bricks. The idea was generated when people noticed, from one hand, that mud has a good resistant in compression but acts weakly under tension and from the other hand, that straw presents a good behavior when stretched. In a common way, many more composite materials were created as people realized that their constructions were benefit from their enhanced properties, such as the combination of wattle (planks or strips of wood) and daub (a composite of mud or clay, straw, gravel, lime, hay, and other substances) or the widely known and used concrete.

Modern Composites

The upgrade from early composites took place around the early 20th century, with the production of the first plastics, such as bakelite and vinyl and the engineering processing of wood in order to make complex products, such as plywood. The breakthrough happened with the invention of fiberglass, a material that showed much stiffer behavior than any other modern composite, while it was especially lightweight and could be folded into the desired shape. The matrix is a polymer substance that holds together the fine threads of glass, that most commonly are woven into various weave types. Many more ameliorations followed the original idea of combining a plastic (or some other kind of matrix) with woven fibres, or even powder and nano-tubes.

1.3 Laminated Composite Materials

A composite laminate is a structure comprised of two or more thin sheets of fibrous material, a.k.a. layers or laminae, that are bonded together with a matrix, a polymer one in most cases. Laminae may be stacked in numerous sequences and/or orientations in order to take a full advantage of the fibres' properties. The final product's properties are mainly defined from the fibres, this is the reason why the fibres' orientation coincides with the stress direction. Depending on the matrix material, there are 3 categories of composite materials: metal matrix composites (MMC), polymer matrix composites (PMC) and ceramic matrix composites (CMC). Considering the reinforcing material, the use of fiberglass, carbon, aramid, boron, basalt or natural fiber (wood, flax, hemp, etc.) is possible, with carbon being the prime preference within many industries because of its exceptional properties.

There are various proposed methodologies for constructing a laminated piece, making the selection for the optimal one a matter of a series of factors, such as the part's geometry, the type of material, the desired precision, the cost etc. Most processes require a mold (sometimes both the positive and the negative mold are needed) so as to end up with the desired geometry, as given from the CAD (*Computer Aided Design).* A brief mentioning of some of the mostly used procedures includes:

- Hand lay-up: The sheets are stacked together, while the matrix is applied with some kind of brush on the upper surface of both the mold and the laminae. The structure is left to be polymerized in ambient conditions. Advantage: Quick process with a decreased cost because there is no need for external power or consumable staff, such as release agents etc. Disadvantage: The curing in an open environment and the hand-application of the matrix may lead to imperfect bonding among the layers, due to air gaps that will be created.
- Vacuum bagging: Brush application of the matrix material upon each layer while stacking all layers together, in order to close up the structure in a bag for being polymerized under vacuum. Advantage: Vacuum permits a sufficiently good bonding among the laminae. Disadvantage: Brush application may create bubbles of air in the

interface among layers, that are considered possible points of failure initiation (*delamination).*

- Vacuum infusion: Similar procedure with the previous one except that the created vacuum is deployed in order to uniformly infuse the matrix via spiral tubes through the whole structure. Advantage: Decreased number of flaws in between the layers, thus better bonding. Disadvantage: The matrix may not be distributed in the entire surface of the part if it is rather complicated, or the tubes are not placed in an appropriate way.
- Autoclave manufacturing: The individual sheets are pre-impregnated with the matrix material, so there is no need for its afterward application. The structure is enclosed in a bag and placed in an autoclave, which is a pressure oven, used for creating vacuum when simultaneously heats up until the appropriate temperature for the polymerization of the matrix. Advantage: heat in parallel to pressure application diminishes the risk of air gaps' creation, while the curing of the matrix takes places in the exact duration that the manufacturer proposes. Disadvantage: augmented cost due to the operational cost of the autoclave and the cost of the specific materials and consumables that must have been fabricated in order to withstand elevated temperatures.

Composite materials have known a wide appreciation within the automotive, airspace, biomechanics and other industries due to their exceptional properties. They combine high specific stiffness and durability, fact that makes them ideal candidates for lightweight designs. Weight decrease is significant factor when designing something for production, due to the fact that it influences a variety of parameters, such as fuel consumption, elevated stresses transmitted through the part's body and/or the used constrained mechanisms, higher friction forces, augmented costs etc.

2. Literature Review

The scope of this chapter is to introduce the reader to the theoretical basis of the individual theories that were combined in order to form of the optimization model. Three separate theories were used, concerning: *Optimization Methodologies, Finite Element Analysis, Laminate Theory.*

It is clarified that the generalized theories are presented in order to provide the basic knowledge of the algorithm's theoretical background, without extensive analysis of the implementation steps, which are explicitly analyzed in chapter 3.

2.1 Optimization Methodologies

Several methodologies have been proposed for optimization of mechanical structures made from conventional materials, that have been subjected to alternations in order to be functional for composite materials, too. Considering the wide range of parameters that affect both design and manufacturing of a composite structure, each optimization methodology should be carefully chosen, in respect to the restrictions that govern each application. Additionally, the percentage of the systems uncertainties must be cautiously assessed, since there is the capability of using either deterministic or stochastic methodologies, that introduce probability factors in the mathematical relations. Another parameter that should be taken into consideration is the available computational capacity, since several mathematical models use numerous high order equations and demand a large number of iterations in order to achieve acceptable convergence.

Some among the most commonly used methodologies are briefly presented hereafter, with more emphasis given to *Structural Optimization,* the general categorization of *Size Optimization.*

2.1.1 Genetic Algorithms (GA)

GA have become very popular over the last years because of their capability of solving complex problems quickly, reliably and accurately. As Malhotra et al. state (Malhotra, et al., 2011) (Press, et al., 1992), they are a subclass of the *Evolutionary Algorithms* (EAs), which are population-based meta heuristic optimization models. The philosophy that governs a GA is inspired from biological mechanisms and the Darwin's theory of *Natural Selection.* They initiate from a parent-population and continuously produce new child-populations, driven by the probability of the existence of an optimum one. The efficiency of a GA is directly affected by its two operators: the crossover, which is the parameter for selecting the appropriate parents to produce an ameliorated child and the mutation, which is a random improvement of the child. The basic steps of the process are presented in an example below:

- Step I: Start - Generate random population of chromosomes, that is, suitable solutions for the problem.
- Step II: Fitness - Evaluate the fitness of each chromosome in the population.

- Step III: New population - Create a new population by repeating the following steps until the new population is complete.
  a) [Selection] Select two parent chromosomes from a population according to their fitness. The better the fitness, the bigger chance to be selected to be the parent.
  b) [Crossover] Using a crossover probability (a certain weight factor), cross over the parents to form new offspring, that is, the child. If no crossover was performed, offspring is the exact copy of parents.
  c) [Mutation] With a mutation probability, mutate new offspring at each locus.
  d) [Accepting] Place new offspring in the new population.
- Step IV: Replace - Use new generated population for a further run of the algorithm.
- Step V: Test - If the end condition is satisfied, stop, and return the best solution in current population.
- Step VI: Loop - Go to step 2.

2.1.2 Multi-objective robust design optimization (MRDO)

MRDO implements the possible existence of uncertainties within the input data to the optimization problem (Padovan, et al., 2003) (Malhotra, et al., 2011). Fluctuate values are very common industrial phenomena in many operational conditions that introduce instabilities to the objective function and result in overestimated optimum values. The target of MRDO is to achieve the best balance between performance and stability of the structure, so, the mathematical model is discretized into maximization of the mean value of the chosen objective function and minimization of its variance. (Padovan, et al., 2003)

$$\max f(x,\sigma) \equiv f: R^n \rightarrow R \tag{1}$$

$$p(x_i) = \frac{2}{\sqrt{2\pi}\sigma_i} e^{\frac{-(x-x_i)^2}{\sigma_i^2}} \tag{2}$$

$$\max f_{mean} = \bar{f} = \sum_{j=1}^{p} \frac{f_i}{p} \tag{3}$$

$$\min \sigma_f = \sum_{j=1}^{p} \frac{(f_i - f)^2}{p-1} \tag{4}$$

An interesting remark is that after solving the model, the designer ends up with a set of solutions (Pareto Front) that represents the optimum compromise between the two different objective functions. The method is also extendible to more than one function to optimize, for example it is possible to improve the lift and drag of an airfoil with fluctuations in the flight speed, without the need of a weighed function to tie the two different performances.

2.1.3 Simulating annealing method (SA)

SA has an analogous relationship with metal annealing process, defined as the heating of a metal alloy, the stabilization of its temperature for a specific time interval in order to gain ductility through stress relieving and recrystallization phenomena. According to William H. Press (Press, et al., 1992) (Tu, et al., 1999), the analogy occurs because the algorithm initially chooses the most apparent and quick solution, the lowest one, identified as a possible local optimum. Then it continues with random, ascent steps, so as to escape the local optima and find the global one. It is composed of two stochastic processes: the first deals with the solutions' generation and the second one with their acceptance. The objective function may be defined within a discrete, configuration space or a continuous one, though a continuous approach contains many levels of difficulties. For practical purposes, the methodology is addressed to problems that involve objective functions that lie within very large, configurational spaces with multiple possible solutions, like the well-known problem of the "Traveling Salesman" (which was mathematically formulated in the 1800s by the Irish mathematician W.R. Hamilton and by the British mathematician Thomas Kirkman). SA method is developed according to the following four basic steps:

1. A description of possible system configurations.

2. A generator of random changes in the configuration; these changes are the "options" presented to the system.

3. An objective function E (analogous to energy) whose minimization is the goal of the procedure.

4. A control parameter T (analogous to temperature) and an annealing schedule which gives the values' alternations, e.g., after how many random changes in configuration is each downward step in T taken, and how large is that step. The meaning of "high" and "low" in this context, and the assignment of a schedule, may require physical insight and/or trial-and-error experiments.

2.1.4 Reliability based design optimization (RBDO)

RBDO is utilized for robust model design, using the mean values of the system's stochastic parameters as its design variables, for solving a non-linear optimization problem, under specified probabilistic constraints, as Jian Tu et al. (Tu, et al., 1999) (Hozic, 2013)state. These constraints represent the system's performance functions, which depend on the variations of its random parameters X=$\left[X_i^T\right]$ ($\forall\ i = 1, .., n$), that may be described by a *Cumulative Distribution Function (CDF),* $F_{X_i}(x_i)$ or a *Probability Density Function (PDF),* $f_{X_i}(x_i)$. Both CDF and PDF are limited my maximum values that derive from structural tolerance bounds, while the system performance criteria are described by the system performance functions, G(x), where the system fails if G(x) < 0. The method also introduces a generalized probability index β_G, which is a non-linear and non-increasing function of the probabilistic performance measure that restrict the performance functions. An RBDO model can be generally defined, according to Enevoldsen and Sorensen as:

$$\begin{aligned} &\text{Minimize} \quad \text{Cost(d)} \\ &\text{Subject to} \quad P_{f,j} = P\big(G_j(x) < 0\big) \le \overline{P_{f,J}}, \ \ \forall\, j = 1,2,..,np \\ &\qquad\qquad\quad d^L \le d \le d^U \end{aligned} \tag{5}$$

where the cost can be any function of the design variable $d = [d_i]^T \equiv [\mu_i]^T$, and $\overline{P_{f,J}}$ is a prescribed failure probability limit.

2.1.5 Structural optimization (SO)

Structural optimization aims to the design of a mechanical structure that will be characterized by the best behavior under specified loads and manufacturing constraints, according to Dzenan Hozic (Hozic, 2013). The mathematical depiction of both system's performance and constraints leads to the desired optimum characteristics, through a computational analysis (*Finite Element Analysis* is the most commonly used structural analysis). The most appropriate performance characteristics may be selected among a wide range of variables, such as the mass, the stiffness (or inversely the compliance), the displacements, the thickness etc. The selection is up to the designer, due to the fact that every single construction depends on specific design requirements, so alternations between the used variables in the equations of the objective function and the constraints is possible, let alone desirable.

Indispensable part of any kind of mathematical model is the implementation of the *design variables* x_i, as Massimiliano Petrucci (Petrucci, 2009) states, where i: the total number of the degrees of freedom governing the design. These are the parameters that indicate the current design of the structure and they are chosen to represent either geometrical (i.e. surface, thickness etc.) or material characteristics (i.e. material density or as far as it concerns a composite material it comes to fiber orientations, ply densities etc.). They are, in fact, the variables that are subject to optimization, operating as multipliers to the performance variables, in a manner of controlling their value variation. Design variables may be either discrete or continuous, meaning that they can choose between isolated values of a given list or move freely in a specified range of values (usually {0,1}).

The general mathematical formulation of a SO problem is done with the use of an objective function and a set of constraints as follows:

$$\left.\begin{aligned} &Min\ or\ Max \qquad F(x,y) \\ &s.t. \qquad\qquad\quad G_k(x,y) \ge 0 \quad \forall\, k \in (1,C) \end{aligned}\right\} \text{(SO)} \tag{6}$$

where C: total number of constraints, **x**: the vector that contains the design variables and **y**: the vector of the variables characterizing the state of the structure as a response of the applied load case (i.e. thickness, displacements, stress etc.)

SO is divided into 3 main problem categorizations: *Shape Optimization, Topology Optimization* and *Size Optimization.*

Shape Optimization deals to find the optimum shape of a mechanical structure, via boundary variation during the optimization process, as Robin Larsson (Larsson, 2016) defines in his master's thesis. The methods that are generally used are either parametric or non-parametric, fact that denotes the use of geometrical parameters (i.e. a radius or triangle's height) or implicit parameters that represent the scalar displacements of the pre-defined design nodes, respectively.

Topology Optimization targets to compute the best possible material distribution, of one determinate material, within any specified design domain. As explained by Bendsoe (Bendsoe, et al., 2003), the pre-defined design domains are further discretized into a finite element mesh, characterized by the design variables that represent the elemental densities ρ_e. Each element fluctuates between solid and void, fact that is mathematically expressed as:

$$\rho_\varepsilon = \begin{cases} 1\,, & if\ \exists\ material\ @\ e \\ 0, & if\ \nexists\ material\ @\ e \end{cases} \tag{7}$$

It arises that the problem is an integer one, due to the fact that the target is to make each ρ_ε equal to either 1, if element e is filled with material or 0, if element e is empty. The integer nature of the mathematical model might cause solving problems, in case it includes a significant number of parameters. Solution malfunctions may be resolved with its conversion to a continuous problem, by using a relaxing method such as the so-called "*SIMP*" method, which introduces a penalty that steers the float values to the discrete 0-1 values.

Size Optimization differs in the way that any design domain, with its enclosed space, is fixed thus the structure may be optimized both in micro and macro-mechanical level, without alternating its topological coordinates, according to Ming Zhou (Zhou, et al., 2009). It is usually related to 2-D continuum plates, made out of composite materials, or truss constructions, defining the elemental layer thicknesses and the beams' cross sections as its design parameters, respectively. It is one of the most common methodologies chosen for problems that contain composite, laminated materials, while aiming to find the optimum composition of the laminate, through continuous alternation of the thickness of every layer, simultaneously for every element. Each lamina is characterized by a specific orientation and an initial "extravagant" thickness, so as to give enough allowance to the optimizer to calculate the final optimum values. The initial plies with the exaggerating thicknesses are defined as "*Superplies".* The selected fiber orientations are based on both the manufacturer's experience and the common standards of the composites' industry (i.e. the most frequently used materials and orientations in the automotive industry are the *Carbon Fiber Reinforced Polymer, a.k.a. CFRP, Unidirectional sheet* and the *CFRP 2x2 twill sheet* at 0^o, 45^o, - 45^o (Alen, et al., 2014)).

The design variables in Size Optimization represent the *ply* (a.k.a layer) thicknesses. Their sum equals to the total degrees of freedom, N^{DOF}, which is given by the product of all elements, N^e, with the number of all laminas, N^l. They are continuous variables, taking float values between [0,1] and they actually are the optimization parameters, by means of leading the various thicknesses to the desired reduction of their initial values.

The objective function is most usually chosen to express the mass minimization, so as to respect the principles of lean engineering. Other possible options are: volume minimization, stiffness maximization (or inversely compliance minimization), stress minimization etc. No matter what kind of objective function would be chosen, constraints, based on every individual engineering problem, are necessary in order to end up with a manufacturable and durable result. Some of the most frequently used engineering constraints are: nodal displacements, maximum mass/compliance/laminate thickness, minimum stiffness/layer thickness.

2.2 Finite Element Analysis

As mentioned in some of the pre-referenced optimization methodologies, *Finite Element Analysis* (*FEA)* of the model may be the preferred or even essential technique to provide the necessary mathematical background for the optimization problem. The structural analysis of a component under specified load cases is most usually realized via a FEA methodology, since it is characterized by computational ease and general applicability, with the capability for implementing a significant variety of loads and constraints in simple, user-friendly and immediate manners. Numerical methods, such as FEA, are the only option for highly non-linear problems, which are difficult to be described analytically. The non-linearity may arise from geometrical complexity, material behaviour or certain boundary conditions

2.2.1 General theoretical background of FEA

As Hutton professes (Hutton, 2004), FEA is an approximate, computational methodology used for confronting structural problems, governed by specific boundary conditions. The physical problem is translated into a mathematical one, under very precise assumptions, leading to differential equations that govern the mathematical nature of the model and finally solved by a numerical FEA procedure. More accurately, the differential equations define the performance of the field variables, the dependent variables that lie within the known domain, restricted from the given boundaries. The derived solution is an approximate one, as stated above, so the mathematical model must be cautiously formulated in order to reflect all physical features that the designer would wish to see in the predicted response. This response offers an insight of the structural problem, representing a specific percentage of accuracy that depends on the refinement of the mathematical model. The resulting performance is mainly interpreted via the occurred stress and strains, but a great number of engineering phenomena could be studied except from material mechanics, such as vibrations, heat conduction, fluid mechanics and electrostatics.

2.2.2 General methodology of FEA

It is actually a piece-wise procedure that can be applied to either one, two or three-dimensional problems. Its basic concept lies on the discretization of the given area (or volume) into non-overlapping parts of simple geometry, defined as *finite elements.* The elements are connected to each other via their nodes, and all considered together constitute the *mesh* of the structure. The number of nodes on each element is a parameter affected by the element's shape (i.e. beam elements for 1-D problems, triangular, rectangular for 2-D problems or brick-shaped, wedge-shaped for 3-D cases) and defines the element's grade (i.e. a triangular element with 6 nodes is defined as a 2^{nd} rate element). The response of each element is expressed in terms of the finite number of degrees of freedom that is related to the physics of the engineering problem. Assemblage of all separate responses together leads to the global performance response of the mathematical model that is chosen to represent our structure.

The analysis is realized in 3 steps: *Pre-processing, Solution, Post-processing.* Pre-processing deals with depiction of the physical model, including the definition of its geometrical features, the representation of the imposed loads and constraints and the mesh generation. The solution part includes all necessary computational steps in order to result in the desired performance characteristics. A numerical method is selected (i.e. the Gauss elimination process) in order to solve the global equation that has derived from the assembly of the individual relationships that govern each element. The solver provides the results in terms of elemental responses, which are combined to express the global response, according to the selected analysis method (i.e. the Galerkin method that was chosen for the current thesis and thus is explained below). Finally, post-processing is about proper visualization of the results. It is very important to have a clear view of the structure's behavior in both global and local levels, so as to be able to correctly compare them with the requirements of the application and decide about its engineering adequacy.

2.2.3 Introduction to the Galerkin Methodology

The Galerkin method will be concisely explained for one-dimensional cases, for ease of the reader, based on the work of Spyros A. Karamanos (Karamanos, 2007). Considering a beam of length L, lied on the x-direction, we select the displacement u(x) as the desired response for visualization. A very common category of engineering problems, that may accurately describe the pre-mentioned case is:

$$\frac{d}{dx}\left(g(x)\frac{du}{dx}\right) - h(x)u = -p(x), \qquad 0 \le x \le L \tag{8}$$

including the boundary conditions:

$$\begin{Bmatrix} u(0) = u_0 \\ u'(L) = q_L \end{Bmatrix} \text{ or } \begin{Bmatrix} u(0) = u_0 \\ u(L) = u_L \end{Bmatrix} \text{ or } \begin{Bmatrix} u(0) = q_0 \\ u(L) = u_L \end{Bmatrix} \text{ or } \begin{Bmatrix} u(0) = q_0 \\ u(L) = q_L \end{Bmatrix}$$

A more specific categorization of the above form that concerns many engineering cases is:

$$\left.\begin{array}{c} EAu'' - ku = -p(x) \\ \begin{Bmatrix} u(0) = 0 \\ u'(L) = q_L \end{Bmatrix} \end{array}\right] \quad \textit{Strong}\text{ form (S)} \tag{9}$$

where: E: elastic modulus of the structure, A: the surface perpendicularly to x-direction and k: a constant that represents the full-body displacement of the model.

The *Weak form* of the above model is computed, because the (S) may not be always easy to be calculated. Introducing a random and acceptable displacement function u*(x) the weak form $(\overline{W})$ may be written as:

$$\left.\begin{array}{c} \int_0^L \big(EAu'' - ku + p(x)\big)u^* dx \\ \begin{Bmatrix} u(0) = 0 \\ u'(L) = q_L \end{Bmatrix} \end{array}\right] \quad (\overline{W}) \tag{10}$$

It is a fact that the solution u(x) of the Strong form is also a solution of the Weak form, for every admissible u*. Extending a bit more the viewpoint of the problem, we write its *Symmetric Weak* (W) form as:

$$\left.\begin{array}{c} \int_0^L EAu'u^{*\prime} dx + \int_0^L kuu^* dx - \int_0^L p(x)u^* dx - EAq_L u^*(L) = 0 \\ \begin{Bmatrix} u(0) = 0 \\ u'(L) = q_L \end{Bmatrix} \end{array}\right] \quad (W) \tag{11}$$

The approximate Galerkin solution is computed based on the (W). We also introduce the *shape functions* N_j(x) ∀ j node (with NUMEL being the total number of nodes), which are defined:

$$N_j(x) = \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}}, x_j \le x \le x_{j+1} \\ \dfrac{x - x_{j+1}}{x_j - x_{j+1}}, x_{j-1} \le x \le x_j \\ \quad 0, \qquad x \le x_{j-1} \\ \quad 0, \qquad x \ge x_{j+1} \end{cases} \tag{12}$$

meaning that:

$$N_i(x_j) = \delta_{ij} \tag{13}$$

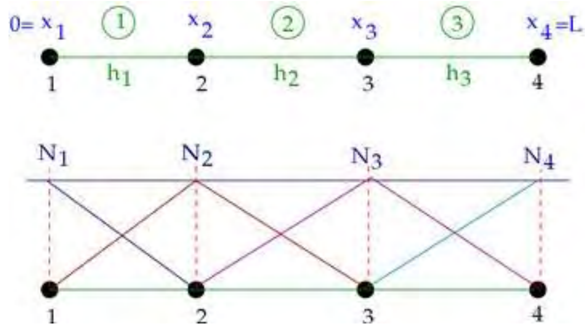


figure 1: Graph of shape functions

As a consequence, the global solution is written as the weighted sum of the separate displacement results on each node, $u_i(x)$, same as the total random, acceptable displacement u*(x):

$$u(x) = \sum_{i=1}^{NUMEL} N_i(x) u_i(x) \tag{14}$$

$$u^*(x) = \sum_{i=1}^{NUMEL} N_i(x) u_i^*(x) \tag{15}$$

It occurs that the (W) relationship may be written as:

$$[K_{ij}]\{u_j\} = \{F_i\} \tag{16}$$

where:

$$K_{ij} = \int_0^L EA\, N_i' N_j' dx + \int_0^L k\, N_i N_j dx \tag{18}$$

$$F_i = \int_0^L p(x) N_i dx + k N_i(L) \tag{17}$$

To sum up, the discretized problem results in demanding the solution of the general equation:

$$[K]\underline{u} = \underline{F} \tag{19}$$

where: $\boldsymbol{K}$: the global symmetric and positively-defined *Stiffness Matrix* and $\boldsymbol{F}$: the global force vector.

The above numerical approximation is similarly applied for both two and three-dimensional problems, besides the one-dimensional ones, but in all cases, the problem results in eq. 12, which constitutes an essential part for the optimization algorithm (in a manner that will be explained in the proceeding chapter).

2.3 Laminate Theory

The *Classical Laminate Theory (CLT)* is the theory that describes the relationship between the loadings (in-plane forces, out-of-plane bending moments and by extension, temperature) and the deformations (in-plane strain and out-of-plane curvatures) by applying the hypothesis of thin *laminates*, according to Christos Kassapoglou (Kassapoglou, 2013). The theory is based on the use of effective, realistic and simplifying assumptions that reduces the three dimensional elastic problem to a two dimensional one.

Prior to introduction of the theory's principles, it is purposive to present some commonly used definitions, in addition to the essential hypotheses, on which CLT is built.

2.3.1 Terminology and assumptions

Ply (or *lamina*): sheet material consisting of continuous fibers, unidirectional or interwoven in various manners (i.e. plain, twill, satin etc.), with homogenous properties that are considered known. An individual ply may be described as isotropic, orthotropic or transversely isotropic.

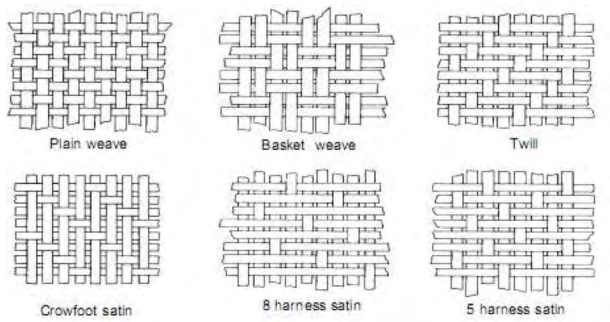


figure 2: Weave types

Laminate: a structure comprised of multiple layers (laminae) bonded together using a matrix-material. The matrix may be ceramic, metal or polymer, with the latter being the most common case.

Isotropic material: is characterized by the same properties all over across its volume

Transversely Isotropic material: is characterized by symmetry about a unique direction, indicated by an axis normal to a plane of isotropy, which sustains the same mechanical properties all over its area.

Orthotropic material: possesses three different, orthogonal planes of symmetry that are described by diverse properties, which are dependent of its three principal directions (specially orthotropic).

Before proceeding with the theory's expansion, it is crucial to state the assumptions that are considered necessary for its validity:

1. The laminate's plies are considered perfectly bonded together and no slip is permitted, so as any displacement phenomena are negligible, yet continuous through the thickness.
2. Each lamina is in the state of plane stress.
3. The laminate deforms according to the Kirchhoff's hypothesis:

- Normals to the mid-plane remain straight
- Normals remain unstretched
- Normals remain normal

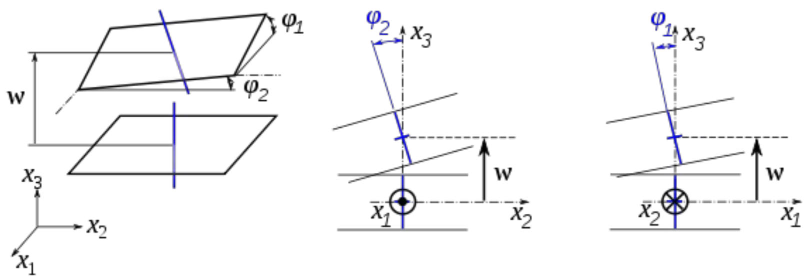


figure 3: Deformation under Kirchhoff's hypothesis

2.3.2 Mathematical model for one ply

The *Generalized Hooke's Law* defines the stress-strain equation as:

$$\underline{\sigma} = [C]\underline{\varepsilon} \tag{20}$$

where: $\underline{\sigma}$: the vector containing the stress components, $[C]$: the *Stiffness Matrix* and $\underline{\varepsilon}$: the vector containing the strain components.

Equivalently, the strain-stress relation is most frequently used:

$$\underline{\varepsilon} = [S]\underline{\sigma} \tag{21}$$

where: $[S]$: the *Compliance Matrix*.

Most composite materials are considered orthotropic, so we limit the analysis to the presentation of their reduced equations and the theoretical relations that connect them. In that case, we proceed with the expansion of the above equations in terms of the engineering constants, as:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{v_{21}}{E_2} & -\frac{v_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{v_{12}}{E_1} & \frac{1}{E_2} & -\frac{v_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{v_{13}}{E_1} & -\frac{v_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} \tag{22}$$

where: E_1: longitudinal Young's modulus (in direction 1), E_2: transverse Young's modulus (in direction 2), v_{ij}: Poisson's ratio that correspond to deformation in j-direction because of a load applied in i-direction.

Note that for an orthotropic material, Poisson's ratios differ in each direction (1, 2, 3), but they are related according to: $\frac{v_{21}}{E_2} = \frac{v_{12}}{E_1}$.

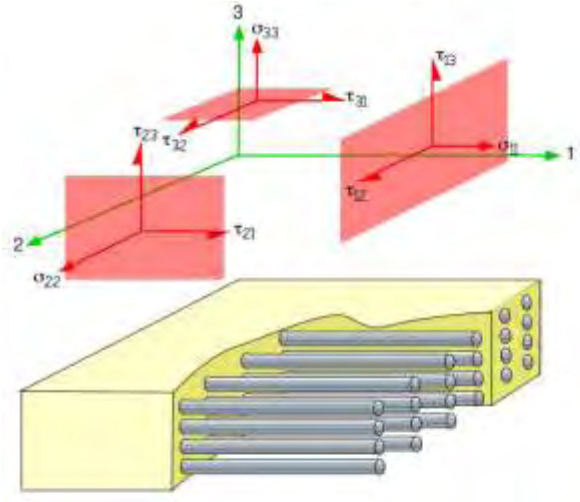


figure 4: Stresses in orthotropic materials

As far as it concerns the Compliance matrix, it is important to note the physical meaning of the remaining S_{ij} terms, because of the significant coupling effect between the applied stress and the resulting deformation:

- S_{11}, S_{22}, S_{33} : represent the extension coupling of $\sigma_{11}, \sigma_{22}, \sigma_{33}$ on the same direction
- S_{44}, S_{55}, S_{66} : represent the shear phenomena on the same plane
- S_{12}, S_{13}, S_{23}: represent the extension-extension coupling also known as the Poisson effect.

Under the consideration of an insignificant plate thickness comparing to its other dimensions, in addition to the non-existence of out-of-plane loads (as stated in the second assumption), we can presume that $\sigma_{33} = \tau_{31} = \tau_{23} = 0$. In that case, the three-dimensional strain-stress relationships reduces to the two-dimensional one, as follows:

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} \tag{23}$$

Or, inversely, in terms of the Stiffness values:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} \tag{24}$$

Q_{ij} are the *Reduced Stiffness* coefficients, which can be expressed in relation to the engineering constants, as:

$$Q_{11} = \frac{E_{11}}{1-v_{12}v_{21}},\ Q_{22} = \frac{E_{22}}{1-v_{12}v_{21}},\ Q_{12} = \frac{v_{12}E_{22}}{1-v_{12}v_{21}} = \frac{v_{21}E_{11}}{1-v_{12}v_{21}},\ Q_{66} = G_{12} \tag{25}$$

It is often necessary to move between the principal coordinates and some orientated coordinates of the lamina, in order to calculate its properties when a load is applied on those non-principal axes. The coordinate system that is aligned with the fiber-direction is considered the principal one (1-2), while any other coordinate system is defined as the off-axis coordinate system (x-y).

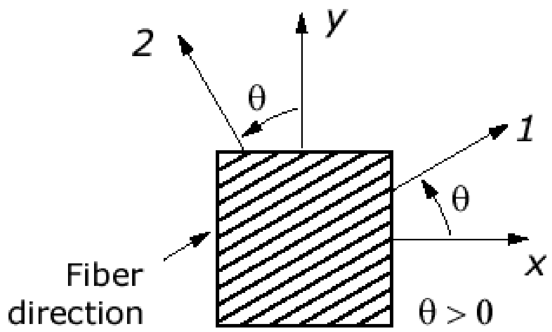


figure 5: Non principal, loading coordinate system

The transformation relationship from the principal to the off-axis coordinates is given below:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} cos^2\theta & sin^2\theta & -2sin\theta cos\theta \\ sin^2\theta & cos^2\theta & 2sin\theta cos\theta \\ -sin\theta cos\theta & sin\theta cos\theta & cos^2\theta - sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{26}$$

Or equivalently, both in terms of stress and strain:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad \& \quad \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix}. \tag{27}$$

where $[T]$: the *Transformation Matrix*

It is also possible to write the inverse relationships, meaning the transformation from the off-axis to the principle directions:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \& \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{28}$$

The previous equations can be substituted in the stress-strain relations in order to obtain the stiffness results of the lamina, according to the equation:

$$[\bar{Q}] = [T]^{-1}[Q][R][T][R]^{-1} \tag{29}$$

where: $[R]$ is the *Reuter Matrix*, defined as:

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{30}$$

and $[\bar{Q}]$ is the *Transformed Reduced Stiffness Matrix* of every lamina, containing the following terms in all of its nine positions, in contrast to the Reduced Stiffness Matrix:

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}cos^4\theta + Q_{22}sin^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta \\
\bar{Q}_{22} &= Q_{11}sin^4\theta + Q_{22}cos^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})sin^2\theta cos^2\theta + Q_{12}(sin^4\theta + cos^4\theta) \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})sin\theta cos^3\theta - (Q_{22} - Q_{12} - 2Q_{66})cos\theta sin^3\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})cos\theta sin^3\theta - (Q_{22} - Q_{12} - 2Q_{66})sin\theta cos^3\theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})sin^2\theta cos^2\theta + Q_{66}(sin^4\theta + cos^4\theta)
\end{aligned} \tag{31}$$

Consequently, the stress-strain relation may be generalized as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{32}$$

or inversely:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\ \bar{S}_{21} & \bar{S}_{22} & \bar{Q}_{26} \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}. \quad (33)$$

2.3.2 Mechanics of a laminate

We proceed with the macromechanical analysis of a laminate, in order to compute the global response of the sum of the bonded-together and differently oriented laminae, under a specified load-case. It is purposive to restate the first theory's assumption, because of its significance for the analysis: the interface between all laminate's plies is governed by the same deformation along the direction of the applied load and the stresses in the transversal direction must be self-equilibrating. In addition, the Kirchhoff's hypothesis leads to ignoring the shear effects and the strain perpendicular to the middle plane, so as:

$$\gamma_{xz} = \gamma_{yz} = 0 \ \& \ \varepsilon_z = 0$$

Considering the displacement in the x-direction as u and the displacement in the y-direction as v, the analogous strains are defined as:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \gamma_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \quad (34)$$

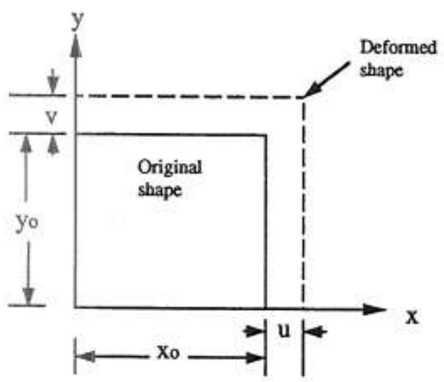


figure 6: Normal displacements

The slope of a surface under bending is given as:

$\frac{\partial w}{\partial x}$ in the x-direction

$\frac{\partial w}{\partial y}$ in the y-direction

The basic principle is that the total displacement of any point across the laminate occurs as the sum of the normal displacements in addition to those provoked by bending, which steers to the equations:

$u = u_0 - z\frac{\partial w_0}{\partial x}$ and $v = v_0 - z\frac{\partial w_0}{\partial y}$ (35)

where: u_0, v_0 are the displacements of the midplane of the laminate and $\frac{\partial w_0}{\partial x}, \frac{\partial w_0}{\partial y}$ are the slopes of the midplane in x and y-direction respectively and z represents the distance of the referenced ply from the middle plane of the laminate.

The combination of the above equations leads to the alternative strain relations:

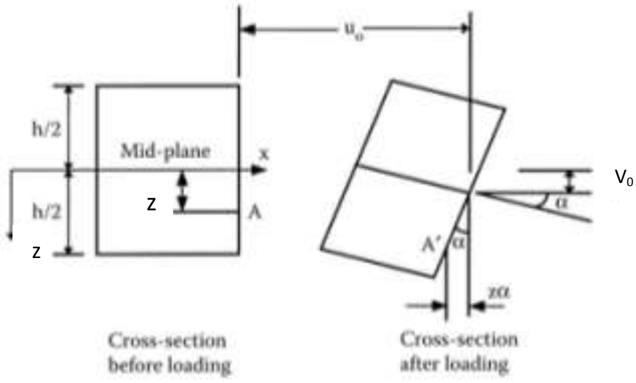


figure 7: Cross section deformation

$$\varepsilon_x = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_0}{\partial x^2}, \varepsilon_y = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_0}{\partial y^2}, \gamma_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w_0}{\partial x \partial y} \quad (36)$$

which can be written in a matrix form as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{xy} \end{Bmatrix} + z\begin{Bmatrix} k^0{}_x \\ k^0{}_y \\ k^0{}_{xy} \end{Bmatrix} \quad (37)$$

where: $k^0{}_x = -\frac{\partial^2 w_0}{\partial x^2}, k^0{}_y = -\frac{\partial^2 w_0}{\partial y^2}$ and $k^0{}_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y}$ are defined as the plate's mid-plane curvatures. $k^0{}_x$ and $k^0{}_y$ express the rate of the slope's change in x and y-direction respectively and $k^0{}_{xy}$ expresses the bending in the x-direction along the y-axis (torsion).

For the k-ply of the laminate the stress-strain relation is written:

$$\{\sigma\}^k = [\bar{Q}]^k\{\varepsilon\}^k \quad (38)$$

So, combining eq. 37 and eq. 38, the relationship between the stress and strains of the k-layer is expressed in terms of the middle-plane's displacements and curvatures as:

(39)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left(\begin{Bmatrix} {\varepsilon^0}_x \\ {\varepsilon^0}_y \\ {\gamma^0}_{xy} \end{Bmatrix} + z \begin{Bmatrix} {k^0}_x \\ {k^0}_y \\ {k^0}_{xy} \end{Bmatrix} \right)$$

It is important to note that the coordinate z of the k-ply refers to its upper surface and that all laminae may be mutually distributed on both sides of the midplane, so as the enumeration is realized in a manner similar to figure below.

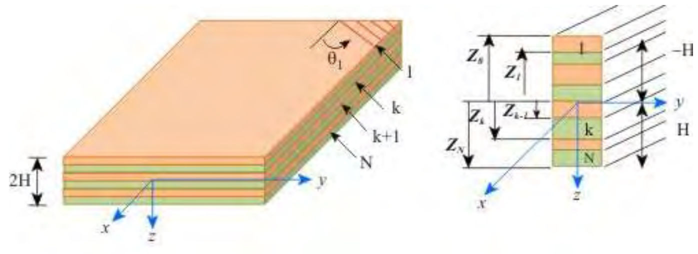
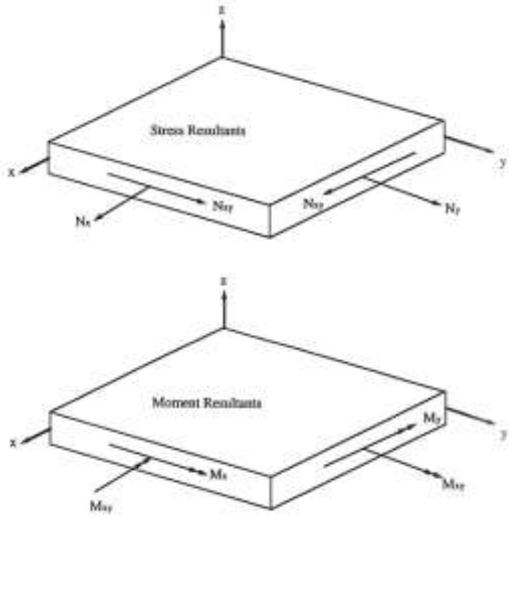


figure 8: Ply enumeration

The final step of CLT demands the connection of forces and moments with the resultant strains and curvatures. We introduce the loading vector **N** as the integral of the through-the-laminate-thickness stresses in each ply and the moments **M** around the midplane as the integral of the occurring stresses multiplied with the moment-arm z, which are mathematically defined as:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} dz = \sum_{k=1}^{N^l} \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^k dz \qquad (40)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} z dz = \sum_{k=1}^{N^l} \int_{-t/2}^{t/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}^k z dz \qquad (41)$$



where: t is the total thickness of the laminate and N^l is the total number of plies.

The above equations can be expressed in terms of stiffness coefficients, using eq. 32:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N^l} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left(\int_{z_{k-1}}^{z_k} \begin{Bmatrix} {\varepsilon^0}_x \\ {\varepsilon^0}_y \\ {\gamma^0}_{xy} \end{Bmatrix} dz + \int_{z_{k-1}}^{z_k} z \begin{Bmatrix} {k^0}_x \\ {k^0}_y \\ {k^0}_{xy} \end{Bmatrix} dz \right) \tag{42}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N^l} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left(\int_{z_{k-1}}^{z_k} \begin{Bmatrix} {\varepsilon^0}_x \\ {\varepsilon^0}_y \\ {\gamma^0}_{xy} \end{Bmatrix} z dz + \int_{z_{k-1}}^{z_k} z^2 \begin{Bmatrix} {k^0}_x \\ {k^0}_y \\ {k^0}_{xy} \end{Bmatrix} dz \right) \tag{43}$$

figure 9:Ply loading and moments

The strains and curvatures of the midplane are constant comparing to z, so they can be excluded from the integrals, fact that permits to perform the simple thickness integrations:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \sum_{k=1}^{N^l} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left(\begin{Bmatrix} {\varepsilon^0}_x \\ {\varepsilon^0}_y \\ {\gamma^0}_{xy} \end{Bmatrix} (z_k - z_{k-1}) + \begin{Bmatrix} {k^0}_x \\ {k^0}_y \\ {k^0}_{xy} \end{Bmatrix} \frac{1}{2}({z_k}^2 - {z_{k-1}}^2) \right) \tag{44}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \sum_{k=1}^{N^l} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}^k \left(\begin{Bmatrix} {\varepsilon^0}_x \\ {\varepsilon^0}_y \\ {\gamma^0}_{xy} \end{Bmatrix} \frac{1}{2}({z_k}^2 - {z_{k-1}}^2) + \begin{Bmatrix} {k^0}_x \\ {k^0}_y \\ {k^0}_{xy} \end{Bmatrix} \frac{1}{3}({z_k}^3 - {z_{k-1}}^3) \right) \tag{45}$$

Gathering together the transformed reduced stiffness matrix terms with the thickness subtractions' parentheses, the new matrices are introduced:

$$[A] = \sum_{k=1}^{N^l} [\bar{Q}_{12}]^k (z_k - z_{k-1}) \tag{48}$$

$$[B] = \frac{1}{2} \sum_{k=1}^{N^l} [\bar{Q}_{12}]^k ({z_k}^2 - {z_{k-1}}^2) \tag{47}$$

$$[D] = \frac{1}{3} \sum_{k=1}^{N^l} [\bar{Q}_{12}]^k ({z_k}^3 - {z_{k-1}}^3) \tag{46}$$

where: $[\boldsymbol{A}]$: *Extensional Stiffness Matrix,* $[\boldsymbol{B}]$: *Coupling Stiffness Matrix,* $[\boldsymbol{D}]$: *Bending Stiffness Matrix.*

Combining all above equations, it is feasible to write the final relation that connects the loads with the laminate's response, in terms of the assembled stiffness matrix:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} {\varepsilon^0}_x \\ {\varepsilon^0}_y \\ {\gamma^0}_{xy} \\ {k^0}_x \\ {k^0}_y \\ {k^0}_{xy} \end{Bmatrix} \tag{49}$$

Or in an equivalent and abbreviated way:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ k^0 \end{Bmatrix} \tag{50}$$

It is important to note that the components of the Coupling Stiffness Matrix $[\boldsymbol{B}]$ are all equal to zero, when it comes to the case of a symmetric laminate, which requires that all plies are equally distributed above and below the middle-plane. This occurs because the geometric midplane coincides with the plate's neutral plane, so that it does not undergo any bending strain that can lead to bending moments.

As a conclusion, the current chapter provided the theoretical substructures for being able to proceed with the optimization algorithm formulation (Koruche, et al., 2015), (Nettles, 1994) and (Roylance, 2000). An introduction of the various optimization techniques, is presented to help with the understanding of the main definitions from the optimization philosophy and specifically the theory that governs the methodology of Size Optimization of composite laminates. Afterwards, explanation of the Finite Element Analysis is necessary for understanding the mathematical tools that are implemented in the algorithm, which have derived from the structural approach of a given engineering application. Finally, brief explanation of the Classical Laminate Theory is considered an indispensable part for comprehension of the way that the distinctive properties and characteristics of composite materials have been integrated into the upcoming optimization approach.

3. Optimization algorithm's breakdown

The scope of this chapter is to present not only the optimization algorithm itself but also all factors that defined its composition: its structure and the steps that were followed during its formulation, all assumptions and compromises that were made in the background. In addition, some parts of the theories presented in the literature review will be further explained in the current chapter, along with all specific equations that originate from them and constitute essential parts for the mathematical models used in the code. Finally, results of specific tested model-cases will be presented, their significance will be discussed, in addition to validation remarks comparing to the results of the same cases that were computed with the aid of the commercial software "Optistruct".

3.1 Structure of the algorithm

3.1.1 General methodology and scope

The algorithm's general methodology to achieve the size optimization of a given laminated structure is the minimization of an objective function, which in this case equals the total mass, in respect of an indispensable constraint that was decided to be the plate's stiffness. The whole procedure is realized via the design variables, $x_{e,k}$, as introduced in chapter *2.1.5*, which actually affect the thickness values of each and every element. The initial thicknesses are given so as to match the superply definition, as stated in chapter *2.1.5* and the ordering of the plies is based on manufacturing experience, but multiple combinations may be inserted as inputs in order to cover a greater range of desired responses. Every optimization cycle results in a specific value for every design variable that, due to the multiplication with the elemental thicknesses and densities for an individual ply ends up with particular results for every element. Summing up the derived elemental thicknesses, we are steered to the global thickness value of the laminated plate as a whole. Because of the fact that $x_{e,k}$ acts on the elemental level of a single lamina, an infinite number of results may occur, from leaving the initial ply as given or diminishing it completely. So, the mathematical tool of the optimization process is in fact focused on the design variables, in a way that is explained thoroughly in the next sub-chapters.

3.1.2 Input parameters

First of all, it must be stated that the whole algorithm is written in *Python.* Python is an interpreted programming language, used for general purposed programming. As a language based on code readability and dynamic typing, it is considered a user-friendly language that provides one of the richest libraries among other languages of the same kind. The main libraries that were used in the composition of our program, are:
Numpy/Sympy/Scipy/Matplotlib that are used for mathematical calculations and plots' extraction, Easygui/Pandas that are used for making the interface between the user and the machine as pleasant as possible and cProfile that provides statistics about the functionality

and efficiency of the code, which led in alternating some statements or functions in order to augment its speed.

The very first step of the optimization code is to define the geometry of each studied case. It is operational for any kind of two-dimensional plate, as long as the coordinates of the geometry's mesh are properly defined. We chose to use quadratic, shell elements for the current analysis, so the *Coordinates' Matrix* ought to include $N^e \times 4$ components, in order to result in fully defined in space nodes. For the completion of the mesh definition, the *Connectivity Matrix* is created, which expresses the manner that each element is connected with its neighboring ones. The connectivity matrix plays a vital role in the finite element analysis, due to the fact that it is the "guide" for the assembly of the individual elements' matrices and vectors into the global ones.

Up to next, the laminate's composition must be designated. As a consequence, the total number of plies, N^l, along with their orientations and engineering properties must be inserted. There is no limitation concerning the type of composite material that makes up a lamina, other than its orthotropic nature, so as to preserve the validity of the pre-mentioned laminate theory's equations throughout the algorithm's formulation. As stated in the Size Optimization's section, the initial thickness of each ply equals an exaggerating value, as the definition of a superply suggests, so there is no need to insert the exact thickness of the purchased material.

3.1.3 Objective function

The function that was selected for the optimization procedure is the total mass of the structure:

$$M = \sum_{e=1}^{N^e} \sum_{k=1}^{N^l} x_{e,k} \times t_k \times dens_k \times A_e \tag{51}$$

where: A_e is the density of the element e, $dens_k$ and t_k are the density and the thickness of the k-ply respectively and $x_{e,k}$ are the design variables.

The optimization problem is particularized into mass minimization, a strategy that is widely used while designing an engineering structure. Mass minimization constitutes a parameter of *Design for Sustainability* (D4S), an eco-design mentality that combines both social-environmental and economic factors during production (Clark, et al., 2009). D4S-based products are characterized with ameliorated engineering properties that lead to extended life-cycle, while the eco-financial effects of the reduction of the raw material's demands contribute to industries' development.

3.1.4 First constraint: nodal displacements

The main constraint is chosen to be a number of specified nodal displacements. The limited motion of a structure is expressed with restriction of the possible movements of the nodes that are located in crucial areas, which are indicated by the designer.

The displacements are calculated based on the methodology of Finite Element Analysis, explained in chapter 2.2.3. As a consequence, the algorithm ends up with requiring the solution of eq. 19:

$$[K]\underline{u} = \underline{F}$$

The $\underline{u}$ vector contains all nodal displacements, occurring in all permittable directions. For a plane problem (with possible motion across x and y-directions) including N^n nodes, the $\underline{u}$ vector is of size $N^n \times 2$. As it is already stated, $[K]$ is the global Stiffness Matrix of the laminate, with a size of $N^n \times N^n$, resulting from the assembly of all separate elemental stiffness matrices, $[K]^{\,e}$, each one with a size of 8×8.

In order to compute an elemental stiffness matrix, the homogenized properties of the sum of all plies are required for each element. This is an essential part due to the fact that the optimization operates on elemental level, meaning that it performs an iterating estimation of each lamina's properties for each and every finite element, in order to end up with the optimum selections. By definition, the term *Homogeneous,* refers to materials that have the same properties all across their volume, characterized as uniform, without irregularities.

Under the homogeneity assumption, the code must calculate the total laminate properties, in terms of the single ply variables, in a manner that enables the continuous alternation of the thickness values. The parametrization of the ply thickness is achieved via the design parameters, suggested by the methodology of Size Optimization. As mentioned in the previous chapter, the number of design variables $x_{e,k}$ equals the number of the total degrees of freedom of the structure, concerning the thickness variations. So, assuming a mesh of N^e elements and N^l laminae, the total number of $x_{e,k}$ equals to $N^e \times N^l$.

The design variables $x_{e,k}$ take float values between [0,1] during the optimization's iterations, steering the elemental thickness from the initial superply thickness to a possible zero-value. A zero-value thickness indicates that the specific element has no need of material, under to forced load case. Therefore, it is clear that an individual ply could be absolutely eliminated of not contributing in any way to the structure's integrity.

So, the homogeneous engineering constants for each oriented lamina are calculated as follows (Farooq, et al., 2017):

$$E_{11} = \frac{1}{A_{11}t^e}, E_{22} = \frac{1}{A_{22}t^e}, G_{12} = \frac{1}{A_{66}t^e}, v_{12} = -\frac{A_{12}}{A_{11}}, v_{21} = -\frac{A_{12}}{A_{22}} \tag{52}$$

where: A_{ij} are the components of the Extensional Stiffness Matrix and t^e is the elemental total thickness.

The Extensional Stiffness Matrix $[A]$ is calculated according to the equations of chapters 2.3.1 and 2.3.2, for the given materials and orientations of each ply. Consequently, the engineering coefficients for an individual element are not only known but inserted in the parametric way needed for the optimization's functionality.

Having obtained the necessary engineering coefficients, the next step is to calculate the elemental stiffness matrices. First of all, an important segregation must be made between the local and global coordinate systems: the global coordinate system includes the (x,y) axis and the nodal enumeration is row-based, while the local coordinates valid for every single element includes the axis (ξ,η) and the nodal enumeration is happening anti-clockwise, as shown in figure below.

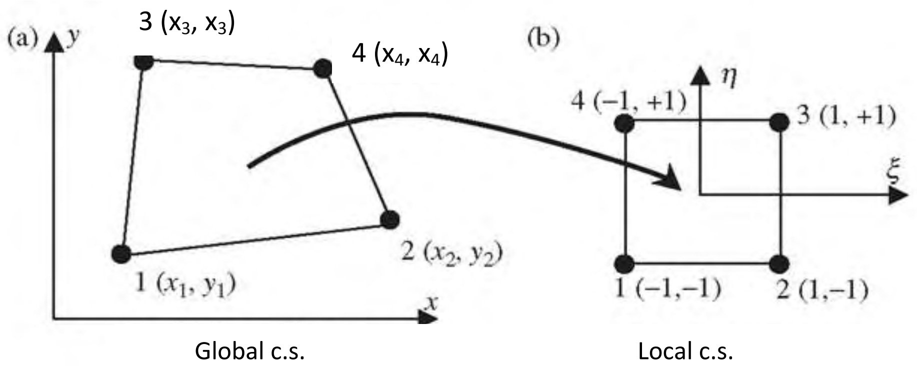


figure 10: Transformation of coordinate systems

The local stiffness matrix for the quadrilateral element stems from the body's *Strain Energy U*, given as:

$$U = \int_V \frac{1}{2}\sigma^T \varepsilon dV = \sum_e t^e \int_{A^e} \frac{1}{2}\sigma^T \varepsilon dA. \tag{53}$$

The vector **ε** contains the two-dimensional strains, as stated in chapter 2.3.2, and is related to the analogous displacements with the differential equations:

$$\varepsilon = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}. \tag{54}$$

The transportation of the various values from the global coordinate system to the local one is realized with the contribution of the *Jacobian Matrix*, $[J]$, defined as:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

The Jacobian components represent an implicit function of derivatives (x,y) expressed in the derivative terms (ξ,η) using the chain rule. But, the coordinates (x,y) of a point within an element must also be expressed in local terms. This is achieved through the *Shape Functions* that are already introduced in chapter 2.2.3, as follows:

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4. \quad (55)$$

For the current used quadrilateral elements, we chose to use the lagrangian shape functions:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) \quad (56)$$

while $N_i = \begin{cases} 1, @\ node\ i \\ 0, @\ the\ rest\ 3\ nodes \end{cases}$.

Given the shape functions, it is also useful to express the local nodal displacements, **q**, in terms of the unknown global displacements u, v:

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7 \quad (58)$$
$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8 \quad (57)$$

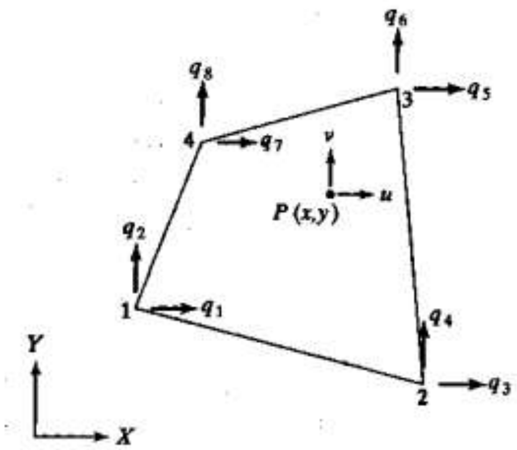


figure 11: Nodal displacements

Combining eq. 55 and eq. 56 the Jacobian matrix can now be written with the following useful form:

$$[J] = \frac{1}{4}\begin{bmatrix} -(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4 & -(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4 \\ -(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4 & -(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4 \end{bmatrix}$$

or equivalently:

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}.$$

As a consequence of the above jacobian considerations, the global displacements' gradients can be re-written as:

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{detJ}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} \tag{59}$$

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{detJ}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}. \tag{60}$$

The combination of the 2 previous equations with eq. 54 leads to the below expression of the strains in terms of the transformed nodal displacements:

$$\varepsilon = [A]\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} \tag{61}$$

where:

$$[A] = \frac{1}{detJ}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{62}$$

We may now proceed with expressing the displacements derivatives as a function of the local displacement vector, q:

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = [G]q$$

where:

$$[G] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

Introducing the *Element Strain-Displacement* matrix, B, which is represented by the relation $[B] = [A][G]$, and afterwards combining eq. 61 and eq. 62, leads to the valuable relation between the strains and the nodal displacements:

$$\varepsilon = [B]q \tag{63}$$

which contributes to the expression of the stresses on terms of the nodal displacements as:

$$\sigma = [D][B]q \tag{64}$$

where $[D]$ is the 3×3 *Constitutive Material* matrix that contributes to the appropriate expression of the structure's response to the applied stresses, operating in a manner similar to $[Q]$, as introduced in chapter 2.3.2. It is formulated using the homogeneous engineering properties that we estimated, based on the classical laminate theory as follows:

$$[D] = \begin{bmatrix} \frac{E_{11}}{1-v_{12}v_{21}} & \frac{v_{21}E_{11}}{1-v_{12}v_{21}} & 0 \\ \frac{v_{12}E_{22}}{1-v_{12}v_{21}} & \frac{E_{22}}{1-v_{12}v_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Finally, the strain energy is expressed as:

$$[U] = \sum_e \frac{1}{2} q^T \left[t^e \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] detJ d\xi d\eta \right] q <=>$$

$$[U] = \sum_e \frac{1}{2} q^T [K]^e q \tag{65}$$

where $[K]^e$ is the 8×8 element *Stiffness Matrix* defined as:

$$[K]^e = t^e \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B] detJ d\xi d\eta \tag{66}$$

At that point where the estimation process of the elemental stiffness matrices was accomplished, last step for the calculation of the global matrix, $[K]$, is to gather the analogous matrices for all given quadrilateral elements and assemble them under the specified manner. This is achieved via the Connectivity Matrix, that was shaped during the first steps of the geometry-mesh definition and is realized under the global coordinate system (x, y). It is a crucial step because nodes shared among neighboring elements must be governed by the summation of values representing each element, but strictly the specific elements that come in contact with a given dispersion.

$$K = \begin{pmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{11}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} & 0 & 0 \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{41}^{(1)} & k_{41}^{(1)} & k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ 0 & 0 & k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ 0 & 0 & k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{pmatrix}$$

The remaining term of equation eq. 19 to be calculated is the global force vector F. Operating in a similar way as the estimation of the global stiffness matrix, the elemental force values are written down, gathered and assembled together to form the global vector. A plane load forced on an element can be tensile, compressive, bending or a combination of them and it may be applied either on its nodes or it can be equally distributed on an edge. i.e. given the tensile force T_x applied on the right edge, l_{23}, of our quadrilateral element, the elemental force vector is written as:

$$F^e = \frac{t_e l_{23}}{2} \{0\ 0\ T_x\ 0\ T_x\ 0\ 0\ 0\}^T \tag{67}$$

or, assuming a nodal, bending load, acting on nodes 2 and 3:

$$F^e = \left\{0\ 0\ 0\ T_y\ 0\ T_y\ 0\ 0\right\}^T \tag{68}$$

So, once all local matrices are calculated, their assembly follows, always based on the information of the connectivity matrix. We end up with the global force vector, F, which is of size N^e.

Eq. 19 can now be solved, in order to provide the desired displacement results. It could be solved either with a numerical approximation or

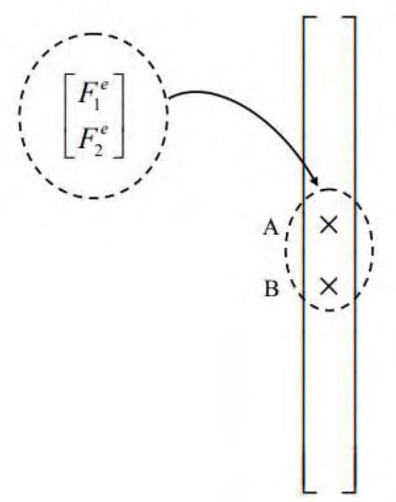


figure 12: Global load vector assembly

with multiplication with the Inverse stiffness matrix as follows:

$$\underline{u} = [K]^{-1}\underline{F}$$

We chose to solve the equation with a custom Gauss Elimination algorithm, shown in Appendix 1, because we noticed on the operational statistics of the code a significantly high memory consumption during calling the Python-made module 'inv(MATRIX)', used for inverse matrices' calculations.

3.1.5 Second constraint: Minimum ply thickness

The second selected constraint is the minimum thickness of each lamina, a necessity arouses due to manufacturability reasons. As introduced in chapter 2.1.5, the optimization algorithm receives the Superplies' thicknesses as the initial thickness parameters, t_k. As a consequence, the initial elemental thickness is:

$$t_e = \sum_{k=1}^{N^l} t_k \tag{69}$$

The thickness of one ply could hypothetically be increased unrestrainedly, by laminating as many material-sheets as the manufacturer intends to. In that way, the superply thickness is considered exaggerating but feasible. Although, a lamina's thickness cannot be decreased less than the minimum thickness of the material, t_k^{MIN} as provided from the supplier.

Therefore, the common minimum thicknesses of the input materials were inserted as ply-level constraints, so as the final resulted laminate could be considered manufacturable:

$$t_{e,k} \leq t_k^{MIN}$$

3.1.6 Python's optimizer

In order to perform the optimization's calculations, after gathering all its necessary inputs, we needed a Python optimizer for linear, constrained objective functions. We chose the use *Scipy.Optimize.Minimize* due to the fact that it not only provides a wide range of possible numerical methods for the computations, but also it permits a parametric implementation of the constraints and bounds of the variables' values, fact that makes the algorithm more adaptable to possible design needs. The formulation of the optimization module is presented in Appendix 2.

3.2 Results' comparison with Optistruct

In order to appreciate the accuracy of the optimization algorithm's results, it was considered necessary to compare them with a common commercial software that performs similar types of analyses. The chosen one was '*Optistruct',* created and owned by Altair Engineering. It is a *Computer Aided Engineering (CAE)* simulation software, used for both linear or non-linear simulation of a grand variety of materials, conventional (i.e. metals) or more particular (i.e. hyperelastic or composites) and for many different cases of static, transient and dynamic loads, noise-vibration-harshness (NVH) or acoustic analyses.

A study case was set up and solved, using both the custom code and Optistruct. The selected loadcase included a bending load on the top-right node and a fixed support to the left edge of the plate.

3.2.1 Detailed description of the Case Study

The geometric model was a thin plate, discretized in 40 quadrilateral, shell finite elements that was comprised of two plies: the first lamina is CFRP twill 3K 2x2 sheet oriented in 0^o degrees and the second is a CFRP UD sheet oriented in 90^o. The materials' properties are shown in the table below:

Material	E_1 (Mpa)	E_2 (Mpa)	G_{12} (Mpa)	v_{12}	v_{21}
CFRP twill 2x2	70000	70000	3740	0,04	0,04
UD CFRP	135000	10000	5000	0,3	0,02

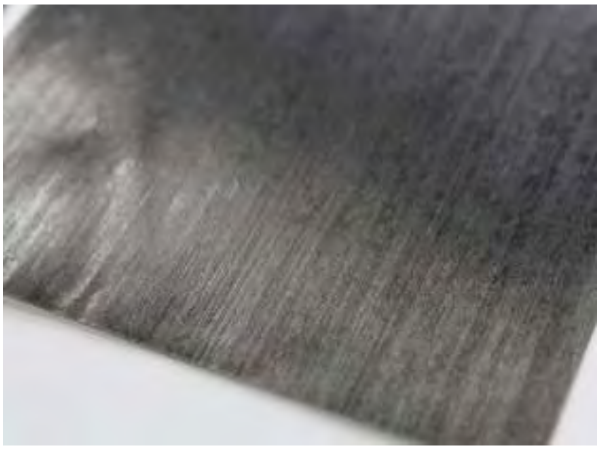


figure 14: CFRP unidirectional

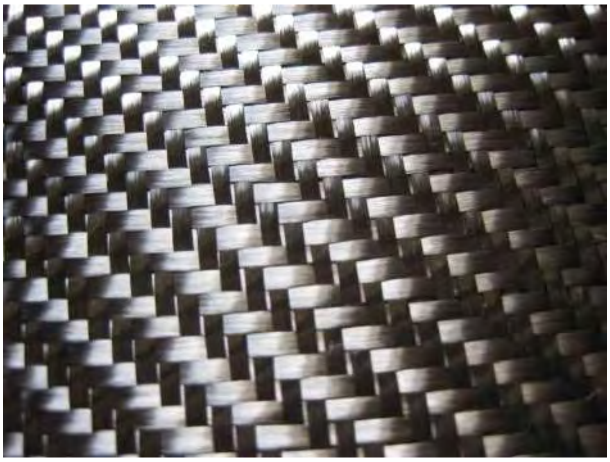


figure 13: CFRP twill 2x2

- The model has 10 finite elements lying in the x-direction and 4 in the y-direction.
- The boundary conditions of the model include a fixed support on the entire left edge of the plate.
- The load is applied in the y-direction, on the top right node and is equal to 5 Newtons.
- Superplies' initial thickness value = 5 mm

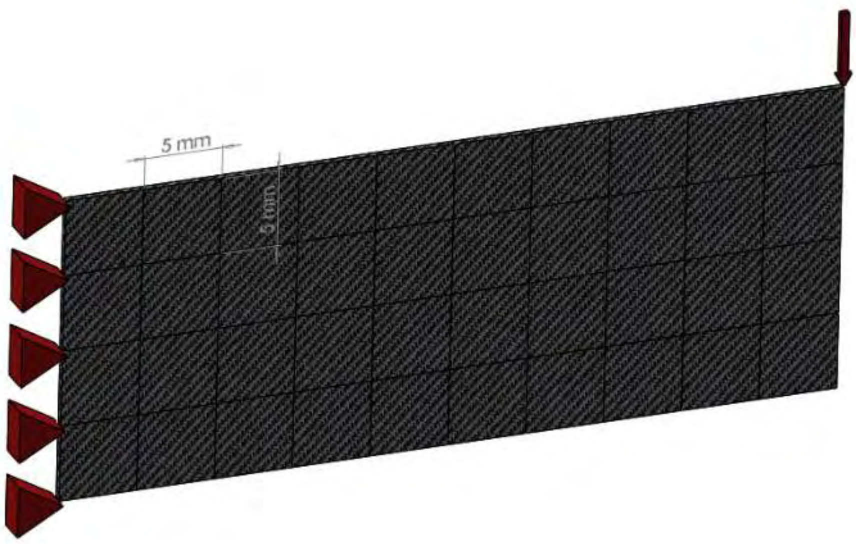


figure 15: CAD model of the plate

figur e 16: CAD mod

el of the plate

The visualization of the results is done through contour plots of both the individual ply thickness and the total laminate's thickness. Contour plots are a useful tool for picturing the fluctuation of the elemental thickness, leading to the creation of *patches*, areas formed of gathered elements with the same or slightly diverse thickness, in order to make the lamination feasible.

We notice that both contour plots of the first ply present a similar and symmetric distribution, indicating that the twill fabric is necessary for more than the half surface of the plate. Most of the required material is gathered in the left of the plate, fact that is expected due to the stresses that arise due to the existing moments.

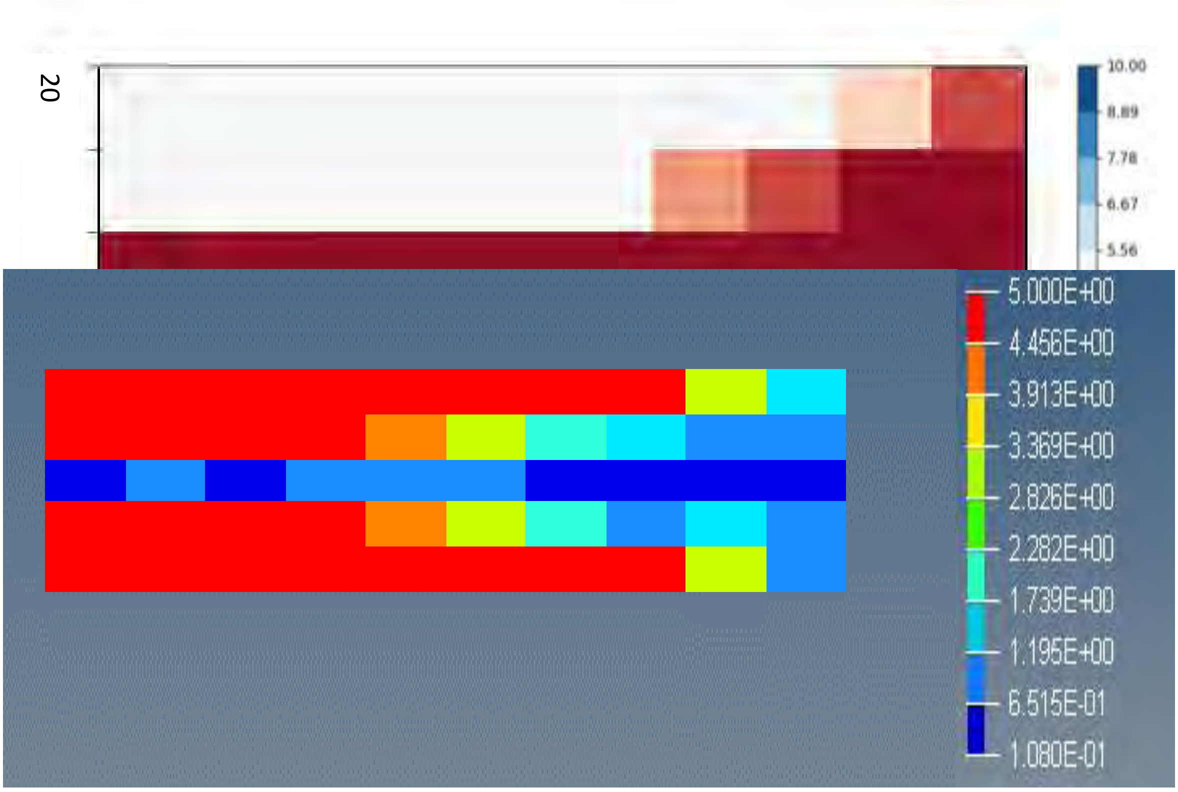


figure 18: Thickness distribution of the 1st ply as resulted from Optitruct

The thickness distribution of the 2nd ply (the UD layer) is shown in the images below. Most of the material is not needed, except from two symmetric areas on the left and one in the middle. This is expected because of the existence of the 1st twill ply, that gathers the stresses in both directions, ending up in relieving the upper ply.

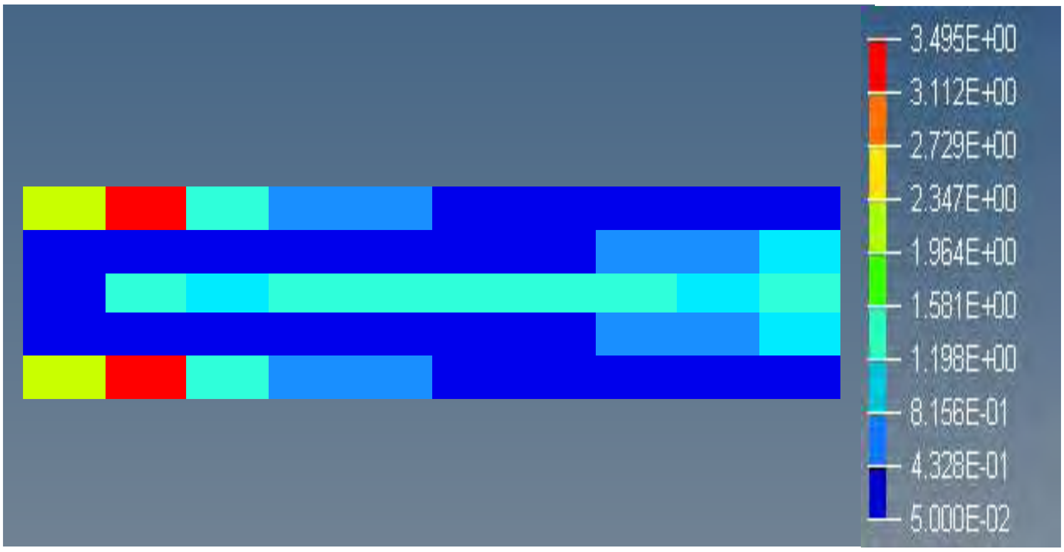
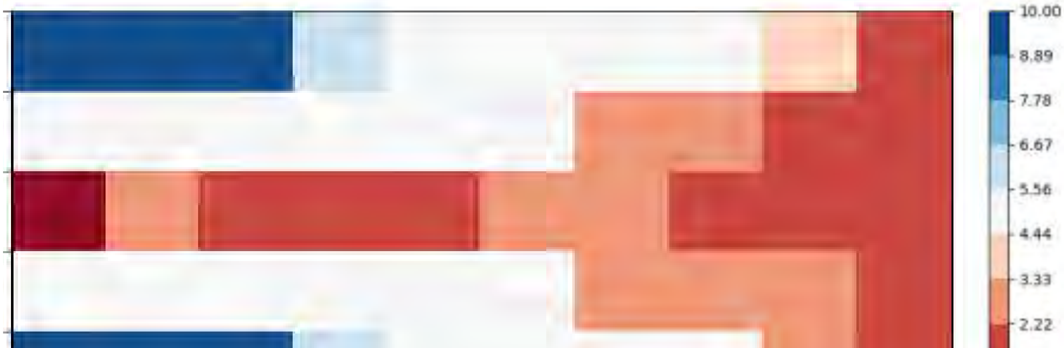


figure 20: Thickness distribution of the 2nd ply as resulted from Optistruct



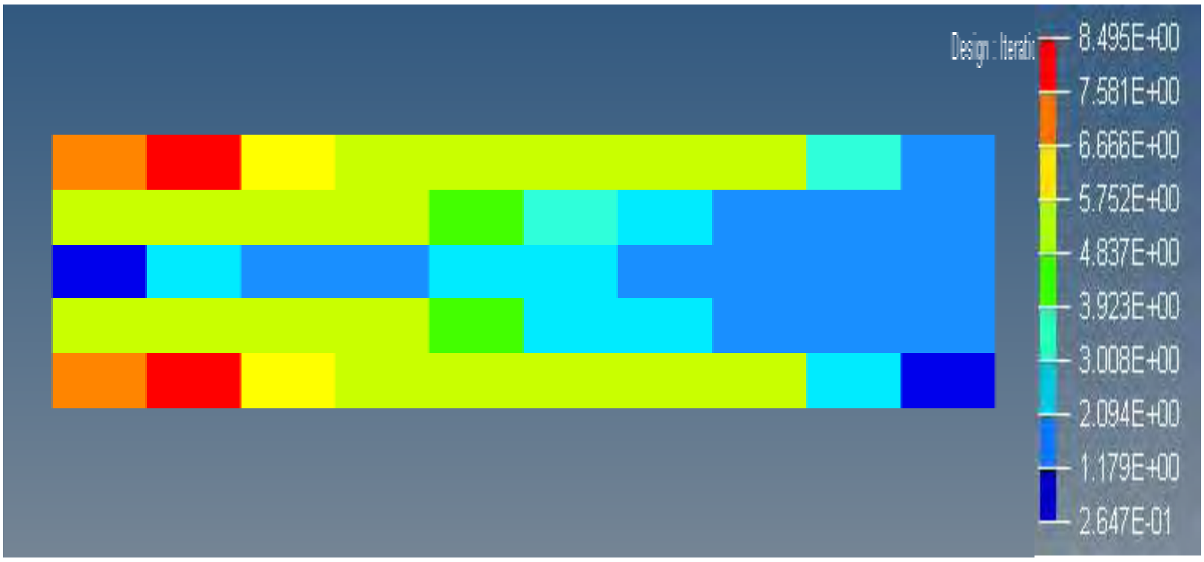


figure 22: Total thickness distribution as resulted from Optistruct

4. Conclusions and Future Work

The scope of this thesis was the composing of an algorithm that would deal with optimizing the size of a laminated structure, via calculation of the elemental thicknesses of each lamina. In order that the algorithm's formulation would be perfectly understood from the reader, it was considered necessary to explain the three basic theories that were used.

In the first chapter, the reader was introduced to the definition of composite materials and laminated structures, including the reasons why they are considered an undeniable asset of the modern industries. We explained the basic manufacturing processes that are used for combining separate materials into unified forms, so as to have a clear image of the manufacturing parameters that may affect the structures as a whole.

Up to next, some common methodologies were presented, in order to give a sight to the many different parameters that may be used for different optimization problems, according to the specific problem's nature. We were focused mostly on structural optimization, the category that includes the Size Optimization, which is the methodology used and thoroughly explained in the current thesis. With the fundamental theoretical and practical tools of SO being understood, we continued with explanation of the theory of Finite Element Analysis, that is an indispensable part of the code, considering that the analysis is realized in elemental level, where specific values must be calculated and be used as inputs for proceeding with the optimization function. The last part of the literature review was the Classical Lamination Theory. It was considered necessary to explain in detail the homogenization process of the elemental material properties that was combined with the Finite Element Analysis in order to result in a formula that initiates from element and ply-level values and ends up with the global values of a laminated structure.

The thesis continues with detailed presentation of the optimization algorithm. The basic technique that is used, including the objective function and the constraints, is explained, as well as all input data that have originated from all pre-mentioned theoretical analyses. Finally, a couple of study cases are presented in order to prove the integrity of the code, by comparison with simulating the same models using a commercial software. The results, as explained above in chapter *3.2*, seemed to be quite similar with the analysis occurred from the external software, so one could adjudge in support of the algorithm's credibility.

As a continuation proposal of the current work, it could be considered useful to modify the algorithm so as to be functional for 3D cases, besides the plate analysis. The philosophy remains the same, but the mathematical tools have to be alternated in order to work for the three dimensional analysis. As a general remark, one could observe that optimization of 3D components made out of composite materials is an all-promising technique that is studied by an all-increasing number of engineering consulting companies, due to the wide demand from numerous kinds of manufacturing industries.

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Appendix 1

Gaussian elimination algorithm written in python:

```
def GaussianElimination(A,b):

  n = len(A)
  M = A

  i = 0
  for x in M:
   x.append(b[i])
   i += 1

  for k in range(n):
   for i in range(k,n):

     if abs(M[i][k]) > abs(M[k][k]):
        M[k], M[i] = M[i],M[k]

     else:
        pass

   for j in range(k+1,n):
       q = M[j][k] / M[k][k]
       for m in range(k, n+1):
          M[j][m] +=  q * M[k][m]

  x = [0 for i in range(n)]

  print("n = ", n)
  print("x = ", x)

  for row in M:

    print(row)

  x[n-1] =float(M[n-1][n])/M[n-1][n-1]
  for i in range (n-1,-1,-1):
    z = 0
    for j in range(i+1,n):
        z = z  + float(M[i][j])*x[j]
    x[i] = float(M[i][n] - z)/M[i][i]

  print x
```

Appendix 2

The optimization formulation for Python's optimizer, *Scipy.Optimize.Minimize:*

```
cons=[]

cons_list=np.array([48])        #This the Constraints' List

for var in cons_list:
    cons.append({'type': 'ineq', 'fun': callsolver(var)})


t0=np.empty(pl*NUMEL)
t0[:]=1                         #Initial values equal 1

bnds=np.zeros((NUMEL*pl,2))
bnds[:,0]=0.000001
bnds[:,1]=1

pr.enable()

#-------------------------MINIMIZER-------------------------------

res=minimize(f,t0,bounds=bnds,constraints=cons,options={'maxiter': 100000})

#-----------------------------------------------------------------
```