University of Thessaly
Department of Civil Engineering
Laboratory of Structural Analysis and Design

The unilateral contact buckling problem of geometrically perfect and imperfect beams

Konstantinos Tzaros
Civil Engineer, M.Sc.

Doctoral Dissertation

Submitted to
the Department of Civil Engineering
in Fulfillment of the Requirements
for the Doctoral Degree

Supervisor: Assoc. Professor Euripidis Mistakidis

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This dissertation is dedicated
to my dearest brother
Charalampos Tzaros
and my dearest friends
Daphne Pantousa and
Kyriaki Georgiadi-Stefanidi

Contents

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1 Introduction

The issue of stability is of great importance in analysis and design of structures due to the fact that structural members may fail before the exhaustion of their material strength. In the area of civil engineering, failure due to stability is mostly encountered in steel structural members under different types of buckling (e.g. flexural buckling, torsional buckling, sway and non-sway buckling of frames etc.). The buckling strength depends on several parameters like, the flexural stiffness of the cross sections, the material yield stress, the boundary conditions, the effective buckling lengths etc. In most cases, the influence of these factors on the buckling strength can be predicted through appropriate methodologies.

Moreover, the ability of a structure to sustain loads which may cause instability effects, depends strongly on the existence of geometric imperfections. In general, geometric imperfections develop in structural members due to a variety of reasons as e.g. manufacturing processes, member handling from the factory to the construction site, etc. It is obvious, that the determination of the type and magnitude of geometric imperfections is a rather difficult task. Taking this fact into account and considering that steel structures are in general sensitive in the presence of geometric imperfections, the stability behaviour is greatly governed by them. For this reason, structural design codes imply the consideration of initial geometric imperfections in the design against buckling.

Even though buckling constitutes a complex problem for the majority of structures, the complexity is increased when unilateral constraints are present. Unilateral boundary conditions are a particular type of supporting conditions where the deflection curve and, consequently, the buckling shape of a structure is obliged to develop in one only direction. This type of buckling is known in the bibliography as constrained buckling. The phenomenon of constrained buckling appears in many engineering applications such as in steel, composite, naval and aerospace structures, in metal forming processes, bioengineering etc.

The treatment of problems involving stability and unilateral contact conditions is usually a difficult task in the field of applied mechanics. The function of the unilateral constraints introduces certain type of nonlinearities in the formulation of the problem, in addition to the inherent material and geometric nonlinearities. Moreover, constraints associated with contact lead to inequalities, making the formulation even more complex. For this reason, computational methods like the Finite Element Method are usually combined with special algorithms in order to handle unilateral contact buckling problems.

The present dissertation deals with the unilateral contact buckling problem of beams. Due to the rapid improvement of modern computer technology, research concerning the development of computational techniques for solving unilateral buckling problems has reached a sufficient level. Therefore, a theoretical study which is able to derive analytical solutions for a class of problems may contribute in many levels to the existing knowledge in the area of the contact buckling problems. The previous fact offered the motivation to study the constrained buckling problem from a different point of view. More specifically, through this dissertation, a theoretical study concerning the contact buckling problem of beams is presented. The study focuses on the calculation of the critical buckling loads of geometrically perfect and imperfect beams in the presence of unilateral supports. This is achieved by means of the fundamental elastic stability theory which is appropriately modified in order to take into account in a mathematically accurate way the unilateral constraints.

A particular feature in the aforementioned mathematical approach is the connection of the buckling problem of beams with a series of theorems and definitions from the area of differential equations and Boundary Value Problems. Under this approach, significant findings concerning the solvability of such problems were revealed. These findings are rather difficult to be discovered by applying computational techniques, proving in some way the importance of the theoretical study in the treatment of complex mechanical problems.

More specifically, the study in the present dissertation is separated into six Chapters. Initially, in Chapter 2, a wide discussion concerning the variety of applications where unilateral contact buckling occurs is displayed. Also in that chapter a review regarding the methodologies and techniques which have been developed until today for the treatment of such problems, is presented.

In the sequel, Chapter 3 covers the issue of buckling for a simply supported beam. In particular, the Boundary Value Problem (BVP) of an axially loaded beam with or without transverse loading is formulated. Through this formulation the concept of “snap” buckling and the concept of “instability” due to the development of extremely large deflections is clearly displayed, leading to the definitions of the **critical** and **instability** loads respectively. Due to the fact that the buckling problem is actually a BVP, the rest part of the chapter is devoted to the mathematical description of the homogeneous and non-homogeneous ordinary BVP. In this description a series of definitions and theorems are given. Especially, the presented Theorem 3.4 is of great importance because it connects the solvability of a non-homogeneous BVP (i.e. buckling of a geometrically imperfect beam) with the solutions of the corresponding homogeneous BVP (i.e. buckling of the geometrically perfect beam). This fundamental theorem is applied in all the cases of constrained buckling examined in this dissertation. In the end of the chapter, specific examples prove that singular points may exist in buckling problems of geometrically imperfect beams. It is essential to notice that these singular points affect the buckling load of a certain structure and are not easy to be detected by the geometrically nonlinear finite element analysis.

In Chapter 4, the unilateral contact buckling problem of a geometrically perfect and imperfect beam in the presence of one intermediate unilateral support, is presented. The homogeneous constrained BVP is initially formulated. The formulation is based on the fundamental elastic stability theory of Euler, appropriately modified in order to take into account the unilateral constraint. The solution is obtained under the separation of the problem into subproblems, according to the possible contact situations (i.e. the constraint may be active, inactive or in neutral contact status). These subproblems are actually the BVPs that correspond to each contact case. The inequality character of these BVPs is maintained in the formulation and in the solution.

Due to the fact that the case of the geometrically perfect beam constitutes an eigenvalue problem, equilibrium is denoted by a bifurcation state and infinite solutions exist. Analytical descriptions are obtained for these solutions which, of course, are valid only under the fulfillment of certain inequality restrictions. In the following part of Chapter 4, the corresponding non-homogeneous BVP is studied. More specifically, first, arbitrary initial geometric imperfections compatible with the function of the unilateral constraint are introduced in the structure. Then, the non-homogeneous constrained BVP is formulated under the same considerations as in the case of the geometrically perfect beam. Following the previous solution procedure, the initial BVP is separated into subproblems, one for each contact situation. The extraction of the solution is not straight forward in this case because, for the different values of the applied load, the problem may be uniquely solvable, unsolvable or solvable with infinite solutions. The issue of solvability is strongly connected with the solutions of the corresponding homogeneous subproblems examined in the case of the geometrically perfect beam. Due to the complexity of the problem, an appropriate calculation procedure is proposed for the treatment of the unilateral buckling of imperfect beams. This procedure is able to detect the critical or singular points in the solution of the problem analytically, without the need to apply any load incrementation scheme. It is noticed that in many cases the calculated buckling load is not the actual load that the beam is able to sustain, due to the fact that the ultimate load is affected by the actual strength of the cross section of the beam. For this reason, in the last part of Chapter 4, the proposed methodology is also equipped with design criteria. Therefore, a complete answer to the matter of unilateral contact buckling of beams with initial imperfections is given.

The unilateral contact buckling problem of the geometrically perfect and imperfect beams is extended in Chapter 5 with the consideration of two unilateral constraints in the formulation of the constrained BVP. Due to the fact that the method presented in Chapter 4 can easily be extended in cases with more than one unilateral supports (the difficulty is attributed only to the extend of the mathematical operations), Chapter 5 considers two unilateral constraints in an “opposite” function mode. The treatment of this specific constrained BVP is based on the methodology described in Chapter 4. Obviously, the main BVP is now separated into nine different constrained subproblems, one for each possible contact case. All the analytically extracted solutions are valid under the fulfillment of certain inequality restrictions which are introduced by the constraints.

The proposed methodology displayed in Chapters 4 and 5 is able to be implemented in a variety of cases. Thus, Chapter 6 and 7 are mainly devoted to the implementation and the better comprehension of the proposed methodology through the demonstration of several examples. The presented examples concern cases with different initial contact situations, various initial geometric imperfections with different shapes and amplitudes and various positions for the unilateral constraints. The aim of the demonstrated examples is to reveal the advantages and the innovative points of the present dissertation.

The arising conclusions and the innovative points of the present research are summarized in the last chapter of the dissertation (Chapter 8). Furthermore, some suggestions for further research are given in that chapter.

Due to the fact that the proposed methodology uses tools from the mathematical area of ordinary differential equations and BVPs, appendices A,B and C at the end of the manuscript provide the fundamentals of the corresponding theories. Most of the information given in the specific appendices concerns definitions and properties of the vector space $L_2(a,b)$ which constitutes a subspace of the well known Hilbert space.

2 Unilateral contact buckling problems in structural analysis - Bibliography Survey

2.1 Introduction

In practical problems, the contact interaction between a deformable structure and the elastic foundation is usually simulated through bilateral boundary conditions. Such models are satisfactory for many engineering applications where the deformable body does not lose the contact with the foundation during the deformation or for cases where the loss of contact is not significant for the overall state of stresses and strains in the system. This result holds for all types of loading (static, dynamic, etc.). A step beyond that simulation is the consideration of models which take into account the mechanical behaviour of soil as elastic subgrade and the interaction between the structure and the soil foundation (see e.g. the works by Vallabhan, 1991a,b; Avramidis and Morfidis, 2005; Morfidis and Avramidis, 2003). In case, where loss of contact occurs, models with bilateral boundary conditions are not reliable and therefore a different type of boundary conditions must be introduced in the formulation of the problem. For example, this could happen when a structure is supported on a foundation which does not provide tensile restraint, such as a plate resting on a rigid foundation. In this case, the plate is possible to detach or debond, but penetration into the foundation is not permitted. Generally, in such cases, it is necessary to consider **unilateral** boundary conditions as part of the solution, since the contact area is not *a priori* known. This type of restraint (or boundary condition) is the well known Signorini-Ficchera unilateral contact condition (Fichera, 1963,1972; Panagiotopoulos, 1985) and has been an area of active research since 1970s. Obviously, the main feature of systems with unilateral constraints is that their displacement in certain positions, is obliged to develop in one only direction.

Problems involving unilateral contact conditions are difficult to be solved due to the inequality character of the arising problems. The complexity of such problems is increased, when the study focuses on the determination of the critical stability points of structures which are vulnerable to buckling. The present dissertation deals with the calculation of such critical points in continuous beams under the presence of unilateral constraints. This issue concerns mostly civil engineering applications, although such simple mechanical systems could be utilized in other engineering applications. Moreover, such models may be used for the simulation, in a simple but reliable way, of more complex systems involving unilateral constraints.

The aim of this chapter is to present engineering applications where unilateral contact buckling phenomena occur, as well as to display the methodologies and techniques which have been used until now for their treatment. Considering the fact that unilateral buckling constitutes a critical type of failure, many researchers have turned their scientific interests in this area.

It is noted that research in the area of unilateral systems usually requires development of numerical techniques due to their complex geometry and their highly nonlinear response. Thus, most of the relevant publications are associated with computational and/or variational methods. However, theoretical studies based on the mathematical treatment of the problem have also been published. In these papers, individual structural members such as plates and beams resting on a tensionless foundation are mainly of interest. Finally, at the end of the chapter, a brief discussion regarding the contribution of the present dissertation to the treatment of the unilateral contact buckling problem is presented.

2.2 Unilateral contact buckling and structural systems

2.2.1 Civil engineering applications

In the area of civil engineering, unilateral contact conditions involving stability phenomena are encountered in steel and composite structures. For example, unilateral local buckling phenomena develop on the thin steel components of composite steel-concrete structures. In these applications concrete plays the role of a restraining medium preventing the free formation of the buckling snap which is forced to develop away from the concrete. Obviously, in this case, the elastic local buckling load is significantly higher than if the rigid medium (concrete) were absent. This issue has been started to be of interest in the last twenty years. More specifically, research in this field has been undertaken by Wright (Wright, 1995), Uy and Bradford (Uy and Bradford, 1995) and Bridge et al (Bridge et al, 1995). The aforementioned researchers studied respectively, the local stability of filled and encased steel sections, composite steel-concrete columns and thin walled tubes filled with concrete. In the sequel, Bradford et al.(Bradford et al, 1998) handled the local buckling problems developing on composite profiled walls while Oehlers et al. (Oehlers et al, 1994) and Uy and Bradford (Uy and Bradford, 1996) studied the unilateral contact buckling problem on composite profiled beams subjected to compressive and bending actions. Recently, a similar work on the unilateral contact buckling problem of lightly profiled skin sheets on concrete-filled composite wall panels and similar members (e.g. composite slabs) has been published by X. Ma et.al (Ma et al, 2006; Ma et al, 2007). In the above publications, computational methods were used, having as a scope to address in an rather approximate way the problem than to treat it in theoretical and complete manner.

Unilateral contact buckling may also develop in applications where thin walled cold formed sections are used, due to their special way of manufacturing. For example, the cold formed steel beam of Fig 2.1 is a particular member which is usually used as platform in scaffold configurations. This type of beams exhibits some particular features, such as the geometry of the cross section and the special embossments developed in the upper plate of the beam. The cross section of this cold formed steel platform is manufactured from a continuous steel sheet which is appropriately folded in order to create two box-shaped compartments at the two edges and a triangular compartment in the middle. The points at which the folded areas come in contact are connected by clinching. In this category of structures the bending loading leads to local buckling phenomena, due to their small thickness and their geometric imperfection sensitivity. The arising local buckling mode at the compressive upper flanges is a constrained type one, due to the existence of the clinching with the other parts of the sheeting which prevents in some way the development of the unconstrained type buckling mode. Fig. 2.2 shows the unilateral contact buckling failure in this type of structures (Tzaros et al 2008; Tzaros and Mistakidis, 2008).

Fig. 2.1. The cross section of the thin walled cold formed steel beam (Tzaros and Mistakidis, 2008).

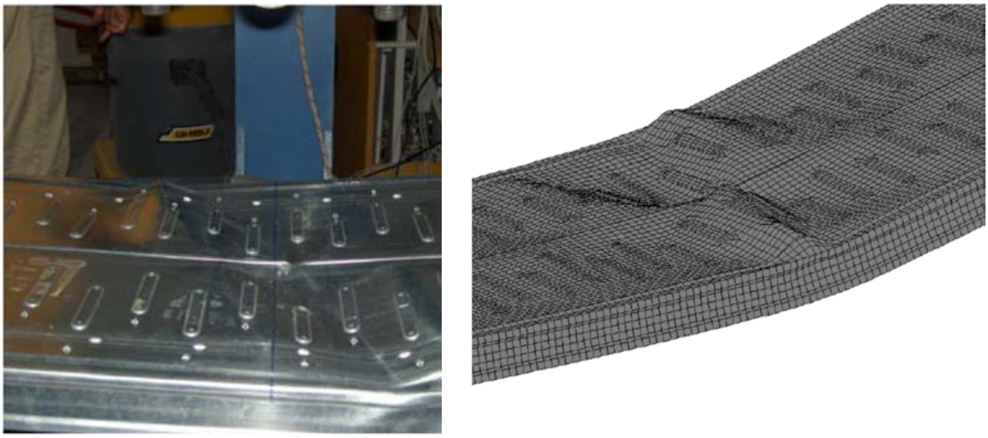

Fig. 2.2. The unilateral contact failure buckling mode of the steel cold formed beam of Fig. 2.1 (Tzaros et al, 2008).

Other examples of unilateral buckling are the upheaval failures of roads and runways (Roorda, 1988) and the buckling of pipelines that are either buried or resting on the seabed (Hobbs, 1981) (Fig. 2.3). Finally, the buckling of rock strata and floating ice sheets should be mentioned.

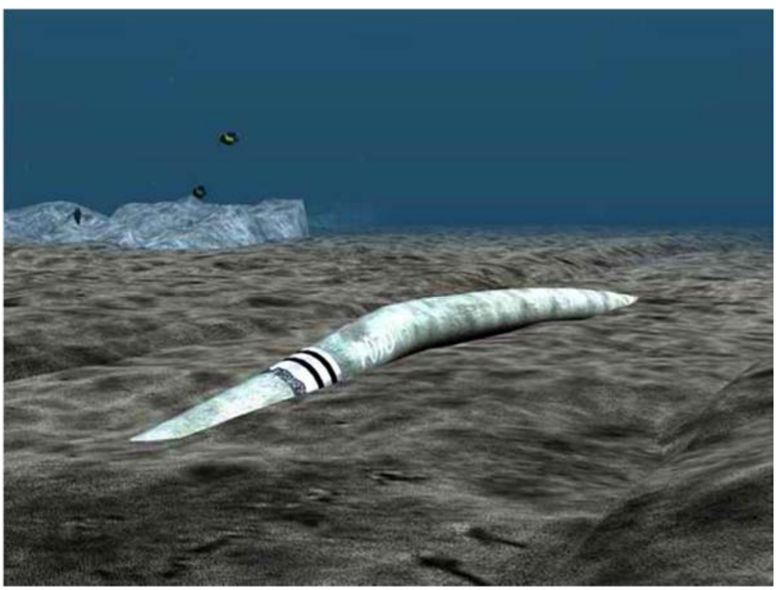

Fig. 2.3. Upheaval buckling of pipelines buried on the seabed (source: Google search).

2.2.2 Mechanical and other engineering applications

There are a lot of applications in the areas of mechanical, aerospace and naval engineering where delamination contact buckling is the predominant type of failure. Structures like aircrafts, automobiles, ships etc. are assembled from components which are being manufactured from laminated materials. For example, Fig. 2.4 shows a delaminated buckling mode of a fiber-metal laminate which is actually a hybrid material consisting of alternating layers of metal and fiber-reinforced prepreg. An example of such material is GLARE (Remmers and Borst, 2001), a combination of aluminum and glass fiber-reinforced epoxy. This type of material is used to construct large parts of the fuselage of the A380 aircraft.

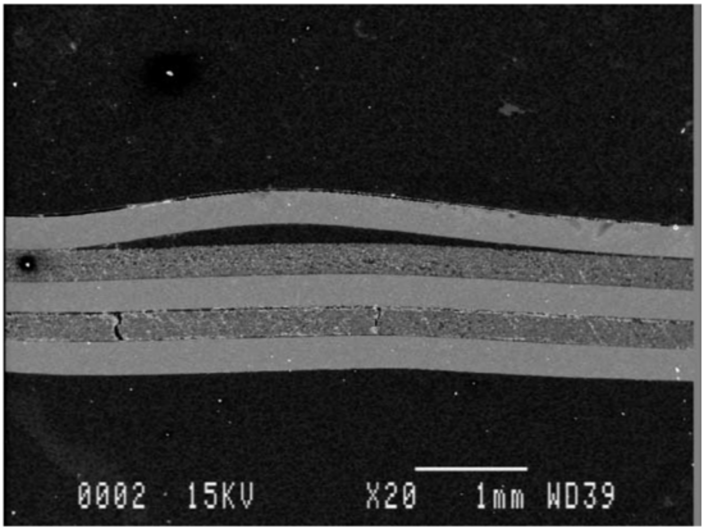

Fig. 2.4. Picture of local unilateralcontact buckling at the top layer of a GLARE specimen after being subjected to a three point bending test (Remmers and Borst, 2001).

Also, in aerospace engineering, a lot of structural components are made of laminated composite plates. The basic problem of these plates is the near-surface delamination (disbond) buckling (Shahwan et al, 1993). Due to the relevantly small thickness ratio of the laminate plate to that of the sublaminate (parent substrate), the sublaminate essentially acts like a rigid surface constraining the out of plane deformations of the plate to be of one sign.

In mechanical engineering, metal forming processes lead to the development of unilateral buckling phenomena in the high stressed metal flanges. These particular problems are even more complex due to the inherent material nonlinearities. Apart from applications encountered in heavy industries, the contact buckling problem at delamination could also arise in thin film coatings such as paints, thermal insulators and electric conductors (Giannakopoulos et al, 1995). These components are used in a variety of applications in microelectronics, aerospace and naval structures, bioengineering etc. It is remarkable that this type of problem arises also in the area of art. More specifically, protecting and restoring old master's paintings is an another area of delamination buckling problem (Giannakopoulos et al, 1995).

2.3 Treatment of unilateral contact buckling problems

2.3.1 Generalities

The most of the previously presented applications where unilateral contact buckling takes place are difficult to be solved, mainly due to the inherent nonlinearities which are introduced by the function of the unilateral constraints and the complex structure geometry. Therefore, in order to study equilibrium and stability of structures with unilateral constraints, two types of nonlinearities, geometric and contact, should be taken into account. The treatment of such problems is based on the variational formulation of the governing differential equations and in the utilization of numerical techniques suitable for handling the arising nonlinear systems of equations. Thus, the first step in the treatment of such problems is the discretization of the continuous system. This is achieved by using an appropriate numerical technique such as the Finite Element Method (FEM), the Rayleigh-Ritz method, the Galerkin method or the Boundary Element Method (BEM). After the discretization of the continuous system, attention should be given in the consideration of the unilateral contact boundary conditions, i.e. to the selection of a proper methodology to handle the contact constraints. As it is well known, the constraints associated with contact are inequalities, thus, particular techniques should be used in order to incorporate contact constraints into the formulation of the problem which, essentially, takes an inequality form. The problem becomes even more complex when the main objective is the computation of the critical stability points of a structure. In this case the traditional path-following methods, like the arc-length method (Riks, 1972; Criesfield, 1981), cannot manage the bifurcation response, since several post critical solutions are likely to develop due to the existence of the unilateral constraints. Thus, these techniques have to be modified appropriately. In the next paragraphs, the most popular techniques associated with the analysis of unilateral systems and the computation of stability points are briefly presented.

2.3.2 Methodologies for the analysis of unilateral systems

In general, there are three options for handling the unilateral buckling problem:

- Transformation of the contact problem into a minimization problem without constraint through the adaption of the usual formulations of structural mechanics (i.e. differentiable functionals and bilateral constraints) to the case of unilateral contact constraints (Adan et al, 1994; Holmes et al, 1999; Silveira and Concalves, 2001; Li and Berger, 2003). The convergence of these procedures, which are of iterative nature, is not guaranted.
- Mathematical programming techniques. This approach allows the solution of the contact problem with or without explicit elimination of the unilateral constraints. The elimination of the constraints is made possible through methods such us Langrange multipliers and the so called "Penalty method" (Simo et al., 1985; Wriggers and Imhof, 1993; Wriggers, 2002; Wriggers and Zavarise, 2004; Fisher and Wriggers, 2005). Besides these techniques, there exist methodologies where the unilateral constraints are maintained in the formulation of the problem. These techniques lead to quadratic programming or linear complementarity problems which can be solved by a variety of mathematical methods, for example, Lemke's or Dantig's algorithms (Lemke, 1968; Ascione and Grimaldi, 1984; Joo and Kwak, 1986; Barbosa, 1986; Silveira, 1995; Koo and Kwak, 1996; Hexiang et al, 1999; Silva et al, 2001; Silveira and Concalves, 2001; Wriggers, 2002; Pereira, 2003; Hollanda and Concalves, 2003). Mathematical programming resembles the force method of structural analysis, thus the local contact loads are set as unknowns and the solution is obtained by minimizing a given function (Chand et al., 1976; Fisher and Melosh, 1987). While this method does not require an iterative solution procedure, the programming and computation time can often be prohibitive.
- Direct substitution of a unilateral buckling mode shape, displacement function or a specified contact surface into the appropriate analysis procedure (i.e. the FEM, the Raleigh-Ritz method, etc.) (Shahwan and Wass, 1993; Ma et al 2006).

2.3.3 Computation of stability points in unilateral contact problems

As it was already mentioned, bifurcation can dominate the response of many structures where unilateral constraints are present. Due to the fact that the bifurcation state leads to solutions which are associated with different branches, several post-critical solutions are possible to appear, because of the different active contact constraints. Thus, special techniques have to be considered in order to calculate the post-critical equilibrium paths. Such algorithms, which have the potential of tracing the complex nonlinear equilibrium paths, have been developed by Wriggers (Wriggers et al, 1987), Stein (Stein et al, 1990), Bjorkman (Bjorkman, 1992), Koo and Kwak (Koo and Kwak, 1996), Silveira and Concalves (Silveira and Concalves, 2001), Tscope et al (Tscope et al, 2003a,b).

It has to be noticed that the treatment of bifurcation problems in the presence of unilateral constraints requires particular considerations (Wriggers, 2006). Near the stability points, the associated eigenvalue problem has to be solved, in order the number of existing branches to be calculated. This is not a simple issue, since the contact constraints are presented by inequalities, thus, in essence, an inequality eigenvalue problem has to be solved. The latter can be approximated by a linearized eigenvalue problem which can be solved utilizing certain results obtained by the significant work of Huy and Werner (Huy and Werner, 1986). The latter is associated with linear variational eigenvalue inequalities.

2.4 The unilateral contact buckling problem of plates and beams

2.4.1 Unilateral contact buckling of plates

A specific type of engineering applications where unilateral contact buckling occurs, employs structures assembled by composite members which, in turn, are composed of several layers. These layers are vulnerable to unilateral buckling under mechanical or thermal loading. This type of buckling can be simulated by treating the distinct layers as elastic plates in a state of unilateral contact. Till now, many researchers have worked on this specific area.

Seide (Seide, 1958) and Do (Do, 1977) were among the first researchers who studied the buckling problem on elastic plates. Seide (Seide, 1958) studied contact effects of infinitely long buckled plates, under simply supported boundary conditions with longitudinal immovable edges. His work has been also extended to a compressive plate on a tensionless rigid foundations by Shahwan and Wass (Shahwan and Wass, 1994; Shahwan and Wass, 1998) and Smith et al (Smith et al, 1999a). The buckling strength of finite size plates with unilateral constraints was considered by Bezine et al. (Bezine et al, 1985), Wright (Wright H.D, 1993) and Smith et al. (Smith et al, 1999a,b) using the finite element method (FEM) and Rayleigh-Ritz approaches. All these studies concerned the cases of linear elastic buckling and they concluded that the constraint increases the buckling load.

Numerical approximations involving stability and postbuckling behavior of plates under unilateral contact constraints imposed by elastic foundation, appear in recent papers by Muradova and Stavroulakis (Muradova and Stavroulakis, 2006), Shen and Li (Shen and Li, 2004), Shen and Yu (Shen and Yu, 2004), Shen and Teng (Shen and Teng, 2004), Hollanda and Concalves (Hollanda and Concalves, 2003). More specifically, Ohtake et al (Ohtake et al, 1980) was among the first researchers who studied the postbuckling behavior of a simply supported square thin plate with unilateral constraints using a finite element scheme coupled with a penalty method. In the sequel, Chai (Chai, 2001) obtained results for the postbuckling behaviour of a clamped thin plate unilaterally constrained by a rigid foundation. On the other hand, Hollanda and Concalves (Hollanda and Concalves, 2003) presented the post buckling analysis of a simply supported thin plate resting on a tensionless elastic foundation. Furthermore, Muradova and Stavroulakis (Muradova and Stavroulakis, 2006) considered the unilateral buckling problem in von Karman plates. The same formulation, based on the higher order shear deformation plate theory with a von Karman type of kinematic nonlinearity, was followed by Shen and Li (Shen and Li, 2004). In their work, the contact postbuckling response of composite laminated plates subjected to thermal loading is for the first time encountered. Additionally, in the same paper, the initial geometric imperfection of the plate is taken into account. In this specific field, assuming von Karman kinematic approximations, Giannakopoulos et al (Giannakopoulos et al, 1995) dealt with the buckling and postbuckling range of a laminated plate which contains delamination. In the latter paper the interaction between postbuckling and local delamination growth is analyzed using an analytical formula for computing the energy release rate along the delamination front. The proposed formulation accounts for geometric nonlinearity and hyperelasticity.

From a more practical point of view, (Wright, 1995), Uy and Bradford (Uy and Bradford, 1996) and Smith et al (Smith et al, 1999b,c) studied the local buckling problem of plates in composite steel-concrete members. Furthermore, results with practical interest have been reported in the work of X. Ma et al (Ma et al, 2006; 2007). In their work, the theoretical and numerical research concerning the initial skin buckling in the behaviour of composite wall panels, was also supported by experiments.

2.4.2 Unilateral contact buckling of beams-Constrained Euler buckling

A lot of practical cases exist, especially in the area of civil engineering, where beam models are suitable in order to describe the buckling behaviour of unilateral systems. Thus, a series of papers associated with the boundary value problem of the buckling of beams under unilateral constraints is encountered in the existing bibliography. Apart from the fact that the findings in these papers could be used for practical purposes, the simplicity of the proposed models provides, among the others, certain advantages. More specifically, analytical formulation and solution of the complex mathematical problem, comparison between theoretical and experimental results, the potential of using the theoretical results as benchmark for numerical methods and quantitatevely characterization of the dependence of the solution on the number and the position of the unilateral constraints, are only some of these advantages.

A significant work with many citations has been presented by Huy and Werner (Huy and Werner, 1985) who studied the linear variational eigenvalue inequalities. Their theoretical work has been applied to the buckling problem of the unilaterally supported beam and it is of great importance because it provides results which can be used to calculate the eigenvalues near stability points in structures with bifurcation response and many possible postbuckling branches.

Earlier, P. Villagio (Villagio, 1978) extended the classical variational theory for eigenvalue problems so as to define the Euler critical load for unilaterally constrained beams. This was achieved by minimizing the Rayleigh quotient on a convex subset of a Hilbert space. In this work, a method for bounding the buckling load in beams with unilateral constraints by comparison with the critical loads obtained by the corresponding unconstrained beams, was proposed. Also, information about the optimal position of the unilateral constraints in order to maximize the critical load as well as the change of the critical load under perturbation of the constraints, was given. Similar research has been presented by M. Papia (Papia, 1988). In his paper a criterion for the calculation of the critical load of continuous beams under unilateral constraints, is proposed. In the latter work, the arising constrained problem is solved by determining the length of the half-wave of the buckled shape, the number of supports involved and their positions with respect to the end of the half-wave. The importance of Papia's work is that it shows the existence of a limit value, below which the critical load is unaffected by the presence of the unilateral supports. The arising results are depicted in a nondimensional diagram which can be used in common practical applications. Results concerning the change of the critical load when beams are resting on elastic foundation has been recently reported by Michalopoulos et al (Michalopoulos et al. 2007). In this specific work, the buckling strength of a cantilevered beam resting on a foundation is investigated and a special fiber-bundle type of beam model is proposed in order to handle the buckling problem.

Recent works on the constrained Euler buckling problem of beams have been published by Domokos et al. (Domokos et al., 1996) and Holmes (Holmes et al., 1999). In these papers the constrained Euler buckling problem of an inextensible beam confined to the plane and subjected to fixed end displacements, is considered. The analytical results from the geometrically nonlinear problem are compared with experimental ones. The experiments have been carried out on steel slender beams. The findings in this research show a rich bifurcation structure with multiple branches in the overall load-displacement curves.

2.5 Contribution to the unilateral contact buckling problem

The present dissertation deals solely with the buckling problem of axially loaded beams in the presence of unilateral supports. As it was already mentioned, beam models may be adequate in the prediction of the critical loads of certain structures. Except that, beam models provide a better insight into the classical mathematical contact buckling problem.

Even though the whole research is focused on the contact buckling problem of beams, many advantages and innovative points exist, which finally contribute to the existing published knowledge in the scientific area of the unilateral contact problems. As it was displayed in the introductory Chapter 1, the main goal of the study is the calculation of the critical buckling loads of geometrically perfect and imperfect beams which are supported by unilateral supports. Through this target the following points are achieved:

- The present work avoids to handle the contact buckling problem numerically. Instead of using variational inequalities for the treatment of the constraints, a more simplified formulation is applied which, nevertheless, maintains the features of a constrained BVP. This formulation is based on the classical Euler's equilibrium method of elastic stability, appropriately modified in order to treat the unilateral constraints.

- The proposed methodology creates a strong connection between the mechanical and the mathematical aspects, of the buckling problem. Taking into account Euler's method and the fact that the buckling problem is actually, from the mathematical point of view, a homogeneous or a non homogeneous BVP, answer to the solvability of the problem could be given just utilizing theorems from the mathematical field of differential equations and BVPs.

- The application of these theorems in various examples localizes the singular points in the obtained solution and specific analytic expressions can be given to them. It is essential to notice that these singular points cannot be located easily in a finite element analysis procedure. An inadequate loading incrementation procedure may "pass" these singular points.

- The calculation of the critical loads and the determination of the singular points is accomplished directly through the utilization of an appropriate calculation procedure which is based on the analytical solutions and the fundamental mathematical theorems, without applying a load incrementation scheme.

- Arbitrary geometrically imperfect systems can be investigated under different initial contact situations.

- Analytical solutions are derived for both the studied cases (i.e. the perfect and imperfect configurations). These solutions can be used as benchmarks in the development of computational methods.

- The particular formulation of the constrained Euler buckling problem of beams in the present dissertation, offers the potential of deriving analytical solutions for a variety of practical problems associated with different end boundary conditions and initial contact conditions. Furthermore, the automatic solution procedure gives the opportunity of studying the influence of the position of the constraints in the value of the critical load.

- The major advantage of the present study, in comparison with related works on the unilateral contact buckling of beams, is that it proposes a methodology of finding the instability and ultimate load of beams in the presence of **initial geometric imperfections.** As the axial loading increases, the bending strength of the beam is decreased due to the interaction between bending moment and axial force. The decrease in the strength is strongly dependent on the shape and magnitude of the initial geometric imperfection. But, except that obvious conclusion, imperfections influence the buckling modes of a beam when unilateral supports are present. Depending on the initial imperfection, the buckling mode as well as the critical buckling load may may be different for two beams with common features (i.e. position of the unilateral supports, material and section properties etc.). This happens due to the fact that each imperfection may activate different contact conditions during the bending deformation. This issue has both theoretical and practical interest and has not been investigated in previous works concerning the unilateral buckling of beams.

3 The BVP of buckling of beams - Mathematical preliminaries

3.1 Introduction

As it was briefly mentioned in Chapter1, the aim of the present dissertation is to propose a certain analytical methodology for the treatment of the buckling problem of continuous beams in the presence of unilateral supports and arbitrary initial geometric imperfections, which is based on the fundamentals of the elastic stability theory. The study requires the formulation of the buckling Boundary Value Problem (BVP), which describes the bending behavior of the beam by means of the second order bending theory. Undoubtedly, the direct consideration of the unilateral constraints in the formulation of the governing differential equations of the BVP, creates a series of mathematical difficulties (Wriggers, 2006). In most practical applications the formulation of the problem in an analytical way is rather impossible, therefore variational methods must be employed in order to solve the BVP numerically in an approximate way. Even in that case, the classical variational techniques confront many difficulties which arise from the unilateral constraints. To this end, evolutionary mathematical theories and algorithms have been proposed by some researchers, leading however to a much more complex formulation (Wriggers et al, 1987; Stein et al, 1990; Bjorkman, 1992; Koo and Kwak, 1996; Silveira and Concalves, 2001; Tscope et al, 2003a,b).

The proposed methodology avoids the numerical treatment of the certain constrained BVP by considering the basic differential equations of a beam in bending and adapting them appropriately in order to take into account the unilateral contact conditions. For this reason, a brief discussion of the classical buckling problem is attempted in the following. Initially, the well-known fourth order governing differential equation is formulated which is based on a theory that takes into account the effect of the deflections and of the unavoidable change of the geometry of the structure on the equilibrium conditions. Therefore, the basic assumptions of the second order bending theory should be given first. Then, the homogeneous and non-homogeneous stability BVP is described. The whole description lies on the limits between the mathematical and the mechanical disciplines. Apart from the classical buckling description dealing mainly with the structural problem which is encountered in the most classical textbooks (Timoshenko and Gere, 1963; Brush and Almorth, 1975; Bazant and Cedolin, 1991), a more mathematical description will be displayed here in order the reader to be acquainted with the mathematical concept of buckling. The latter includes theorems related with the general BVP of ordinary differential equations which will be very helpful in the following chapters. More details concerning the mathematical description of the BVP can be found in Appendix A.

3.2 The basic assumptions of the second order bending theory

In order to calculate the critical load of axially loaded beams in the framework of the second-order bending theory (linear elastic stability theory), the fundamental equilibrium equations are formulated in the deformed configuration (geometric nonlinearity). The formulation of the governing differential equation (which includes the equilibrium equations, the constitutive law and the compatibility law) lies on the following assumptions (Timoshenko and Gere, 1963; Bazant and Cedolin, 1991; Kounadis, 1997):

- The material of the under consideration mechanical system is supposed to be homogeneous, isotropic and linear elastic obeying the Hooke's law.
- The stress-strain law is the same, both in tension and compression.
- The Bernoulli-Navier assumption holds, i.e. during bending the cross-section of the beam remains plane and normal to the deformed axis.
- The transverse loading passes through the shear center of the cross-section and is parallel to one of the centroidal inertia axes. Therefore, twisting or torsion of the considered cross-section about the axis of the member is avoided.
- The problem is formulated within the context of the theory of small deflections. More specifically, the axial strain ε attributed to the axial displacement u and the transverse displacement w are considered small enough compared to the cross-section dimensions. Therefore, the following relations hold for the axial strain ε and the curvature k of the beam:

$$\varepsilon = \frac{du(x)}{dx} = u'(x) \tag{3.1}$$

$$k = -\frac{d^2 w(x)}{dx^2} = -w''(x) \tag{3.2}$$

It is noted that the description of the curvature k of the beam according to relation (3.2), has been extracted with the additional assumption that the beam is incompressible (or inextensional) i.e. the length of the beam does not change during the bending deformation. This assumption is used only for the description of the curvature of the beam so that the simple formula of relation (3.2) is extracted. More complicated descriptions of the curvature are encountered in the nonlinear theory of elastic stability (Trogger and Steindl, 1991). It also has to be noticed that in the above relations x' and x'' denote the first and second derivatives.

- The shear deformation is neglected.
- The critical buckling loads are calculated considering only the bending deformation. The possible axial deformations arising from the buckling phenomenon are neglected[1].

3.3 The reference systems for the analysis

For the formulation of the governing differential equations which describe the bending behavior of beams according to the second-order bending theory, the reference system of Fig. 3.1 is considered. The beam is separated by the unilateral constraint into two spans. The two spans are equipped with the coordinate systems x_1, w_1 and x_2, w_2 as it is shown in Fig. 3.1, where x_1, x_2 measure the position along the axis of the beam and w_1, w_2 denote the transverse deflections of the beam in the two spans. The positive internal forces are also displayed in Fig. 3.1.

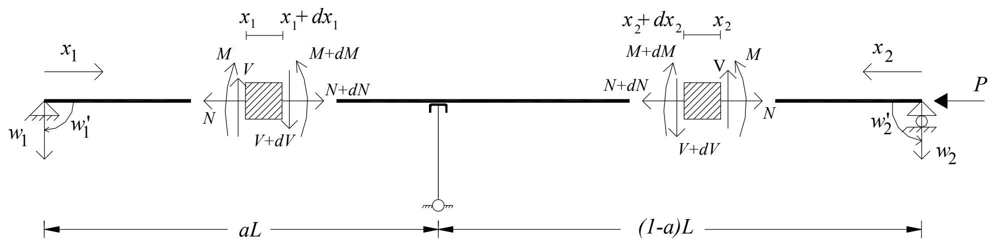

Fig. 3.1 The considered conventions for the positive displacements and internal forces.

[1]In reality, at the moment that the buckling phenomenon occurs, the bending deflection is accompanied by an axial deformation. This axial deformation is neglected in the formulation of the governing differential equation in the sense of equilibrium of forces (Euler's equilibrium method). It should be noticed that if an energy approach with respect to the same principles of the elastic stability theory is applied, then the consideration of the axial deformation is mandatory in order the equilibrium equation which arises from the energy criterion to be valid (obviously the axial force should have the potential of producing work).

3.4 The concept of buckling for a simply supported beam – Formulation of the BVP - Bifurcation state

Let us consider the slender simply supported geometrically perfect beam (i.e. without any initial geometric imperfection) of Fig. 3.2, subjected to an axial compressive load P. Let E be the Young's modulus of elasticity and I the moment of inertia of the cross section with respect to the axis of bending (in plane bending).

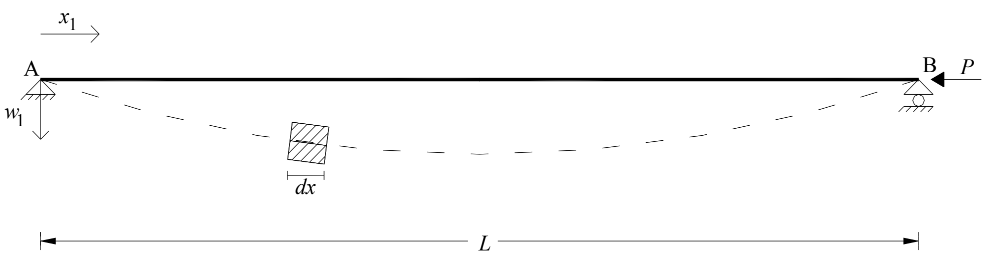

Fig. 3.2 The buckling problem of the geometrically perfect beam.

According to the principles of the first order bending theory, the beam will be stressed and shortened axially due to the action of the compressive load. As it is well known, there exists a certain value of the compressive load P (critical load) for which the geometry of the beam will take a curved shape (buckling shape) different from the straight line configuration. In order to calculate this critical load and the corresponding buckling shape, it is assumed that the beam can be in equilibrium in a curved deformed configuration. In this state, the governing differential equation describing the bending behavior can be formulated. If this equation admits a solution for a certain value of the compressive load P, then the initial assumption is correct and, as a result, the beam has the ability to buckle and equilibrate in a curved configuration.

Let us consider the infinitesimal element of Fig.3.3 with length dx in its deformed configuration. For convenience, the shear components are taken into account by means of the forces V which are perpendicular to the undeformed axis of the beam. For this element three sets of equilibrium equations can be derived:

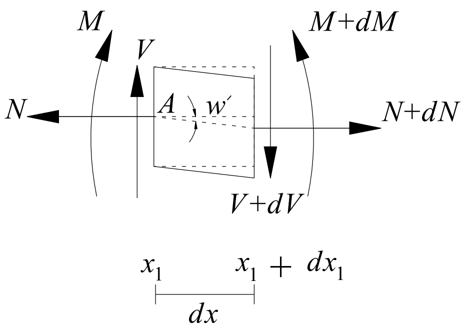

Fig. 3.3 Body force diagram for an infinitesimal element dx of the beam.

- Equilibrium of horizontal forces:

$$\mathrm{N} + dN - N = 0 \Rightarrow dN = 0 \Rightarrow N(x) = \text{constant} \tag{3.3}$$

Applying the boundary condition at the right end (roll support) of the beam, the following equation is derived:

$$N(L) = -P \Rightarrow N(x) = -P \tag{3.4}$$

- Equilibrium of vertical forces:

$$V + dV - V = 0 \Rightarrow dV = 0 \Rightarrow V(x) = \text{constant} \tag{3.5}$$

- Equilibrium of moments at the point A:

$$M - (M + dM) + (V + dV)dx - (N + dN)\frac{dw}{dx}dx = 0 \Rightarrow$$

$$\Rightarrow -dM + Vdx + dVdx - N\frac{dw}{dx}dx - dN\frac{dw}{dx}dx = 0 \tag{3.6}$$

Neglecting in the above equation the terms with higher order differentials, the following equation is derived:

$$-dM + Vdx - N\frac{dw}{dx}dx = 0 \Rightarrow V = \frac{dM}{dx} + N\frac{dw}{dx} \tag{3.7}$$

Differentiating twice the two parts of equation (3.7) and taking into account relations (3.4) and (3.5), the above equation is transformed into the following ordinary second order homogeneous differential equation:

$$\frac{d^2M(x)}{dx^2}+P\frac{d^2w(x)}{dx^2}=0\,. \tag{3.8}$$

Recall now the following basic relation from the theory of elasticity which connects the bending moment $M(x)$ and the second derivative of the transverse displacement $w(x)$:

$$M(x)=-EI\frac{d^2w(x)}{dx^2}\,. \tag{3.9}$$

Using the above, the differential equation (3.8) is transformed into a general differential equation of fourth order which takes into account indirectly the linear elastic constitutive law and the compatibility conditions and is therefore able to describe the bending behavior of the beam:

$$\frac{d^4w(x)}{dx^4}+\frac{P}{EI}\frac{d^2w(x)}{dx^2}=0\,. \tag{3.10}$$

The ordinary homogeneous fourth order differential equation (3.10) constitutes the governing equation for beams in bending in the framework of the second-order bending theory. It is noticed that this equation is very convenient for the analysis of systems with various boundary conditions. The general solution of the above equation has the form:

$$w(x)=A\cos kx+B\sin kx+Cx+D \tag{3.11}$$

where:

$$k=\sqrt{\frac{P}{EI}}\neq 0\,. \tag{3.12}$$

The constant coefficients A,B,C,D of equation (3.11) are determined by applying the boundary conditions of the problem. In the studied case the transverse deflection and the bending moment at the two ends of the beam are equal to zero. Therefore, the following relations can be used to determine the coefficients A,B,C,D:

- Zero bending moment at the position of the support (point A)

$$EIw''(0) = 0 \Rightarrow A = 0 \tag{3.13}$$

- Zero vertical displacement at the position of the support (point A)

$$w(0) = 0 \Rightarrow A + D = 0 \Rightarrow D = 0 \tag{3.14}$$

- Zero bending moment at the position of the rolling support (point B)

$$w''(L) = 0 \Rightarrow B \sin kL = 0 \tag{3.15}$$

- Zero vertical displacement at the position of the rolling support (point B)

$$w(L) = 0 \Rightarrow B \sin kL + CL = 0 \Rightarrow C = 0 \tag{3.16}$$

The demand to have a non trivial[2] solution results to $B \neq 0$. Consequently:

$$\sin kL = 0 \,. \tag{3.17}$$

Equation (3.17) has an infinite number of solutions having the following form:

$$k = \frac{n\pi}{L} \Rightarrow P_n = \frac{n^2\pi^2 EI}{L^2}, \quad n = 1,2,3...... \tag{3.18}$$

Therefore, infinite values of axial loads exist, for which the beam can be in equilibrium in a curved deformed configuration different from the straight line one. The phenomenon for which the beam suddenly jumps from the initial straight line equilibrium configuration to a new curved one, is called buckling. Relation (3.17) constitutes the buckling equation while relation (3.18) gives the buckling loads (eigenvalues). The corresponding buckling shapes (eigenmodes) are practically the infinite solutions of the governing differential equation (3.10), calculated individually for each eigenvalue from relation (3.11). In the framework of the elastic stability theory, only the buckling loads and the shape of the buckling modes can be determined. The magnitude of the deflections which develop at the moment that the “violent” buckling occurs cannot be calculated by this specific theory. From an engineering point of view, only the smallest buckling load (for $n = 1$), the so-called Euler critical load, is of interest in real applications. Fig. 3.4

[2] The case of the trivial solution corresponds to the obvious zero solution, i.e. $A = B = C = D = 0$ which describes the undeformed configuration.

represents the equilibrium path of the beam as a function of the transverse deflection w. For $P < \frac{\pi^2 EI}{L^2} = P_{cr}$ the beam can be in equilibrium in the straight line configuration. For $P = P_{cr} = \frac{\pi^2 \mathrm{EI}}{L^2}$ two different equilibrium modes exist (bifurcation state), the straight line mode, which is unstable and the curved mode[3].

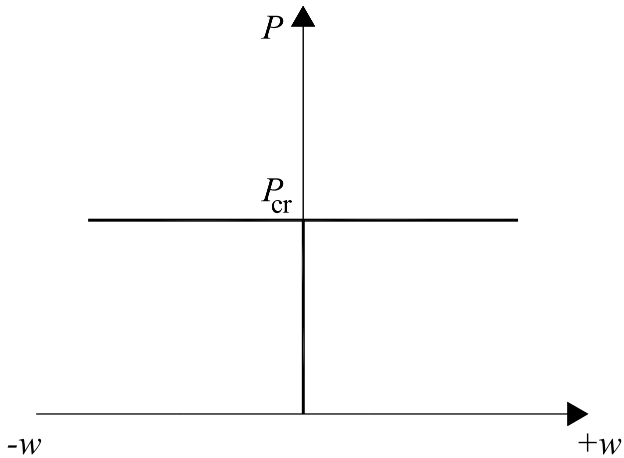

Fig. 3.4 Equilibrium path of a geometrically perfect beam .

3.5 The critical buckling load of a simply supported beam with transverse loading

In the previous paragraph the studied beam was assumed to be geometrically perfect (i.e. without any initial geometric imperfections). Furthermore, the beam was subjected only to axial loading without any kind of transverse loading acting on it. Obviously, a different behavior is expected in case where geometric imperfections are present or when the beam is subjected to transverse loading.

Let us consider the same slender geometrically perfect simply supported beam of (Fig. 3.2). The beam is subjected to axial compressive load P considered as variable and to a constant uniformly distributed load q (Fig. 3.5).

[3]The elastic stability theory has not the ability to decide if the curved mode is stable or unstable. This question can be answered either numerically, using geometric nonlinear finite element analysis (Bathe,1996;), either with the application of superior analytical theories of static stability (e.g. Trogger and Steindl, 1991) or dynamic stability (e.g. Sophianopoulos, 1996) .

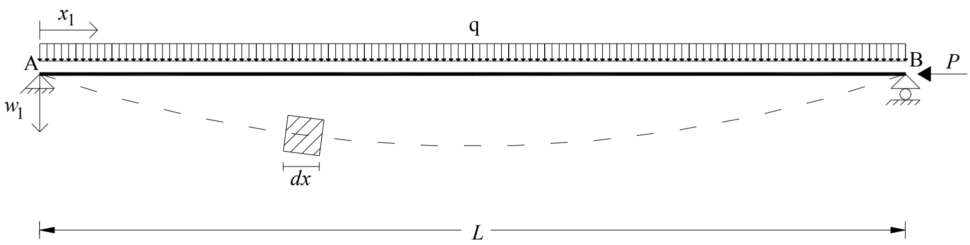

Fig. 3.5 The buckling problem of the geometrically perfect beam subjected to axial and transverse loading.

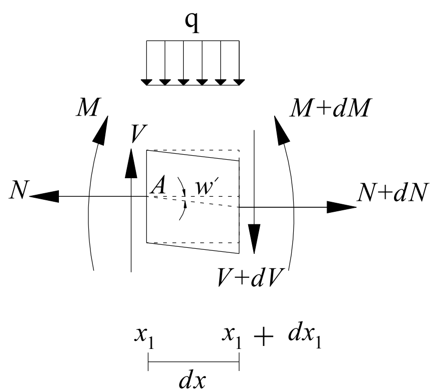

Fig. 3.6 Body force diagram for an arbitrary finite element dx of the beam (case with transverse loading).

Taking into account the contribution of the transverse load (Fig.3.6), the new equilibrium equations can be written as:

- Equilibrium of vertical forces

$$V + dV - V + qdx = 0 \Rightarrow \frac{dV}{dx} = -q \tag{3.19}$$

- Equilibrium of moments at point A

$$M-(M+dM)+(V+dV)dx-(N+dN)\frac{dw}{dx}dx-qdx\frac{dx}{2}$$
$$\Rightarrow -dM+Vdx+dVdx-N\frac{dw}{dx}dx-dN\frac{dw}{dx}dx-qdx\frac{dx}{2}=0 \tag{3.20}$$

Concerning the equilibrium of the horizontal forces, relations (3.3) and (3.4) are still valid. Neglecting the terms with high order differentials, (3.20) yields the following equation:

$$-dM+Vdx-N\frac{dw}{dx}=0\Rightarrow V=\frac{dM}{dx}+N\frac{dw}{dx}. \tag{3.21}$$

Obviously, differentiating twice the two parts of equation (3.21) and taking into account relations (3.4) and (3.19), the above equation is transformed into the following ordinary second order non-homogeneous differential equation:

$$-\frac{d^2M(x)}{dx^2}+P\frac{d^2w(x)}{dx^2}=q. \tag{3.22}$$

Following the procedure of the previous paragraph, relation (3.22) leads to the following governing fourth order non-homogeneous differential equation:

$$EI\frac{d^4w(x)}{dx^4}+P\frac{d^2w(x)}{dx^2}=q. \tag{3.23}$$

The above equation is an ordinary non-homogeneous differential equation of fourth order. Therefore, the solution of the latter results from the addition of a particular solution to the solution of the corresponding homogeneous equation (general solution). The general solution has the form of relation (3.11) while the particular solution is given by the next equation:

$$w_p=Gx^2, \tag{3.24}$$

where the coefficient G can be determined so that $w_p=Gx^2$ satisfies equation (3.23). Therefore, the substitution of relation (3.24) into (3.23) gives:

$$G=\frac{q}{2P}. \tag{3.25}$$

Then, the coefficients of the general solution (3.11) can be determined through the utilization of the boundary conditions of the problem:

$$w''(0)=0 \Rightarrow A=\frac{qEI}{P^2} \tag{3.26}$$

$$w(0)=0 \Rightarrow A+D=0 \Rightarrow D=\frac{-qEI}{P^2} \tag{3.27}$$

$$w''(L)=0 \Rightarrow Bk^2\sin kL=\frac{q}{P}(1-\cos kL) \tag{3.28}$$

$$w(L)=0 \Rightarrow A\cos kL+B\sin kL+CL+D+\frac{q}{2P}L^2$$

$$\Rightarrow C=\frac{1}{L}\left(-A\cos kL-B\sin kL-D-\frac{q}{2P}L^2\right). \tag{3.29}$$

A very interesting conclusion arises from the above equations connected with the mathematical aspects of the problem. For any value of the axial load $P<\frac{\pi^2\mathrm{EI}}{L^2}$ the certain boundary value problem has a **unique** solution which can be expressed as:

$$w(x)=\frac{q}{2P}\cos kx+\frac{qEI}{P^2}\left(\frac{1-\cos kL}{\sin kL}\right)\sin kx+$$

$$+\frac{q}{2LP}\left((1-\cos kL)\left(1-\frac{2EI}{P}\right)-L^2\right)x+\frac{q}{2P}\left(L^2-1\right). \tag{3.30}$$

This results from the fact that for any value of $kL<\pi$ (e.g. $P<\frac{\pi^2\mathrm{EI}}{L^2}$) the boundary conditions are satisfied and therefore the coefficients A,B,C,D can be determined uniquely. When the parameter k takes the value $\frac{\pi}{L}$ (e.g. $P=\frac{\pi^2\mathrm{EI}}{L^2}$), it is noticed that the boundary condition (3.28) cannot be satisfied for any value of $B\neq 0$, due to the fact that the term of the right hand side is always positive and cannot become zero. Therefore, the studied BVP is not solvable for this certain value of load. It is interesting to notice that this value of load is solution of the corresponding homogeneous problem and more specifically, it constitutes the critical load for which the geometrically perfect beam buckles instantaneously.

Studying the problem from a more engineering point of view, it is clearly concluded that when the axial load P approaches the aforementioned critical value, the second term of equation (3.30) tends to infinity while the other terms have a finite value. Thus, the values of the deflection curve tend to infinity. In this case, the beam does not buckle instantaneously (bifurcation state) but it deflects progressively. When the load tends to the critical load, the deflections become disproportionately large, indicating the failure of the beam. As it will be shown in the following paragraphs, similar conclusions can be inferred for the case of the geometrically imperfect beam which also leads to a non-homogeneous BVP. Generally, it seems that a strong connection between the homogeneous boundary value problem and the non-homogeneous one, exists. This connection will be more clearly demonstrated in the next section.

3.6 Mathematical description of the homogeneous and non-homogeneous ordinary BVP- Essential theorems and definitions

3.6.1 Generalities

In the previous sections two types of mechanical problems were studied, where the bending and stability behavior of the system is dominated by the presence of the axial compressive force. The formulation of these problems in the framework of the second-order bending theory led to a certain differential equation of fourth order with an equal number of boundary conditions, which should be fulfilled. It was clearly concluded that some problems may have a solution and others may have not. Furthermore, in some cases infinite solutions may appear. On the other hand, the mechanical behavior of systems described by non-homogeneous differential equations, seems to be affected by the solutions of the corresponding homogeneous problems.

The governing differential equations (3.10) and (3.23) together with the linear homogeneous boundary conditions can be put in the general form (Rektorys, 1994):

$$\mathrm{M}(w) - \lambda \mathrm{N}(w) = f(x) \tag{3.31}$$

$$a_{i0} w(a) + b_{i0} w(b) + a_{i1} w'(a) + b_{i1} w'(b) + ... + a_{i,2m-1} w^{(2m-1)}(a) + b_{i,2m-1} w^{(2m-1)}(b) = 0\,,$$
$$i = 1,2,3....,2m \quad .(3.32)$$

In the above, a,b are the boundaries of a closed interval $[a,b]$ and $a_{i0}, b_{i0}, ...$ are real constants. Moreover $w^{(2m-1)}$ denotes the $2m-1$ derivative of the function w. Also, the function $f(x): \mathbb{R} \to \mathbb{R}$ describes the non-homogeneous term. Relation (3.31) is the governing differential equation while equations (3.32) represent the boundary conditions which should be equal in number to the order of the differential equation so that the whole problem is well posed. In equation (3.31) $\mathrm{M}(w), \mathrm{N}(w)$ are self-adjoint[4] expressions with respect to the differential operators M, N having orders $2m$ and $2n$ respectively, with $m > n$. These operators are defined in the domain[5] D_A and have the following form:

$$\mathrm{M}(w) = \sum_{k=0}^{m} (-1)^k \left[h_k(x) w^{(k)} \right]^{(k)} = (-1)^m \left[h_m(x) w^{(m)} \right]^{(m)} +$$

$$+ (-1)^{m-1} \left[h_{m-1}(x) w^{(m-1)} \right]^{(m-1)} + ... + (-1)^1 \left[h_1(x) w' \right]' + h_0(x) w \tag{3.33}$$

$$\mathrm{N}(w) = \sum_{k=0}^{n} (-1)^k \left[g_k(x) w^{(k)} \right]^{(k)} = (-1)^n \left[g_n(x) w^{(n)} \right]^{(n)} +$$

$$+ (-1)^{n-1} \left[g_{n-1}(x) w^{(n-1)} \right]^{(n-1)} + ... + (-1)^1 \left[g_1(x) w' \right]' + g_0(x) w . \tag{3.34}$$

Relations (3.31)-(3.34) refer to a closed interval $\left[a,b\right]$. In this interval the real functions $h_k(x), g_k(x)$ are continuous and have k continuous derivatives. Additionally, $h_m(x) \neq 0, \mathrm{g}_n(x) \neq 0$ in the interval $\left[a,b\right]$. The boundary conditions described by equations (3.32) are supposed to be linearly independent and the real

[4]Consider the linear ordinary differential equation $f_n(x)w^{(n)} + f_{n-1}(x)w^{(n-1)} + ... + f_1(x)w' + f_0(x)w = f(x)$, briefly in the form $L(w) = f(x)$, where L is a linear differential operator of the $n-th$ order. Expression $K(w) = (-1)^n \left[f_n(x) w \right]^{(n)} + (-1)^{n-1} \left[f_{n-1}(x) w \right]^{(n-1)} + ... + \left[f_2(x) w \right]'' - \left[f_1(x) w \right]' + f_0(x) w$ constitutes the adjoint expression to expression $L(w)$ (and K is the adjoint differential operator to the operator L). If $L(w) = K(w)$ for each n times differentiable function w, then expression $L(w)$ (and the differential operator L) is called self-adjoint (Rektorys, 1994; Rektorys, 1999 Kovach, 1984).

[5] D_A is a subset of the space C^{2m} of all the continuous functions including their derivatives up to the order of $2m$ in a closed region $\overline{\Omega}$.

constant coefficients $a_{i0}, b_{i0}, \ldots$ are not simultaneously equal to zero for any of the equations (3.32).

Therefore, if the real number λ in equation (3.31) is given, the problem of finding the solution $w(x)$ of equation (3.31) satisfying the boundary conditions (3.32), is called a Boundary Value Problem (BVP). In case where $f(x) \neq 0$, the BVP is called non-homogeneous while in the opposite case it is called homogeneous.

Example 3.1

The equation (3.10) in section 3.4 can be written as:

$$\frac{d^4 w(x)}{dx^4} + \frac{P}{EI}\frac{d^2 w(x)}{dx^2} = 0 \, . \tag{3.35}$$

The above equation has the form of the general equation (3.31) where:

$$\mathrm{M}(w) = \frac{d^4 w(x)}{dx^4} = w^{(4)}(x) \tag{3.36a}$$

$$\mathrm{N}(w) = -\frac{d^2 w(x)}{dx^2} = w''(x) \tag{3.36b}$$

$$\lambda = \frac{P}{EI} \, . \tag{3.36c}$$

The differential operators M, N are of $4th$ and $2nd$ order respectively, thus $2m = 4$ and $2n = 2$. Therefore, using relations (3.33),(3.34) it is proved that the expressions $\mathrm{M}(w)$, $\mathrm{N}(w)$ of (3.36) are self-adjoint expressions of $4th$ and $2nd$ order respectively, due to the fact that the following relations hold:

$$\mathrm{M}(w) = \sum_{k=0}^{m} (-1)^k \left[h_k(x) w^{(k)} \right]^{(k)} = \sum_{k=0}^{2} (-1)^k \left[h_k(x) w^{(k)} \right]^{(k)} =$$

$$= (-1)^1 [h_1(x) w']' + (-1)^2 [h_2(x) w'']'' = (-1)^1 [0 \cdot w']' + (-1)^2 [1 \cdot w'']'' = \frac{d^4 w(x)}{dx^4} \tag{3.37}$$

$$\mathrm{N}(w) = \sum_{k=0}^{n} (-1)^k \left[g_k(x) w^{(k)} \right]^{(k)} = \sum_{k=0}^{1} (-1)^k \left[g_k(x) w^{(k)} \right]^{(k)} =$$

$$= (-1)^1 [g_1(x) w']' = (-1)^1 [1 \cdot w']' = -\frac{d^2 w(x)}{dx^2}, \tag{3.38}$$

with $h_1(x) = 0$, $h_2(x) = 0$, $h_3(x) = 0$.

In the following paragraphs several theorems and definitions related with the homogeneous and the non-homogeneous BVP will be referred. These theorems mostly concern the solvability and the properties of the derived solutions. As it will be clearly demonstrated, there is a strong connection between the homogeneous and non-homogeneous BVP. The following theorems will be very useful in the formulation of the proposed methodology for solving the unilateral contact buckling problem of a continuous beam in the presence of initial geometric imperfection (Chapters 4,5, and 7).

3.6.2 The homogeneous BVP

3.6.2.1 Eigenvalues and eigenfunctions

When the function $f(x)$ of equation (3.31) is equal to zero, the latter differential equation takes the form:

$$\mathrm{M}(w) - \lambda \mathrm{N}(w) = 0 . \tag{3.39}$$

In this case the homogeneous problem defined by (3.39) and the boundary conditions, (3.32) have to be solved. It is obvious that for $\lambda = 0$ the BVP has the "zero" solution $w(x) = 0$, which constitutes the trivial solution of the certain problem and it is, of course, of no interest. If there exists $\lambda \neq 0$ so that the BVP (3.39),(3.32) has solutions different from the trivial one, then these values of λ are called eigenvalues, the corresponding solutions constitute the eigenfunctions and the BVP is referred as the"eigenvalue problem". Definitions and theorems will be given in the following which concerns the solvability of the general BVP and the properties of eigenvalues and eigenfunctions.

3.6.2.2 Definitions and theorems

Initially, the symmetric and positive eigenvalue problem will be defined. To this end, the comparison (or test, or trial) functions (Rektorys, 1975; Bathe, 1996; Gosz, 2005) should be introduced.

Definition 3.1

A real function $w: \mathbf{R} \to \mathbf{R}$ is called a comparison function of the eigenvalue problem (3.39),(3.32) if it has $2m$ continuous derivatives in the interval $[a,b]$ and satisfies the given boundary conditions (3.32). Thus, the comparison function belongs to the set $D_A = \{w : w \in C^{(2m)}(\bar{\Omega})\}$. The comparison functions are sufficiently smooth and satisfy the given boundary conditions. It is noticed that there is no need for a comparison function to satisfy the differential equation (3.39).

Definition 3.2

The eigenvalue problem (3.39), (3.32) is called symmetric, if for any comparison functions $u(x)$, $v(x)$ the following relations are fulfilled:

$$\int_a^b [u\mathrm{M}(v) - v\mathrm{M}(u)]dx = 0 \tag{3.40}$$

$$\int_a^b [u\mathrm{N}(v) - v\mathrm{N}(u)]dx = 0\,. \tag{3.41}$$

Definition 3.3

The eigenvalue problem (3.39), (3.32) is called positive, if for any non-zero comparison function $u(x)$ the following relations hold:

$$\int_a^b u\mathrm{M}(u)dx > 0 \tag{3.42}$$

$$\int_a^b u\mathrm{N}(u)dx > 0\,. \tag{3.43}$$

In general, it is easy to prove if a certain BVP is symmetric and positive just applying the rule of integration by parts. The following example is a typical one. In cases where this is rather difficult, the well known Green's formula may be applied.

Example 3.2

The eigenvalue problem of section 3.4 is considered again here. As it was shown in Example 3.1 this eigenvalue problem can be written in the general form of equation (3.31) considering the self-adjoint expressions of relations (3.36a) and (3.36b). For any comparison functions u, v (which, of course, satisfy the boundary conditions (3.13)-(3.16)), the following relations hold:

$$\int_0^L [u\mathrm{M}(v) - v\mathrm{M}(u)]dx = \int_0^L u\mathrm{M}(v)dx - \int_0^L v\mathrm{M}(u)dx =$$

$$= \int_0^L u(v''')'dx - \int_0^L v(u''')'dx = \left[uv'''\right]_0^L - \int_0^L u'(v'')'dx - \left[vu'''\right]_0^L + \int_0^L v'(u'')'dx =$$

$$= 0 - \int_0^L u'(v'')'dx - 0 + \int_0^L v'(u'')'dx = -\left[u'v''\right]_0^L + \int_0^L u''v''dx + \left[v'u''\right]_0^L - \int_0^L v''u''dx =$$

$$= 0 + \int_0^L u''v''dx + 0 - \int_0^L v''u''dx = 0 \quad \forall\ u, v \in D_A \tag{3.44}$$

$$\int_0^L [u\mathrm{N}(v) - v\mathrm{N}(u)]dx = \int_0^L u\mathrm{N}(v)dx - \int_0^L v\mathrm{N}(u)dx =$$

$$= \int_0^L u(-v')'dx - \int_0^L v(-u')'dx = -\left[uv'\right]_0^L + \int_0^L u'v'dx + \left[vu'\right]_0^L - \int_0^L v'u'dx =$$

$$= -0 + \int_0^L u'v'dx + 0 - \int_0^L v'u'dx = 0 \quad \forall\ u, v \in D_A\ . \tag{3.45}$$

Due to the fact that equations (3.44) and (3.45) are fulfilled, the examined eigenvalue problem is symmetric. In order to prove that it is also positive, relations (3.42), (3.43) should additionally be fulfilled. For the specific case treated here and for any non-zero comparison function $u \in D_A$, it holds:

$$\int_0^L uM(u)dx = \int_0^L uu^{(4)}dx = \int_0^L (u'')^2 dx > 0 \quad \forall\ u \neq 0 \in D_A \tag{3.46}$$

$$\int_0^L uN(u)dx = \int_0^L u(-u'')dx = \int_0^L (u')^2 dx > 0 \quad \forall\, u \neq 0 \in D_A . \tag{3.47}$$

Thus, the eigenvalue problem of section 3.4 is symmetric and positive.

Theorem 3.1

If the eigenvalue problem is symmetric, then the eigenfunctions $w_1(x), w_2(x)$, corresponding to different eigenvalues λ_1, λ_2 are orthogonal in the so-called generalized sense, i.e.

$$\int_a^b w_1(x) w_2(x) dx = 0, \;\; \lambda_1 \neq \lambda_2 . \tag{3.48}$$

Theorem 3.2

If the eigenvalue problem is positive, then it can have only positive eigenvalues.

Theorem 3.3

If the eigenvalue problem is symmetric and positive, then there exists a countable set of positive, mutually different eigenvalues, thus $\lambda_n \to \infty$ for $n \to \infty$.

3.6.3 The non-homogeneous BVP

Consider the BVP (3.31), (3.32) where the function $f(x)$ of the right-hand side has non-zero values for each value of the variable x . Then, the BVP is called non-homogeneous. The following theorem that concerns the solvability of the problem holds.

Theorem 3.4

Let a real number λ in (3.30) be given. Then:

- If this value λ is not an eigenvalue of the corresponding homogeneous problem (3.39), (3.32), then the given non-homogeneous problem has exactly one solution for every arbitrary right-hand side function $f(x)$.
- If this value λ is an eigenvalue of the corresponding homogeneous problem (3.39), (3.32) then the given non-homogeneous problem is in general not solvable. It is solvable (but not uniquely) if and only if the function $f(x)$ is orthogonal to every eigenfunction ϕ corresponding to that λ, thus if the following equation holds for every such eigenfunction.

$$(f,\phi)=\int_a^b f(x)\phi(x)dx=0\,^{6}. \tag{3.49}$$

Example 3.3

It is assumed that the beam of Fig. 3.2 is imperfect i.e. it has an initial deflection with a maximum amplitude equal to α_0. It is considered that the initial deflected configuration can be described by the following relation:

$$w_0(x)=a_0\sin\frac{\pi x}{L},\quad x\in[0,L]. \tag{3.50}$$

It is assumed that this initial deflected configuration is not accompanied by the development of internal stresses (hence the term "initial"). Therefore, in this specific case the internal bending moment (included e.g. in the first term of relation (3.8)) is caused only by the additional deflection beyond $w_0(x)$. Denoting by $w(x)$ the total deflection of the beam, the governing differential equation can be written as:

$$EI\frac{d^4(w(x)-w_0(x))}{dx^4}+P\frac{d^2w(x)}{dx^2}=0\Rightarrow EI\frac{d^4w(x)}{dx^4}+P\frac{d^2w(x)}{dx^2}=EI\frac{d^4w_0(x)}{dx^4}\,. \tag{3.51}$$

Substituting equation (3.50) into (3.51) and dividing all the terms with the constant rigidity EI, the following differential equation is obtained:

$$\frac{d^4w(x)}{dx^4}+\frac{P}{EI}\frac{d^2w(x)}{dx^2}=a_0\left(\frac{\pi}{L}\right)^4\sin\frac{\pi x}{L}\,. \tag{3.52}$$

The solution of equation (3.52) is yielded by the addition of a particular solution to the solution of the corresponding homogeneous equation (general solution). The general solution has the form of relation (3.11) while the particular solution is given by the following equation:

[6] This integral represents the scalar product of two functions in the metric space $L_2(a,b)$ of the square integrable functions (in the Lebesgue sense). For more details about this space and the relative properties see Apppendix A.

$$w_p = \left(\frac{EIa_0 \left(\frac{\pi}{L}\right)^4}{EI\left(\frac{\pi}{L}\right)^4 - P\left(\frac{\pi}{L}\right)^2} \right) \sin\frac{\pi x}{L} = a_0 \left(\frac{1}{1 - \frac{P}{\frac{\pi^2 \mathrm{EI}}{L^2}}} \right) \sin\frac{\pi x}{L} \tag{3.53}$$

where the following term is termed as “magnification factor”:

$$F = \left(\frac{1}{1 - \frac{P}{\frac{\pi^2 EI}{L^2}}} \right). \tag{3.54}$$

The “magnification factor” is encountered in the most structural textbooks and codes concerning stability, where it is actually used to approximate the stress and strain state of the system by means of the corresponding magnitudes obtained by the first order bending theory.

Then, the coefficients of the general solution (3.11) can be determined by means of the boundary conditions, thus:

$$w(0) = 0 \Rightarrow A + D = 0 \Rightarrow D = 0 \tag{3.55}$$

$$w(0) = 0 \Rightarrow A + D = 0 \Rightarrow D = 0 \tag{3.56}$$

$$w''(L) = 0 \Rightarrow B \sin kL + a_0 \left(\frac{1}{1 - \frac{P}{\frac{\pi^2 \mathrm{EI}}{L^2}}} \right) \sin\frac{\pi L}{L} = 0 \Rightarrow$$

$$\Rightarrow B\sin kL + a_0\left(\frac{1}{1-\frac{P}{\frac{\pi^2 \mathrm{EI}}{L^2}}}\right)\cdot 0 = 0 \tag{3.57}$$

$$w(L) = 0 \Rightarrow B\sin kL + CL = 0\,. \tag{3.58}$$

Applying the theorem (3.4), it is apparent that $B = C = 0$ for any value of the axial load $P < \frac{\pi^2 \mathrm{EI}}{L^2}$ (i.e. $k < \frac{\pi}{L}$). Moreover, the certain boundary value problem has a unique solution which can be expressed as:

$$w(x) = w_p(x) = a_0\left(\frac{1}{1-\frac{P}{\frac{\pi^2 \mathrm{EI}}{L^2}}}\right)\sin\frac{\pi x}{L}\,. \tag{3.59}$$

When the parameter k takes the value $\frac{\pi}{L}$ (i.e. $P = \frac{\pi^2 \mathrm{EI}}{L^2}$) which constitutes the first eigenvalue of the corresponding homogeneous BVP, the problem will either have infinite solutions or it will be unsolvable. It is obvious that for this specific value of load, the term $B\sin kL$ in relations (3.57) and (3.58) becomes zero (and hence $C = 0$), while the boundary condition (3.57) cannot be satisfied because the denominator of the magnification factor becomes zero. Thus, the problem is not solvable for every value of B. This can also be proved by taking the scalar product between the function $f(x) = a_0\left(\frac{\pi}{L}\right)^4 \sin\frac{\pi x}{L}$ and the first eigenfunction $\phi_1(x) = b_0 \sin\frac{\pi x}{L}$ which corresponds to the first eigenvalue $P = \frac{\pi^2 \mathrm{EI}}{L^2}$:

$$(f(x),\phi_1) = \int_0^L w_0(x)\phi_1(x)dx = a_0 b_0\left(\frac{\pi}{L}\right)^4 \int_0^L \left(\sin\frac{\pi x}{L}\right)^2 =$$

$$= a_0 b_0 \left(\frac{\pi}{L}\right)^4 \left[0.5x - 0.25\frac{L}{\pi}\sin\frac{2\pi x}{L}\right]_0^L = a_0 b_0 \left(\frac{\pi}{L}\right)^4 \left[0.5L - 0 - 0 + 0\right] =$$

$$= a_0 b_0 \left(\frac{\pi}{L}\right)^4 L \neq 0 . \tag{3.60}$$

Therefore, the criterion (3.49) does not hold. Thus the non-homogeneous problem is not solvable. For $k \to \frac{\pi}{L} \Rightarrow P \to \frac{\pi^2 EI}{L^2}$ the deflection curve takes infinite values and the beam develops extremely large deflections.

Untill now two different types of instability have been displayed. The first one corresponds to the "snap" buckling when the axial compressive load of the structure is **taking** a critical value while the second one corresponds to the development of disproportionate large deflections, when the axial load **approaches** a critical value. Moreover, the first type of instability results from a homogeneous BVP while the later results from a non-homogeneous BVP. In the framework of the present dissertation these two types of loading, which both lead to instability should be appropriately distinguished. The load which lead the structure to "snap" buckling is termed as "critical load" while the axial load which leads to the development of extremely large deflections is termed as "instability load".

Example 3.4

Let us consider herein the previous example, where now, instead of an imperfection described by the first eigenmode of the corresponding homogeneous BVP, the second eigenmode is considered. According to the findings of the previous example, the imperfect beam of the studied example is expected to be solvable for all the values of load which do not constitute eigenvalues of the corresponding homogeneous BVP (i.e the axially loaded simply supported beam), while for values of load P which constitute eigenvalues an appropriate examination should be done with respect to Theorem 3.4. For this specific values the certain problem may be either unsolvable or solvable with infinite solutions.
The initial deflected configuration can be described by the following relation:

$$w_0(x) = b_0 \sin\frac{2\pi x}{L} \qquad x \in [0, L]. \tag{3.61}$$

According to the procedure followed in Example 3.3, the following differential equation is obtained:

$$\frac{d^4w(x)}{dx^4}+\frac{P}{EI}\frac{d^2w(x)}{dx^2}=16b_0\left(\frac{\pi}{L}\right)^4\sin\frac{2\pi x}{L}. \tag{3.62}$$

The above differential equation has the form of equation (3.31) where:

$$\mathrm{M}(w)=\frac{d^4w(x)}{dx^4} \tag{3.63}$$

$$\mathrm{N}(w)=-\frac{d^2w(x)}{dx^2} \tag{3.64}$$

$$f(x)=16b_0\left(\frac{\pi}{L}\right)^4\sin\frac{2\pi x}{L} \tag{3.65}$$

$$\lambda=\frac{P}{EI}=k^2. \tag{3.66}$$

The solution of equation (3.62) is obtained by the addition of a particular solution to the solution of the corresponding homogeneous equation (general solution). The general solution has the form of relation (3.11) while the particular solution is given by the following equation:

$$w_p(x)=b_0\left(\frac{1}{1-\dfrac{P}{\dfrac{4\pi^2\mathrm{EI}}{L^2}}}\right)\sin\frac{2\pi x}{L}=b_0F_2\sin\frac{2\pi x}{L}, \tag{3.67}$$

where the "magnification factor" is given by:

$$F_2=\left(\frac{1}{1-\dfrac{P}{\dfrac{4\pi^2\mathrm{EI}}{L^2}}}\right). \tag{3.68}$$

Then, the coefficients of the general solution (3.11) can be determined by means of the boundary conditions of the problem. At this point a strict investigation of the solvability of the studied non-homogeneous BVP with respect to the different values of the axial load should be made. More specifically, in the previous example the obtained solution (3.59) was valid for all the values of the axial load P in the open interval $(0,\frac{\pi^2 EI}{L^2})$. When the axial load was approaching the value $\frac{\pi^2 EI}{L^2}$, the values of the solution of the BVP were developing the tendency to become extremely large, while for this specific value the BVP was unsolvable. The same findings were obtained either through the direct application of the boundary conditions or through the utilization of the basic Theorem 3.4. In the example considered now, a rather unusual, situation is arising just applying the boundary conditions of the problem. The interesting findings are also verified by the utilization of the Theorem 3.4. Therefore, let us consider the following problem:

Find $w(x) \in L_2(0,L)$ such that equation (3.62) and the following boundary conditions are satisfied:

$$w''(0) = 0 \Rightarrow A = 0 \tag{3.69}$$

$$w(0) = 0 \Rightarrow A + D = 0 \Rightarrow D = 0 \tag{3.70}$$

$$w''(L) = 0 \Rightarrow -Bk^2 \sin kL - b_0 \left(\frac{1}{1 - \frac{P}{\frac{4\pi^2 \mathrm{EI}}{L^2}}} \right) \left(\frac{2\pi}{L} \right)^2 \sin \frac{2\pi L}{L} = 0 \Rightarrow$$

$$\Rightarrow -Bk^2 \sin kL - b_0 \left(\frac{1}{1 - \frac{P}{\frac{4\pi^2 \mathrm{EI}}{L^2}}} \right) \left(\frac{2\pi}{L} \right)^2 \cdot 0 = 0$$

(3.71)

$$w(L)=0 \Rightarrow B\sin kL + CL + b_0 \frac{1}{1-\frac{P}{\frac{4\pi^2 EI}{L^2}}} \cdot \sin\frac{2\pi L}{L} = 0\,, \tag{3.72}$$

where the load P belongs to the open interval $(0,\frac{4\pi^2 EI}{L^2})$. It is easily inferred that for any value of the axial load $P \in (0,\frac{4\pi^2 EI}{L^2}) / \left\{\frac{\pi^2 EI}{L^2}\right\}$ the unknown coefficients B, C become zero, therefore the deflection curve $w(x)$ is uniquely determined and is equal to the particular solution $w_p(x)$ (equation (3.67)). When the value of load P approaches to the critical value $\frac{4\pi^2 EI}{L^2}$, then the deflection curve is taking extremely large values. For this specific value, the non-homogeneous BVP is unsolvable due to the fact that an indeterminate form is arising ("$0\cdot\infty$") and thus the boundary conditions cannot be fulfilled. The above conclusion results also and by the utilization of the basic Theorem 3.4. More specifically, the eigenvalues of the homogeneous problem (the BVP of the simply supported beam which has been already discussed in paragraph 3.4) do not belong in the open interval $P \in (0,\frac{4\pi^2 EI}{L^2}) / \left\{\frac{\pi^2 EI}{L^2}\right\}$. Therefore the certain non-homogeneous BVP is solved uniquely. For the specific value $\frac{4\pi^2 EI}{L^2}$, which is the second eigenvalue of the homogeneous BVP, the problem is unsolvable due to the fact that the eigenmode corresponding to that eigenvalue is not orthogonal to the function $f(x)$ of the right part of equation (3.62). This can be proved easily by taking the scalar product of the two functions:

$$(f(x),\phi_2) = \int_0^L f(x)\phi_2(x)dx = \int_0^L \left(16b_0\left(\frac{\pi}{L}\right)^4 \sin\frac{2\pi x}{L}\right)\left(m_0 \sin\frac{2\pi x}{L}\right)dx =$$

$$=16b_0 m_0\left(\frac{\pi}{L}\right)^4 \int_0^L \left(\sin\frac{2\pi x}{L}\right)^2 = 16b_0 m_0\left(\frac{\pi}{L}\right)^4 \left[0.5x - 0.125\frac{L}{\pi}\sin\frac{4\pi x}{L}\right]_0^L =$$

$$=16b_0 m_0\left(\frac{\pi}{L}\right)^4 \left[0.5L - 0 - 0 + 0\right] = 8b_0 m_0 \frac{\pi^4}{L^3} \neq 0\,. \tag{3.73}$$

Comparing this example with the previous one, the studied BVP problem reveals a very interesting point. Inside the open interval $(0,\frac{4\pi^2 EI}{L^2})$ there exists the value $\frac{\pi^2 EI}{L^2}$ which constitutes the first eigenvalue of the corresponding homogeneous BVP. Therefore, according to the fundamental Theorem 3.4, for this specific value the studied non-homogeneous BVP will be either unsolvable or solvable, but not uniquely. Taking the scalar product between the function $f(x)$ and the eigenmode corresponding to that eigenvalue, we have:

$$(f(x),\phi_1)=\int_0^L f(x)\phi_1(x)dx=\int_0^L\left(16b_0\left(\frac{\pi}{L}\right)^4\sin\frac{2\pi x}{L}\right)\left(r_0\sin\frac{\pi x}{L}\right)dx=$$

$$=16b_0r_0\left(\frac{\pi}{L}\right)^4\int_0^L\left(\sin\frac{2\pi x}{L}\right)\left(\sin\frac{\pi x}{L}\right)=0\,. \tag{3.74}$$

Therefore, it is proved that the two functions, $f(x)$, ϕ_1 are orthogonal and, consequently, the non-homogeneous BVP is solvable. This conclusion also results from the boundary conditions (3.71) and (3.72). More specifically, for $P=\frac{\pi^2 EI}{L^2}$ the latter give respectively:

$$B\sin kL=0\Rightarrow B\sin\frac{\pi}{L}L=0\Rightarrow B\cdot 0=0 \tag{3.75}$$

$$CL=0\Rightarrow C=0\,. \tag{3.76}$$

Equation (3.75) is satisfied for every real number B, therefore the non-homogeneous BVP (3.62), (3.69)-(3.72), has infinite solutions. These infinite solutions are obtained for arbitrary choice of the coefficient B:

$$w(x) = B\sin\frac{\pi x}{L} + b_0\left(\frac{1}{1-\frac{\frac{\pi^2 \mathrm{EI}}{L^2}}{\frac{4\pi^2 \mathrm{EI}}{L^2}}}\right)\sin\frac{2\pi x}{L} = B\sin\frac{\pi x}{L} + \frac{4}{3}b_0\sin\frac{2\pi x}{L}\,. \tag{3.77}$$

Summarizing, the investigation for the solvability of the non-homogeneous BVP (3.62), the following results are obtained:

a. For $P \in (0, \frac{4\pi^2 EI}{L^2})/\left\{\frac{\pi^2 EI}{L^2}\right\}$ the BVP is uniquely solvable and the function $w(x)$ is given by the following relation:

$$w(x) = w_p(x) = b_0\left(\frac{1}{1-\frac{P}{\frac{4\pi^2 \mathrm{EI}}{L^2}}}\right)\sin\frac{2\pi x}{L} = b_0 F_2 \sin\frac{2\pi x}{L} \quad x \in [0, L] \tag{3.78}$$

b. For $P = \frac{\pi^2 EI}{L^2}$ the BVP has infinite solutions:

$$w(x) = B\sin\frac{\pi x}{L} + \frac{4}{3}b_0\sin\frac{2\pi x}{L} \quad x \in [0, L] \tag{3.79}$$

c. For $P = \frac{4\pi^2 EI}{L^2}$ the BVP is unsolvable.

It is obvious from the above results, that the solution of the non-homogeneous BVP appears singularities for the values $P = \frac{\pi^2 EI}{L^2}$ and $P = \frac{4\pi^2 EI}{L^2}$, which constitute the first and the second eigenvalues of the corresponding homogeneous BVP. The effect of this singular behavior is actually the same, i.e. buckling occurs for both cases. The difference is associated with the type of the instability. More specifically, the value $P=\frac{\pi^2 EI}{L^2}$ corresponds to an eigenvalue which is orthogonal to the function of the right hand side of the fundamental equation (3.62), thus, the BVP has infinite solutions for that value. In this case, instability occurs for this **specific** value of load. On the other hand, for $P=\frac{4\pi^2 EI}{L^2}$ the problem is unsolvable due to the fact that the eigenmode corresponding to that eigenvalue is not orthogonal to the function of the right hand side of the fundamental equation (3.62). In this case instability occurs as the value of the applied load **approaches** the specifc eigenvalue. In the following chapters the load for which singular behaviour arises in the solution of the non-homogeneous BVP, is termed as instability load in both the previous cases.

4 The unilateral buckling of beams – Part 1

4.1 Introduction

In the previous chapter the classical BVP of buckling of beams was discussed and several mathematical theorems were displayed. The present chapter concerns the calculation of the critical buckling load of a beam in which, apart from the classical bilateral support conditions, unilateral constraints are also present. The existence of the unilateral supports leads to a unilateral contact buckling problem. Problems of that type are usually handled using computational techniques based on variational formulations of the governing differential equations (see Chapter 2).

In this dissertation an analytical approach is developed which can be applied in common practical problems. The proposed methodology is based on the linear elastic stability theory, appropriately extended in order to take into account the unilateral constraints. The presented method concerns mostly beams which are considered as geometrically imperfect. However, as it was intuitively inferred in the previous chapter, the corresponding bifurcation problem is important and has to be handled as well.

More specifically, the considered beam is separated into parts by the unilateral supports. For each part of the considered beam, a fourth-order differential equation is constructed, arising from equilibrium and describing the bending behavior of the beam. Then, applying the boundary conditions at the ends of the divided parts, a total BVP is formulated. The BVP is then equipped with certain restrictions yielded by the unilateral constraints.

In the following paragraphs the formulation of the proposed methodology will be displayed. Without losing generality, the BVP of a simply supported beam with two unequal spans and one intermediate unilateral support, subjected to axial compressive load is formulated and investigated in the present chapter. The same procedure may be followed also in the case that the beam is equipped with more than one unilateral constraints, with the difference that the arising mathematical operations become more complicated.

Initially, the elastic contact buckling problem of the perfect structure is formulated. For this homogeneous constrained BVP, the eigenvalues (critical buckling loads) and the eigenmodes (buckling shapes) are extracted. In the next part of the chapter, arbitrary initial geometric imperfections are introduced in the structure and the non-homogeneous constrained BVP is examined. For the imperfect beam, the instability load and the corresponding buckling shape can be calculated by following an effective algorithm without having the necessity to use an incremental loading procedure.

Due to the fact that the present research aims also at deriving practical solutions for real life applications, a wide discussion is devoted to the consideration of the actual strength of the beam under axial compression and bending.

The present chapter deals only with the formulation of the proposed methodology. The implementation of the described procedures is carried out in Chapters 6 and 7 through the demonstration of a series of numerical examples.

4.2 Formulation of the elastic contact buckling problem of a geometrically perfect continuous beam

4.2.1 Formulation

A geometrically perfect (i.e. without any initial geometric imperfections) beam with an intermediate unilateral constraint is considered, subjected to an axial compressive load (Fig. 4.1). The beam is divided into two spans, Span I and Span II, having lengths aL and $(1-a)L$ respectively, where L is the total length of the beam. The two spans are equipped with the coordinate systems x_1, w_1 and x_2, w_2 as it is shown in Fig. 4.1. Here, x_1, x_2 measure the position along the axis of the beam and w_1, w_2 denote the transverse deflections of the beam in the two spans.

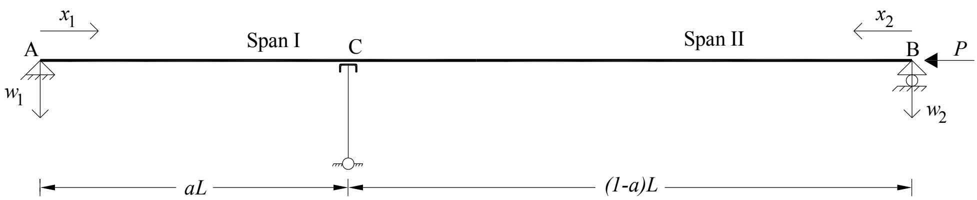

Fig. 4.1 The buckling problem of the beam with the unilateral support.

The positive internal forces follow the conventions given in Fig. 4.2. For the description of the bending behaviour of the beam, the Euler's equilibrium method can be applied as it was demonstrated in Section 3.4, leading to a fourth-order homogeneous differential equation for the two spans of the beam:

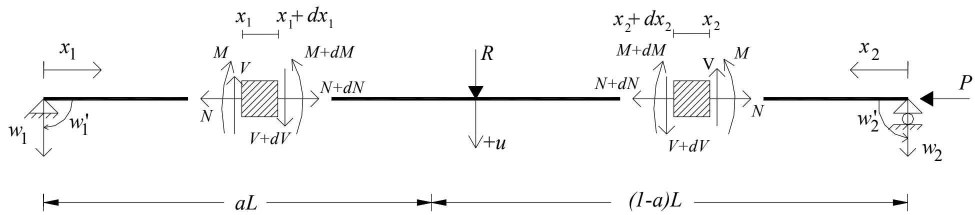

Fig. 4.2 The considered conventions for the positive displacements and internal forces.

$$\frac{d^4 w_1(x_1)}{dx_1{}^4} + k^2 \frac{d^2 w_1(x_1)}{dx_1{}^2} = 0 \quad x_1 \in [0, aL] \tag{4.1}$$

$$\frac{d^4 w_2(x_2)}{dx_2{}^4} + k^2 \frac{d^2 w_2(x_2)}{dx_2{}^2} = 0 \quad x_2 \in [0, (1-a)L]. \tag{4.2}$$

The parameter k is given by relation (3.12) where E denotes the Young's modulus of the material of the beam, I denotes the moment of inertia of the beam's cross-section for in plane bending and P is the axial compressive load applied on the beam.

If a non-trivial solution for the above equations exists, the beam can be in equilibrium in a bended configuration different from the straight line one (bifurcation equilibrium state). Therefore, the solution of the above equations gives the transverse deflections w_1, w_2 of the beam at any point, as a function of the compressive load P. The boundary conditions of the problem are formulated taking into account the essential boundary conditions, the natural boundary conditions and the unilateral contact conditions at the point of the unilateral support.

1. Essential boundary conditions

- Zero vertical displacement at the positions of the classical supports (points A,B):

$$w_1(0) = 0 \tag{4.3}$$

$$w_2(0) = 0 \tag{4.4}$$

- Common vertical displacement at the points of the unilateral support (point C):

$$w_1(aL) = w_2((1-a)L) = u \tag{4.5}$$

2. Natural boundary conditions

- Common rotation at the position of the unilateral support (point C):

$$-w_1'(aL) = w_2'((1-a)L) \tag{4.6}$$

- Zero bending moment at the positions of the bilateral supports (points A,B):

$$-EIw_1''(0) = 0 \tag{4.7}$$

$$-EIw_2''(0) = 0 \tag{4.8}$$

- Moment equilibrium at the position of the unilateral support (point C):

$$-EIw_1''(aL) + EIw_2''((1-a)L) = 0 \tag{4.9}$$

- Forces equilibrium at the position of the unilateral support (point C):

  In order to formulate the boundary conditions that correspond to the unilateral constraint, the support reaction R should be considered with an unknown value (Fig. 4.3). Obviously, the existence of this reaction force R depends on whether the unilateral constraint is active or not.

$$\left[-EIw_1'''(aL) - Pw_1'(aL)\right] + \left[-EIw_2'''((1-a)L) - Pw_2'((1-a)L)\right] = R \tag{4.10}$$

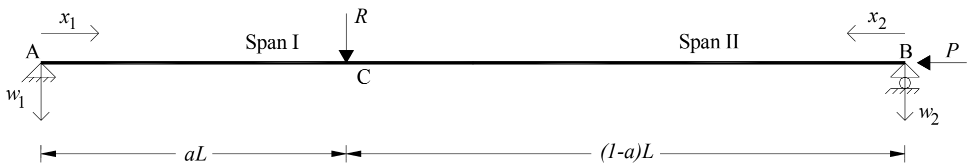

Fig. 4.3 The indirect consideration of the function of the unilateral support through the unknown reaction force *R*.

3. Unilateral contact boundary conditions at the position of the unilateral constraint
   The unilateral constraint may be represented through the following inequality conditions (Panagiotopoulos, 1985):

$$w_1(aL) = w_2((1-a)L) = u \leq 0 \tag{4.11}$$

$$R \leq 0 \tag{4.12}$$

$$R \cdot u = 0 \ . \tag{4.13}$$

Inequalities (4.11) and (4.12) together with the complementarity condition (4.13) express that either the displacement or the reaction may develop, with a negative value. Thus, the homogeneous constrained BVP describing the buckling problem of the continuous beam with the intermediate unilateral constraint is formulated as:

Problem $\mathscr{BP}_1$:

Find $w(x) = \begin{Bmatrix} w(x_1) \ x_1 \in [0, aL] \\ w(x_2) \ x_2 \in [0, (1-a)L] \end{Bmatrix} \in L_2(0,L)$ [7] of the system of differential equations (4.1), (4.2) such as the equality boundary conditions (4.3)-(4.10), the inequality constraints (4.11)-(4.12) and the complementarity condition (4.13) are satisfied.

It has to be noticed that the fundamental equations (4.55) and (4.56) do not take into account the stress in the lateral direction (Euler-Bernoulli beam assumption). Therefore, they can also be derived by the consideration of the one-dimensional von Karman model (Washizu, 1968). Moreover, these types of models are suitable for pre-buckling analysis under the assumption of infinitesimal deformations. For problems where large deformations have to be consider, different types of beam models should be taken into account (Gao, 1996; Gao, 2000). Such models lead to nolinear ordinary differential equations and are suitable for the analysis and simulation of a variety of mechanical applications. However, bifurcation problems of that type are very difficult to be solved analytically and, therefore, computational methods are usually employed.

[7] The vector space $L_2(0,L)$ is a subspace of the well known Hilbert space. The space $L_2(0,L)$ constitutes the space of square integrable functions (in the Lebesgue sense). Further details are given in Appendix A.

4.2.2 *Solution of the Problem $\mathscr{BP}_1$ for all the possible contact situations*

The BVP of Problem $\mathscr{BP}_1$ consists of homogeneous fourth order differential equations, which have, respectively, general solutions of the following form:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 \tag{4.14}$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 \, . \tag{4.15}$$

The full expression of the deflection curve of the continuous beam is given after the determination of the coefficients $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2$. These coefficients are calculated through the boundary conditions of the problem. However, due to the presence of the inequality conditions, the calculation of the above coefficients requires an appropriate examination for all the possible contact situations. Obviously, the solution of the BVP of the perfect structure depends on the different contact situations which can occur. There exist three possible deformed configurations compatible with the unilateral constraint. The first corresponds to the situation where the unilateral constraint is inactive. In this case the contact reaction force $R = 0$ and the transverse deflection $u < 0$. When $R < 0$ the unilateral constraint is active, therefore the transverse deflection u is equal to zero. Finally, there exists the limit situation, where the beam is in contact with the constraint without producing any reaction force ($R = 0, \; u = 0$).

Applying the boundary conditions (4.3), (4.4) and (4.7), (4.8) and using the derivatives of the deflection curves which are given in Table 4.1, the following algebraic equations are obtained:

Deflection curve	$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1$	$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2$
First Derivative	$w_1'(x_1) = -A_1 k \sin kx_1 + B_1 k \cos kx_1 + C_1$	$w_2(x_2) = -A_2 k \sin kx_2 + B_2 k \cos kx_2 + C_2$
Second Derivative	$w_1''(x_1) = -A_1 k^2 \cos kx_1 - B_1 k^2 \sin kx_1$	$w_2''(x_2) = -A_2 k^2 \cos \lambda x_2 - B_2 k^2 \sin kx_2$
Third Derivative	$w_1'''(x_1) = A_1 k^3 \sin kx_1 - B_1 k^3 \cos kx_1$	$w_2'''(x_2) = A_2 k^3 \sin kx_2 - B_2 k^3 \cos kx_2$

Table 4.1 The derivatives of the deflection curves.

$$w_1(0) = 0 \Rightarrow A_1 + D_1 = 0 \tag{4.16}$$

$$w_2(0) = 0 \Rightarrow A_2 + D_2 = 0 \tag{4.17}$$

$$-EIw_1''(0) = 0 \Rightarrow -A_1k^2 = 0 \Rightarrow A_1 = 0 \quad (k \neq 0 \text{ for } P \neq 0) \tag{4.18}$$

$$-EIw_2''(0) = 0 \Rightarrow -A_2k^2 = 0 \Rightarrow A_2 = 0 \quad (k \neq 0 \text{ for } P \neq 0)\,. \tag{4.19}$$

Obviously, the previous inequalities yield $A_1 = A_2 = D_1 = D_2 = 0$. As a result, the rest of the boundary conditions are transformed to the following equations which constitute an algebraic system with respect to the coefficients B_1, B_2, C_1, C_2:

$$w_1(aL) = w_2((1-a)L) = u \Rightarrow$$

$$\Rightarrow B_1 \sin kaL - B_2 \sin k(1-a)L + C_1 aL - C_2(1-a)L = 0 \tag{4.20}$$

$$-EIw_1''(aL) + EIw_2''((1-a)L) = 0 \Rightarrow B_1 \sin kaL - B_2 \sin k(1-a)L = 0 \tag{4.21}$$

$$-w_1'(aL) = w_2'((1-a)L) \Rightarrow B_1 k \cos kaL + B_2 k \cos k(1-a)L + C_1 + C_2 = 0 \tag{4.22}$$

$$\left[-EIw_1'''(aL) - Pw_1'(aL)\right] + \left[-EIw_2'''((1-a)L) - Pw_2'((1-a)L)\right] = R \Rightarrow$$

$$B_1 \cos kaL + B_2 \cos k(1-a)L = \frac{R}{EIk^3}\,. \tag{4.23}$$

The above linear system, can be written in a matrix form as:

$$\begin{bmatrix} \sin(kaL) & -\sin(k(1-a)L) & aL & -(1-a)L \\ \sin(kaL) & -\sin(k(1-a)L) & 0 & 0 \\ k\cos kaL & k\cos k(1-a)L & 1 & 1 \\ \cos kaL & \cos k(1-a)L & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{R}{EIk^3} \end{bmatrix} \tag{4.24}$$

Considering the above, the initial problem $\mathscr{BP}_1$ is transformed into the following one:

Problem $\mathcal{BP}_{1\text{-}a}$:

Find a solution of the algebraic system of equations (4.24) with respect to the unknown coefficients B_1, C_1, B_2, C_2, such that restrictions (4.11)-(4.13) are satisfied.

The solution of the Problem $\mathcal{BP}_{1\text{-}a}$ is obtained under the separation of the problem into subproblems according to the possible contact situation (i.e. the constraint may be active, inactive or in neutral contact status).

4.2.2.1 Inactive constraint, $R=0$ and $u<0$

In the case where the unilateral constraint is inactive (i.e. $R=0,\ u<0$) the following problem has to be solved:

Problem $\mathcal{BP}_{1\text{-}a,1}$:

Find a solution of the algebraic system of equations (4.20)-(4.23) or equivalently of the system (4.24) with respect to the unknown coefficients B_1, C_1, B_2, C_2 , such that the following restrictions are satisfied:

$$R=0 \tag{4.25}$$

$$u<0. \tag{4.26}$$

The above BVP produces infinite number of solutions. More specifically, the study of the homogeneous linear system (4.24) under the certain contact conditions gives:

$$C_1 = C_2 = 0. \tag{4.27}$$

The demand of having non-trivial solution leads to:

$$B_1 = B_2 \frac{\sin(k(1-a)L)}{\sin(kaL)} \neq 0, \tag{4.28}$$

where B_2 is chosen arbitrarily, and to the following buckling equation:

$$\cot(kaL) + \cot(k(1-a)L) = 0. \tag{4.29}$$

The above equation is a transcendental algebraic one, having infinite solutions[8]. The existence of solution for this specific contact situation indicates a curved deformed equilibrium configuration different from the initial straight line one (bifurcation). The buckling equation (4.29) gives the eigenvalues of the BVP which are the **critical loads** of the problem under consideration. Moreover, the implementation of equations (4.14), (4.15) for each eigenvalue gives the corresponding eigenmodes:

$$w_1(x_1) = B_1 \sin kx_1,\ \ x_1 \in\ [0, aL] \tag{4.30}$$

$$w_2(x_2) = B_2 \sin kx_2,\ \ x_2 \in\ [0,(1-a)L]\,. \tag{4.31}$$

From the set of all the functions which can be created for arbitrary values of the coefficient B_2 (coefficient B_1 is calculated from relation (4.28)), only the functions which fulfill restrictions (4.25) and (4.26) are admissible. Therefore, the coefficients B_1 and B_2 should, in turn, satisfy the following inequalities:

$$B_1 \sin(kaL) < 0 \tag{4.32a}$$

or, equivalently:

$$B_2 \sin(k(1-a)L) < 0\ \ . \tag{4.32b}$$

4.2.2.2 Active constraint, $R < 0$ and $u = 0$

When the unilateral constraint is active (i.e. $R < 0$ and $u = 0$) the following problem has to be solved:

Problem $\mathcal{BP}_{1\text{-}a,2}$:

Find a solution of the algebraic system of equations (4.24) with respect to the unknown coefficients B_1, C_1, B_2, C_2, such that the following restrictions are satisfied:

$$R < 0 \tag{4.33}$$

$$u = 0\,. \tag{4.34}$$

[8]The values of the roots of transcendental equations can be derived easily by using numerical methods such as the iterative Newton method. Further details can be found in Appendix C.

The study of the linear algebraic system (4.24) with respect to the coefficients B_1, C_1, B_2, C_2 leads to:

$$B_1 = B_2 \frac{\sin(k(1-a)L)}{\sin(kaL)} \neq 0 \tag{4.35}$$

$$C_1 = \frac{-B_1 \sin(kaL)}{aL} \tag{4.36}$$

$$C_2 = \frac{aC_1}{1-a} \tag{4.37}$$

and to the following buckling equation:

$$\cot(kaL) + \cot(k(1-a)L) = \frac{1}{k(1-a)aL}\,. \tag{4.38}$$

The buckling equation (4.38) is also a transcendental one, producing the critical loads (eigenvalues) of the continuous beam in the case that the constraint is active. The corresponding eigenmodes are the buckle deflection curves which are determined substituting, for each eigenvalue k, relations (4.35)-(4.37) into equations (4.14), (4.15):

$$w_1(x_1) = B_1 \sin kx_1 + C_1 x_1 \qquad x_1 \in [0, aL] \tag{4.39}$$

$$w_2(x_2) = B_2 \sin kx_2 + C_2 x_2 \qquad x_2 \in [0, (1-a)L]\,. \tag{4.40}$$

The above solution is valid only if the restriction introduced by inequality (4.33) is satisfied. Therefore, the coefficient B_2 must be chosen appropriately so that the following inequality condition, yielded from relation (4.33), is fulfilled:

$$B_1 \cos(kaL) + B_2 \cos(k(1-a)L) < 0\,. \tag{4.41}$$

4.2.2.3 Neutral contact status, $R = 0$ and $u = 0$

This is a special case, where simultaneously the reaction force R and the transverse displacement u in the position of the unilateral constraint are equal to zero. For this case, the following problem has to be solved:

Problem $\mathcal{BP}_{1\text{-}a,3}$:

Find a solution of the algebraic system of equations (4.24) with respect to the unknown coefficients B_1, C_1, B_2, C_2, such that the following restrictions are satisfied:

$$R = 0 \tag{4.42}$$

$$u = 0\,. \tag{4.43}$$

The mathematical demand of the above equations leads to:

$$C_1 = C_2 = 0 \tag{4.44}$$

$$B_1 = -B_2 \frac{\cos[k(1-a)L]}{\cos kaL} \neq 0\,. \tag{4.45}$$

In this particular contact situation the infinite number of eigenvalues (critical loads) is calculated directly through the following formula:

$$P_{(\rho+n)} = \frac{(\rho+n)^2 \pi^2 EI}{L^2}, \quad \rho, n \in \mathbf{Z}\,. \tag{4.46}$$

It is noticed that the demand of having simultaneously the contact force and the common displacement in the position of the unilateral support equal to zero, can be produced only when the parameter a (Fig. 4.1) can be expressed as the ratio $a = \dfrac{n}{\rho + n}$, thus a must be a rational number. Then the eigenmodes that constitute solutions of Problem $\mathcal{BP}_{1\text{-}a,3}$ and correspond to the eigenvalues calculated by equation (4.46) are the following:

$$w_1(x_1) = B_1 \sin kx_1 \quad x_1 \in [0, aL] \tag{4.47}$$

$$w_2(x_2) = B_2 \sin kx_2 \quad x_2 \in [0, (1-a)L]\,. \tag{4.48}$$

From all the functions which can be generated from relations (4.47) and (4.48) for arbitrary values of the coefficient B_2, only those which satisfy restrictions (4.42) and (4.43) are admissible.

4.2.3 Calculation of the critical load

In order to determine the critical buckling load of an axially loaded geometrically perfect beam in the presence of one intermediate unilateral support, the following steps are considered:

a. A sufficient number of eigenvalues is calculated for each contact case through the respective buckling equations (i.e. equations (4.29), (4.38) and (4.46) for the inactive, active and neutral contact cases respectively).

b. From the set of the calculated eigenvalues, only the eigenvalues which produce eigenmodes compatible with the unilateral constraint are accepted.

c. The smallest acceptable eigenvalue is the critical one. For this eigenvalue, the critical load and the buckling mode of the beam is determined by means of equations (3.12) and (4.14), (4.15) respectively.

4.3 Formulation of the elastic contact buckling problem of a geometrically imperfect continuous beam

4.3.1 Formulation

In a similar way as described for the geometrically perfect continuous beam, the BVP of a simply supported continuous geometrically imperfect beam with two unequal spans and one intermediate unilateral constraint subjected to an axial compressive load, can be also formulated. The initial shape of the imperfect beam is assumed to be described by a Fourier sine series having the following form:

$$w_0(x) = \sum_{r=1}^{n} b_r \sin\left(\frac{r\pi x}{L}\right), \quad x \in [0, L] . \tag{4.49}$$

In the above relation, L is the total length of the beam and w_0 are the initial deflections of the imperfection. The arbitrary initial geometric imperfection has to be compatible with the unilateral constraint, thus, the following inequality should be satisfied:

$$w_0(aL) \leq 0 . \tag{4.50}$$

Due to the fact that the proposed methodology divides the beam into parts, the initial geometric imperfection has to be separated into two functions, one for each span of the beam (Span I and Span II):

$$w_{1,0}(x_1) = \sum_{r=1}^{n} b_r \sin\left(\frac{r\pi x_1}{L}\right), \quad x_1 \in [0, aL] \tag{4.51}$$

$$w_{2,0}(x_2) = -\sum_{r=1}^{n} b_r \sin\left(\frac{r\pi x_2}{L}\right)(-1)^r, \quad x_2 \in [0,(1-a)L]. \tag{4.52}$$

Inequality (4.50) imposes that the Fourier coefficients b_r of the above relations should, in turn, satisfy the following inequalities:

$$\sum_{r=1}^{n} b_r \sin(r\pi a) \leq 0 \tag{4.53}$$

or equivalently,

$$\sum_{r=1}^{n} b_r \sin(r\pi(1-a)) \geq 0 \ . \tag{4.54}$$

Then, for each part of the beam of Fig. 4.1, a fourth-order linear non-homogeneous ordinary differential equation can be constructed respectively that describes the bending behaviour of the beam. These relations are similar to equation (3.52) which was derived in Example 3.3 of Chapter 3. Differentiating four times the functions (4.51), (4.52) of the initial imperfections, the following equations are obtained:

$$\frac{d^4 w(x_1)}{dx_1^{\,4}} + k^2 \frac{d^2 w(x_1)}{dx_1^{\,2}} = \sum_{r=1}^{n} b_r (\frac{r\pi}{L})^4 \sin(\frac{r\pi x_1}{L}), \quad x_1 \in [0, aL] \tag{4.55}$$

$$\frac{d^4 w(x_2)}{dx_2^{\,4}} + k^2 \frac{d^2 w(x_2)}{dx_2^{\,2}} = -\sum_{r=1}^{n} b_r (\frac{r\pi}{L})^4 \sin(\frac{r\pi x_2}{L})(-1)^r, \quad x_2 \in [0,(1-a)L]. \tag{4.56}$$

The parameter k is given by equation (3.12). The solution of the above equations gives the transverse deflection w at each point of the beam as a function of the axial compressive load P. Consequently, the contact elastic buckling problem of the geometrically imperfect continuous beam with one intermediate unilateral constraint is formulated as the following non-homogenous constrained BVP:

Problem $\mathcal{BI}_1$:

Find $w(x)=\begin{Bmatrix} w(x_1)\; x_1\in[0,aL] \\ w(x_2)\; x_2\in[0,(1-a)L] \end{Bmatrix}\in L_2(0,L)$ of the system of differential equations (4.55), (4.56) such that the equality boundary conditions (4.3)-(4.10), the inequality constraints (4.11)-(4.12) and the complementarity condition (4.13) are fulfilled.

4.3.2 *Solution of the Problem $\mathcal{BI}_1$ of the geometrically imperfect beam for all the possible contact situations*

For the geometrically imperfect structure, the solution of the Problem $\mathcal{BI}_1$ is not a simple issue due to the fact that the non-homogeneous constrained BVP may be unsolvable, uniquely solvable or solvable with infinite solutions, for all the possible values of the axial loading P. An answer concerning the solvability of the Problem $\mathcal{BI}_1$ can be derived from the application of the fundamental Theorem 3.4 of Chapter 3. From the demonstration of the Examples 3.3 and 3.4 of Chapter 3 it is also concluded that the solvability of the fundamental Problem $\mathcal{BI}_1$ is strongly connected with the type of the initial imperfection and the eigenvalues of the corresponding homogeneous constrained BVP, which in turn, depend on the position of the unilateral support and the initial contact conditions (i.e. whether an initial gap between the beam and the unilateral support exists, after the introduction of the imperfection).

Therefore, due the complexity of problem and the strong dependence of the solution on the aforementioned factors, a general closed solution for each case cannot be derived. Although each problem should be treated individually, a flexible algorithm is proposed in the present dissertation which is able to give the solution after certain steps which are based on a unified solution formula. The concept of this algorithm is based on the fundamental question: “is the certain value of the axial load P in equations (4.55) and (4.56), for which the solution of the BVP is seeked, an eigenvalue of the corresponding eigenvalue problem ? ”. The answer to this question, as it will be displayed in the following paragraphs, leads to the solution of the problem.

In the studied case of the imperfect beam, the critical state is usually denoted by disproportionate large transverse deflections, developing when the axial compressive load P is approaching a certain value, the so-called instability load (denoted by P_i). Notice that, as it was stated in Chapter 3 (see Example 3.4), instability can also occur “suddenly” when the load takes the critical value for which the non-homogeneous BVP has infinite solutions. Due to the existence of the initial imperfection, the beam has a bending deflection even from the initial stage of the loading, in contrast with the perfect structure where the critical state is indicated by an instantaneous passing from the stable initial straight line configuration to the curved deformed configuration (bifurcation equilibrium state).

The solution of Problem $\mathcal{BI}_1$ of paragraph 4.3.1 is a superposition of a general solution and of a particular solution related with the type of the initial imperfection, i.e.:

$$w(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{n} b_r F_r \sin(\frac{r\pi x_1}{L}), \quad x_1 \in [0, aL] \tag{4.57}$$

$$w(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 - \\ -\sum_{r=1}^{n} b_r F_r \sin(\frac{r\pi x_2}{L})(-1)^r, \quad x_2 \in [0, (1-a)L]. \tag{4.58}$$

It is noticed that the solutions w_1, w_2 give the total transverse deflections of the beam, i.e. the initial deflections are included in them. In the above solutions the terms F_r are functions of the axial compressive load P, the so-called "magnification factors":

$$F_r = \frac{1}{1 - \dfrac{P}{P_r}}. \tag{4.59}$$

In the above, P_r are the eigenvalues of the elastic contact buckling problem of the simply supported beam, i.e.:

$$P_r = \frac{r^2 \pi^2 EI}{L^2}. \tag{4.60}$$

The coefficients of the general solution are calculated through the boundary conditions (relations (4.3)-(4.10)) of the problem. The boundary conditions (4.3)-(4.10) are transformed into the following equations:

$$w_1(0) = 0 \Rightarrow A_1 + D_1 = 0 \tag{4.61}$$

$$w_2(0) = 0 \Rightarrow A_2 + D_2 = 0 \tag{4.62}$$

$$-EIw_1''(0) = 0 \Rightarrow -A_1 k^2 = 0 \Rightarrow A_1 = 0 \quad (k \neq 0 \text{ for } P \neq 0) \tag{4.63}$$

$$-EIw_2''(0)=0 \Rightarrow -A_2k^2=0 \Rightarrow A_2=0 \quad (k \neq 0 \text{ for } P \neq 0)\,. \tag{4.64}$$

Therefore, $D_1 = D_2 = 0$ and the rest of the boundary conditions take the form:

$$w_1(aL) = w_2((1-a)L) = u \Rightarrow B_1 \sin kaL + C_1 aL + \sum_{r=1}^{n} b_r F_r \sin r\pi a =$$

$$= B_2 \sin(k(1-a)L) + C_2(1-a)L - \sum_{r=1}^{n} b_r F_r \sin r\pi(1-a)(-1)^r \tag{4.65}$$

$$+EIw_1''(aL) = EIw_2''((1-a)L) \Rightarrow -B_1k^2 \sin kaL - \sum_{r=1}^{n} b_r F_r (\frac{r\pi}{L})^2 \sin(r\pi a) =$$

$$= -B_2k^2 \sin k(1-a)L + \sum_{r=1}^{n} b_r F_r (\frac{r\pi}{L})^2 \sin(r\pi(1-a))(-1)^r \tag{4.66}$$

$$-w_1'(aL) = w_2'((1-a)L) \Rightarrow -[B_1 k \cos kaL + C_1 + \sum_{k=1}^{n} b_r F_r (\frac{r\pi}{L}) \cos(r\pi a)] =$$

$$= B_2 k \cos k(1-a)L + C_2 - \sum_{r=1}^{n} b_r F_r (\frac{r\pi}{L}) \cos(r\pi(1-a))(-1)^r \tag{4.67}$$

$$\left[-EIw_1'''(aL) - Pw_1'(aL)\right] + \left[-EIw_2'''((1-a)L) - Pw_2'((1-a)L)\right] = R \Rightarrow$$

$$\Rightarrow [-w_1'''(aL) - w_2'''((1-a)L)] = \frac{R}{EI} \Rightarrow B_1k^3 \cos kaL + \sum_{r=1}^{n} b_r F_r (\frac{r\pi}{L})^3 \cos(r\pi a) +$$

$$+B_2 \cos k(1-a)L - \sum_{r=1}^{n} b_r F_r (\frac{r\pi}{L})^3 \cos(r\pi(1-a))(-1)^r = \frac{R}{EI}\,. \tag{4.68}$$

Equations (4.65)-(4.68) are used in order to determine the unique values of the coefficients B_1, B_2, C_1, C_2 . The above equations are transformed in the following equations, which constitute an algebraic system with respect to the unknown coefficients B_1, B_2, C_1, C_2.

$$B_1 \sin kaL + C_1 aL = B_2 \sin k(1-a)L + C_2(1-a)L \tag{4.69}$$

$$B_1 \sin kaL = B_2 \sin k(1-a)L \tag{4.70}$$

$$B_1 k \cos kaL + B_2 k \cos k(1-a)L + C_1 + C_2 = 0 \tag{4.71}$$

$$B_1 \cos kaL + B_2 \cos k(1-a)L = \frac{R}{EIk^3}. \tag{4.72}$$

Notice that relations (4.69)-(4.72) have been derived using the following formulas:

$$\sum_{r=1}^{n} b_r F_r \sin(r\pi(1-\alpha))(-1)^r = -\sum_{r=1}^{n} b_r F_r \sin(r\pi a) \tag{4.73}$$

$$\sum_{r=1}^{n} b_r F_r \cos(r\pi(1-\alpha))(-1)^r = \sum_{r=1}^{n} b_r F_r \cos(r\pi a). \tag{4.74}$$

Now, the fundamental constrained BVP $\mathcal{BI}_1$ can be modified to the following problem:

Problem $\mathcal{BI}_{1\text{-}a}$:

Find a solution of the algebraic system of equations (4.69)-(4.72) with respect to the unknown coefficients B_1, C_1, B_2, C_2, so that restrictions (4.11)-(4.13) are satisfied.

Obviously, the values of these coefficients are different for each contact situation. Due to the inequality conditions (4.11), (4.12) and the complementarity condition (4.13), an examination for all the possible contact situations is required. Thus, the solution of the Problem $\mathcal{BI}_{1\text{-}a}$ is obtained under the separation of the problem into subproblems according to the possible contact situation (i.e. the constraint may be active, inactive or in neutral contact status).

4.3.2.1 Case of inactive constraint and of neutral contact status, $R = 0$ and $u < 0$

In the case where the unilateral constraint is inactive, the normal contact force is equal to zero. Then, the following problem has to be solved:

Problem $\mathcal{BI}_{1\text{-}a,1}$:

Find a solution of the algebraic system of equations (4.69)-(4.72) with respect to the unknown coefficients B_1, C_1, B_2, C_2, so that the following restrictions are satisfied:

$$R = 0 \tag{4.75}$$

$$u \leq 0\,. \tag{4.76}$$

Applying to the BVP the necessary conditions $R = 0$ and $u \leq 0$, the following relations are obtained:

$$w_1(aL) = B_1 \sin kaL + C_1 aL + \sum_{r=1}^{n} b_r F_r \sin r\pi a \leq 0 \tag{4.77}$$

$$w_2((1-a)L) = B_2 \sin(k(1-a)L) + C_2(1-a)L + \sum_{r=1}^{n} b_r F_r \sin r\pi a \leq 0 \tag{4.78}$$

$$B_1 \cos kaL + B_2 \cos k(1-a)L = \frac{R}{EIk^3} = 0\,. \tag{4.79}$$

From both relations (4.69) and (4.70), it is derived that:

$$C_1 aL = C_2(1-a)L \Rightarrow C_1 = \frac{C_2(1-a)}{a}\,. \tag{4.80}$$

Additionally, from equations (4.71) and (4.77), the following relation arises:

$$C_1 = -C_2\,. \tag{4.81}$$

Obviously, in order equations (4.80), (4.81) to be satisfied simultaneously, the following condition should hold:

$$C_1 = C_2 = 0\,. \tag{4.82}$$

Then, equations (4.70) and (4.71) formulate the following 2×2 linear homogeneous system with respect to the unknowns B_1, B_2:

$$\begin{bmatrix} \sin(kaL) & -\sin(k(1-a)L) \\ \cos(kaL) & \cos(k(1-a)L) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.83}$$

The above system admits obviously the zero solution (i.e. $B_1 = B_2 = 0$). Also, if the determinant of the system is equal to zero, infinite solutions may exist for B_1, B_2, and consequently, for the non-homogeneous BVP of the imperfect beam. Here:

$$D_{2\times 2}=\begin{vmatrix}\sin(kaL) & -\sin(k(1-a)L)\\ \cos(kaL) & \cos(k(1-a)L)\end{vmatrix}=$$

$$=\sin(kaL)\cos(k(1-a)L)+\cos(kaL)\sin(k(1-a)L)=\sin kL\,. \qquad (4.84)$$

In order to investigate the solvability of the studied BVP, Theorem 3.4 of Chapter 3 should be applied. More specifically, let us consider the value of $k=\dfrac{\pi}{L}$ which fulfills equation (4.84) and also constitutes eigenvalue of the corresponding homogeneous BVP (i.e. solution of the geometrically perfect structure). This value gives for (4.84) $D_{2\times 2}=0$. According to the aforementioned theorem, the non-homogeneous BVP is solvable (but not uniquely) if and only if the function of the right hand side of equation (4.55) (termed as $f(x)$ in the following relation) is orthogonal to every eigenfunction corresponding to that value of k, thus in essence, if the following equation holds for every value $a_0 \in \square^*$:

$$(f(x),a_0\sin\frac{\pi x_1}{L})=\int_0^{aL}\left[\sum_{r=1}^{n}b_r\left(\frac{r\pi}{L}\right)^4\sin(\frac{r\pi x_1}{L})\right]a_0\sin\frac{\pi x_1}{L}dx=0,\ (a_0\neq 0,\ b_1\neq 0)\ .(4.85)$$

The expansion of the above integral leads to:

$$\int_0^{aL}\left[\sum_{r=1}^{n}b_r\left(\frac{r\pi}{L}\right)^4\sin(\frac{r\pi x_1}{L})\right]a_0\sin\frac{\pi x_1}{L}dx_1=\left(b_1\left(\frac{\pi}{L}\right)^4a_0\right)\cdot\int_0^{aL}\sin(\frac{\pi x_1}{L})\sin\frac{\pi x_1}{L}dx_1+$$

$$+\left(b_2\left(\frac{2\pi}{L}\right)^4\sin(\frac{\pi x_1}{L})a_0\right)\cdot\int_0^{aL}\sin(\frac{2\pi x_1}{L})\sin(\frac{\pi x_1}{L})dx_1+.....$$

$$....+\left(b_n\left(\frac{n\pi}{L}\right)^4\sin(\frac{n\pi x_1}{L})a_0\right)\cdot\int_0^{aL}\sin(\frac{n\pi x_1}{L})\sin(\frac{\pi x_1}{L})dx_1=$$

$$=\left(b_1\left(\frac{\pi}{L}\right)^4a_0\right)\cdot\int_0^{aL}\sin(\frac{\pi x_1}{L})\sin(\frac{\pi x_1}{L})dx+0=\left(b_1\left(\frac{\pi}{L}\right)^4a_0\right)\int_0^{aL}\left(\sin(\frac{\pi x_1}{L})\right)^2dx=$$

$$=\left(b_1\left(\frac{\pi}{L}\right)^4a_0\right)\left[0.5x-0.25\frac{L}{\pi}\sin\frac{2\pi x_1}{L}\right]_0^{aL}=$$

$$=\left(b_1\left(\frac{\pi}{L}\right)^4 a_0\right)(0.5aL-0.25\frac{L}{\pi}\sin(2\pi a))\neq 0 \quad \forall a\in(0,1)\,. \tag{4.86}$$

Relation (4.86) verifies that the function of the right hand side of equation (4.55) is *not* orthogonal to none of the eigenfunctions which correspond to the eigenvalue $k=\frac{\pi}{L}$. Therefore, for values of loading inside the interval $(0,\frac{\pi^2 EI}{L^2})$ the BVP has a unique solution. However, it cannot be asserted that a unique solution can be extracted for all the values of loading. It is noticed that in the above example it was assumed that $b_1\neq 0$. Depending on the position of the unilateral constraint, the eigenmodes of the homogeneous BVP may not be orthogonal to the function of the right hand side of equation (4.55) (which corresponds to the function $f(x)$ of Theorem 3.4). Then, according to Theorem 3.4, this remark indicates that the non-homogeneous BVP accepts infinite solutions.

Leaving for the moment this interesting conclusion and considering values of the parameter k in equations (4.55) and (4.56) such that the BVP has a unique solution, the algebraic system (4.83) yields only the zero solution, thus $B_1=B_2=0$. This means that the solution of the under study non-homogeneous BVP consists only of the particular solution, i.e:

$$w_1(x_1)=\sum_{r=1}^{n} b_r F_r \sin(\frac{r\pi x_1}{L}), \quad x_1\in[0,aL] \tag{4.87}$$

$$w_2(x_2)=-\sum_{r=1}^{n} b_r F_r \sin(\frac{r\pi x_2}{L})(-1)^r, \quad x_2\in[0,(1-a)L]\,. \tag{4.88}$$

The above solution is valid only if the restriction introduced by the following inequality is satisfied:

$$w_1(aL)=\sum_{r=1}^{n} b_r F_r \sin r\pi a\leq 0 \quad \forall\, P\geq 0 \tag{4.89}$$

or equivalently,

$$w_2((1-a)L)=\sum_{k=1}^{n} b_r F_r \sin(r\pi(1-a))(-1)^r\geq 0 \quad \forall\, P\geq 0\,. \tag{4.90}$$

The previous inequality implies that a specific value of the axial load P that makes the inequality untrue is possible to exist. This value of load causes the development of the reaction force R and is termed as P_c. The situation for which $P = P_c$ corresponds to the neutral contact status situation. This particular value of loading is calculated from the limit case where (4.89) (or equivalently (4.90), holds as equality, i.e.

$$w_1(aL) = \sum_{r=1}^{n} b_r F_r \sin r\pi a = 0 \quad , \tag{4.91}$$

or equivalently,

$$w_2((1-a)L) = \sum_{k=1}^{n} b_r F_r \sin(r\pi(1-a))(-1)^r = 0 \,. \tag{4.92}$$

These latter equations derive an $(n-1)$ order polynomial algebraic equation with respect to the variable P. Obviously, negative or complex values of load P cannot be admissible solutions. Also it should be pointed out that more than one real positive solutions may exist, from which only the one with the smallest value is of interest.

4.3.2.2 Case of active constraint, $R < 0$ and $u = 0$

In case where the unilateral constraint is active the normal contact force is $R < 0$, and the following problem has to be solved:

Problem $\mathscr{BI}_{1\text{-}a,2}$:

Find a solution of the algebraic system of equations (4.69)-(4.72) with respect to the unknown coefficients B_1, C_1, B_2, C_2, so that the following restrictions are satisfied:

$$R < 0 \tag{4.93}$$

$$u = 0 \,. \tag{4.94}$$

Applying these restrictions to the relative boundary conditions, the following relations are obtained:

$$w_1(aL) = B_1 \sin kaL + C_1 aL + \sum_{r=1}^{n} b_r F_r \sin r\pi a = 0 \tag{4.95}$$

$$w_2((1-a)L) = B_2 \sin(k(1-a)L) + C_2(1-a)L + \sum_{r=1}^{n} b_r F_r \sin r\pi a = 0 \tag{4.96}$$

$$B_1 \cos kaL + B_2 \cos k(1-a)L = \frac{R}{EIk^3} < 0\,. \tag{4.97}$$

Relations (4.95), (4.96) lead to the determination of the coefficients C_1, C_2 :

$$C_1 = \frac{-B_1 \sin kaL - \sum_{r=1}^{n} b_r F_r \sin r\pi a}{aL} \tag{4.98}$$

$$C_2 = \frac{-B_2 \sin(k(1-a)L) - \sum_{r=1}^{n} b_r F_r \sin r\pi a}{(1-a)L}\,, \tag{4.99}$$

while equations (4.72), (4.73) formulate a linear algebraic system with respect to the coefficients B_1, B_2 :

$$\begin{bmatrix} \sin(kaL) & -\sin(k(1-a)L) \\ -k\cos(kaL) & \text{-}k\cos(k(1-a)L) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C_1 + C_2 \end{bmatrix}. \tag{4.100}$$

The determinant of the above linear system, gives:

$$D_{2\times 2} = \begin{vmatrix} \sin(kaL) & -\sin(k(1-a)L) \\ -k\cos(kaL) & \text{-}\cos k(k(1-a)L) \end{vmatrix} =$$

$$= -k\sin(kaL)\cos(k(1-a)L) - k\cos(kaL)\sin(k(1-a)L) = -k\sin kL \tag{4.101}$$

Obviously, the solvability of the non-homogeneous BVP depends on the different values of the parameter k in equations (4.55) and (4.56). Considering that the above system has a unique solution (i.e. either the values of the parameter k are not eigenvalues of the corresponding homogeneous BVP or they are eigenvalues which, however, produce eigenmodes not orthogonal to the functions of the right hand side of equations (4.55), (4.56)), and after a sequel of mathematical operations, the coefficients B_1, B_2 can be determined as:

$$B_1 = \frac{\left[\sum_{r=1}^{n} b_r F_r \sin[r\pi a]\right]\sin[k(1-a)L]}{a(1-a)kL\sin[kL] - \sin[kaL]\sin[k(1-a)L]} \quad (4.102)$$

[9]

$$B_2 = \frac{B_1 \sin[kaL]}{\sin[k(1-a)L]}. \quad (4.103)$$

Clearly, the above solution is valid only if the restriction introduced by the inequality condition (4.93), is satisfied, i.e.:

$$B_1 \cos[kaL] + B_2 \cos[k(1-a)L] < 0 \quad \forall P \geq 0. \quad (4.104)$$

If a value of the load P exists so that the left side of inequality (4.104) tends to zero, then the beam develops the tendency to be separated from the unilateral constraint. This axial load is termed as P_s . Obviously, the admissible values of P_s should belong into the set of positive real numbers ($\mathbb{R}^{+}$). From all the admissible solutions, only the smallest value is of interest.

4.3.3 *Singularities and infinite solutions*

As it was intuitively inferred in the previous paragraphs, depending on the type of the initial imperfection and on the position of the unilateral constraint, the following cases may appear for an axially loaded geometrically imperfect beam when the axial load takes values in an arbitrary open interval (P_1, P_2).

- If inside the interval (P_1, P_2) does not exist an eigenvalue of the corresponding homogeneous BVP, then the non-homogeneous BVP has a unique solution. This solution can be determined according to the formulas derived in paragraphs 4.3.2.1 and 4.3.2.2.

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[9]It is essential to notice that the denominator of relation (4.100) is actually the buckling equation (4.38) for the corresponding contact situation of the homogeneous BVP, due to the fact that

$$\cot[kaL] + \cot[k(1-a)L] = \frac{\sin[kL]}{\sin[kaL]\sin[k(1-a)L]}.$$

- If inside the interval (P_1, P_2) an eigenvalue of the corresponding homogeneous BVP exists, for which the corresponding eigenmode is *not* orthogonal to the function of the right hand side of equations (4.55) and (4.56), then the non-homogeneous BVP is unsolvable.
- If inside the interval (P_1, P_2) an eigenvalue of the corresponding homogeneous BVP exists, for which the corresponding eigenmode is orthogonal to the function of the right hand side of equations (4.55) and (4.56), then the non-homogeneous BVP has infinite solutions for the specific eigenvalue. The infinite solutions can be easily determined for the particular eigenvalues, by means of the boundary conditions.

It is then obvious that in the last two cases, the deflection curve appears certain “singularities” for these specific eigenvalues. The values of load which lead to these “singularities” cause instability and, therefore, have to be detected.

4.3.4 Calculation of the instability load

In the previous paragraphs the formulation and the solution of the unilateral contact elastic buckling problem of a geometrically imperfect beam with an intermediate unilateral constraint, was presented. This solution is actually the elastic deflection curve of the beam which is dependent on the axial loading, the type of the initial imperfection and the position of the unilateral support along the axis of the beam. As it was aforementioned, the critical equilibrium state for an imperfect beam is usually denoted by disproportionate large values of the deflection curve, when the loading tends to the value of the instability load ($P \to P_i$). The calculation of this load is accomplished via the determination of the poles[10] of the function representing the deflection curve. This type of instability results from the fact that for the instability load the non-homogeneous BVP is unsolvable. However, instability can also occur when the load becomes equal to the critical eigenvalue, for which the non-homogeneous BVP has infinite solutions. Depending on the contact situation (e.g. active constraint, inactive constraint or neutral condition) the poles of the deflection curve are the eigenvalues of the buckling equations of the corresponding bifurcation problem, a topic that has been addressed in Section 4.2. Due to the different contact cases which can be developed during the bending deformation, the determination of the instability load is not a simple issue. For this reason, the following convenient calculation procedure is proposed.

Initially, according to Section 4.2, the eigenvalues for each contact case (i.e. active, inactive and neutral contact status) are extracted. Then,

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[10] As a pole of a function is defined the point for which the limit of the function about this point, tends to infinity.

Step 1

The initial imperfection is applied and the deflection at the position of the unilateral support is examined (i.e. if $u<0$ or if $u=0$).

Step 2

If $u<0$, then relations (4.87)-(4.90) hold. It is then checked whether a valid value P_c exists, yielded by relations (4.91) or (4.92) when they hold as equalities.

a) If these relations do not produce a valid value for P_c , then it is sure that the beam will never come in contact with the unilateral support. In this case the procedure continues with Step 5.

b) If these relations produce a valid value for P_c, it is then examined if an eigenvalue of the corresponding BVP of the perfect structure, located inside the interval $[0, P_c]$, exists. If such an eigenvalue does not exist, then the beam is able to sustain more loading till the unstable equilibrium state and the procedure continues with Step 4. If such an eigenvalue exists, then it is checked if the latter produces eigenmodes orthogonal to the particular each time function of the right hand side part of the fundamental equation (3.31).

   - If yes, then the deflection curve has infinite solutions for this certain value of loading and thus a singularity point appears. In this case the beam buckles "suddenly" and the instability load is actually this eigenvalue.
   - If not, then this eigenvalue is also the instability load of the beam due to the fact that as the applied load approaches the specific eigenvalue the deflections of the beam take extremely large values.

In both cases:

$$P_{in} = P_{cr,eig}^{(I)} \qquad (4.105)$$

Step 3

If $u=0$, then the beam is in contact with the constraint and the relations of paragraph 4.3.2.2 hold. It is then checked, whether a valid value P_s exists, yielded by relation (4.104) when it holds as equality ($R=0$). In this case the beam develops the tendency to be separated from the unilateral support.

a) If relation (4.104) does not produce a valid value for P_s, then it is sure that the beam will never lose contact with the unilateral support. In this case the procedure is continued with Step 4.

b) If relation (4.104) produces a valid value for P_s, then it is examined if there exists an eigenvalue, $P^{(A)}_{cr,eig}$, of the corresponding homogeneous BVP of the perfect structure, located inside the interval $[0, P_s]$, that either produces disproportionate large deflections or leads to a singular case expressed through a different function of the deflection curve.

- If not, the procedure continues with Step 5.
- If yes, then this eigenvalue constitutes the instability load of the beam and the procedure is terminated. In this case,

$$P_{in} = P^{(A)}_{cr,eig} \tag{4.106}$$

Step 4

Having reached in this step, the deflection curve is in contact with the unilateral support and will remain in contact till the maximum value of the loading, which leads to unstable equilibrium state, has been attained, therefore, $u = 0$ and $R < 0$. Now the relations of paragraph 4.3.2.2 hold. Two different cases can be distinguished, depending on the previous steps which led to this specific situation. More specifically:

a) Case of arriving in Step 4 from Step 2b

In this case the beam has come in contact with the unilateral support for $P = P_c$ and has the ability to sustain more loading. For loads $P > P_c$, the bending behaviour of the beam is described by the set of equations (4.95)-(4.104). The instability load is equal to the critical eigenvalue ($P^{(A)}_{cr,eigen}$) which corresponds to the existing contact case (i.e. the case of the active constraint). The type of instability can be examined through the utilization of Theorem 3.4.

b) Case of arriving in Step 4 from Step 3a

As in the previous case, the beam will either undergo disproportionate large deflections as the applied load approaches the critical eigenvalue of the corresponding bifurcation problem ($P^{(A)}_{cr,eigen}$), or lose its stability suddenly as the load takes a value equal to that eigenvalue. Similarly, the type of instability can be examined through the utilization of Theorem 3.4.

In both cases of Step 4:

$$P_{in} = P_{cr,eig}^{(A)} \tag{4.107}$$

Step 5

The deflection curve is not in contact with the unilateral support and this situation will not change till the unstable equilibrium state will be attained, therefore $u \leq 0$ and $R = 0$ and relations (4.85),(4.86) hold. In this case the determination of the instability load is based on the following:

a) Case of arriving in Step 5 from Step 2a
In this case the bending behaviour of the beam is described by equations (4.87) and (4.88) until the unstable equilibrium state. The instability load is the critical eigenvalue (P_{cr_eig}) which corresponds to the case of the inactive constraint, i.e the Euler load of the simply supported beam:

$$P_{in} = P_{cr,eig}^{(I)} = \frac{\pi^2 EI}{L^2}. \tag{4.108}$$

b) Case of arriving in Step 5 from Step 3a

In case where the beam separates from the unilateral support (Step 3a), two different cases can be occur depending on the value of P_s :

- If $P_s < P_{cr,eig}^{(I)}$ (where $P_{cr,eig}^{(I)}$ is the critical eigenvalue of the existing contact case, i.e the case of the inactive constraint), then the beam is able to sustain more loading until that critical value is reached.
- If $P_s > P_{cr,eig}^{(I)}$, then the beam cannot stay in equilibrium and the deflections of the beam are accompanied by an abrupt decrease of the applied load to lower values.

In both cases the instability load is equal to the critical eigenvalue $P_{cr,eig}^{(I)}$.

$$P_{in} = P_{cr,eig}^{(I)} \tag{4.109}$$

Moreover, the type of instability has to be checked, as in the previous cases.

The solution procedure which has been described in this section is also displayed in the flow-chart of Fig.4.3 and will be clearly demonstrated in the examples treated in Chapter 7.

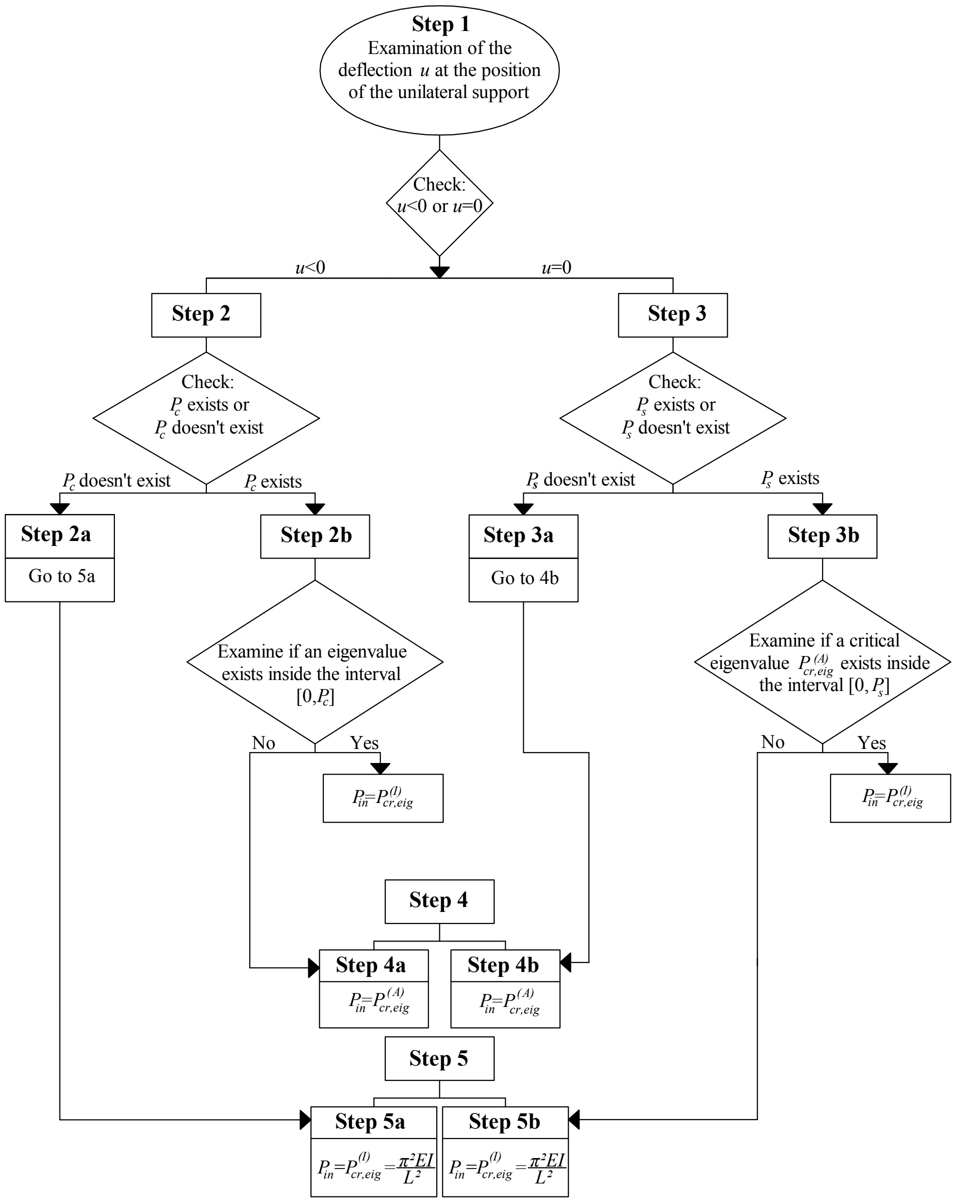

Fig. 4.3 The flow chart diagram of the proposed calculation procedure.

4.4 Failure axial load of the continuous beam considering the cross-section strength

4.4.1 Generalities

Even though the methodology described in the previous section is able to handle the unilateral contact elastic buckling problem of continuous beams with geometric initial imperfections, from a practical point of view, the calculated instability load is not the ultimate load that the beam can sustain. As it is obvious from the previous paragraphs, two types of loads causing instability have been calculated so far. The first one, the critical load, is the smallest eigenvalue (compatible with the restriction introduced by the unilateral constraint) calculated from the homogeneous BVP of the perfect continuous beam. This load surely does not reflect the realistic buckling load of the beam, due to the fact that it does not take into account the inevitable initial geometric imperfections which are present in real structures. For this reason, a second type of load, the instability load, was determined in Section 4.3 for imperfect continuous beams. Nevertheless, this load cannot be considered as the maximum load that the beam can sustain when a real design case is under study, due to the fact that it does not take into account the cross-section actual strength. Clearly, as the loading increases, the transverse deflections of the beam increase disproportionately, leading eventually to the exhaustion of the ultimate strength of the cross-sections of the beam. Moreover, the exhaustion of the capacity of the cross-section should be examined under the coexistence of the compressive axial load with the second order bending moment, which, in turn, depends on the magnitude of the developed deflections. Therefore, the ultimate load of a real life problem depends on the specific shape of the initial imperfections, their amplitudes and on the strength of the actual cross-section of the beam. For the determination of this ultimate load, the proposed method should be equipped with design criteria, connected with the strength of the cross-sections. Without losing generality, it is assumed that the beams which are considered in the present dissertation are made of steel and, therefore, the provisions of Eurocode 3 (EN 1993.01.01 (2005)) apply. As a result, the design criterion is related with the bending moment resistance $M_{N.Rd}$ which is computed considering also the effect of the axial compressive force P. More specifically, at each point of the beam the following relation should be fulfilled:

$$M_{Ed} \leq M_{N.Rd}\,, \tag{4.110}$$

where M_{Ed} is the design second order bending moment that can be determined through the following equations:

$$M_{Ed}(x_1) = -EIw''_{1,p}(x_1) = -EI(w''_1(x_1) - w''_{1,0}(x_1)) \quad x_1 \in [0, aL] \tag{4.111}$$

$$M_{Ed}(x_1) = -EIw''_{2,p}(x_1) = -EI(w''_2(x_2) - w''_{2,0}(x_2)) \quad x_2 \in [0, (1-a)L], \tag{4.112}$$

where, $w''_{1,p}$ and $w''_{2,p}$ are the second derivatives of the elastic transverse deflection which are attributed solely to the axial loading P. The calculation of the bending moment resistance $M_{N.Rd}$ depends on the type of the cross-section and the appropriate relations can be found in Eurocode 3.

4.4.2 *Calculation of the second-order bending moment as a function of the unilateral constraint conditions*

Using the expressions obtained in Section 4.3 for the deflections w_1 and w_2, the following equations are obtained that give the function of the bending moment when the constraint is inactive and active, respectively.

a. Inactive unilateral constraint and neutral contact status, i.e. $R = 0$ and $u \leq 0$:

$$M_{Ed}(x_1) = -EI\left[\sum_{k=1}^{n} b_r \left(\frac{r\pi}{L}\right)^2 (1 - F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right] \quad x_1 \in [0, aL] \tag{4.113}$$

$$M_{Ed}(x_2) = -EI\left[\sum_{k=1}^{n} b_r \left(\frac{r\pi}{L}\right)^2 (F_r - 1)\sin\left(\frac{r\pi x_2}{L}\right)(-1)^k\right] \quad x_2 \in [0, aL] \tag{4.114}$$

b. Active unilateral constraint, i.e. $R < 0$ and $u = 0$:

$$M_{Ed}(x_1) = -EI\left[-B_1 k^2 \sin(kx_1) + \sum_{r=1}^{n} b_r \left(\frac{r\pi}{L}\right)^2 (1 - F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right], \quad x_1 \in [0, aL] \tag{4.115}$$

$$M_{Ed}(x_2) = -EI[-B_2 k^2 \sin(kx_2) + \sum_{r=1}^{n} b_r \left(\frac{r\pi}{L^2}\right)(F_r - 1)\sin\left(\frac{r\pi x_2}{L}\right)(-1)^r, \quad x_2 \in [0, (1-a)L]. \tag{4.116}$$

5 The unilateral buckling of beams - Part 2

5.1 Introduction

In the previous chapter the contact buckling problem of a beam with one intermediate unilateral constraint was treated. The presented methodology can be extended in order to solve the same problem when more than one unilateral constraints are likely to be present in the considered beam. In general, there is no difficulty in the formulation of the latter contact problem, apart from the complexity of the mathematical operations which are increased.

The present chapter deals with the contact buckling problem of beams when the functioning of the unilateral constraints is such that their reactions are likely to have opposite signs (Fig. 5.1). For the simplification of the required calculations and without compromising generality, the contact buckling problem of a beam with two intermediate unilateral constraints in an “opposite” functioning mode is considered here. The proposed methodology can be extended in order to handle more than two intermediate constraints. For the formulation of the problem, assumptions similar to those of the contact buckling problem of Chapter 4 will be used. Again, the described methodology will be an extension of Euler’s equilibrium method.

This study concerns two different types of the considered structure, i.e. the perfect beam and the geometrically imperfect beam. For the two individualy studied cases, analytical solutions are derived which can be applied in several different classes of contact buckling problems. In the present Chapter, only the formulation and the solution of the aforementioned problem are presented. The range of applications which the described methodology has the potential to handle, is presented in the next chapter through the demonstration of several examples.

5.2 Formulation of the elastic contact buckling problem of a geometrically perfect continuous beam with two opposite functioning unilateral supports

5.2.1 *Formulation*

A geometrically perfect (i.e. without any initial geometric imperfections) beam with two intermediate unilateral constraints having opposite functions is considered, subjected to an axial compressive load (Fig.5.1). The beam is divided into three spans, Span I, Span II and Span III, having lengths aL , bL and cL

respectively, where L is the total length of the beam. The three spans are equipped with the coordinate systems x_1, w_1 , x_2, w_2 and x_3, w_3 as it is shown in Fig. 5.1. The coordinates x_1, x_2, x_3 measure the position along the axis of the beam in each span and w_1, w_2, w_3 denote the corresponding transverse deflections.

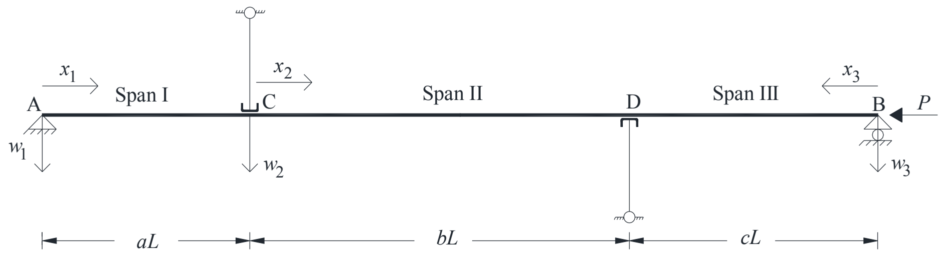

Fig. 5.1 The buckling problem of a beam with two intermediate unilateral constraints having opposite functions.

The positive internal forces are assumed to follow the positive directions of Fig. 3.1. For the description of the bending behaviour of the beam, the Euler's equilibrium method can be applied, as it was already explained in Section 3.4, leading to a fourth-order homogeneous differential equation for each span of the beam:

$$\frac{d^4 w_1(x_1)}{dx_1^{\,4}} + k^2 \frac{d^2 w_1(x_1)}{dx_1^{\,2}} = 0 \quad x_1 \in [0, aL] \tag{5.1}$$

$$\frac{d^4 w_2(x_2)}{dx_2^{\,4}} + k^2 \frac{d^2 w_2(x_2)}{dx_2^{\,2}} = 0 \quad x_2 \in [0, bL] \tag{5.2}$$

$$\frac{d^4 w_3(x_3)}{dx_3^{\,4}} + k^2 \frac{d^2 w_3(x_3)}{dx_3^{\,2}} = 0 \quad x_3 \in [0, cL] \; . \tag{5.3}$$

The parameter k is given by relation (3.12). If a non-trivial solution for the above equations exists, the beam can be in equilibrium in a curved configuration different from the straight line one (bifurcation equilibrium state). Therefore, the solution of the above equations gives the transverse deflections w_1, w_2, w_3 of the beam at any point, as a function of the compressive load P. The boundary conditions of the problem are formulated taking into account the essential boundary conditions, the natural boundary conditions and the unilateral contact conditions at the points of the unilateral supports. For the certain case with the two intermediate unilateral supports which produce reactions with opposite signs, twelve boundary conditions and two sets of restrictions exist. More specifically:

1. Essential boundary conditions

- Zero vertical displacement at the positions of the classical supports (points A,B):

$$w_1(0) = 0 \tag{5.4}$$

$$w_3(0) = 0 \tag{5.5}$$

- Common vertical displacement at the unilateral support at point C:

$$w_1(aL) = w_2(0) = u_1 \tag{5.6}$$

- Common vertical displacement at the unilateral support at point D:

$$w_2(bL) = w_3(cL) = u_2 \tag{5.7}$$

2. Natural boundary conditions

- Common rotation at the position of the unilateral support at point C:

$$w_1'(aL) = w_2'(0) \tag{5.8}$$

- Common rotation at the position of the unilateral support at point D:

$$-w_2'(bL) = w_3'(cL) \tag{5.9}$$

- Zero bending moment at the positions of the two ends of the beam (points A,B):

$$-EIw_1''(0) = 0 \tag{5.10}$$

$$-EIw_3''(0) = 0 \tag{5.11}$$

- Moment equilibrium at the position of the unilateral support at point C:

$$-EIw_1''(aL) + EIw_2''(0) = 0 \tag{5.12}$$

- Moment equilibrium at the position of the unilateral support at point D:

$$-EIw_2''(bL) + EIw_3''(cL) = 0 \tag{5.13}$$

- Forces equilibrium at the position of the unilateral support (points C and D)

  In order to formulate the boundary conditions that correspond to the unilateral constraints, the support reaction of the unilateral constraint at point C (denoted as R_1) and the support reaction of the unilateral constraint at point D (denoted as R_2) should be considered with an unknown value (Fig. 5.2). Obviously, the existence of this reaction force R_i, $(i = 1,2)$ depends on whether the unilateral constraint is active or not. In any case, the equilibrium of forces at these two points is written as:

$$\left[-EIw_1'''(aL) - Pw_1'(aL)\right] - \left[-EIw_2'''(0) - Pw_2'(0)\right] = R_1 \tag{5.14}$$

$$\left[-EIw_2'''(bL) - Pw_2'(bL)\right] + \left[-EIw_2'''(cL) - Pw_3'(cL)\right] = R_2 \tag{5.15}$$

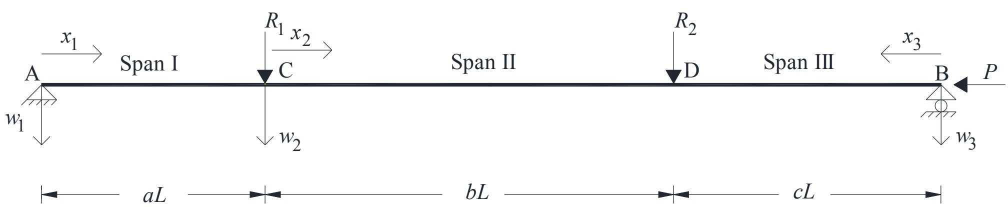

Fig 5.2 The considered positive directions for the unknown reaction forces R_1 and R_2 of the unilateral supports.

3. Unilateral contact boundary conditions at the positions of the unilateral constraints

   The unilateral constraint at point C may be represented through the following inequality conditions (Panagiotopoulos, 1985):

$$w_1(aL) = w_2(0) = u_1 \geq 0 \tag{5.16}$$

$$R_1 \geq 0 \tag{5.17}$$

$$R_1 \cdot u_1 = 0\,. \tag{5.18}$$

Similarly, the unilateral constraint at point D may be represented through the following inequality conditions:

$$w_2(bL) = w_3(cL) = u_2 \le 0 \tag{5.19}$$

$$R_2 \le 0 \tag{5.20}$$

$$R_2 \cdot u_2 = 0\,. \tag{5.21}$$

Thus, the unilateral contact buckling problem of the continuous beam with the two intermediate unilateral constraints with opposite functions is formulated as:

Problem $\mathcal{BP}_2$:

Find $w(x) = \left\{ \begin{array}{l} w(x_1),\ x_1 \in [0, aL] \\ w(x_2),\ x_2 \in [0, bL] \\ w(x_3),\ x_3 \in [0, cL] \end{array} \right\} \in L_2(0,L)$ of the system of differential equations (5.1)- (5.3) such as the equality boundary conditions (5.4)-(5.15), the inequality constraints (5.16), (5.17), (5.19), (5.20), and the complementarity conditions (5.18), (5.21) are fulfilled.

5.2.2 Examination of the BVP for all the possible contact situations

The BVP discussed in the previous paragraph consists of homogeneous fourth order differential equations, which have, respectively, general solutions of the following form:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 \tag{5.22}$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 \tag{5.23}$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 \tag{5.24}$$

The full expression of the deflection curve of the continuous beam is given after the determination of the coefficients $A_1,\ B_1,\ C_1,\ D_1,\ A_2,\ B_2,\ C_2,\ D_2,\ A_3,\ B_3,\ C_3,\ D_3$ These coefficients are calculated through the boundary conditions of the problem. However, due to the presence of the inequality conditions, the calculation of the above coefficients requires an appropriate examination for all the possible contact situations. Any of the two constraints may be active, inactive or in a neutral contact status (both $u_i = 0$ and $R_i = 0$). Obviously, there are nine different possible contact combinations which are presented in Table 5.1.

Contact Case	Unilateral constraint placed at point C	Unilateral constraint placed at point D	Contact Status
CC1	Active ($R_1 > 0,\ u_1 = 0$)	Active ($R_2 < 0,\ u_2 = 0$)	A-A
CC2	Inactive ($R_1 = 0,\ u_1 > 0$)	Inactive ($R_2 = 0,\ u_2 < 0$)	I-I
CC3	Active ($R_1 > 0,\ u_1 = 0$)	Inactive ($R_2 = 0,\ u_2 < 0$)	A-I
CC4	Inactive ($R_1 = 0,\ u_1 > 0$)	Active ($R_2 < 0,\ u_2 = 0$)	I-A
CC5	Active ($R_1 > 0,\ u_1 = 0$)	Neutral Contact Status ($R_2 = 0,\ u_2 = 0$)	A-N
CC6	Inactive ($R_1 = 0,\ u_1 > 0$)	Neutral Contact Status ($R_2 = 0,\ u_2 = 0$)	I-N
CC7	Neutral Contact Status ($R_1 = 0,\ u_1 = 0$)	Active ($R_2 < 0,\ u_2 = 0$)	N-A
CC8	Neutral Contact Status ($R_1 = 0,\ u_1 = 0$)	Inactive ($R_2 < 0,\ u_2 = 0$)	N-I
CC9	Neutral Contact Status ($R_1 = 0,\ u_1 = 0$)	Neutral Contact Status ($R_2 = 0,\ u_2 = 0$)	N-N

Table 5.1 The nine possible different contact combinations of the beam with two opposite functioning unilateral supports.

Applying the boundary conditions at the ends of the beam, i.e. the equations (5.4), (5.5), (5.10) and (5.11), the following relations are obtained:

$$w_1(0) = 0 \Rightarrow A_1 + D_1 = 0 \tag{5.25}$$

$$w_3(0) = 0 \Rightarrow A_3 + D_3 = 0 \tag{5.26}$$

$$-EIw_1''(0) = 0 \Rightarrow -A_1 k^2 = 0 \Rightarrow A_1 = 0 \quad (k \neq 0 \text{ for } P \neq 0) \tag{5.27}$$

$$-EIw_3''(0)=0 \Rightarrow -A_3k^2=0 \Rightarrow A_3=0 \quad (k\neq 0 \text{ for } P\neq 0)\,. \tag{5.28}$$

Obviously, $A_1=A_3=D_1=D_3=0$. Then, applying the boundary conditions at the points of the unilateral constraints (points C and D) an algebraic 8×8 system of equations with respect to the coefficients $B_1, C_1, A_2, B_2, C_2, D_2, B_3, C_3$, is formulated:

$$w_1(aL)=w_2(0)=u_1 \Rightarrow B_1\sin kaL + C_1 aL = A_2 + D_2 \tag{5.29}$$

$$-EIw_1''(aL)+EIw_2''(0)=0 \Rightarrow A_2 = B_1\sin kaL \tag{5.30}$$

$$w_1'(aL)=w_2'(0) \Rightarrow B_1 k\cos kaL + C_1 = B_2 k + C_2 \tag{5.31}$$

$$\left[-EIw_1'''(aL)-Pw_1'(aL)\right]-\left[-EIw_2'''(0)-Pw_2'(0)\right]=R_1 \Rightarrow$$

$$\Rightarrow B_1\cos kaL - B_2 = \frac{R_1}{EIk^3} \tag{5.32}$$

$$\begin{aligned} w_2(bL)=w_3(cL)=u_2 \Rightarrow A_2\cos kbL + B_2\sin kbL + C_2 bL + D_2 &= \\ = B_3\sin kcL + C_3 cL & \end{aligned} \tag{5.33}$$

$$-EIw_2''(bL)+EIw_3''(cL)=0 \Rightarrow A_2\cos kbL + B_2\sin kbL = B_3\sin kcL \tag{5.34}$$

$$-w_2'(bL)=w_3'(cL) \Rightarrow A_2 k\sin kbL - B_2 k\cos kbL - C_2 = B_3 k\cos kcL + C_3 \tag{5.35}$$

$$\left[-EIw_2'''(bL)-Pw_2'(bL)\right]+\left[-EIw_3'''(cL)-Pw_3'(cL)\right]=R_2 \Rightarrow$$

$$\Rightarrow -A_2\sin kbL + B_2\cos kbL + B_3\cos kcL = \frac{R_2}{EIk^3}\,. \tag{5.36}$$

The above algebraic system can be written in matrix form as follows:

$$\begin{bmatrix} \sin kaL & aL & -1 & 0 & 0 & -1 & 0 & 0 \\ \sin kaL & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ k\cos kaL & 1 & 0 & -k & -1 & 0 & 0 & 0 \\ \cos kaL & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos kbL & \sin kbL & bL & 1 & -\sin kcL & -cL \\ 0 & 0 & \cos kbL & \sin kbL & 0 & 0 & -\sin kcL & 0 \\ 0 & 0 & k\sin kbL & -k\cos kbL & -1 & 0 & -k\cos kcL & -1 \\ 0 & 0 & -\sin kbL & \cos kbL & 0 & 0 & \cos kcL & 0 \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ C_1 \\ A_2 \\ B_2 \\ C_2 \\ D_2 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{R_1}{EIk^3} \\ 0 \\ 0 \\ 0 \\ \dfrac{R_2}{EIk^3} \end{bmatrix} \tag{5.37}$$

Finally, the non-homogeneous constrained BVP $\mathcal{BP}_2$ can be replaced equivalently by the following problem:

Problem $\mathcal{BP}_{2\text{-}a}$

Find a solution of the algebraic system of equations (5.37) with respect to the unknowns B_1, C_1, A_2, B_2, C_2, D_2, B_3, C_3 so that the restrictions (5.16)-(5.21) are satisfied.

Obviously, the values of these coefficients are different for each contact situation. Due to the inequality conditions (5.16), (5.17), (5.19), (5.20) and the complementarity conditions (5.18), (5.21), an examination for all the possible contact situations is required. Thus, the solution of the Problem $\mathcal{BP}_{2\text{-}a}$ is obtained under the separation of the problem into subproblems, according to the possible contact situations (see Table 5.1).

5.2.2.1 Active constraints, $R_1 > 0\,, u_1 = 0$ and $R_2 < 0\,, u_2 = 0$

In the case where the unilateral constraints are active, the displacements u_1, u_2 at the points C, D respectively, are equal to zero. Therefore, the following problem has to be solved:

Problem $\mathcal{BP}_{2\text{-}a,1}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.38}$$

$$R_1 > 0 \tag{5.39}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.40}$$

$$R_2 < 0\,. \tag{5.41}$$

Applying to equation (5.29) the demand of having zero displacement at point C and taking into account relation (5.30), the coefficients C_1, D_2 can be calculated as a function of the unknown coefficient B_1, while from relation (5.33) the coefficient C_3 is calculated as a function of the unknown coefficient B_3 , i.e.:

$$C_1 = \frac{-B_1 \sin kaL}{aL} \tag{5.42}$$

$$D_2 = -B_1 \sin kaL \tag{5.43}$$

$$C_3 = \frac{-B_3 \sin kcL}{cL}\,. \tag{5.44}$$

Substituting relation (5.34) into equation (5.33) and taking into account equation (5.43), the coefficient C_2 is determined:

$$C_2 = \frac{B_1 \sin kaL - B_3 \sin kcL}{bL}\,. \tag{5.45}$$

Then, substituting relations (5.44) and (5.45) into the boundary equation (5.31), the coefficient B_2 can be calculated as a function of the coefficients B_1 and B_3 :

$$B_2 = B_1 \cos kaL - B_1 \frac{a+b}{kabL} \sin kaL + B_3 \frac{\sin kcL}{kbL}\,. \tag{5.46}$$

Till now, all the unknown coefficients have been calculated as functions of the unknowns B_1 and B_3. Obviously, substituting all the above calculated coefficients into the unused relations (5.33) and (5.35), a 2×2 algebraic system with respect to the unknowns B_1 and B_3 is formulated. More specifically, after the appropriate mathematical operations, the following equations are obtained:

$$u_2 = 0 \Rightarrow A_2 \cos kbL + B_2 \sin kbL + C_2 bL + D_2 = 0 \Rightarrow$$

$$\Rightarrow B_1\left[\sin\left[k(a+b)L\right]-\frac{a+b}{kabL}\sin kaL\sin kbL\right]+B_3\left[\frac{\sin kcL\sin kbL}{kbL}-\sin kcL\right]=0 \quad (5.47)$$

$$A_2 k\sin kbL - B_2 k\cos kbL - C_2 = B_3 k\cos kcL + C_3 \Rightarrow$$

$$B_1[-k\cos\left[k(a+b)L\right]+\frac{a+b}{abL}\sin kaL\cos kbL-\frac{\sin kaL}{bL}]+$$

$$+B_3[-\frac{\sin kcL\cos kbL}{bL}-k\cos kcL+\frac{c+b}{bcL}\sin kcL]=0\,. \quad (5.48)$$

In order to have a non-trivial solution for the specific BVP, the determinant of the above homogeneous algebraic system of equations (5.47)-(5.48) with respect to B_1 and B_3 should be equal to zero, thus:

$$D_{2\times 2}=0\Rightarrow\frac{a+b}{abL}\sin\left[kaL\right]\sin\left[k(b+c)L\right]+\frac{b+c}{bcL}\sin\left[kcL\right]\sin\left[k(a+b)L\right]=$$

$$=k\sin kL+\frac{2\sin\left[kaL\right]\sin\left[kcL\right]}{bL}+\frac{\sin\left[kaL\right]\sin\left[kbL\right]\sin\left[kcL\right]}{kabcL^2}\,. \quad (5.49)$$

Equation (5.49) constitutes the **buckling equation** for the studied contact case of the active constraints. The certain equation belongs to the family of the trigonometric transcendental equations producing infinite eigenvalues k_n. Obviously, infinite pairs of the unknowns coefficients B_1 and B_3 exist, for which the treated BVP has a non-trivial solution. Due to the fact that the BVP is valid under certain restrictions which are imposed by the "unilateral" status of the intermediate supports, the coefficients B_1 and B_3 should be appropriately calculated , so that restrictions (5.38)-(5.41) are satisfied.

It is noticed that one of the aforementioned coefficients (e.g. B_1) may be chosen arbitrarily while the other is calculated either from equation (5.47) or equation (5.48). The solution of the BVP where the two intermediate constraints are considered as active is then given by relations (5.22)-(5.24) just substituting the above calculated coefficients.

5.2.2.2 Inactive constraints, $R_1 = 0,\ u_1 > 0$ and $R_2 = 0,\ u_2 < 0$

When the unilateral constraints are simultaneously inactive, the following problem has to be solved:

Problem $\mathscr{BP}_{2\text{-}a,2}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 > 0 \tag{5.50}$$

$$R_1 = 0 \tag{5.51}$$

$$w_2(bL) = w_3(cL) = u_2 < 0 \tag{5.52}$$

$$R_2 = 0\,. \tag{5.53}$$

The reaction forces R_1, R_2 are equal to zero, thus equations (5.32) and (5.36) are transformed to the following:

$$B_1 \cos kaL - B_2 = \frac{R_1}{EIk^3} = 0 \Rightarrow B_2 = B_1 \cos kaL \tag{5.54}$$

$$-A_2 \sin kbL + B_2 \cos kbL + B_3 \cos kcL = \frac{R_2}{EIk^3} = 0 \Rightarrow$$

$$\Rightarrow B_3 \cos kcL = A_2 \sin kbL - B_2 \cos kbL\,. \tag{5.55}$$

Substituting equation (5.30) into (5.29) and equation (5.54) into (5.31), the following relations are obtained respectively:

$$C_1 aL = D_2 \tag{5.56}$$

$$C_1 = C_2\,. \tag{5.57}$$

In a similar way, the substitution of equation (5.55) into (5.35) gives:

$$C_2 = -C_3 = C_1\,. \tag{5.58}$$

Then, taking into account the boundary equation (5.33) in combination with equation (5.34) and substituting into them the above calculated coefficients, the following equality arises:

$$C_2 bL + D_2 = C_3 cL \Rightarrow C_1(a+b+c)L = 0 \Rightarrow C_1 = 0\,. \tag{5.59}$$

In the above, the obvious relation $a+b+c=1$ was used. Due to the fact that coefficient C_1 is equal to zero, coefficients C_2, C_3, D_2 are also taking zero values. Then, taking into account equations (5.30) and (5.54), relations (5.33) and (5.34) reduce to an algebraic system with respect to the unknown coefficients B_1 and B_3. After a series of mathematical operations this system takes the following matrix form:

$$\begin{bmatrix} \cos[k(a+b)L] & \cos kcL \\ \sin[k(a+b)L] & -\sin kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.60}$$

In order the under study BVP to have a non-trivial solution, the determinant of the above system should be equal to zero, thus:

$$D_{2\times 2} = \begin{vmatrix} \cos[k(a+b)L] & \cos kcL \\ \sin[k(a+b)L] & -\sin kcL \end{vmatrix} = 0 \Rightarrow \sin kL = 0 \Rightarrow k_n = \frac{n\pi}{L}, \quad n = 1,2,3.... \tag{5.61}$$

Equation (5.61) constitutes the **buckling equation** of the certain contact condition. It is essential to notice that some of the eigenvalues k_n are producing non admissible deflection curves. This results from the following restrictions which should be satisfied simultaneously:

$$u_1 > 0 \Rightarrow B_1 \sin(n\pi\alpha) > 0, \quad n = 1,2,3,....... \tag{5.62}$$

$$u_2 < 0 \Rightarrow B_1 \sin(n\pi(a+b)) < 0, \quad n = 1,2,3........ \tag{5.63}$$

The determination of the coefficient B_3 is achieved through the usage of one of the equations of the algebraic system (5.60) after the arbitrary (but compatible with the restrictions (5.50)-(5.53)) selection of the coefficient B_3. The corresponding eigenmodes of the studied BVP are the buckle deflection curves which are determined substituting the calculated coefficients for each admissible eigenvalue k_n, into equations (5.22)-(5.24).

As it will be shown in the following paragraphs, the buckling equation (5.61) is also encountered in all the contact cases where the reaction forces of the supports are equal to zero (i.e. in cases 6,8,9, of Table 5.1). Clearly, each contact case with respect to the corresponding restrictions formulates its own set of admissible solutions which are obtained from the solution of the buckling equation (5.61).

5.2.2.3 Active constraint at point C and inactive constraint at point D, $R_1 > 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 < 0$

When the constraint at point D is inactive while the constraint at point C is active, the following problem has to be solved:

Problem $\mathcal{BP}_{2\text{-}a,3}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.64}$$

$$R_1 > 0 \tag{5.65}$$

$$w_2(bL) = w_3(cL) = u_2 < 0 \tag{5.66}$$

$$R_2 = 0\,. \tag{5.67}$$

The reaction force R_1 and the vertical displacement u_1 are taking zero values, thus the corresponding boundary equations (5.36) and (5.29) are transformed in the following relations:

$$A_2 \sin kbL - B_2 \cos kbL = B_3 \cos kcL \tag{5.68}$$

$$B_1 \sin kaL + C_1 aL = 0 \Rightarrow C_1 = \frac{-B_1 \sin kaL}{aL} \tag{5.69}$$

$$A_2 + D_2 = 0 \Rightarrow D_2 = -A_2\,. \tag{5.70}$$

Considering now equation (5.30), coefficient D_2 can be written as:

$$D_2 = -B_1 \sin kaL\,. \tag{5.71}$$

Then, substituting relation (5.58) into the boundary equation (5.36) it arises that:

$$C_2 = -C_3\,. \tag{5.72}$$

In a similar way, substituting equation (5.34) into equation (5.33) and using the above relations (5.71) and (5.72) the following equation is obtained:

$$C_2 bL + D_2 = C_3 cL \Rightarrow C_3 = \frac{-B_1 \sin kaL}{(b+c)L} \tag{5.73}$$

from which the unknown coefficient C_3 can be determined.
Coefficient B_2 can be determined from the boundary equation (5.31) as a function of the unknown coefficient B_1:

$$B_1 k \cos kaL + C_1 = B_2 k + C_2 \Rightarrow B_2 = B_1 \left[\cos kaL - \frac{\sin kaL}{k(b+c)aL} \right] \tag{5.74}$$

Obviously, equations (5.68) and (5.34) formulate an algebraic system with respect to the unknowns B_1, B_3 which can be written in matrix form:

$$\begin{bmatrix} \dfrac{\sin kaL \cos kbL}{ka(b+c)L} - \cos[k(a+b)L] & -\cos kcL \\ \sin\left[k(a+b)L\right] - \dfrac{\sin kaL \sin kbL}{ka(b+c)L} & -\sin kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.75}$$

In order the studied BVP to have a non-trivial solution, the determinant of the above system should be equal to zero. A series of mathematical operations leads to the buckling equation of the considered contact case:

$$D_{2\times 2} = \begin{vmatrix} \dfrac{\sin kaL \cos kbL}{ka(b+c)L} - \cos[k(a+b)L] & -\cos kcL \\ \sin\left[k(a+b)L\right] - \dfrac{\sin kaL \sin kbL}{ka(b+c)L} & -\sin kcL \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \sin kL = \frac{\sin kaL \sin[k(b+c)L]}{ka(b+c)L}. \tag{5.76}$$

Obviously, the calculated eigenvalues k_n should produce eigenmodes which satisfy the restrictions (5.64)-(5.67). As a result, the calculated coefficients should fulfill the following inequalities:

$$R_1 > 0 \Rightarrow B_1 \frac{\sin kaL}{ka(b+c)L} > 0 \tag{5.77}$$

$$u_2 < 0 \Rightarrow B_1\left[\sin[k(a+b)L] - \frac{\sin kaL \sin kbL}{ka(b+c)L} - \frac{c}{b+c}\sin kaL\right] < 0 \,. \tag{5.78}$$

After the choice of a value for the coefficient B_1, coefficient B_3 may be determined through the system (5.75). Then, the eigenmodes of the perfect structure for the certain contact status are constructed just substituting the calculated coefficients into equations (5.22)-(5.24).

5.2.2.4 Active constraint at point D and inactive constraint at point C $R_1 = 0$, $u_1 > 0$ and $R_2 < 0$, $u_2 = 0$

In this contact case the problem is stated as:

Problem $\mathscr{BP}_{2\text{-}a,4}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 > 0 \tag{5.79}$$

$$R_1 = 0 \tag{5.80}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.81}$$

$$R_2 < 0 \,. \tag{5.82}$$

Obviously, this case is similar to the previous one due to the symmetry of the problem, thus the buckling equation results easily after the cyclic permutation of the parameters a, b, c. Therefore, the corresponding buckling equation can be written as:

$$\sin kL = \frac{\sin kcL \sin[k(a+b)L]}{kc(a+b)L}, \tag{5.83}$$

while the corresponding coefficients are calculated from the following relations:

$$A_2 = B_1 \sin kaL \tag{5.84}$$

$$B_2 = B_1 \cos kaL \tag{5.85}$$

$$C_1 = \frac{-B_1 \sin\left[k\left(a+b\right)L\right]}{\left(a+b\right)L} \tag{5.86}$$

$$C_2 = C_1 \tag{5.87}$$

$$C_3 = \frac{a+b}{c} C_1 \tag{5.88}$$

$$D_2 = C_1 aL\,. \tag{5.89}$$

The coefficients B_1 and B_3 are determined using the equations of the following algebraic system:

$$\begin{bmatrix} \dfrac{\sin[k(a+b)L]}{(a+b)cL} - k\cos[k(a+b)L] & -k\cos kcL \\ \sin\left[k(a+b)L\right] & -\sin kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.90}$$

Similarly to the previous discussed contact case, the solutions of the buckling equation (5.83) should satisfy the restrictions (5.79)-(5.82). The corresponding eigenmodes can be calculated by relations (5.22)-(5.24) and should, in turn, satisfy the following restrictions:

$$R_2 < 0 \Rightarrow B_1 \cos\left[k\left(a+b\right)L\right] + B_3 \cos kcL < 0 \tag{5.91}$$

$$u_1 > 0 \Rightarrow B_1 \left[\sin kaL - \frac{aL}{a+b}\sin[k(a+b)L]\right] > 0\,. \tag{5.92}$$

5.2.2.5 Active constraint at point C and neutral contact status condition for the constraint at point D, $R_1 > 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 = 0$

In this certain contact condition the following problem has to be solved:

Problem $\mathcal{BP}_{2\text{-}a,5}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.93}$$

$$R_1 > 0 \tag{5.94}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.95}$$

$$R_2 = 0\,. \tag{5.96}$$

Due to the fact that $R_2 = 0$, $u_2 = 0$ and $u_1 = 0$, and following the same procedure as in the previous case, the implementation of the boundary conditions (5.29), (5.30) and (5.35), (5.36), gives:

$$D_2 = C_1 aL \tag{5.97}$$

$$C_1 = \frac{-B_1 \sin kaL}{aL} \tag{5.98}$$

$$C_2 = -C_3\,. \tag{5.99}$$

Then, substituting relation (5.34) into (5.33) and taking into account the above equation, it results that:

$$C_2 bL + D_2 = C_3 cL \Rightarrow C_2 = -C_1 \frac{a}{b+c}\,. \tag{5.100}$$

Therefore, coefficient B_2 can be calculated using equation (5.31) as a function of the coefficient B_1, i.e.:

$$B_1 \cos kaL + C_1 = B_2 k + C_2 \Rightarrow B_2 = B_1 \cos kaL - \frac{B_1 \sin kaL}{ka(b+c)L}\,. \tag{5.101}$$

It is then obvious that equations (5.34) and (5.36) formulate an algebraic system with respect to the unknowns B_1 and B_3 . More specifically, the system in matrix form can be written as:

$$\begin{bmatrix} \sin[k(a+b)L] - \dfrac{\sin kaL \sin kbL}{\lambda a(b+c)L} & -\sin kcL \\ \cos[k(a+b)L] - \dfrac{\sin kaL \cos kbL}{ka(b+c)L} & \cos kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.102}$$

In order to have a non-trivial solution, the determinant of the above system should be equal to zero, thus:

$$\begin{bmatrix} \sin[k(a+b)L] - \dfrac{\sin kaL \sin kbL}{ka(b+c)L} & -\sin kcL \\ \cos[k(a+b)L] - \dfrac{\sin kaL \cos kbL}{ka(b+c)L} & \cos kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \sin kL = \frac{\sin kaL \sin[k(b+c)L]}{ka(b+c)L} . \tag{5.103}$$

Equation (5.103) constitutes the **buckling equation** of the studied contact case. Admissible eigenvalues are only those that produce eigenmodes compatible with restrictions (5.93)-(5.96).

5.2.2.6 Inactive constraint at point C and neutral contact status condition for the constraint at point D, $R_1 = 0$, $u_1 > 0$ and $R_2 = 0$, $u_2 = 0$

For this contact case the following problem has to be solved:

Problem $\mathcal{BP}_{2\text{-}a,6}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 > 0 \tag{5.104}$$

$$R_1 = 0 \tag{5.105}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.106}$$

$$R_2 = 0 . \tag{5.107}$$

The solution procedure of the algebraic system (5.37) with respect to restrictions (5.104)-(5.107) leads to the following relations for the determination of the unknown coefficients:

$$B_2 = B_1 \cos kaL \tag{5.108}$$

$$A_2 = B_1 \sin kaL \tag{5.109}$$

$$C_1 = C_2 = C_3 = D_2 = 0 . \tag{5.110}$$

The buckling equation arises from the following system with respect to the unknowns B_1 and B_3.

$$\begin{bmatrix} \sin[k(a+b)L] & -\sin kcL \\ \cos[k(a+b)L] & \cos kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{5.111}$$

For the above algebraic system, the demand of having non-trivial solution leads to:

$$\sin kL = 0 \Rightarrow k_n = \frac{n\pi}{L}, \quad n = 1,2,3,.... \tag{5.112}$$

Obviously, the calculated eigenvalues should satisfy the equality and inequality restrictions which are introduced by the studied contact case, i.e. the following restrictions:

$$u_1 > 0 \Rightarrow B_1 \sin kaL > 0 \Rightarrow B_1 \sin n\pi a > 0 \tag{5.113}$$

$$u_2 = 0 \Rightarrow \sin n\pi(a+b) = 0. \tag{5.114}$$

For the admissible eigenvalues, the buckling eigenmodes can then be determined, after the substitution of the calculated coefficients into equations (5.22)-(5.24).

5.2.2.7 Active constraint at point D and neutral contact status condition for the constraint at point C, $R_1 = 0$, $u_1 = 0$ and $R_2 < 0$, $u_2 = 0$

In this contact case, the following problem has to be solved:

Problem *BP*$_{2\text{-}a,7}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.115}$$

$$R_1 = 0 \tag{5.116}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.117}$$

$$R_2 = 0. \tag{5.118}$$

The unknown coefficients are given by the following relations:

$$B_2 = B_1 \cos kaL \tag{5.119}$$

$$A_2 = B_1 \sin kaL \tag{5.120}$$

$$C_1 = \frac{-B_1 \sin kaL}{aL} \tag{5.121}$$

$$C_1 = C_2 \tag{5.122}$$

$$C_3 = \frac{a+b}{c} C_1 \tag{5.123}$$

$$D_2 = C_1 aL \ . \tag{5.124}$$

The last two unknown coefficients B_1 and B_3 are calculated by the following algebraic system:

$$\begin{bmatrix} \sin[k(a+b)L] & -\sin kcL \\ k\cos[k(a+b)L] - \dfrac{\sin kaL}{caL} & k\cos kcL \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ . \tag{5.125}$$

The demand of having a non-trivial solution for the studied contact case, requires the determinant of the system (5.125) to be equal to zero. The latter leads to the buckling equation for the specific contact case:

$$\sin kL = \frac{\sin kaL \sin kcL}{kacL} \ . \tag{5.126}$$

The roots of the above equation should satisfy the restrictions (5.115)-(5.118) in order to be accepted. Then, the eigenmodes for the admissible corresponding eigenvalues are calculated through the substitution of the above formulas into the equations (5.22)-(5.24).

5.2.2.8 Inactive constraint at point D and neutral contact status condition for the constraint at point C, $R_1 = 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 < 0$

This contact case is similar to the one treated in paragraph 5.2.2.6. More specifically, the following problem has to be solved:

Problem $\mathscr{BP}_{2\text{-}a,8}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.127}$$

$$R_1 = 0 \tag{5.128}$$

$$w_2(bL) = w_3(cL) = u_2 < 0 \tag{5.129}$$

$$R_2 = 0\,. \tag{5.130}$$

All the unknown coefficients are calculated from relations (5.108)-(5.112). Obviously, the same buckling equation (equation (5.112)) is derived also for this case. However, the roots of this equation should now satisfy the following restrictions:

$$u_1 = 0 \Rightarrow B_1 \sin kaL > 0 \Rightarrow \sin n\pi a = 0 \tag{5.131}$$

$$u_2 < 0 \Rightarrow \sin n\pi(a+b) < 0\,. \tag{5.132}$$

5.2.2.9 Neutral contact status condition for the constraints at the point C,D, $R_1 = 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 = 0$

This limit case corresponds to the neutral contact status condition for both the unilateral constraints. Therefore, the following problem has to be solved:

Problem $\mathscr{BP}_{2\text{-}a,9}$

Find a solution of the algebraic system of equations (5.37) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.133}$$

$$R_1 = 0 \tag{5.134}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.135}$$

$$R_2 = 0\,. \tag{5.136}$$

The buckling loads for this contact case are derived from the same buckling equation (5.112), as in the previous case. The admissible buckling loads should now satisfy the following two requirements:

$$u_1 = 0 \Rightarrow B_1 \sin kaL = 0 \Rightarrow \sin n\pi a = 0 \tag{5.137}$$

$$u_2 = 0 \Rightarrow \sin n\pi(a+b) = 0\,. \tag{5.138}$$

5.2.3 Calculation of the critical buckling load

In order to calculate the critical buckling load of a geometrically perfect beam with two unilateral supports functioning in opposite directions, the following steps should be followed:

- Initially, a sufficient number of eigenvalues is calculated for all the contact cases. The unknown eigenvalues are determined through the utilization of the buckling equations which correspond to each contact situation.
- Due to the fact that the obtained eigenvalues may produce eigenmodes which are not compatible with the unilateral constraints, only the eigenvalues which satisfy the required restrictions for each contact case are accepted. Recall that for the same eigenvalue, the arbitrary choice of the coefficient B_1 results actually to a scaling of the corresponding eigenmode, producing actually an infinite number of similar buckling curves, having different amplitudes. Therefore, depending on the value of B_1, some of them may be acceptable while others not.
- The smallest eigenvalue from the set of the accepted eigenvalues is the critical one. For this eigenvalue the critical buckling load and the corresponding buckling mode can be determined by applying equations (3.12) and (5.22)-(5.24) respectively.

5.3 Formulation of the elastic contact buckling problem of a geometrically imperfect continuous beam with two opposite functioning unilateral supports

5.3.1 Formulation

In a similar way as described for the geometrically perfect beam, the BVP of a simply supported geometrically imperfect beam with two intermediate unilateral supports with opposite functions, subjected to an axial compressive load, can also be formulated. The initial shape of the imperfect beam is assumed to be described by a Fourier sine series having the following form:

$$w_0(x) = \sum_{r=1}^{n} g_r \sin\left(\frac{r\pi x}{L}\right), \quad x \in [0, L] \; . \tag{5.139}$$

In the above relation, L is the total length of the beam and w_0 are the initial deflections due to the existence of the imperfection. The arbitrary initial geometric imperfection has to be compatible with the unilateral constraints, thus, the following inequalities should be satisfied:

$$w_0(aL) \geq 0 \tag{5.140}$$

$$w_0((a+b)L) \leq 0 \; . \tag{5.141}$$

Due to the fact that the proposed methodology divides the beam into parts, the initial geometric imperfection has to be separated into three functions, one for each span of the beam (Span I, Span II and Span III):

$$w_{1,0}(x_1) = \sum_{r=1}^{n} g_r \sin\left(\frac{r\pi x_1}{L}\right), \quad x_1 \in [0, aL] \tag{5.142}$$

$$w_{2,0}(x_2) = \sum_{r=1}^{n} g_r \sin\left(\frac{r\pi (x_2 + aL)}{L}\right), \quad x_2 \in [0, bL] \, . \tag{5.143}$$

$$w_{3,0}(x_3) = -\sum_{k=1}^{n} g_r \sin\left(\frac{r\pi x_3}{L}\right)(\text{-}1)^{\mathrm{r}}, \quad x_3 \in [0, cL] \, . \tag{5.144}$$

Inequalities (5.140) and (5.141) impose that the Fourier coefficients g_r of the above relations should, in turn, satisfy the following inequalities:

$$\sum_{r=1}^{n} g_r \sin(r\pi a) \geq 0 \tag{5.145}$$

and

$$\sum_{r=1}^{n} g_r \sin(r\pi (a+b)) \leq 0 \; . \tag{5.146}$$

Equivalently, the above can be written in the form

$$\sum_{r=1}^{n} g_r \sin(r\pi c)(-1)^r \geq 0 \quad . \tag{5.147}$$

Then, for each part of the beam of Fig. 5.2, a fourth-order linear non-homogeneous ordinary differential equation can be constructed respectively, that describes the bending behaviour of the beam. This equation should be similar to equations (4.55), (4.56) which were derived in the previous chapter for the contact buckling problem of beams with one intermediate unilateral support. Differentiating four times the functions (5.142)-(5.144) of the initial imperfection and substituting into the basic differential equations, the following equations are obtained:

$$\frac{d^4 w(x_1)}{dx_1^4} + k^2 \frac{d^2 w(x_1)}{dx_1^2} = \sum_{r=1}^{n} g_r \left(\frac{r\pi}{L}\right)^4 \sin(\frac{r\pi x_1}{L}), \quad x_1 \in [0, aL] \tag{5.148}$$

$$\frac{d^4 w(x_1)}{dx_1^4} + k^2 \frac{d^2 w(x_1)}{dx_1^2} = \sum_{r=1}^{n} g_r \left(\frac{r\pi}{L}\right)^4 \sin(\frac{r\pi (x_2 + aL)}{L}), \quad x_2 \in [0, bL] \tag{5.149}$$

$$\frac{d^4 w(x_3)}{dx_3^4} + k^2 \frac{d^2 w(x_3)}{dx_3^2} = -\sum_{r=1}^{n} g_r \left(\frac{r\pi}{L}\right)^4 \sin(\frac{r\pi x_3}{L})(-1)^r, \quad x_3 \in [0, cL]. \tag{5.150}$$

The parameter k is given by equation (3.12). The solution of the above equations gives the transverse deflection w at each point of the beam, as a function of the axial compressive load P. Therefore, the contact elastic buckling problem of the geometrically imperfect beam with two intermediate unilateral constraints having opposite functions, can be formulated as:

Problem $\mathscr{BI}_2$

Find $w(x) = \left\{ \begin{matrix} w(x_1),\ x_1 \in [0, aL] \\ w(x_2),\ x_2 \in [0, bL] \\ w(x_3),\ x_3 \in [0, cL] \end{matrix} \right\} \in L_2(0, L)$ of the system of differential equations (5.148)-(5.150) such that the equality boundary conditions (5.4)-(5.15), the inequality constraints (5.16), (5.17), (5.19), (5.20), and the complementarity conditions (5.18), (5.21) are fulfilled.

5.3.2 *Solution of the BVP of the geometrically imperfect beam for all the possible contact situations*

As it was already inferred previously in Chapter 4, the solution of the fundamental problem $\mathcal{BJ}_2$ is rather complicated due to the fact that there exist values of loading P for which the function of the deflection curve presents singularities. More specifically, depending on the value of load P the problem $\mathcal{BJ}_2$ may be uniquely solvable, unsolvable or solvable with infinite solutions. A criterion in order to decide about the solvability of problem $\mathcal{BJ}_2$ is offered through the application of the fundamental Theorem 3.4.

Supposing that the values of load P in equations (5.148)-(5.150) are not eigenvalues of the corresponding homogeneous BVP, then the non-homogeneous BVP is expected to have one unique solution that gives the deflection curve of the beam. This solution depends on the type and the amplitude of the initial imperfections considered, in combination with the position of the unilateral support and the initial contact conditions.

In this case the arising solution of the non-homogeneous BVP of paragraph 5.3.1 is a superposition of a general solution and of a particular solution related with the type of the initial imperfection, i.e.:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{n} g_r F_r \sin(\frac{r\pi x_1}{L}) , \quad x_1 \in [0, aL] \tag{5.151}$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{n} g_r F_k \sin(\frac{r\pi (x_2 + aL)}{L}) , \quad x_2 \in [0, bL] \tag{5.152}$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 - \\ - \sum_{r=1}^{n} g_r F_r \sin(\frac{r\pi x_3}{L})(-1)^r , \quad x_3 \in [0, cL] . \tag{5.153}$$

It is noticed that the solutions w_1, w_2, w_3 give the total transverse deflections of the beam, i.e. the initial deflections are included in them. The terms F_r and P_r have already been defined in Chapter 4 (relations (4.59) and (4.60)). The coefficients of the general solution are calculated through the boundary conditions (relations (5.4)-(5.15) of the problem). Applying, the boundary conditions at the ends of the beam it is proved that:

$$w_1(0) = 0 \Rightarrow A_1 + D_1 = 0 \tag{5.154}$$

$$w_3(0)=0 \Rightarrow A_3+D_3=0 \quad 5.155)$$

$$-EIw_1''(0)=0 \Rightarrow -A_1k^2=0 \Rightarrow A_1=0 \ \ (\mathrm{k}\neq 0 \text{ for } P\neq 0) \quad (5.156)$$

$$-EIw_3''(0)=0 \Rightarrow -A_3k^2=0 \Rightarrow A_3=0 \ \ (\mathrm{k}\neq 0 \text{ for } P\neq 0)\,. \quad (5.157)$$

Therefore, $D_1=D_3=0$. As the above coefficients take zero values, the functions of the deflections curve can be simplified. In order the mathematical operations of the rest of the boundary conditions to be implemented in a comprehensive way, the required derivatives of certain functions are also displayed below:

- Deflection curve $w_1,\ x_1\in[0,aL]$

$$w_1(x_1)=B_1\sin kx_1+C_1x_1+\sum_{r=1}^{n}g_rF_r\sin(\frac{r\pi x_1}{L}) \quad (5.158)$$

$$w_1'(x_1)=B_1k\,\mathrm{os}\,kx_1+C_1+\sum_{r=1}^{n}g_rF_r(\frac{r\pi}{L})\cos(\frac{r\pi x_1}{L}) \quad (5.159)$$

$$w_1''(x_1)=-B_1k^2\sin kx_1-\sum_{r=1}^{n}g_rF_r(\frac{r\pi}{L})^2\sin(\frac{r\pi x_1}{L}) \quad (5.160)$$

$$w_1'''(x_1)=-B_1k^3\cos kx_1-\sum_{r=1}^{n}g_rF_r(\frac{r\pi}{L})^3\cos(\frac{r\pi x_1}{L}) \quad (5.161)$$

- Deflection curve $w_2,\ x_2\in[0,bL]$

$$w_2(x_2)=A_2\cos kx_2+B_2\sin kx_2+C_2x_2+D_2+\sum_{r=1}^{n}g_rF_r\sin(\frac{r\pi(x_2+aL)}{L}) \quad (5.162)$$

$$w_2'(x_2)=-A_2k\sin kx_2+B_2k\cos kx_2+C_2+\sum_{r=1}^{n}g_rF_r(\frac{r\pi}{L})\cos(\frac{r\pi(x_2+aL)}{L}) \quad (5.163)$$

$$w_2''(x_2)=-A_2k^2\cos kx_2-B_2k^2\sin kx_2-\sum_{r=1}^{n}g_rF_r(\frac{r\pi}{L})^2\sin(\frac{r\pi(x_2+aL)}{L}) \quad (5.164)$$

$$w_2'''(x_2) = A_2 k^3 \sin kx_2 - B_2 k^3 \cos kx_2 - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^3 \cos(\frac{r\pi(x_2 + aL)}{L}) \tag{5.165}$$

- Deflection curve $w_3,\ x_3 \in [0, cL]$

$$w_3(x_3) = B_3 \sin kx_3 + C_3 x_3 - \sum_{r=1}^{n} g_r F_r \sin(\frac{r\pi x_3}{L})(-1)^r \tag{5.166}$$

$$w_3'(x_3) = B_3 k \cos kx_3 + C_3 - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L}) \cos(\frac{r\pi x_3}{L})(-1)^r \tag{5.167}$$

$$w_3''(x_3) = -B_3 k^2 \sin kx_3 + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^2 \sin(\frac{r\pi x_3}{L})(-1)^r \tag{5.168}$$

$$w_3'''(x_3) = -B_3 k^3 \cos kx_3 + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^3 \cos(\frac{r\pi x_3}{L})(-1)^r \tag{5.169}$$

Using the above relations, the following equations which describes the boundary conditions at the positions of the unilateral supports can be constructed. More specifically, the boundary conditions (5.6)-(5.9) and (5.12)-(5.15) of paragraph 5.2.1 can be organized as follows:

- Boundary conditions at point C

$$w_1(aL) = w_2(0) \Rightarrow B_1 \sin kaL + C_1 aL + \sum_{r=1}^{n} g_r F_r \sin(r\pi a) =$$

$$= A_2 + D_2 + \sum_{r=1}^{n} g_r F_r \sin(r\pi a) \Rightarrow B_1 \sin kaL + C_1 aL = A_2 + D_2 \tag{5.170}$$

$$w_1'(aL) = w_2'(0) \Rightarrow$$

$$\Rightarrow B_1 k \cos kaL + C_1 + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L}) \cos(r\pi a) = B_2 k + C_2 + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L}) \cos(r\pi a) \Rightarrow$$

$$\Rightarrow B_1 k \cos kaL + C_1 = B_2 k + C_2 \tag{5.171}$$

$$-EIw_1''(aL) = -EIw_1''(0) \Rightarrow$$

$$\Rightarrow -B_1 k^2 \sin kaL - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^2 \sin(r\pi a) = -A_2 k^2 - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^2 \sin(r\pi a) \Rightarrow$$

$$\Rightarrow A_2 = B_1 \sin kaL \tag{5.172}$$

$$[-EIw_1'''(aL) - Pw_1'(aL)] - [-EIw_2'''(0) - Pw_2'(0)] = R_1 \Rightarrow -EIw_1'''(aL) + EIw_2'''(0) = R_1 =$$

$$\Rightarrow B_1 k^3 \cos kaL + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^3 \cos(r\pi a) - B_2 k^3 - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^3 \cos(r\pi a) = \frac{R_1}{EI} \Rightarrow$$

$$\Rightarrow B_1 \cos kaL - B_2 = \frac{R_1}{k^3 EI} \tag{5.173}$$

- Boundary conditions at point D

$$w_2(bL) = w_3(cL) \Rightarrow$$

$$\Rightarrow A_2 \cos kbL + B_2 \sin kbL + C_2 bL + D_2 + \sum_{r=1}^{n} g_r F_r \sin(r\pi(a+b)) =$$

$$= B_3 \sin kcL + C_3 cL - \sum_{r=1}^{n} g_r F_r \sin(r\pi c)(-1)^r \tag{5.174}$$

$$-w_2'(bL) = w_3'(cL) \Rightarrow$$

$$\Rightarrow A_2 k \sin kbL - B_2 k \cos kbL - C_2 - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L}) \cos(r\pi(a+b)) =$$

$$= B_3 k \cos kcL + C_3 - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L}) \cos(r\pi c)(-1)^r \tag{5.175}$$

$$-EIw_2''(bL) = -EIw_3''(cL) \Rightarrow$$

$$\Rightarrow -A_2 k^2 \cos kbL - B_2 k^2 \sin kbL - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^2 \sin(r\pi(a+b)) =$$

$$= -B_3 k^2 \sin kcL + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^2 \sin(r\pi c)(-1)^r \tag{5.176}$$

$$[-EIw_2'''(bL) - Pw_2'(bL)] + [-EIw_3'''(cL) - Pw_3'(cL)] = R_2 \Rightarrow$$

$$\Rightarrow -A_2 k^3 \sin kbL + B_2 k^3 \cos kbL + \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^3 \cos(r\pi(a+b)) +$$

$$+ B_3 k^3 \cos kcL - \sum_{r=1}^{n} g_r F_r (\frac{r\pi}{L})^3 \sin(r\pi c)(-1)^r = \frac{R_2}{EI} \tag{5.177}$$

It has to be noticed that the complex equations (5.174)-(5.177) can be worked out in order to take a simpler form leading to the following relations[11]:

$$A_2 \cos kbL + B_2 \sin kbL + C_2 bL + D_2 = B_3 \sin kcL + C_3 cL \tag{5.178}$$

$$A_2 \cos kbL + B_2 \sin kbL = B_3 \sin kcL \tag{5.179}$$

$$A_2 k \sin kbL - B_2 k \cos kbL - C_2 = B_3 k \cos kcL + C_3 \tag{5.180}$$

$$-A_2 \sin kbL + B_2 \cos kbL + B_3 \cos kcL = \frac{R_2}{EIk^3} \tag{5.181}$$

Finally, the eight in number boundary equations (5.170)-(5.173) and (5.178)-(5.181) formulate an algebraic system with respect to the unknown coefficients $B_1, C_1, A_2, B_2, C_2, D_2, B_3, C_3$. Therefore, the solution of the initial BVP $\mathcal{BI}_2$ can be derived through the solution of the following problem:

Problem $\mathcal{BI}_{2\text{-}a}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) with respect to the unknowns $B_1, C_1, A_2, B_2, C_2, D_2, B_3, C_3$ such that the restrictions (5.16)-(5.21) are satisfied.

For the solution of this problem, a similar procedure as in the case of corresponding homogeneous BVP is adopted for the determination of the unknown coefficients. As a result, the following paragraphs, present only the final solutions without displaying the complicated sequences of the mathematical operations. The solution of the fundamental Problem $\mathcal{BI}_{2\text{-}a}$ derives analytical solutions for all the aforementioned nine contact cases (see Table 5.1). Each solution satisfies the corresponding contact condition for which it has been calculated.

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[11]For the appropriate mathematical operations used for the formulation of relations (5.178)-(5.181), the following equalities have been considered:

$$\sin(r\pi(a+b)) = \sin(r\pi(1-c)) = -(-1)^r \sin r\pi c$$

$$\cos(r\pi(a+b)) = \cos(r\pi(1-c)) = (-1)^r \cos r\pi c$$

5.3.2.1 Active constraints, $R_1 > 0$, $u_1 = 0$ and $R_2 < 0$, $u_2 = 0$

When both of the constraints are active, the following problem has to be solved:

Problem $\mathscr{BJ}_{2\text{-}a,1}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.182}$$

$$R_1 > 0 \tag{5.183}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.184}$$

$$R_2 < 0\,. \tag{5.185}$$

For the case where the two unilateral constraints are both active, the solution of the differential equations (5.148)-(5.150) for each value k which does not constitute an eigenvalue of the corresponding homogeneous problem is given by the equations (5.151)-(5.153). The unknown coefficients of the deflection curves of equations (5.151)-(5.153) are calculated by means of the following relations:

$$C_1 = \frac{-B_1 \sin kaL - \sum_{r=1}^{n} g_r F_r \sin r\pi a}{aL} \tag{5.186}$$

$$C_2 = \frac{-B_3 \sin kcL + B_1 \sin kaL + \sum_{r=1}^{n} g_r F_r \sin r\pi a + \sum_{r=1}^{n} g_r F_r \sin r\pi c(-1)^r}{bL} \tag{5.187}$$

$$C_3 = \frac{-B_3 \sin kcL + \sum_{r=1}^{n} g_r F_r \sin r\pi c(-1)^r}{cL} \tag{5.188}$$

$$A_2 = B_1 \sin kaL \tag{5.189}$$

$$D_2 = -B_1 \sin kaL - \sum_{r=1}^{n} g_r F_r \sin r\pi a \tag{5.190}$$

$$B_2 = B_1 \cos kaL - B_1 \frac{a+b}{kabL} \sin kaL - \frac{a+b}{kabL} \sum_{r=1}^{n} g_r F_k \sin r\pi a +$$

$$+ B_3 \frac{\sin kcL}{kbL} - \frac{1}{kbL} \sum_{r=1}^{n} g_r F_r \sin r\pi c(-1)^r \ . \tag{5.191}$$

The above relations are functions of the unknown coefficients B_1 and B_3. The latter are calculated through the following algebraic system which results from the application of the same boundary conditions used in the corresponding bifurcation problem:

$$\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{5.192}$$

where:

$$K_1 = \sin[k(a+b)L] - \frac{a+b}{kabL} \sin kaL \sin kbL \tag{5.193a}$$

$$K_2 = \frac{\sin kcL \sin kbL}{kbL} - \sin kcL \tag{5.193b}$$

$$K_3 = -\cos[k(a+b)L] + \frac{a+b}{kabL} \sin kaL \cos kbL - \frac{\sin kaL}{kbL} \tag{5.193c}$$

$$K_4 = -\cos kcL + \frac{b+c}{kbcL} \sin kcL - \frac{\sin kcL \cos kbL}{kbL} \tag{5.193d}$$

$$X_1 = \frac{(a+b)\sin kbL}{kabL} \sum_{r=1}^{n} g_r F_r \sin r\pi a + \frac{\sin kbL}{kbL} \sum_{r=1}^{n} g_r F_r (\sin r\pi c)(-1)^r \tag{5.194a}$$

$$X_2 = \frac{b+c}{kbcL} \sum_{r=1}^{n} g_r F_r (\sin r\pi c)(-1)^r + \frac{1}{kbL} \sum_{r=1}^{n} g_r F_r \sin r\pi a -$$

$$- \frac{a+b}{kabL} \cos kbL \sum_{r=1}^{n} g_r F_r \sin r\pi a - \frac{\cos kbL}{kbL} \sum_{r=1}^{n} g_r F_r (\sin r\pi c)(-1)^r \ . \tag{5.194b}$$

The solution of the above system gives:

$$B_3 = -B_1 \cdot \Lambda_1 + \Lambda_2 + \Lambda_3 \ , \tag{5.195}$$

where:

$$\Lambda_1 = \frac{\sin[k(a+b)L] - \dfrac{(a+b)}{kabL}\sin kaL \sin kbL}{\dfrac{1}{kbL}\sin kcL \sin kbL - \sin kcL} \tag{5.196a}$$

$$\Lambda_2 = \frac{\dfrac{(a+b)}{kabL}\sin kbL \sum_{r=1}^{n} g_r F_r \sin r\pi a}{\dfrac{1}{kbL}\sin kcL \sin kbL - \sin kcL} \tag{5.196b}$$

$$\Lambda_3 = \frac{\dfrac{1}{kbL}\sin kbL \sum_{r=1}^{n} g_r F_r (\sin r\pi c)(-1)^r}{\dfrac{1}{kbL}\sin kcL \sin kbL - \sin kcL} \tag{5.196c}$$

and:

$$B_1 \cdot \Pi_1 = \Pi_2 + \Pi_3 + \Pi_4 , \tag{5.197}$$

where:

$$\Pi_1 = -\cos[k(a+b)L] + \frac{(a+b)}{kabL}\sin kaL \cos kbL - \frac{\sin kaL}{kbL} - \Lambda_1 \cdot K_4 \tag{5.198a}$$

$$\Pi_2 = -\Lambda_2 \cdot K_4 \tag{5.198b}$$

$$\Pi_3 = -\Lambda_3 \cdot K_4 \tag{5.198c}$$

$$\begin{aligned} \Pi_4 = {} & \frac{b+c}{kbcL}\sum_{r=1}^{n} g_r F_r (\sin r\pi c)(-1)^r + \frac{1}{kbL}\sum_{r=1}^{n} g_r F_r \sin r\pi a - \\ & - \frac{a+b}{kabL}\cos kbL \sum_{r=1}^{n} g_r F_r \sin r\pi a - \frac{\cos kbL}{kbL}\sum_{r=1}^{n} g_r F_r (\sin r\pi c)(-1)^r \ . \end{aligned} \tag{5.198d}$$

For the values of the parameter k which constitute eigenvalues of the corresponding bifurcation problem (i.e. the homogeneous BVP), the studied non-homogeneous problem is either unsolvable or it has infinite solutions. Obviously, as aforementioned, the issue of solvability is strongly connected with the type of the eigenvalues and the shape of the initial geometric imperfection. Therefore, a general answer cannot be given for each case. This remark will be more clear in the examples that will be treated in the next chapter.

5.3.2.2 Inactive constraints, $R_1 = 0, u_1 > 0$ and $R_2 = 0, u_2 < 0$

In the case of inactive constraints, the following problem has to be solved:

Problem $\mathcal{BI}_{2\text{-}a,2}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 > 0 \tag{5.199}$$

$$R_1 = 0 \tag{5.200}$$

$$w_2(bL) = w_3(cL) = u_2 < 0 \tag{5.201}$$

$$R_2 = 0\,. \tag{5.202}$$

In this case, the examination of the non-homogeneous BVP formulated by equations (5.148)-(5.150) with the boundary conditions (5.4)-(5.15) and the restrictions (5.199)-(5.202) leads to zero values for all the unknown coefficients. Thus, the particular solution is the solution of the problem. This unique solution is valid for all the values of the parameter k except the values which are eigenvalues of the corresponding homogeneous problem (problem of paragraph 5.2.2.2). For the latter, the solvability issue has to be investigated by means of Theorem 3.4.

5.3.2.3 Active constraint at point C and inactive constraint at point D, $R_1 > 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 < 0$

When the constraint at point C is considered as active while the constraint at point D is considered as inactive, the following problem has to be solved:

Problem $\mathcal{BJ}_{2\text{-}a,3}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.203}$$

$$R_1 > 0 \tag{5.204}$$

$$w_2(0) = w_3(cL) = u_2 < 0 \tag{5.205}$$

$$R_2 = 0\,. \tag{5.206}$$

Following the same solution procedure which as in the corresponding case of paragraph 5.2.2.3 and assuming that the studied constrained non-homogeneous BVP is solved for all the values of load P which are different from the eigenvalues of the corresponding homogeneous constrained BVP, the unknown coefficients of the deflection curves are calculated through the following relations:

$$C_1 = \frac{-B_1 \sin kaL - \sum_{r=1}^{n} g_r F_r \sin r\pi a}{aL} \tag{5.207}$$

$$C_2 = \frac{B_1 \sin kaL + \sum_{r=1}^{n} g_r F_r \sin r\pi a}{(b+c)L} \tag{5.208}$$

$$C_3 = -C_2 \tag{5.209}$$

$$A_2 = B_1 \sin kaL \tag{5.210}$$

$$D_2 = -B_1 \sin kaL - \sum_{r=1}^{n} g_r F_r \sin r\pi a \tag{5.211}$$

$$B_2 = B_1 \cos kaL - \frac{\left[\sum_{r=1}^{n} g_r F_r \sin r\pi a + B_1 \sin kaL\right]}{ka(b+c)L} \tag{5.212}$$

$$B_3 = \frac{\left[\sum_{r=1}^{n} g_r F_r \sin r\pi a\right] \dfrac{\sin kaL}{ka(b+c)L}}{\sin kL - \dfrac{\sin kaL}{a} \dfrac{\sin\left[k(b+c)L\right]}{k(b+c)L}} . \tag{5.213}$$

The above coefficients are functions of the coefficient B_1 which in turn, is calculated by the following relation:

$$B_1 = \frac{\left[\sum_{r=1}^{n} g_r F_r \sin r\pi a\right] \dfrac{\sin\left[k(b+c)L\right]}{ka(b+c)L}}{\sin kL - \dfrac{\sin kaL}{a} \dfrac{\sin\left[k(b+c)L\right]}{k(b+c)L}} \tag{5.214}$$

Obviously, the above solution is valid only if the restrictions of relations (5.203)-(5.206) are fulfilled. It is also reminded that this solution has been derived under the assumption that the values of load P are not eigenvalues of the corresponding homogeneous constrained BVP.

5.3.2.4 Active constraint at point D and inactive constraint at point C, $R_1 = 0$, $u_1 > 0$ and $R_2 < 0$, $u_2 = 0$

In this contact case, the problem to be solved is stated as:

Problem $\mathcal{BI}_{2\text{-}a,4}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 > 0 \tag{5.215}$$

$$R_1 = 0 \tag{5.216}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.217}$$

$$R_2 < 0 . \tag{5.218}$$

The unknown coefficients are calculated by the following relations:

$$B_2 = B_1 \cos kaL \tag{5.219}$$

$$A_2 = B_1 \sin kaL \tag{5.220}$$

$$C_1 = \frac{\sum_{r=1}^{n} g_r F_r \sin r\pi c(-1)^r - B_3 \sin kcL}{(a+b)L} \tag{5.221}$$

$$C_2 = C_1 \tag{5.222}$$

$$C_3 = C_1 \frac{(a+b)}{c} \tag{5.223}$$

$$D_2 = C_1 aL \tag{5.224}$$

$$B_3 = \frac{-\frac{\sin\left[k(a+b)L\right]}{c(a+b)L} \sum_{r=1}^{n} g_r F_r \sin r\pi c(-1)^r}{k \sin kL - \frac{\sin\left[k(a+b)L\right]}{(a+b)} \frac{\sin kcL}{cL}} \tag{5.225}$$

$$B_1 = \frac{-\frac{\sin kcL}{c(a+b)L} \sum_{r=1}^{n} g_r F_r \sin r\pi c(-1)^r}{k \sin kL - \frac{\sin\left[k(a+b)L\right]}{(a+b)} \frac{\sin kcL}{cL}}. \tag{5.226}$$

After the calculation of the above coefficients, the deflection curve of the beam can be determined through the direct substitution of these calculated values into relations (5.151)-(5.153). This solution is valid only if restrictions (5.215)-(5.218) are satisfied.

5.3.2.5 Active constraint at point C and neutral contact status condition for the constraint at point D, $R_1 > 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 = 0$

In this particular contact case where the beam is marginally in contact with the constraint at point D (i.e. without producing any reaction contact force) while simultaneously the constraint at point C is assumed to be active, the problem to be solved is stated as:

Problem $\mathcal{BI}_{2\text{-}a,5}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.227}$$

$$R_1 > 0 \tag{5.228}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.229}$$

$$R_2 = 0\,. \tag{5.230}$$

The unknown coefficients can be calculated through the following relations:

$$A_2 = B_1 \sin kaL \tag{5.231}$$

$$B_2 = B_1 \cos kaL + C_1 \frac{1}{k(b+c)} \tag{5.232}$$

$$C_1 = \frac{-B_1 \sin kaL - \sum_{r=1}^{n} g_r F_r \sin r\pi a}{aL} \tag{5.233}$$

$$C_2 = -C_1 \frac{a}{b+c} \tag{5.234}$$

$$C_3 = -C_2 \tag{5.235}$$

$$D_2 = C_1 aL \tag{5.236}$$

$$B_3 = -B_1 \frac{\cos\left[k(a+b)L\right]}{\cos kcL} + \frac{B_1 \sin kaL + \sum_{r=1}^{n} g_r F_r \sin r\pi a}{k(b+c)aL} \frac{\cos kbL}{\cos kcL} \tag{5.237}$$

$$B_1 = \frac{\left[\sum_{r=1}^{n} g_r F_r \sin r\pi a\right]\left[\sin kbL + \cos kbL \tan kcL\right]}{ka(b+c)L\left[\sin\left[k(a+b)L\right] + \cos\left[k(a+b)L\right]\tan kcL - \dfrac{\sin kaL\left[\sin kbL + \cos kbL \tan kcL\right]}{ka(b+c)L}\right]}. \tag{5.238}$$

The previous solution is accepted only if the restrictions introduced by relations (5.227)-(5.230) are fulfilled.

5.3.2.6 Inactive constraint at point C and neutral contact status condition for the constraint at point D, $R_1 = 0$, $u_1 > 0$ and $R_2 = 0$, $u_2 = 0$

In this particular contact case the beam is marginally in contact with the constraint at point D (i.e. without producing any reaction contact force) while at the same time the constraint at point C is considered as inactive. Therefore, the following problem has to be solved:

Problem $\mathcal{BI}_{2\text{-}a,6}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 > 0 \tag{5.239}$$

$$R_1 = 0 \tag{5.240}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.241}$$

$$R_2 = 0\,. \tag{5.242}$$

In this contact case the investigation of the certain constrained BVP leads to zero values for all the unknown coefficients. Thus, the solution of the problem consists only of the particular solution (relations (5.151)-(5.153)).

5.3.2.7 Active constraint at point D and neutral contact status condition for the constraint at point C, $R_1 = 0$, $u_1 = 0$ and $R_2 < 0$, $u_2 = 0$

In this particular contact case the beam is marginally in contact with the constraint at point C (i.e. without producing any reaction contact force) while at the same time the constraint at point D is assumed to be active. Therefore, the following problem has to be solved:

Problem $\mathcal{BI}_{2\text{-}a,7}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.243}$$

$$R_1 = 0 \tag{5.244}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.245}$$

$$R_2 < 0\,. \tag{5.246}$$

The unknown coefficients can be calculated through the following relations:

$$A_2 = B_1 \sin kaL \tag{5.247}$$

$$B_2 = B_1 \cos kaL \tag{5.248}$$

$$C_1 = \frac{-B_1 \sin kaL - \sum_{r=1}^{n} g_r F_r \sin r\pi a}{aL} \tag{5.249}$$

$$C_2 = C_1 \tag{5.250}$$

$$C_3 = C_1 \frac{a+b}{c} \tag{5.251}$$

$$D_2 = C_1 aL \tag{5.252}$$

$$B_3 = \frac{\frac{\sin\left[k(a+b)L\right]}{caL}\left[\sum_{r=1}^{n} g_r F_r \sin r\pi a\right]}{k \sin kL - \frac{\sin kaL \sin kcL}{caL}} \tag{5.253}$$

$$B_1 = \frac{\frac{\sin kcL}{caL}\left[\sum_{r=1}^{n} g_r F_r \sin r\pi a\right]}{k \sin kL - \frac{\sin kaL \sin kcL}{caL}}\,. \tag{5.254}$$

5.3.2.8 Inactive constraint at point D and neutral contact status condition for the constraint at point C, $R_1 = 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 < 0$

In this particular contact case the beam is marginally in contact with the constraint at point C (i.e. without producing any reaction contact force) while at the same time the constraint at point D is considered as inactive. Therefore, the following problem has to be solved:

Problem $\mathcal{BJ}_{2\text{-}a,8}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.255}$$

$$R_1 = 0 \tag{5.256}$$

$$w_2(bL) = w_3(cL) = u_2 < 0 \tag{5.257}$$

$$R_2 = 0\,. \tag{5.258}$$

In this contact case the investigation of the certain constrained BVP leads also to zero values for all the unknown coefficients. Thus, the solution of the problem consists only of the particular solution (of relations (5.151)-(5.153)) which of course, should satisfy the restrictions of relations (5.255)-(5.258).

5.3.2.9 Neutral contact status condition for the constraints at points C, D, $R_1 = 0$, $u_1 = 0$ and $R_2 = 0$, $u_2 = 0$

This situations consists a very particular contact situation whereby the beam is marginally in contact with the constraints at points C, D without producing any reaction forces. Obviously the following problem has to be solved:

Problem $\mathcal{BJ}_{2\text{-}a,9}$

Find a solution of the algebraic system of equations (5.170)-(5.173) and (5.178)-(5.181) such that the following restrictions are satisfied:

$$w_1(aL) = w_2(0) = u_1 = 0 \tag{5.259}$$

$$R_1 = 0 \tag{5.260}$$

$$w_2(bL) = w_3(cL) = u_2 = 0 \tag{5.261}$$

$$R_2 = 0\,. \tag{5.262}$$

Obviously, the investigation of this constrained BVP leads to zero values for all the unknown coefficients. Thus, the solution of the problem consists only of the particular solution (relations (5.151)-(5.153)) which should satisfy restrictions (5.247)-(5.250).

5.3.3 Calculation of the instability load

The determination of the instability load of a geometrically perfect beam with two unilateral supports functioning in opposite direction, is based on the solution of the constrained non-homogeneous BVP which was formulated in Section 5.3.1. For the solution of that problem, the initial constraint BVP has to be separated into specific constrained subproblems, each of them corresponding to different contact case. The latter, depending on the value of the applied load P, may be uniquely solvable, unsolvable or solvable with infinite solutions. More specifically:

- For values of the applied load P, which do not constitute eigenvalues of the corresponding bifurcation problem the studied subproblem is uniquely solvable and the solution can be derived by applying the appropriate equations of Section 5.3.2
- For values of the applied load P, which constitute eigenvalues of the corresponding bifurcation problem, the studied subproblem is either unsolvable or solvable but not uniquely. For both the previous cases, the behaviour of the initial non-homogeneous BVP is considered as singular and the critical eigenvalue of the corresponding contact condition constitutes the instability load.

Depending on the type of the eigenvalue and on the initial imperfection, the singular behaviour reveals two different types of instability. If the problem is unsolvable (see Theorem 3.4) then the beam develops disproportionate large deflections as the applied load **approaches** this critical eigenvalue. If the problem is solvable then infinite solutions exist for that eigenvalue. In that case the beam buckles "violently" for this value of load.

In order to determine the instability load of a certain geometrically imperfect beam, the following steps should be followed. These steps are practically similar to the ones described thoroughly in Section 4.3.4 for the contact buckling problem of beams with one intermediate unilateral support. Due to the fact that for the case of the two unilateral supports more contact cases exist, the following proposed methodology for the determination of the instability load is displayed in simpler form:

Step 1

The initial imperfection is applied and the deflection at the position of the unilateral supports (points C and D) is examined.

Step 2

Depending on the initial contact conditions of Step 1, it is checked which of the presented in Table 5.1 contact cases is valid. Then, for this contact case, for values of the applied load P which do not constitute eigenvalues of the corresponding bifurcation problem, the initial constrained non-homogeneous BVP has a unique solution, i.e. the deflection curve of the beam can be described by applying the appropriate equations of Section 5.3.2. The description of the deflection curve according to the previous equations may be not valid for each value of load P.

a) If a value of the applied load exists, for which the inequality restrictions of the existing contact case are satisfied as equalities, then the beam develops the tendency to change contact status. Practically, this means that for this specific value of load one of the following cases is arised:

- The beam develops the tendency to come in contact with the unilateral support at point C (Fig. 5.1), thus this load is termed as P_c^C.
- The beam develops the tendency to come in contact with the unilateral support at point D (Fig. 5.1), thus this load is termed as P_c^D.
- The beam develops the tendency to be separated from the unilateral support at point C, thus this load is termed as P_s^C.
- The beam develops the tendency to be separated from the unilateral support at point D, thus this load is termed as P_s^D.

Depending on the initial imperfection and on the positions of the unilateral supports the above cases may be combined. Then, it is examined if inside the interval $[0, P_i^M]$ ($i = s$ or $i = c$ and $M = C$ or $M = D$) the critical eigenvalue ($P_{cr,eig}$) of the corresponding bifurcation problem exists. If yes, then this value of load is the instability load of the beam, i.e:

$$P_{in} = P_{cr,eig}^{(C-S)}\,.^{12} \tag{5.263}$$

If such an eigenvalue does not exist, then the instability load should be seeked outside the interval $[0, P_i^M]$. This means that instability phenomena will occur in a different contact case, for different values of the applied load. The calculation procedure continues with Step 3.

b) If such a value of load does not exist (i.e. the beam does not change contact status for none of the values of the applied load P), then the instability load is

[12] The abbreviation "C-S" in relation (5.263) means "Contact status" and indicates one of the nine different contact states, i.e. A-A, I-A, A-I, etc. (see Table 5.1)

equal to the critical eigenvalue of the corresponding bifurcation problem for the current contact case (similar relation to equation (5.263)).

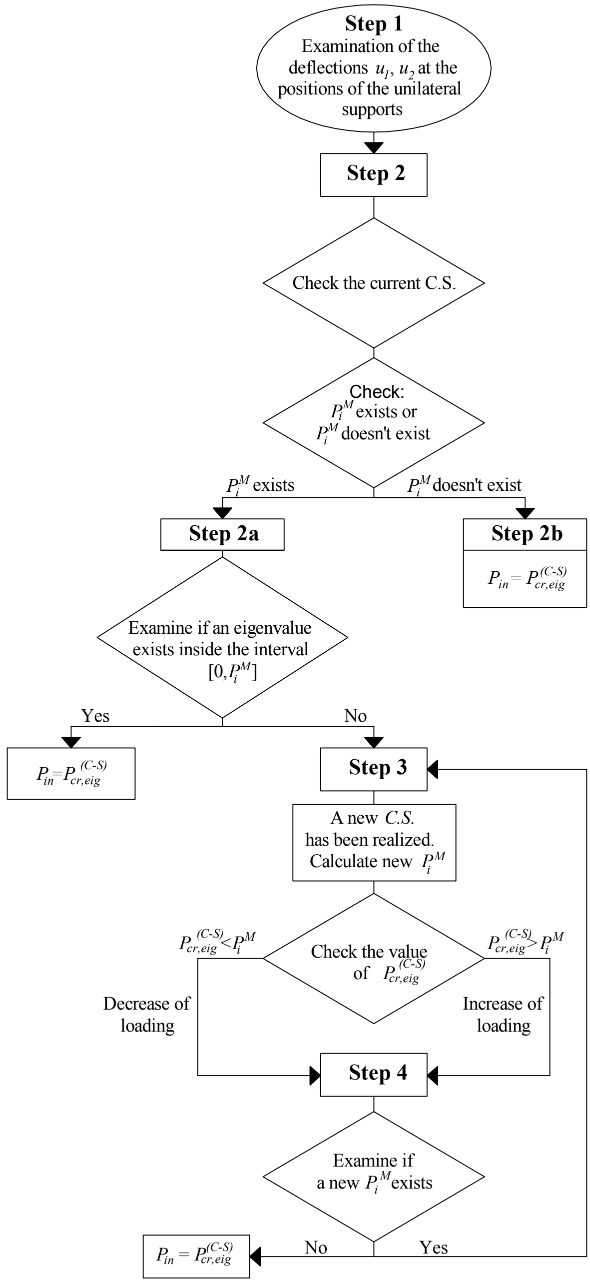


Fig. 5.3 The flow chart diagram of the proposed calculation procedure.

Step 3

The beam has changed contact status. For the new contact situation two cases exist:

a) If the critical eigenvalue of the existing contact status is smaller than P_i^M, then the beam cannot stay in equilibrium and the deflections of the beam are accompanied by a decrease of the applied load to lower values. In that case the procedure continues with step 4.

b) If the critical eigenvalue of the existing contact status is greater than P_i^M, then the beam is able to sustain more loading and the procedure also continues with step 4.

Step 4

For the existing contact situation, it is examined, if a value of load exists, for which the beam changes contact status. If such a value exists then the procedure continues with the application of Step 3 for the new contact case. If such a value does not exist then the instability load of the beam is equal to the critical eigenvalue which corresponds to the current contact case.

The above calculation procedure is presented schematically in the flow chart of Fig. 5.3.

5.3.4 Calculation of the second-order bending moment as a function of the unilateral constraint conditions

The design second order bending moment of a beam with two unilateral supports functioning in opposite direction, can be determined through the following equations:

$$M_{Ed}(x_1) = -EIw''_{1,p}(x_1) = -EI(w''_1(x_1) - w''_{1,0}(x_1)) , \quad x_1 \in [0, aL] \tag{5.264}$$

$$M_{Ed}(x_2) = -EIw''_{2,p}(x_1) = -EI(w''_2(x_2) - w''_{2,0}(x_2)) , \quad x_2 \in [0, bL] \tag{5.265}$$

$$M_{Ed}(x_3) = -EIw''_{3,p}(x_3) = -EI(w''_3(x_3) - w''_{3,0}(x_3)) , \quad x_3 \in [0, cL], \tag{5.266}$$

where, $w''_{1,p}$, $w''_{2,p}$ and $w''_{3,p}$ are the second derivatives of the elastic transverse deflections which are attributed solely to the axial loading P. Using the expressions obtained in Section 5.3 for the deflections w_1, w_2 and w_3, the following equations are obtained that give the function of the bending moment for the various contact cases may arise.

$$M_{Ed}(x_1) = -EI\left[-B_1 k^2 \sin(kx_1) + \sum_{r=1}^{n} g_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right], x_1 \in [0, aL] \quad (5.267)$$

$$M_{Ed}(x_2) = -EI\Bigg[-A_2 k^2 \cos(kx_2) - B_2 k^2 \sin(kx_2) + \\ + \sum_{r=1}^{n} g_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi(x_2+aL)}{L}\right)\Bigg], \quad x_2 \in [0, bL] \quad (5.268)$$

$$M_{Ed}(x_3) = -EI\left[-B_3 k^2 \sin(kx_3) + \sum_{r=1}^{n} g_r \left(\frac{r\pi}{L}\right)^2 (F_r - 1)\sin\left(\frac{r\pi x_3}{L}\right)(-1)^r\right], \\ x_3 \in [0, cL] \quad (5.269)$$

The determination of the coefficients A_2, B_1, B_2, B_3 in the above equations depends on the existing contact condition and is accomplished through the utilization of the relations of Section 5.3 for the respective contact case.

6 Geometrically perfect beams - Examples

6.1 Introduction

The aim of this chapter is to present several examples related with the buckling problem of geometrically perfect beams in the presence of unilateral supports. As it was stated in the previous chapters, this problem belongs to the family of constrained homogeneous BVPs. The latter produce infinite solutions which, in turn, should satisfy the required restrictions. In the following examples the critical values of the axial load (eigenvalues) and the corresponding deflection curves (eigenmodes) are calculated for a variety of different configurations. The solution of these problems is accomplished through the methodology which has been introduced in Chapters 4 and 5.

More specifically, for an axially loaded beam with one or two opposite functioning unilateral supports, the buckling loads and the corresponding buckling shapes are determined through the following steps:

- Calculation of the eigenvalues (buckling loads) through the corresponding for each contact case **buckling equation,** according to the relations given in Chapters 4 and 5. Due to the fact that the eigenvalues are infinite, only the first ten eigenvalues are calculated in the presented examples for each contact case.
- For the calculated eigenvalues, the corresponding eigenmodes are then determined. The latter are calculated through the direct determination of the unknown coefficients $A_2, B_1, B_2, B_3, C_1, C_2, C_3, D_2$, utilizing the appropriate relations given in Chapters 4 and 5. Then, the obtained eigenmodes should satisfy the restrictions which are introduced by the contact case for which they are calculated. The eigenvalues for which the corresponding eigenmodes do not satisfy the related restrictions are excluded.
- The critical buckling load is the one corresponding to the smallest admissible eigenvalue.

6.2 Presentation of the considered examples

In the following paragrhaphs examples concerning the methodology of calculating the buckling load and the buckling mode of geometrically perfect beams subjected to axial load in the presensce of unilateral supports, are presented. More specifically, Section 6.3 deals with the buckling problem of a beam with a single intermediate unilateral support while in Section 6.4 beams with two unilateral supports functioning in opposite direction are treated.

In Section 6.3, the presented example is a simple one. The derived results are used also to support the solution of the examples presented in the next Chapter (Section 7.3). The second category of examples (Section 6.4), deals with special cases where interesting situation arise. In particular, Example 1 shows a trifurcation equilibrium state, i.e. the critical load corresponds to three different equilibrium configurations. The first one corresponds to the straight line configuration while the other two are curved configurations corresponding to different contact cases.

In Example 2 a non symmetric beam is treated, while the 3rd example of Section 6.4 shows a special case where the two unilateral supports are very close the one to the other. In this particular example, the deflections of the beam due to buckle, resemble a virtual clamped support placed at the vicinity of the unilateral supports.

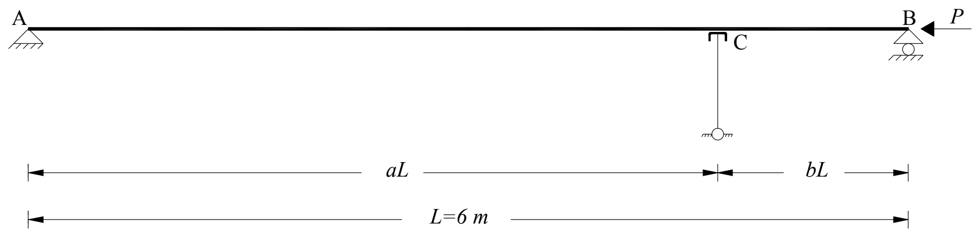
Example 4 shows how the critical load and the critical buckling mode change if the left unilateral support of Example 1 remains in the same position while the other unilateral support moves to the right. This case derives a critical buckling mode with smaller critical load than the buckling loads of Examples 1 and 2, due to the fact that the effective buckling length has increased.

Example 5 is also particular. More specifically, if one of the unilateral supports come very close to the roll support, then the two close supports may be substituded by a single support which exhibits free or fixed rotation. This type of support can be considered as a unilateral clamped support. The free rotation corresponds to the case where the initial unilateral support is active, while the fixed rotation corresponds to the opposite case.

6.3 Geometrically perfect beams with one unilateral support

6.3.1 Example 1

A geometrically perfect beam with a length of 6 m supported by one intermediate unilateral support is considered, subjected to an axial compressive load (Fig. 6.1). For this beam, the buckling loads and the corresponding buckling modes will be determined considering in plane bending, according to the theory and relations displayed in Chapter 4.



Data

E=210000 Mpa

I=8360 cm^4

aL=4.7 m, bL=1.3 m

Fig. 6.1 The geometrically perfect beam treated in Example 1.

- **Calculation of the eigenvalues (buckling loads)**

Two different contact conditions exist for the beam of Fig. 6.1. The first one corresponds to the case where the unilateral constraint at point C is inactive while the second one corresponds to the case where the unilateral support is active. For each contact situation the corresponding eigenvalues are calculated through the utilization of the corresponding buckling equations (equations (4.29) and (4.38)). By applying the latter the values of Table 6.1 are obtained.

	Inactive Constraint $R=0$ $u<0$	Active Constraint $R<0$ $u=0$
k_1^{cc}	0.5236	0.8784
k_2^{cc}	1.0472	1.4986
k_3^{cc}	1.5708	2.0826
k_4^{cc}	2.0944	2.6136
k_5^{cc}	2.6180	3.1043

Table 6.1 The first five eigenvalues for each contact case.

- **Calculation of the admissible eigenmodes**

The eigenvalues which are derived from buckling equations (4.29) and (4.38) may produce eigenmodes which are not compatible with the unilateral constraints. In order these eigenvalues to be accepted, the corresponding eigenmodes should satisfy the restrictions which are imposed by each contact situation. The eigenvalues of Table 6.1 are all acceptable due to the fact that they fulfill the required restrictions. It is recalled herein that only the shape (and not the magnitude) of the obtained eigenmodes can be calculated. Thus, the determination of the latter is based on an arbitrary selection of the coefficient B_2. Table 6.2 presents the first five admissible eigenvalues of the under study constrained buckling problem, while Fig. 6.2 presents schematically the corresponding eigenmodes.

Contact Case	Accepted Eigenvalue	B_2	Functions of the eigenmodes
Inactive Constraint	$k_1=0.5236$	<0	
Active Constraint	$k_2=0.8784$	<0	$w(x_1)=B_1\sin kx_1+C_1x_1,\ x_1\in[0,aL]$
Inactive Constraint	$k_3=1.0472$	<0	$w(x_2)=B_2\sin kx_2+C_2x_2,\ x_2\in[0,bL]$
Active Constraint	$k_4=1.4986$	<0	
Inactive Constraint	$k_5=1.5708$	<0	

Table 6.2 The first ten admissible eigenvalues and the corresponding eigenmodes.

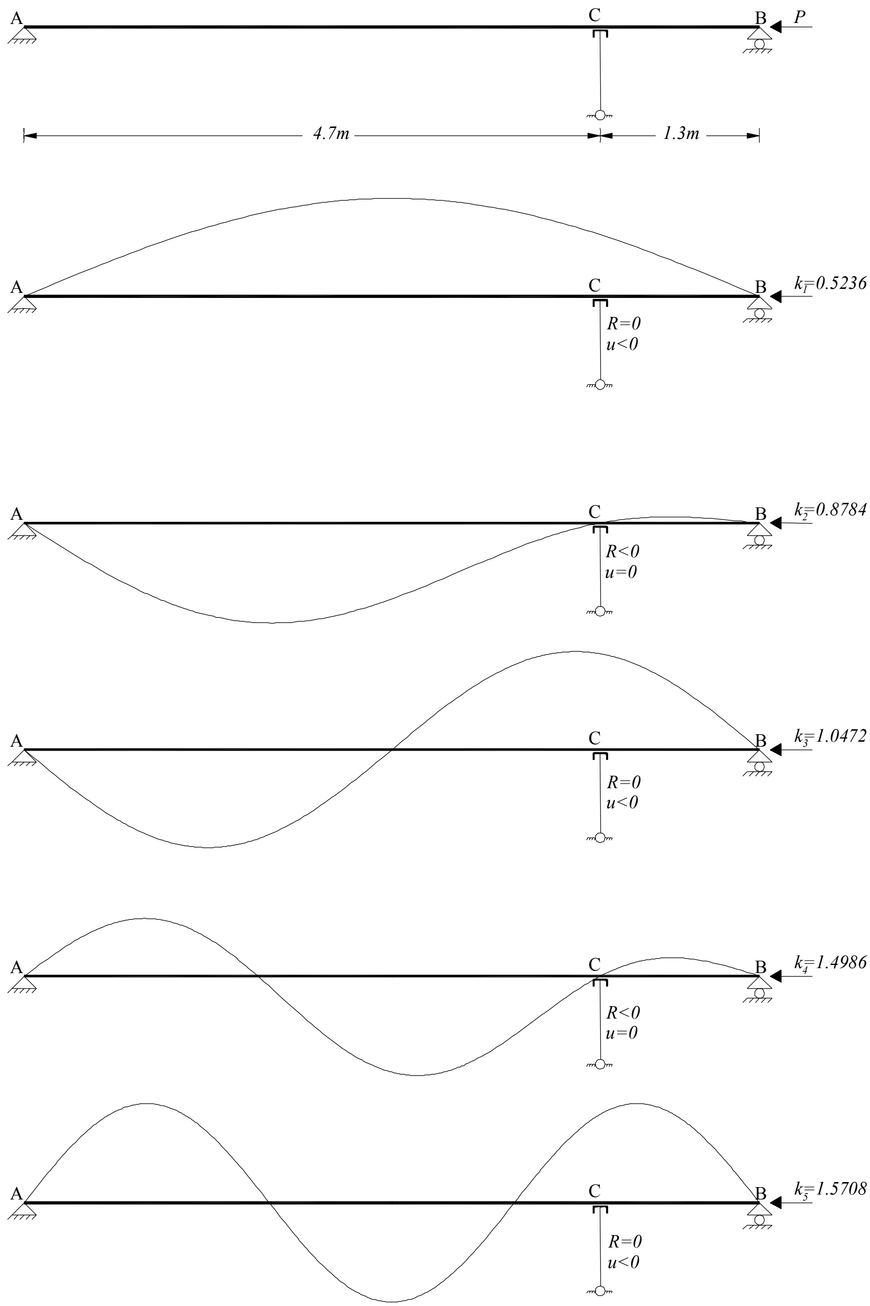
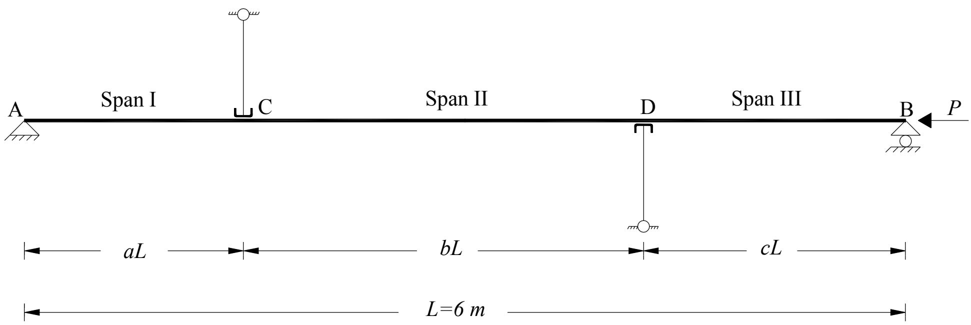


Fig. 6.2 The first five eigenmodes of the beam of Example 1.

6.4 Geometrically perfect beams with two opposite functioning unilateral supports

6.4.1 Example 1

A geometrically perfect beam with a length of 6 m, supported by two unilateral supports is considered, subjected to an axial compressive load (Fig. 6.3). For this beam, the buckling loads and the corresponding buckling modes will be determined considering in plane bending, according to the theory and relations displayed in Chapter 5.



<u>*Data*</u>

E=210000 Mpa

I=8360 cm^4

aL=1.5 m, bL=3 m, cL=1.5 m

Fig. 6.3 The geometrically perfect beam treated in Example 1 of Section 6.4.1.

- **Calculation of the eigenvalues (buckling loads)**

For each contact situation (see Table 5.1) the corresponding eigenvalues are calculated through the utilization of the corresponding buckling equations. Due to the fact that these equations are of transcendental type, a numerical procedure for solving nonlinear algebraic equations should be considered. For the purposes of the present dissertation, the Newton’s method is applied through the Mathematica software package. Table 6.3 displayes the first ten eigenvalues (from the lowest to the highest) for each contact case[13].

Contact Case	CC1 $R_1>0$ $u_1=0$ $R_2<0$ $u_2=0$	CC2 $R_1=0$ $u_1>0$ $R_2=0$ $u_2<0$	CC3 $R_1>0$ $u_1=0$ $R_2=0$ $u_2<0$	CC4 $R_1=0$ $u_1>0$ $R_2<0$ $u_2=0$	CC5 $R_1>0$ $u_1=0$ $R_2=0$ $u_2=0$	CC6 $R_1=0$ $u_1>0$ $R_2=0$ $u_2=0$	CC7 $R_1=0$ $u_1=0$ $R_2<0$ $u_2=0$	CC8 $R_1=0$ $u_1=0$ $R_2=0$ $u_2<0$	CC9 $R_1=0$ $u_1=0$ $R_2=0$ $u_2=0$
Contact Status	A-A	I-A	A-I	I-A	A-N	I-N	N-A	N-I	N-N
k_1^{cc}	1.4978	0.5236	**0.9022**	**0.9022**	0.9022	0.5236	2.0944	0.5236	0.5236
k_2^{cc}	2.0944	**1.0472**	1.5294	1.5294	1.5294	1.0472	2.5471	1.0472	1.0472
k_3^{cc}	2.5751	1.5708	2.0944	2.0944	2.0944	1.5708	3.2886	1.5708	1.5708
k_4^{cc}	**2.9956**	2.0944	2.5870	2.5870	2.5870	2.0944	3.5855	2.0944	**2.0944**
k_5^{cc}	3.6347	2.6180	**3.0943**	**3.0943**	3.0943	2.6180	4.1888	2.6180	2.6180
k_6^{cc}	4.1888	**3.1416**	3.6461	3.6461	3.6461	3.1416	4.6704	3.1416	3.1416
k_7^{cc}	4.6887	3.6652	4.1888	4.1888	4.1888	3.6652	5.3219	3.6652	3.6652
k_8^{cc}	**5.1502**	4.1888	4.6959	4.6959	4.6959	4.1888	5.715	4.1888	**4.1888**
k_9^{cc}	5.7403	4.7124	**5.2077**	**5.2077**	5.2077	4.7124	6.2832	4.7124	4.7124
k_{10}^{cc}	6.2832	**5.236**	5.7472	5.7472	5.7472	5.2360	7.8227	5.236	5.236

Table 6.3 The first ten eigenvalues for each contact case.

- **Calculation of the admissible eigenmodes**

The eigenvalues presented in Table 6.3 are accepted only if the corresponding eigenmodes satisfy the restrictions which correspond to each contact situation. The computation of the admissible eigenmodes is based on a special worksheet code which has been developed for this reason. The input of this program constists of the basic data of Fig. 6.3 and the eigenvalues extracted from the Mathematica software. It is reminded that only the shape (and not the magnitude) of the obtained eigenmodes can be calculated. Thus, the determination of the latter is based on an

[13] The abbreviation CC which is encountered in the columns of Table 6.1 corresponds to the term "contact case". The eigenvalues of this table refer to each contact case, thus they are represented with the superscript "cc".

arbitrary selection of the coefficient B_1, which in turn should result to eigenmodes compatible with the constraints. The rest coefficients are calculated by means of the appropriate relations which are given in Chapter 5 for each contact case. The admissible for each contact case eigenvalues are presented in a bold font in Table 6.3. Table 6.4 presents[14,15] the first ten admissible eigenvalues and the corresponding eigenmodes of the under study constrained buckling problem, while Fig. 6.4 and Fig. 6.5 show the graphical representation of the eigenmodes. The fourth column of Table 6.4 show the appropriate sign of B_1 in order the restrictions which are introduced by the constraints to be satisfied.

Contact Case	Contact Status	Accepted Eigenvalue	B_1	Functions of the eigenmodes
CC3	A-I	$k_1 = 0.9022$	>0	$w(x_1) = B_1 \sin kx_1 + C_1 x_1,\ x_1 \in [0,1.5]$
CC4	I-A	$k_2 = 0.9022$	>0	
CC2	I-I	$k_3 = 1.0472$	>0	
CC9	N-N	$k_4 = 2.0944$	<0	$w(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2,\ x_2 \in [0,3]$
CC1	A-A	$k_5 = 2.9956$	<0	
CC3	A-I	$k_6 = 3.0943$	<0	$w(x_3) = B_3 \sin kx_3 + C_3 x_3,\ x_3 \in [0,1.5]$
CC4	I-A	$k_7 = 3.0943$	<0	
CC2	I-I	$k_8 = 3.1416$	<0	
CC9	N-N	$k_9 = 4.1888$	>0	
CC1	A-A	$k_{10} = 5.1502$	>0	

Table 6.4 The first ten admissible eigenvalues and the corresponding eigenmodes.

The buckilng load is calculated from the smallest admissible eigenvalue of Table 6.4.

[14] It is noted that the eigenvalues presented in Table 6.4 are the eigenvalues of the initial constrained BVP while the eigenvalues of Table 6.3 are solutions of the constrained subproblems that correspond to each contact case. Thus the superscript “cc” has been removed from the description.

[15] In the second column of Table 6.4 the contact status of each contact case is presented by means of the abbreviation “K-L”. The first letter (K) in that abbreviation describes the contact status of the unilateral constraint at point C, while the second describes the contact status of the unilateral constraint at point D. The active status of each of the constraints is termed as A, the inactive as I and the neutral as N.

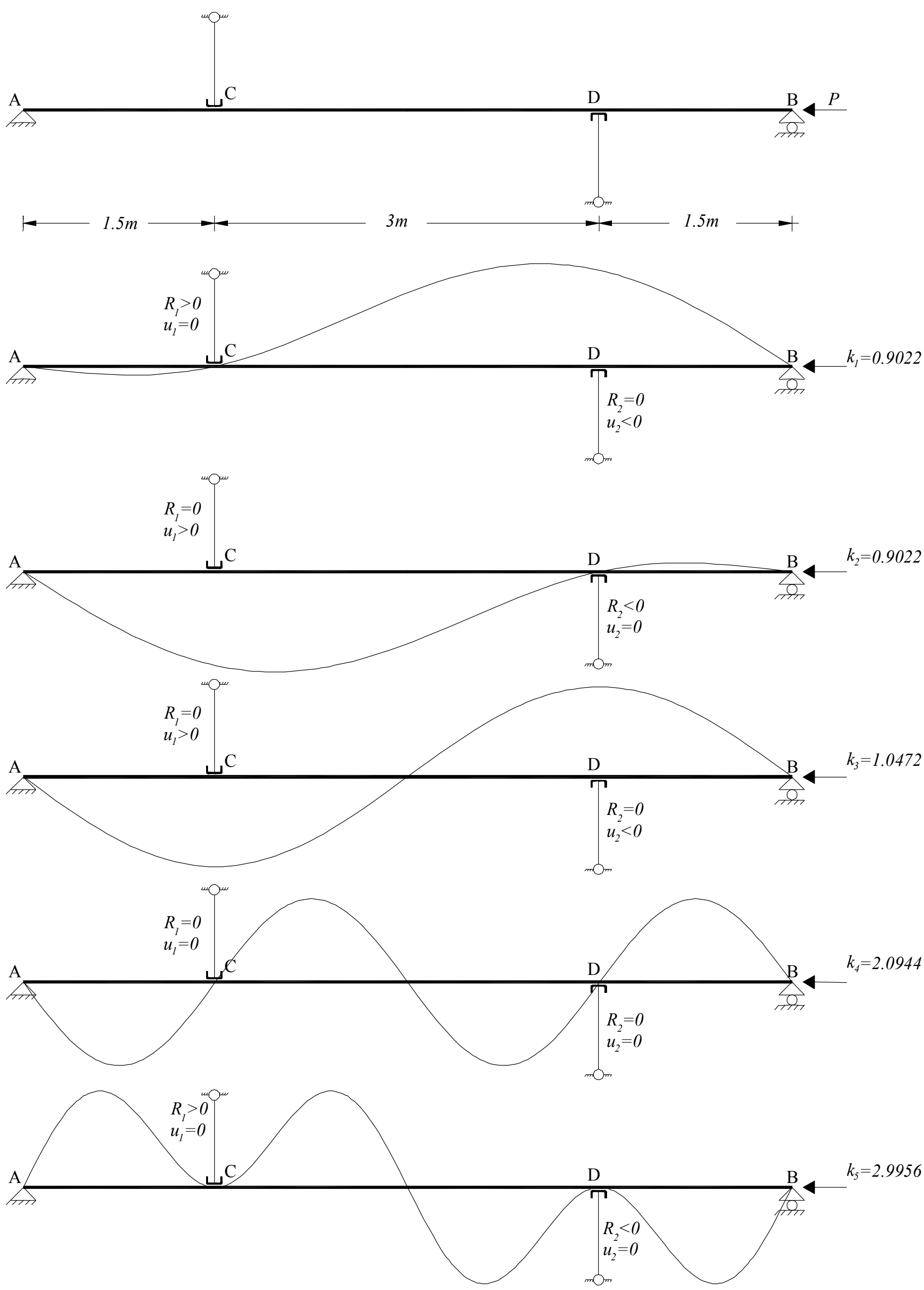


Fig. 6.4 The first five eigenmodes of the beam of Example 1 of Section 6.4.1.

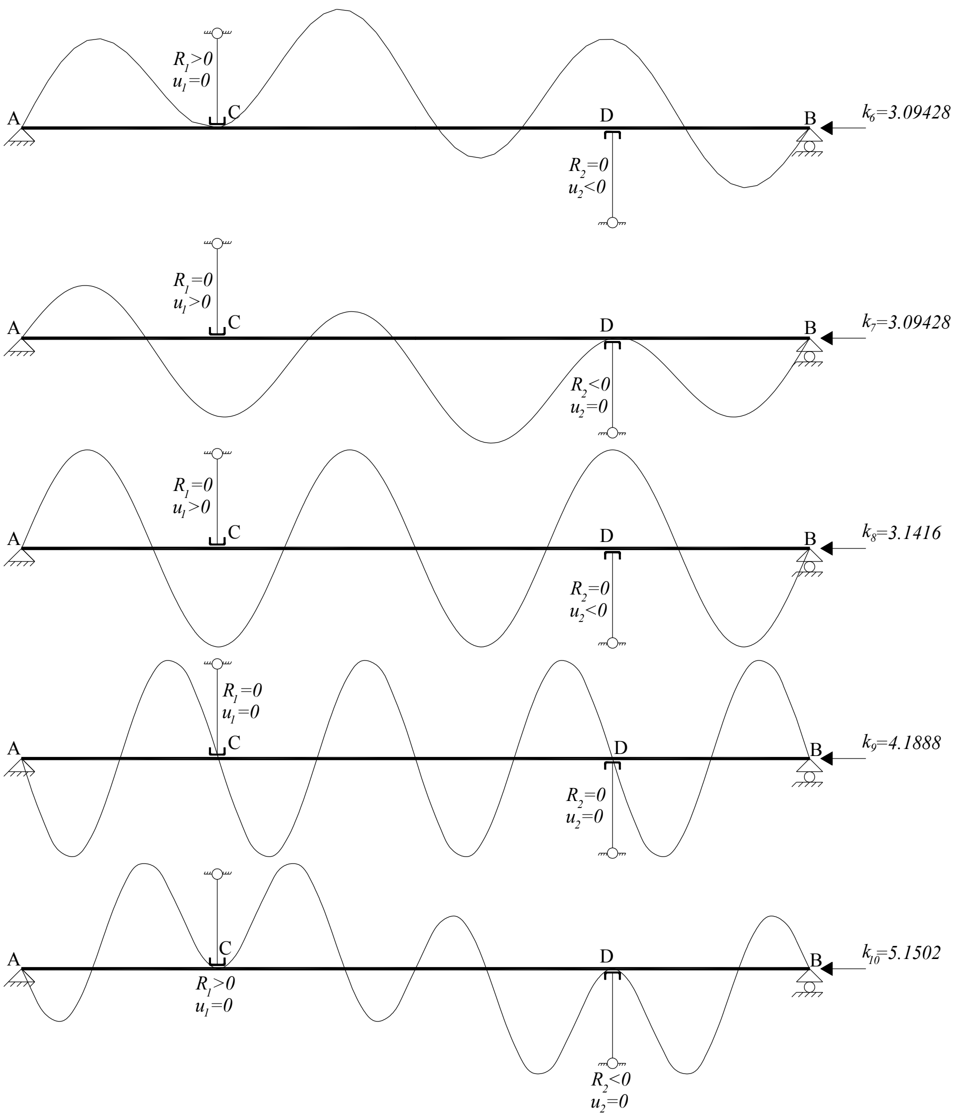


Fig. 6.5 Superior eigenmodes of the beam of Example 1 of Section 6.4.1.

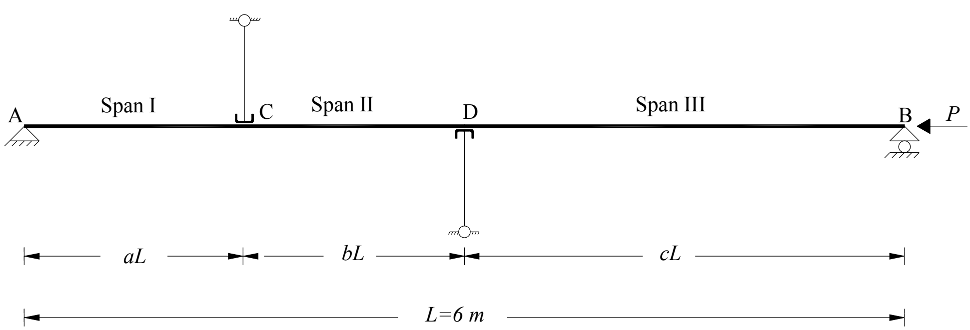
The critical buckling load corresponds to the smallest admissible eigenvalue, thus it is calculated by the following relation:

$$P_{cr} = k_1^2 EI = 0.9022^2 \times 17556 = 14289.96\ kN \quad . \tag{6.1}$$

It is noticed that for this load, due to the symmetry of the structure, the beam can be in equilibrium in two different curved configurations. The first one corresponds to the contact case where the constraint at point C is active and at the same time the constraint at point D is inactive, while the other corresponds to the exactly opposite contact situation. Obviously, the buckling response of the beam is dominated by a trifurcation state of equilibrium. For this critical load value the beam can be in equilibrium either in the straight line configuration or in one of the two different curved configurations. The same buckling behaviour (trifurcation) appears also for other eigenvalues as it is clearly figured in Table 6.1 (e.g. $k_6 = k_7 = 3.0943$).

6.4.2 Example 2

Here, the beam of Fig. 6.6 is considered. The difference with the previous one is the position of the unilateral support at point D, which now is in the middle of the beam (Fig. 6.6). For this case the buckling loads and the corresponding buckling modes will be calculated.



<u>*Data*</u>

E=210000 Mpa

I=8360 cm^4

aL=1.5 m, bL=3 m, cL=1.5 m

Fig. 6.6 The geometrically perfect beam of Example 2.

- **Calculation of the eigenvalues (buckling loads)**

Table 6.5 displays the first ten eigenvalues (from the lowest to the highest) of each contact case.

Conatct Case	CC1 $R_1>0$ $u_1=0$ $R_2<0$ $u_2=0$	CC2 $R_1=0$ $u_1>0$ $R_2=0$ $u_2<0$	CC3 $R_1>0$ $u_1=0$ $R_2=0$ $u_2<0$	CC4 $R_1=0$ $u_1>0$ $R_2<0$ $u_2=0$	CC5 $R_1>0$ $u_1=0$ $R_2=0$ $u_2=0$	CC6 $R_1=0$ $u_1>0$ $R_2=0$ $u_2=0$	CC7 $R_1=0$ $u_1=0$ $R_2<0$ $u_2=0$	CC8 $R_1=0$ $u_1=0$ $R_2=0$ $u_2<0$	CC9 $R_1=0$ $u_1=0$ $R_2=0$ $u_2=0$
Contact Status	A-A	I-A	A-I	I-A	A-N	I-N	N-A	N-I	N-N
k_1^{cc}	**1.3089**	0.5236	**0.9022**	1.0472	0.9022	0.5236	1.0472	0.5236	0.5236
k_2^{cc}	2.0944	1.0472	**1.5294**	**1.4978**	1.5294	**1.0472**	1.6541	1.0472	1.0472
k_3^{cc}	**2.4587**	**1.5708**	2.0944	2.0944	2.0944	1.5708	2.0944	1.5708	1.5708
k_4^{cc}	**3.0827**	2.0944	**2.5870**	**2.5751**	2.5870	2.0944	2.6823	2.0944	**2.0944**
k_5^{cc}	3.6316	**2.6180**	**3.0943**	3.14159	3.0943	2.6180	3.1416	2.6180	2.6180
k_6^{cc}	4.1888	3.1416	3.6461	3.6347	3.6461	3.1416	3.6187	3.1416	3.1416
k_7^{cc}	4.6859	3.6652	4.1888	4.1888	4.1888	3.6652	4.1888	3.6652	3.6652
k_8^{cc}	**5.1872**	4.1888	4.6959	4.6887	4.6959	4.1888	4.6804	4.1888	**4.1888**
k_9^{cc}	**5.6835**	4.7124	**5.2077**	5.2360	5.2077	4.7124	5.2360	4.7124	4.7124
k_{10}^{cc}	6.2832	5.2360	**5.7472**	**5.7403**	5.7472	5.2360	5.7857	5.2360	5.2360

Table 6.5 The first ten eigenvalues for each contact case.

- **Calculation of the admissible eigenmodes**

The eigenvalues presented in Table 6.5 are acceptable only if the corresponding eigenmodes satisfy the restrictions which correspond to each contact situation. Table. 6.6 presents the first ten admissible eigenvalues and the corresponding eigenmodes of the under study constrained buckling problem, while Fig. 6.7 and Fig. 6.8 show the eigenmodes graphically.

Contact Case	Contact Status	Accepted Eigenvalue	B_1	Function of the eigenmodes
CC3	A-I	$k_1 = 0.9022$	>0	
CC6	I-N	$k_2 = 1.0472$	>0	$w(x_1) = B_1 \sin kx_1 + C_1 x_1,\ x_1 \in [0,1.5]$
CC1	A-A	$k_3 = 1.3089$	>0	
CC4	I-A	$k_4 = 1.4978$	>0	
CC3	A-I	$k_5 = 1.5294$	>0	$w(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2,\ x_2 \in [0,1.5]$
CC2	I-I	$k_6 = 1.5708$	>0	
CC9	N-N	$k_7 = 2.0944$	>0	
CC1	A-A	$k_8 = 2.4587$	<0	$w(x_3) = B_3 \sin kx_3 + C_3 x_3,\ x_3 \in [0,3]$
CC4	I-A	$k_9 = 2.5751$	<0	
CC3	A-I	$k_{10} = 2.5870$	<0	

Table 6.6 The first ten admissible eigenvalues and the corresponding eigenmodes.

Obviously, the critical buckling load is calculated by the following relation:

$$P_{cr} = k_1^2 EI = 0.9022^2 \times 17556 = 14289.96\ kN\ , \tag{6.2}$$

and it is the same as in Example 1. It is noted that the effective buckling length of the beam, which corresponds to the contact case where the unilateral support at point C is active, while the unilateral support at point D is inactive (CC3), did not change due to the movement of the latter to the left. Moreover, this contact case remains the critical. On the contrary, the effective buckling length of the beam which corresponds to the opposite contact case (CC4) has been decreased (now it is equal to the half length of the beam). Therefore, the buckling load corresponding to this contact case has been increased.

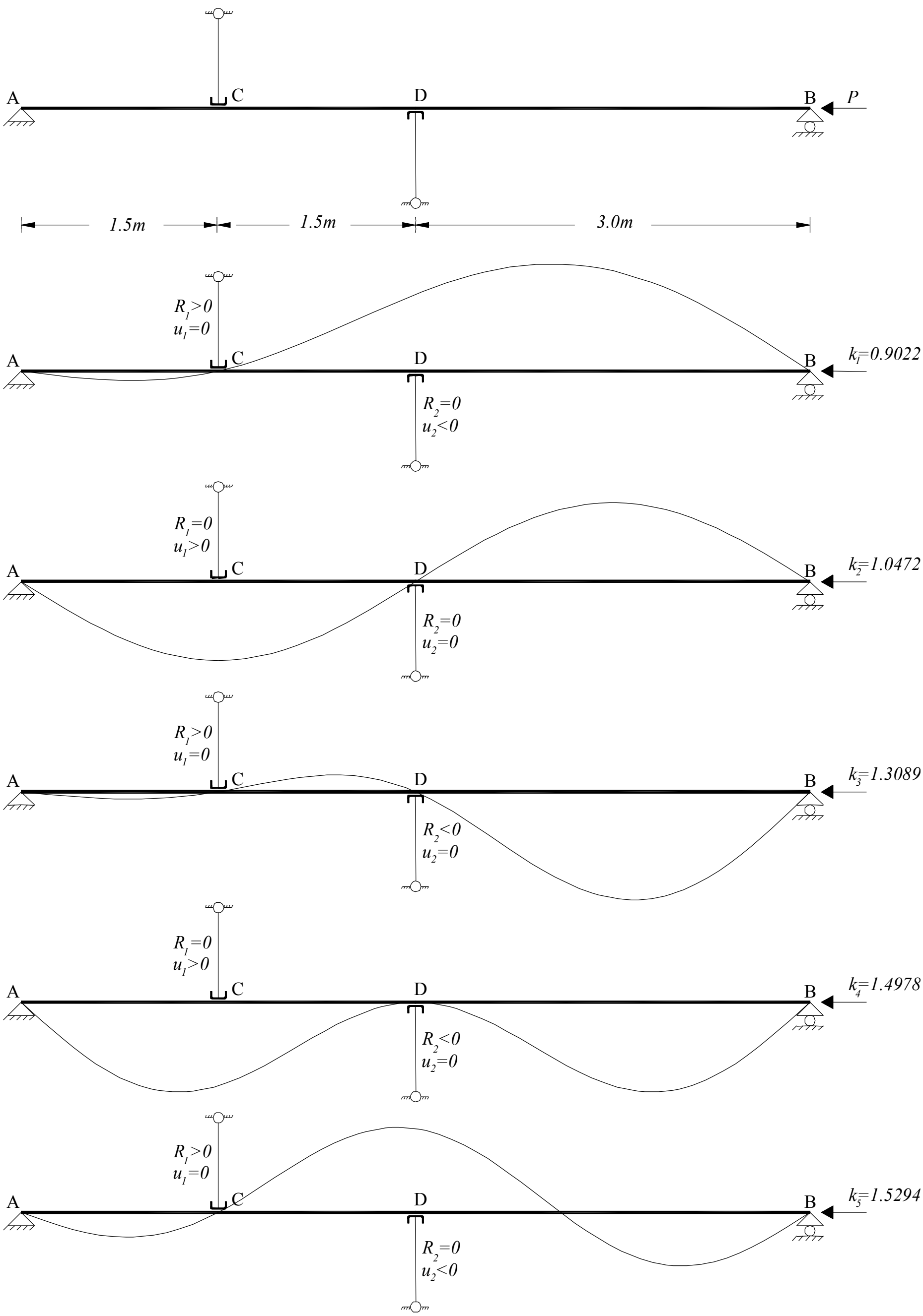


Fig. 6.7 The first five eigenmodes of the beam of Example 2.

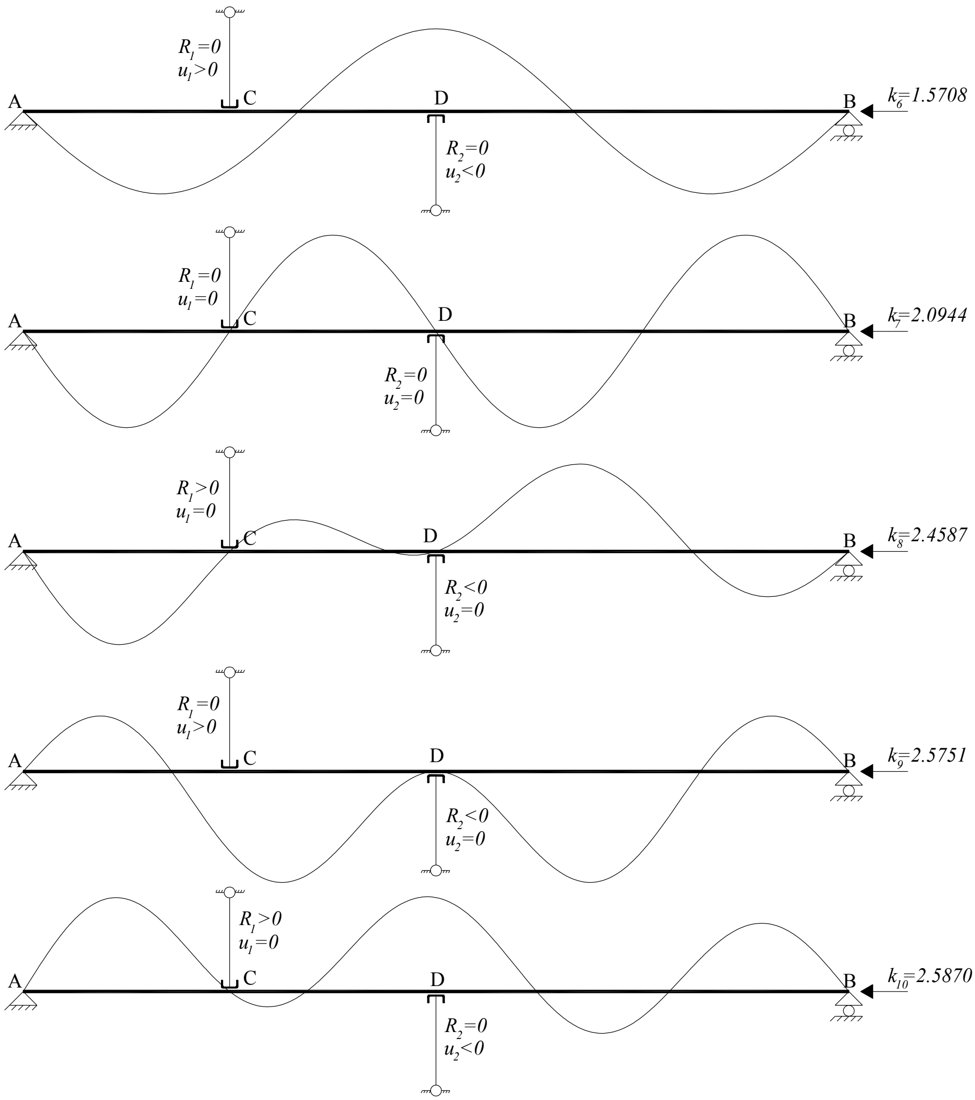
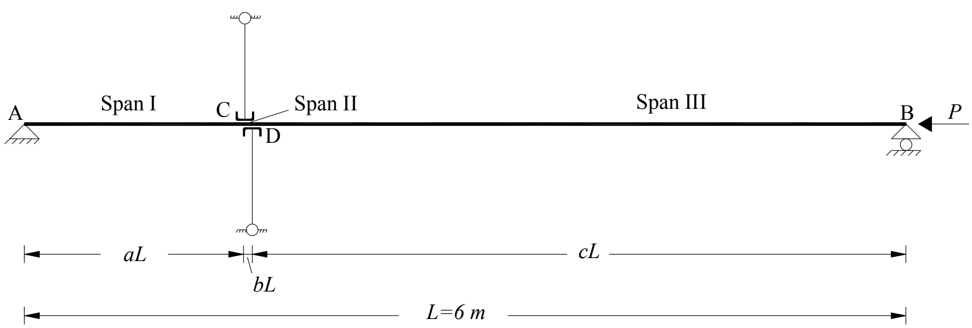


Fig. 6.8 Superior eigenmodes of the beam of Example 2.

6.4.3 Example 3

The following example is a particular one because it provides the potential of evaluating the reliability of the relations derived in Chapter 5, through the comparison of the results with already known ones. This is accomplished via the movement of the unilateral support at point D, in the vicinity of the other unilateral support at point C (Fig. 6.9). In this way, the two unilateral supports, being very close the one to the other, may (in certain cases) lead to the development of a pair of opposite in sign forces or, equivalently, to a reaction bending moment. This, in turn, may be considered to originate from a fictitious clamped support placed in the vicinity of points C and D. Of course, due to the unilateral character of the supports, this clamped support is activated only under certain conditions (i.e. when both the unilateral supports in C and D are active).



<u>*Data*</u>

E=210000 Mpa

I=8360 cm^4

aL=1.5 m, bL=0.05 m, cL=4.45 m

Fig. 6.9 The geometrically perfect beam of Example 3.

- **Calculation of the eigenvalues (buckling loads)**

For each contact situation (see Table 5.1) the corresponding eigenvalues are calculated through the utilization of the corresponding buckling equations. Table 6.7 displays the first ten eigenvalues (from the lowest to the highest) for each contact case.

Contact Case	CC1 $R_1>0$ $u_1=0$ $R_2<0$ $u_2=0$	CC2 $R_1=0$ $u_1>0$ $R_2=0$ $u_2<0$	CC3 $R_1>0$ $u_1=0$ $R_2=0$ $u_2<0$	CC4 $R_1=0$ $u_1>0$ $R_2<0$ $u_2=0$	CC5 $R_1>0$ $u_1=0$ $R_2=0$ $u_2=0$	CC6 $R_1=0$ $u_1>0$ $R_2=0$ $u_2=0$	CC7 $R_1=0$ $u_1=0$ $R_2<0$ $u_2=0$	CC8 $R_1=0$ $u_1=0$ $R_2=0$ $u_2<0$	CC9 $R_1=0$ $u_1=0$ $R_2=0$ $u_2=0$
Contact Status	A-A	I-A	A-I	I-A	A-N	I-N	N-A	N-I	N-N
k_1^{cc}	**1.0060**	0.5236	**0.9022**	0.9083	0.9022	0.5236	0.9053	0.5236	0.5236
k_2^{cc}	**1.7296**	1.0472	**1.5294**	1.5364	1.5294	1.0472	1.5331	1.0472	1.0472
k_3^{cc}	**2.4412**	1.5708	2.0944	**2.0936**	2.0944	1.5708	2.0944	1.5708	1.5708
k_4^{cc}	**2.9627**	2.0944	2.5870	**2.5801**	2.5870	2.0944	2.5837	2.0944	2.0944
k_5^{cc}	**3.1497**	2.6180	**3.0943**	3.0973	3.0943	2.6180	3.0959	2.6180	2.6180
k_6^{cc}	**3.8555**	3.1416	**3.6461**	3.6531	3.6461	3.1416	3.6500	3.1416	3.1416
k_7^{cc}	**4.5605**	3.6652	4.1888	**4.1873**	4.1888	3.6652	4.1888	3.6652	3.6652
k_8^{cc}	**5.0930**	4.1888	4.6959	**4.6891**	4.6959	4.1888	4.6927	4.1888	4.1888
k_9^{cc}	**5.2672**	4.7124	**5.2077**	5.2106	5.2077	4.7124	5.2092	4.7124	4.7124
k_{10}^{cc}	**5.9701**	5.2360	**5.7472**	5.7540	5.7472	5.236	5.7513	5.2360	5.236

Table 6.7 The first ten eigenvalues for each contact case.

- **Calculation of the admissible eigenmodes**

The eigenvalues of Table 6.7 are accepted only if the corresponding eigenmodes satisfy the restrictions which correspond to each contact situation. Table 6.8 presents the first ten admissible eigenvalues and the corresponding eigenmodes of the under study constrained buckling problem, while Fig. 6.10 and Fig. 6.11 show the graphical representation of the eigenmodes.

Contact Case	Contact Status	Accepted Eigenvalue	B_1	Functions of the eigenmodes
CC3	A-I	$k_1 = 0.9022$	>0	
CC1	A-A	$k_2 = 1.0060$	>0	$w(x_1) = B_1 \sin kx_1 + C_1 x_1,\ x_1 \in [0, 1.5]$
CC3	A-I	$k_3 = 1.5294$	>0	
CC1	A-A	$k_4 = 1.7296$	>0	
CC4	I-A	$k_5 = 2.0936$	>0	$w(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2,\ x_2 \in [0, 0.05]$
CC1	A-A	$k_6 = 2.4412$	>0	
CC4	I-A	$k_7 = 2.5801$	<0	
CC3	A-I	$k_8 = 3.0943$	<0	$w(x_3) = B_3 \sin kx_3 + C_3 x_3,\ x_3 \in [0, 4.45]$
CC1	A-A	$k_9 = 3.1497$	<0	
CC3	A-I	$k_{10} = 3.6461$	<0	

Table 6.8 The first ten admissible eigenvalues and the corresponding eigenmodes.

Due to the fact that the length bL is very small, the critical buckling mode is the same as in the previous examples, thus:

$$P_{cr} = k_1^2 EI = 0.9022^2 \times 17556 = 14289.97\ kN\ . \tag{6.3}$$

However, as the unilateral support at point D moves very close to point C, the contact case where the two supports are active (CC1) corresponds now to the second buckling mode. This eigenmode corresponds actually to the critical eigenmode of the buckling problem of a beam with length equal to cL and clamped support at point C. As it is well known, the critical load for this case is given by the following relation:

$$k_{cr} \approx \frac{\pi}{(0.7L)} = \frac{\pi}{(0.7 \times 4.5)} = 0.9973\ . \tag{6.4}$$

Obviously, the value $k_2 = 1.006$ is very close to the above value and tends to be equal with that when the unilateral support at point D approaches marginally the unilateral support at point C.

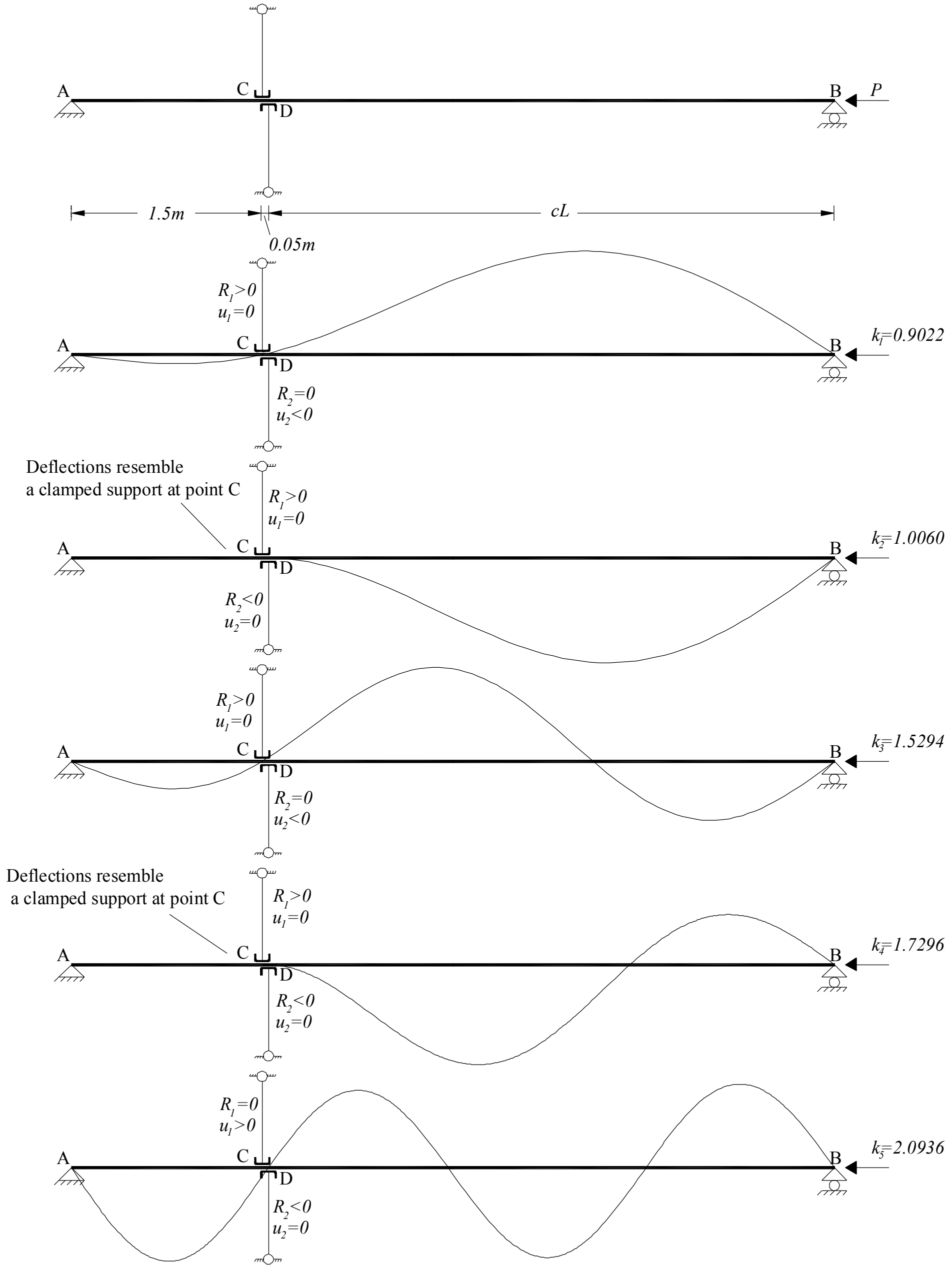


Fig. 6.10 The first five eigenmodes of the beam of Example 3.

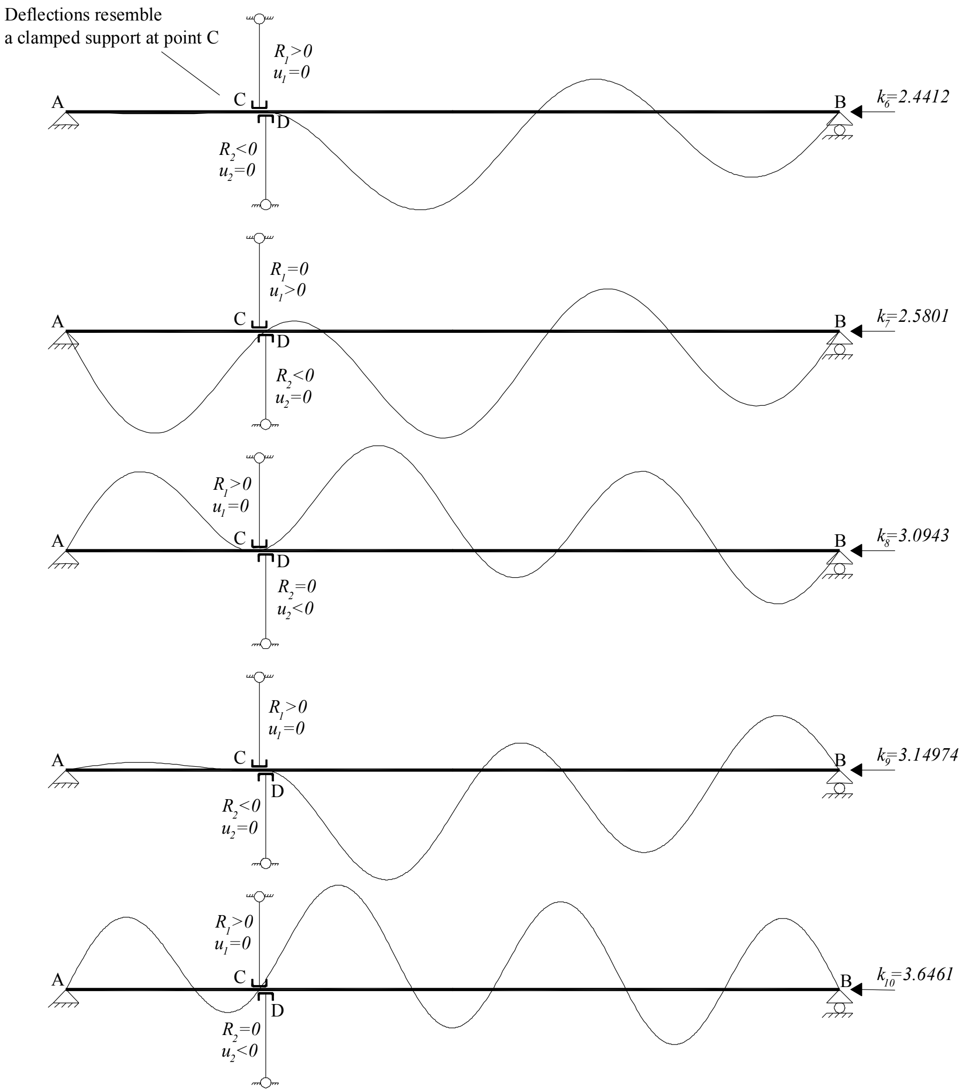
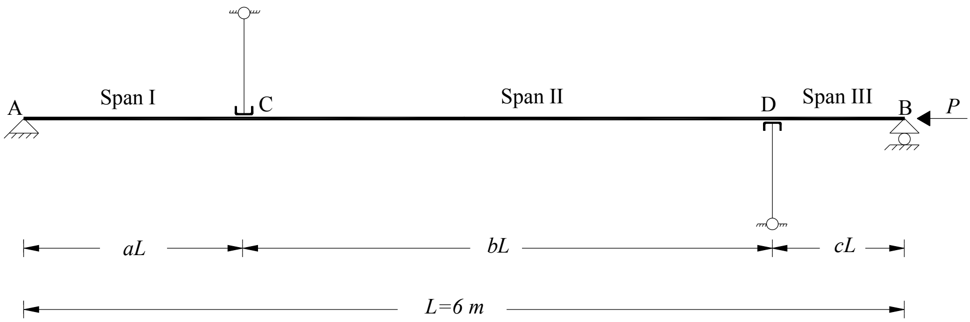


Fig. 6.11 Superior eigenmodes of the beam of Example 3.

6.4.4 Example 4

The aim of the present example is to show how the critical load changes if the support at point D moves to the right (Fig. 6.12). It is expected that an eigenmode based on the case where the support at point D is active while at the same time the unilateral support at point C is inactive will be the critical one, due to the fact that this contact situation corresponds to the greatest effective buckling length.



Data

E=210000 Mpa

I=8360 cm^4

aL=1.5 m, bL=3.60 m, cL=0.90 m

Fig. 6.12 The geometrically perfect beam of Example 4.

- **Calculation of the eigenvalues (buckling loads)**

For each contact situation (see Table 5.1) the corresponding eigenvalues are calculated through the utilization of the respective buckling equations. Table 6.9 displays the first ten eigenvalues (from the lowest to the highest) for each contact case.

Contact Case	CC1 $R_1>0$ $u_1=0$ $R_2<0$ $u_2=0$	CC2 $R_1=0$ $u_1>0$ $R_2=0$ $u_2<0$	CC3 $R_1>0$ $u_1=0$ $R_2=0$ $u_2<0$	CC4 $R_1=0$ $u_1>0$ $R_2<0$ $u_2=0$	CC5 $R_1>0$ $u_1=0$ $R_2=0$ $u_2=0$	CC6 $R_1=0$ $u_1>0$ $R_2=0$ $u_2=0$	CC7 $R_1=0$ $u_1=0$ $R_2<0$ $u_2=0$	CC8 $R_1=0$ $u_1=0$ $R_2=0$ $u_2<0$	CC9 $R_1=0$ $u_1=0$ $R_2=0$ $u_2=0$
Contact Status	A-A	I-A	A-I	I-A	A-N	I-N	N-A	N-I	N-N
k_1^{cc}	1.3955	0.5236	**0.9022**	**0.8338**	0.9022	0.5236	0.9053	0.5236	0.5236
k_2^{cc}	**2.0027**	**1.0472**	1.5294	1.4308	1.5294	1.0472	1.5331	1.0472	1.0472
k_3^{cc}	**2.5339**	1.5708	2.0944	**2.0128**	2.0944	1.5708	2.0944	1.5708	1.5708
k_4^{cc}	3.0927	2.0944	**2.5870**	**2.5827**	2.5870	2.0944	2.5837	**2.0944**	2.0944
k_5^{cc}	3.6461	**2.6180**	3.0943	3.1355	3.0943	2.6180	3.0959	2.6180	2.6180
k_6^{cc}	**4.1669**	3.1416	3.6461	3.6637	3.6461	3.1416	3.6501	3.1416	3.1416
k_7^{cc}	**4.6422**	3.6652	4.1888	**4.1688**	4.1888	3.6652	4.1888	3.6652	3.6652
k_8^{cc}	5.1791	4.1888	**4.6959**	**4.6727**	4.6959	4.1888	4.6927	**4.1888**	4.1888
k_9^{cc}	**5.7051**	**4.7124**	5.2077	5.1946	5.2077	4.7124	5.2092	4.7124	4.7124
k_{10}^{cc}	**6.2715**	5.2360	5.7472	**5.7314**	5.7472	5.2360	5.7513	5.2360	5.2360

Table 6.9 The first ten eigenvalues for each contact case.

- **Calculation of the admissible eigenmodes**

The eigenvalues of Table 6.9 are acceptable only if the corresponding eigenmodes satisfy the restrictions which correspond to each contact situation. Table. 6.10 presents the first ten admissible eigenvalues and the corresponding eigenmodes of the under study constrained buckling problem, while Fig. 6.13 and Fig. 6.14 show the graphical representation of the eigenmodes.

Contact Case	Contact Status	Accepted Eigenvalue	B_1	Functions of the eigenmodes
CC4	I-A	$k_1 = 0.8338$	>0	$w(x_1) = B_1 \sin kx_1 + C_1 x_1,\ \ x_1 \in [0,1.5]$
CC3	A-I	$k_2 = 0.9022$	>0	
CC2	I-I	$k_3 = 1.0472$	>0	
CC1	A-A	$k_4 = 2.0027$	>0	
CC4	I-A	$k_5 = 2.0128$	>0	$w(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2,\ \ x_2 \in [0,3.60]$
CC8	N-I	$k_6 = 2.0944$	>0	
CC1	A-A	$k_7 = 2.5339$	<0	
CC4	I-A	$k_8 = 2.5827$	<0	$w(x_3) = B_3 \sin kx_3 + C_3 x_3,\ \ x_3 \in [0,0.90]$
CC3	A-I	$k_9 = 2.5870$	<0	
CC2	I-I	$k_{10} = 2.6180$	<0	

Table 6.10 The first ten admissible eigenvalues and the corresponding eigenmodes.

It is now noticed that when the unilateral support at point D is moving to the right, the critical buckling mode corresponds to the case where the unilateral support at point D is active, while simultaneously the other unilateral support is inactive. The effective buckling length of the beam for this contact case (CC4) is the greatest. Therefore the critical load is now given by the following equation:

$$P_{cr} = k_1^{\,2} EI = 0.8338^2 \times 17556 = 12205.32\ kN\,. \tag{6.5}$$

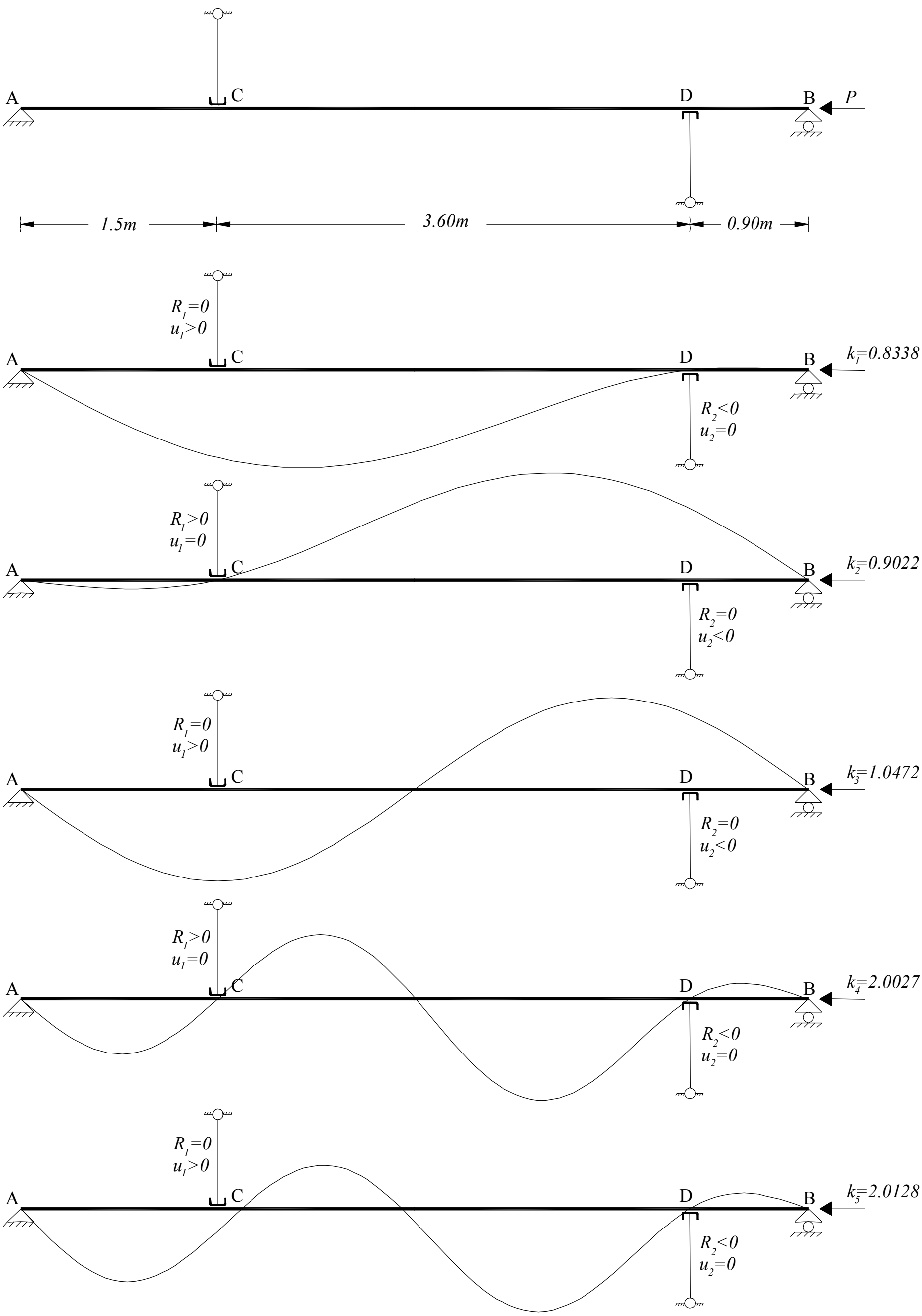


Fig. 6.13 The first five eigenmodes of the beam of *Example 4*.

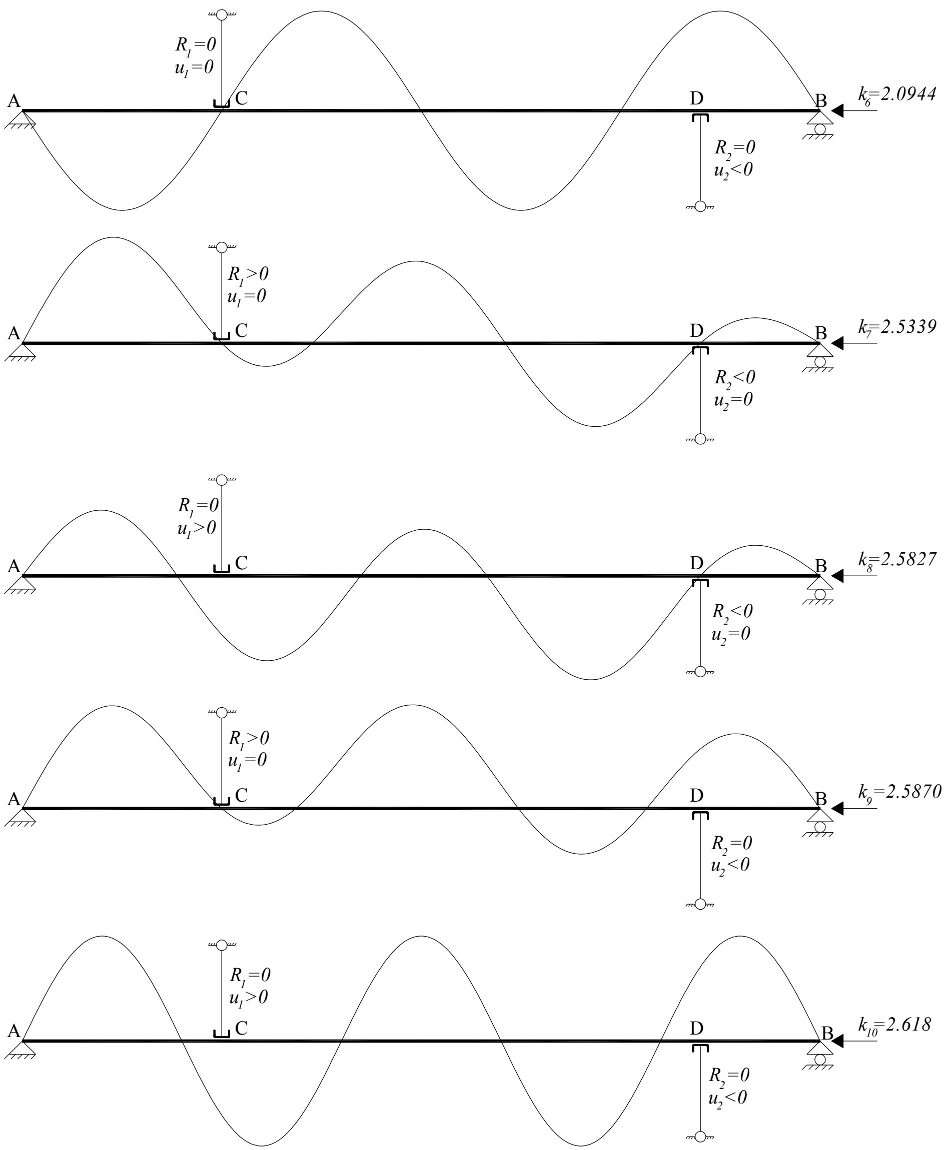
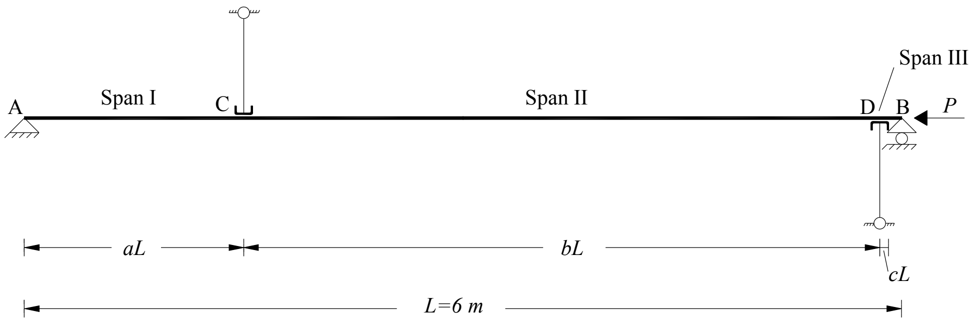


Fig. 6.14 Superior eigenmodes of the beam of Example 4.

6.4.5 Example 5

The following example is also a particular one because it refers to a special case where the unilateral support at point D has come very close to the roll support at point B (Fig. 6.15).



<u>*Data*</u>

E=210000 Mpa

I=8360 cm^4

aL=1.5 m, bL=4.35 m, cL=0.15 m

Fig. 6.15 The geometrically perfect beam of Example 5.

It is noticed that the two close supports at points B and D may be substituted by a single support which exhibits either free or fixed rotation. The free rotation corresponds to the case where the initial unilateral support at point D is active, while the fixed rotation corresponds to the opposite case. This special support can be considered as a **unilaterally clamped support.** A proposal for sketching such a support is presented in Fig. 6.16

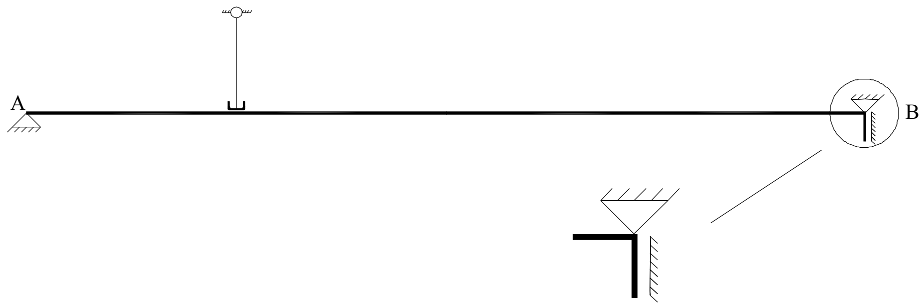


Fig. 6.16 A proposal for sketching unilaterally clamped support.

- **Calculation of the eigenvalues (buckling loads)**

For each contact situation (see Table 5.1) the corresponding eigenvalues are calculated through the utilization of the respective buckling equations. Table 6.11 displays the first ten eigenvalues (from the lowest to the highest) for each contact case.

Contact Case	CC1 $R_1>0$ $u_1=0$ $R_2<0$ $u_2=0$	CC2 $R_1=0$ $u_1>0$ $R_2=0$ $u_2<0$	CC3 $R_1>0$ $u_1=0$ $R_2=0$ $u_2<0$	CC4 $R_1=0$ $u_1>0$ $R_2<0$ $u_2=0$	CC5 $R_1>0$ $u_1=0$ $R_2=0$ $u_2=0$	CC6 $R_1=0$ $u_1>0$ $R_2=0$ $u_2=0$	CC7 $R_1=0$ $u_1=0$ $R_2<0$ $u_2=0$	CC8 $R_1=0$ $u_1=0$ $R_2=0$ $u_2<0$	CC9 $R_1=0$ $u_1=0$ $R_2=0$ $u_2=0$
Contact Status	A-A	I-A	A-I	I-A	A-N	I-N	N-A	N-I	N-N
k_1^{cc}	1.2638	0.5236	**0.9022**	**0.7617**	0.9022	0.5236	2.0944	0.5236	0.5236
k_2^{cc}	**1.8179**	**1.0472**	1.5294	1.3095	1.5294	1.0472	4.0834	1.0472	1.0472
k_3^{cc}	**2.3545**	1.5708	2.0944	**1.8482**	2.0944	1.5708	4.1888	1.5708	1.5708
k_4^{cc}	2.8923	2.0944	**2.5870**	**2.3842**	2.5870	2.0944	4.2997	**2.0944**	2.0944
k_5^{cc}	**3.3970**	**2.6180**	3.0943	2.9188	3.0943	2.6180	6.2832	2.6180	2.6180
k_6^{cc}	3.9862	3.1416	**3.6461**	**3.4528**	3.6461	3.1416	8.1674	3.1416	3.1416
k_7^{cc}	4.5176	**3.6652**	4.1888	3.9862	4.1888	3.6652	8.3776	3.6652	3.6652
k_8^{cc}	**5.0077**	4.1888	4.6959	4.5195	4.6959	4.1888	8.5987	**4.1888**	4.1888
k_9^{cc}	5.5752	4.7124	**5.2077**	**5.0524**	5.2077	4.7124	10.4720	4.7124	4.7124
k_{10}^{cc}	**6.1120**	**5.2360**	5.7472	5.5852	5.7472	5.236	12.5664	5.236	5.2360

Table 6.11 The first ten eigenvalues for each contact case.

- **Calculation of the admissible eigenmodes**

The eigenvalues of Table 6.11 are acceptable only if the corresponding eigenmodes satisfy the restrictions which correspond to each contact situation. Table. 6.12 presents the first ten admissible eigenvalues and the corresponding eigenmodes of the under study constrained buckling problem, while Fig. 6.17 and Fig. 6.18 show the graphical representation of the eigenmodes.

Contact Case	Contact Status	Accepted Eigenvalue	B_1	Function of the eigenmodes
CC4	I-A	$k_1 = 0.7617$	>0	
CC3	A-I	$k_2 = 0.9022$	>0	$w(x_1) = B_1 \sin kx_1 + C_1 x_1,\ x_1 \in [0, 1.5]$
CC2	I-I	$k_3 = 1.0472$	>0	
CC1	A-A	$k_4 = 1.8179$	>0	
CC4	I-A	$k_5 = 1.8483$	>0	$w(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2,\ x_2 \in [0, 4.35]$
CC8	N-I	$k_6 = 2.0944$	>0	
CC1	A-A	$k_7 = 2.3545$	<0	
CC4	I-A	$k_8 = 2.3842$	<0	$w(x_3) = B_3 \sin kx_3 + C_3 x_3,\ x_3 \in [0, 0.15]$
CC3	A-I	$k_9 = 2.5870$	<0	
CC2	I-I	$k_{10} = 2.618$	<0	

Table 6.12 The first ten admissible eigenvalues and the corresponding eigenmodes.

As it clearly results from Table 6.12, the critical buckling mode occurs when the unilateral support at point D is active, while at the same time the unilateral support at point C is inactive, i.e.:

$$P_{cr} = k_1^{\,2} EI = 0.7617^2 \times 17556 = 10185.76\ kN \tag{6.6}$$

The corresponding critical eigenmode is actually the critical eigenmode of the buckling problem of a beam with length equal to L and fixed restraint conditions at point B. As it is well known, the critical load for this case is given by the following relation:

$$k_{cr} \cong \frac{\pi}{(0.7L)} = \frac{\pi}{(0.7 \times 6)} = 0.748\,. \tag{6.7}$$

Obviously, the value $k_{cr} = 0.748$ is very close to the value $k_1 = 0.7617$. The accuracy is affected by the length cL. As $c \to 0$, the obtained critical value $k_1 \to k_{cr}$.

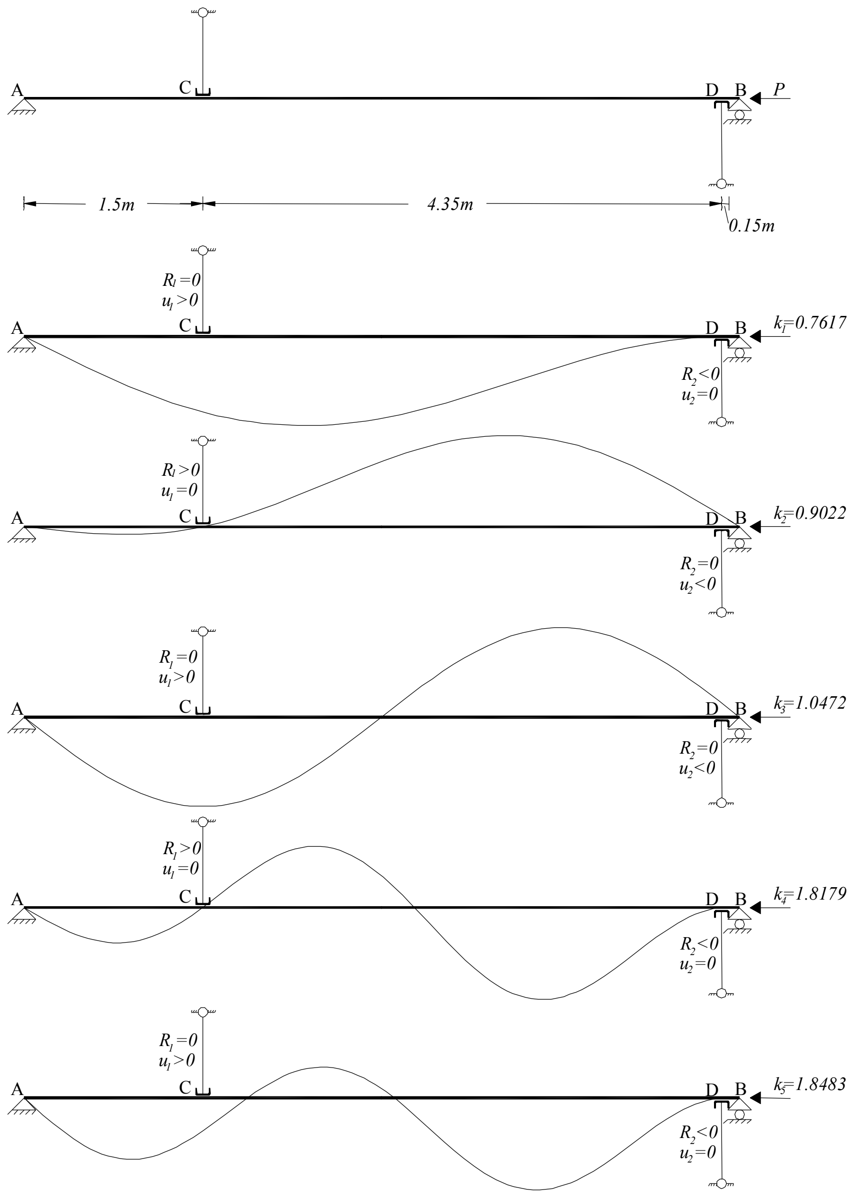


Fig. 6.17 The first five eigenmodes of the beam of Example 5.

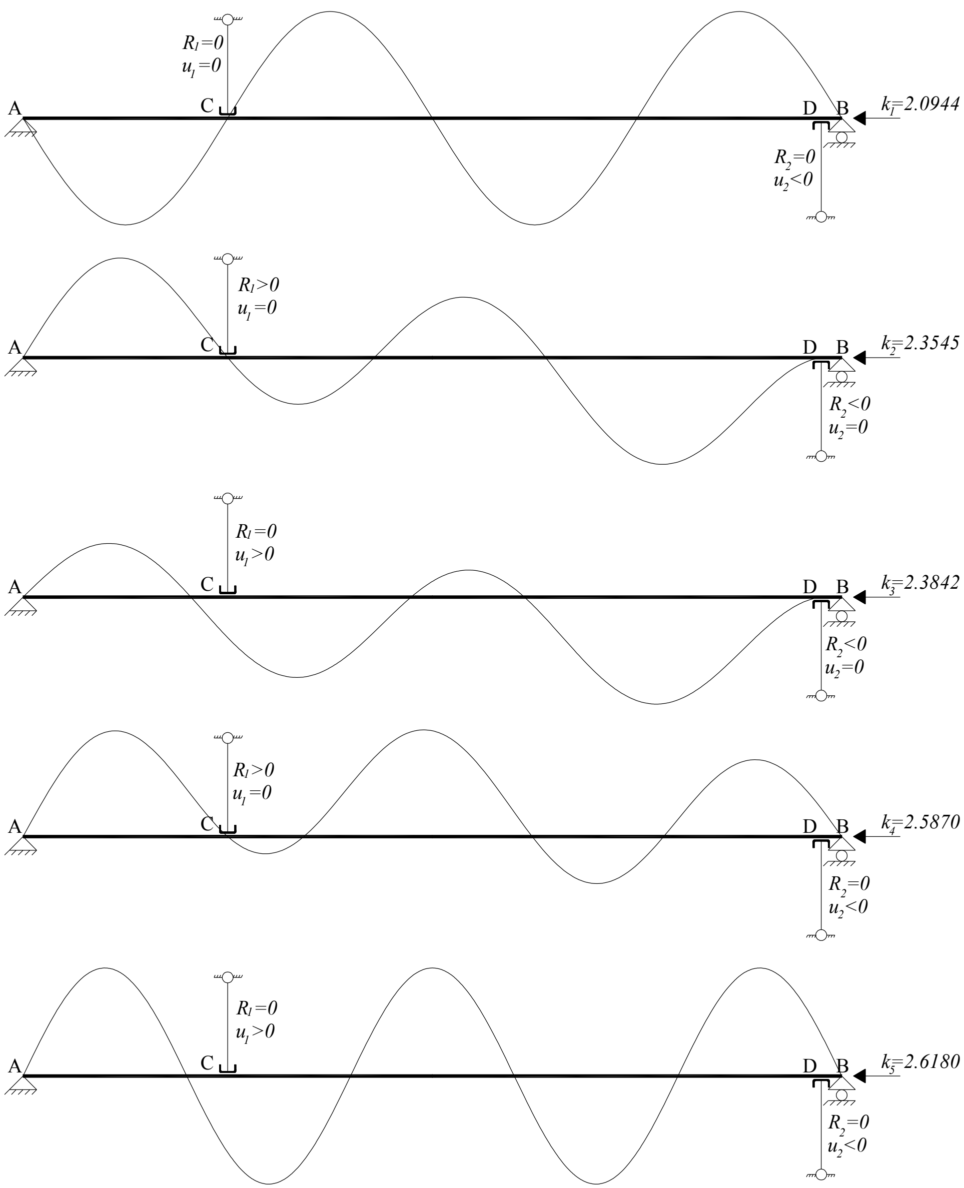


Fig. 6.18 Superior eigenmodes of the beam of Example 5.

7 Geometrically imperfect beams - Examples

7.1 Introduction

The main contribution of the present dissertation is the treatment, through an analytical mathematical way, of the buckling problem of geometrically imperfect beams in the presence of unilateral supports. The proposed methodology which was described thoroughly in Chapters 4 and 5, gives the potential of calculating the instability load of geometrically imperfect beams with one intermediate unilateral support or with two intermediate unilateral supports functioning in an opposite direction.

In the present chapter several examples are demonstrated, covering different cases. Beams with various types of imperfect shapes and initial contact conditions are treated. In the first series of the presented examples (Section 7.3) the instability load of a beam with one intermediate unilateral support is calculated for different initial contact conditions, according to the mathematical theory introduced in Chapter 4. In Section 7.4, beams with two unilateral supports functioning in an opposite direction are treated, with various types of initial geometric imperfections. In the presented examples different contact conditions appear during the bending deformation, offering in that way a better comprehension of the implementation of the proposed methodology. For the implementation of the demonstrated examples the following steps are considered in both cases.

- Initial imperfections are introduced in the structure. The imperfect shape of the beam is described by a Fourier sine series.

- Depending on the number of the unilateral supports (one or two), the calculation procedure for the determination of the instability load introduced in Chapter 4 (see Section 4.3.4) or Chapter 5 (Section 5.3.3) is followed.

In most of the presented examples, the instability load is calculated considering elastic material behaviour. Nevertheless, another contribution of the present study is the ability to consider in the calculations the real strength of the cross-sections of the beams, which offers the opportunity of dealing with practical applications, where the beam fails after the exhaustion of its bending strength. For this reason, in the last example of each of the following sections, the ultimate load of the beam is calculated considering the actual strength according to provisions of Eurocode 3 for steel beams.

7.2 Review of the presented examples

In the following paragraphs, the buckling problem of geometrically imperfect beams is under study for specific cases. More specifically, Section 7.3 displays four examples which concern the unilateral contact buckling problem of beams with one unilateral support.

In the first one (Fig. 7.1), the imperfect beam is not initially in contact with the unilateral support. Due to the increase of loading, the beam will touch the unilateral support for a specific value of load, which is termed as P_c. Finally, the beam will buckle when the load approaches a specific value which corresponds to the first eigenvalue of the buckling problem of the geometrically perfect beam (Fig. 7.2).

In the second example (Fig. 7.3), different contact conditions hold in the beginning of loading. In particular, the beam is in contact with the unilateral support before the beginning of loading and will remain in contact until the moment that the applied load reaches a specific value. For this value of load, the beam tends to be separated from the unilateral support. It is then proved, that the beam does not have the potential to be in equilibrium and thus, the deflections of the beam are accompanied by a sudden decrease in the values of load. Eventually, the beam buckles according to the first eigenmode of the simply supported beam (case of inactive constraint) (Fig. 7.4).

The third example of this category (Fig. 7.5) is similar to the previous one with the difference that instead of using an arbitrary imperfection, a specific superior eigenmode of the corresponding bifurcation problem is considered as the initial imperfection. In this case, the beam tends to be separated from the unilateral support (i.e. to change contact status) for a value of load which is approximately equal to the buckling load of the corresponding bilateral problem (Fig. 7.5)

In the end of Section 7.3, the problem of Example 1 is extended (Fig. 7.1) in order to cover the case of the material nonlinearity. More specifically, the aim of Example 4 is to determine the ultimate load of the imperfect beam of Example 1, when strength criteria are taken into account. Moreover, a parametric investigation is performed in order to investigate the effect of the amplitude of the imperfection to the ultimate load of the beam.

Section 7.4 presents five different examples of beams with two unilateral supports functioning in an opposite direction. In the first one (Fig. 7.11), the beam is not initially in contact with any of the unilateral supports. For a certain value of load the beam will come in contact with one of the unilateral supports. That new contact status, where the one of the unilateral supports is active while the other is inactive, will remain until the moment that the applied load will approach a certain value for which extremely large deflections are developed in the structure (Fig 7.12).

The second example of Section 7.4 (Fig. 7.13) is particular, because the contact status changes many times during the bending deformation. More specifically, the beam is initially in contact with one of the unilateral supports. For a specific value of load, loss of contact occurs, leading to a different contact case. For greater values of loading the beam will come in contact with the other unilateral support. In the sequel, the beam has the potential to sustain more loading and eventually to come in contact for a second time with the other unilateral support. Finally, the beam buckles for a buckling mode which corresponds to the case where both constraints are active (Fig 7.14).

The third example in Section 7.4 (Fig. 7.15) is also particular, because a lot of different contact conditions are alternated until the beam reaches an instability state. The innovative aspect with respect to the previous example, is that an abrupt decrease of loading takes place during the bending deformation (Fig 7.16).

The fourth example (Fig. 7.17) concerns the case where the beam is initially in contact with both constraints. However, the arising buckling mode corresponds to the contact status where one of the unilateral supports is active while the other is inactive (Fig. 7.18).

In the end of Section 7.4, the problem of Example 1 (Fig. 7.11) is extended in order to determine the ultimate load of the specific imperfect beam under the consideration of the actual strength of the cross-section of the beam (Example 5).

7.3 Geometrically imperfect beams with one unilateral support

7.3.1 Example 1

The continuous beam of Fig. 7.1 is considered, which has a total length of 6m and stiffness rigidity equal to $EI = 16989\ kNm^2$. The beam is divided into two unequal spans by a unilateral contact support which is placed at a distance of 4.7 m from the left end of the beam. The initial imperfection (Fig.7.1) of the beam is supposed to be described by a Fourier sine series of five terms and is given by the following relation:

$$w_0(x) = \sum_{r=1}^{5} b_r \sin\frac{r\pi x}{6}, \quad x \in [0,6], \tag{7.1}$$

while the Fourier coefficients are given in Table 7.1.

Fourier coefficients of the imperfection function
$b_1 = 0.00125$
$b_2 = 0.00250$
$b_3 = 0.00330$
$b_4 = 0.00650$
$b_5 = 0.00125$

Table. 7.1 The Fourier coefficients for the function of the geometric initial imperfection.

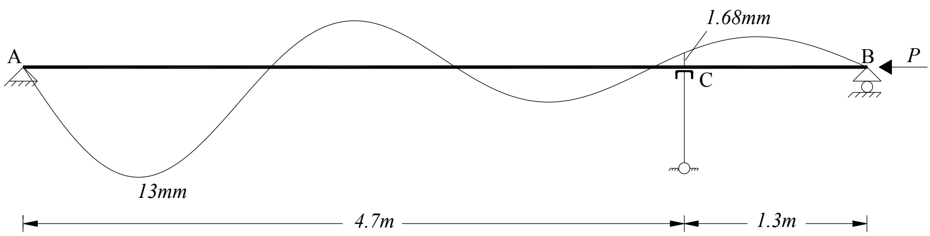


Fig. 7.1 The imperfect beam of Example 1 of Section 7.3.1.

The initial imperfection does not violate the intermediate unilateral constraint due to the fact that it satisfies the requirement of inequality (4.50). It is noted that a gap equal to 1.68 mm exists at the position of the unilateral support. The specific imperfect beam is subjected to an axial compressive load P leading to bending deformation. The bending behaviour of the above problem is described through the equations (4.55) and (4.56), the boundary conditions (4.61)-(4.68) and the restrictions (4.11)-(4.13). Due to the fact that the solution of the non-homogeneous constrained BVP is connected with the eigenvalues of the corresponding bifurcation problem, Table 7.2 presents the critical eigenvalues for each contact case.

Contact Case	Eigenvalue	Critical load $P_{cr,eig}$
Inactive constraint	$k_2 = 0.5236$	$P_{cr,eig}^{(1)} = 4657.65\ kN$
Active constraint	$k_2 = 0.8784$	$P_{cr,eig}^{(2)} = 13108.87\ kN$
Inactive constraint	$k_3 = 1.0472$	$P_{cr,eig}^{(3)} = 18630.31\ kN$

Table 7.2 The critical eigenvalues of the corresponding bifurcation problem for each contact case.

The required calculations for the determination of the eigenvalues of Table 7.2 are not presented herein because they have been already displayed in Example 1 of Section 6.3. In order to determine the instability load of the beam and the final buckling shape, the following steps are followed according to the calculation procedure of Section 4.3.4:

- Application of Steps 1 and 2

Initially, it is checked if the beam is in contact with the unilateral support. In the beginning of loading the beam is not in contact with the unilateral support thus, the constraint at point C is inactive ($u<0,\ R=0$). For this contact situation the equations of section 4.3.2.1 hold. Therefore, for values of the load P which do not constitute eigenvalues of the corresponding homogeneous BVP and at the same time satisfy the required restrictions, the deflection curve of the beam is described by the following relations:

$$w_1(x_1)=\sum_{r=1}^{5} b_r F_r \sin\left(\frac{r\pi x_1}{6}\right),\quad x_1\in[0,4.7] \tag{7.2}$$

$$w_2(x_2)=-\sum_{r=1}^{5} b_r F_r \sin\left(\frac{r\pi x_2}{6}\right)(-1)^r,\quad x_2\in[0,1.3]. \tag{7.3}$$

Then, it should be examined if a value of the load P which makes (4.87) untrue, exists. If such a value exists, then the beam will come in contact with the unilateral support and a contact reaction force will appear. The load P that corresponds to the moment that the beam will marginally come in contact with the unilateral constraint, without producing any reaction force (neutral contact status), is calculated through equations (4.89) or (4.90). The calculations for the case treated here give $P_c=3391.365\ \text{kN}$. In the sequel, it has to be checked if for this specific contact status **(inactive constraint)** the corresponding homogeneous BVP produces an eigenvalue inside the interval $[0,P_c]$. From Table 7.2 it is obtained the value, $P_{cr,eig}^{(1)}=4657.65\ kN$ which of course, does not lie inside the interval $[0,P_c]$. Therefore, the applied load can be increased and the deflection curve of the beam will be described by a different set of equations, as follows.

- Application of Step 4

For load values $P\geq P_c$ which do not constitute eigenvalues of the corresponding homogeneous BVP (i.e. eigenvalues of the buckling problem of the geometrically perfect beam for the case of the active constraint) and at the same time do not violate the required restriction of this specific contact situation, the set of equations of Section 4.3.2.2 holds. The deflection curve of the beam is then given by the following equations:

$$w_1(x_1) = B_1 \sin kx_1 + C_1 x_1 + \sum_{r=1}^{5} b_r \sin\left(\frac{r\pi x_1}{6}\right), \quad x_1 \in [0, 4.7] \tag{7.4}$$

$$w_2(x_2) = B_2 \sin kx_2 + C_2 x_2 - \sum_{r=1}^{5} b_r \sin\left(\frac{r\pi x_2}{6}\right)(-1)^r, \quad x_2 \in [0, 1.3]. \tag{7.5}$$

The coefficients B_1, B_2, C_1, C_2 are calculated through equations (4.96), (4.97), (4.100) and (4.101). It is noted that the deflections of the beam tend to infinity when the value of load P approaches the value:

$$P_{cr,eig}^{(2)} = 13108.87\ kN. \tag{7.6}$$

The load $P_{cr,eig}^{(2)}$ constitutes the first eigenvalue of the corresponding homogeneous BVP for the specific contact case **(active constraint)**. Additionally, the set of equations (7.5) and (7.6) is valid for every $P \in [3391.365, 13108.87)$ i.e. restriction (4.102) holds for each value inside this interval. Therefore, the instability load of the structure for the given imperfection is equal to:

$$P_i = P_{cr,eig}^{(2)} = 13108.87\ kN. \tag{7.7}$$

Fig. 7.2 presents the deflections of the beam for characteristic values of the applied axial load P.

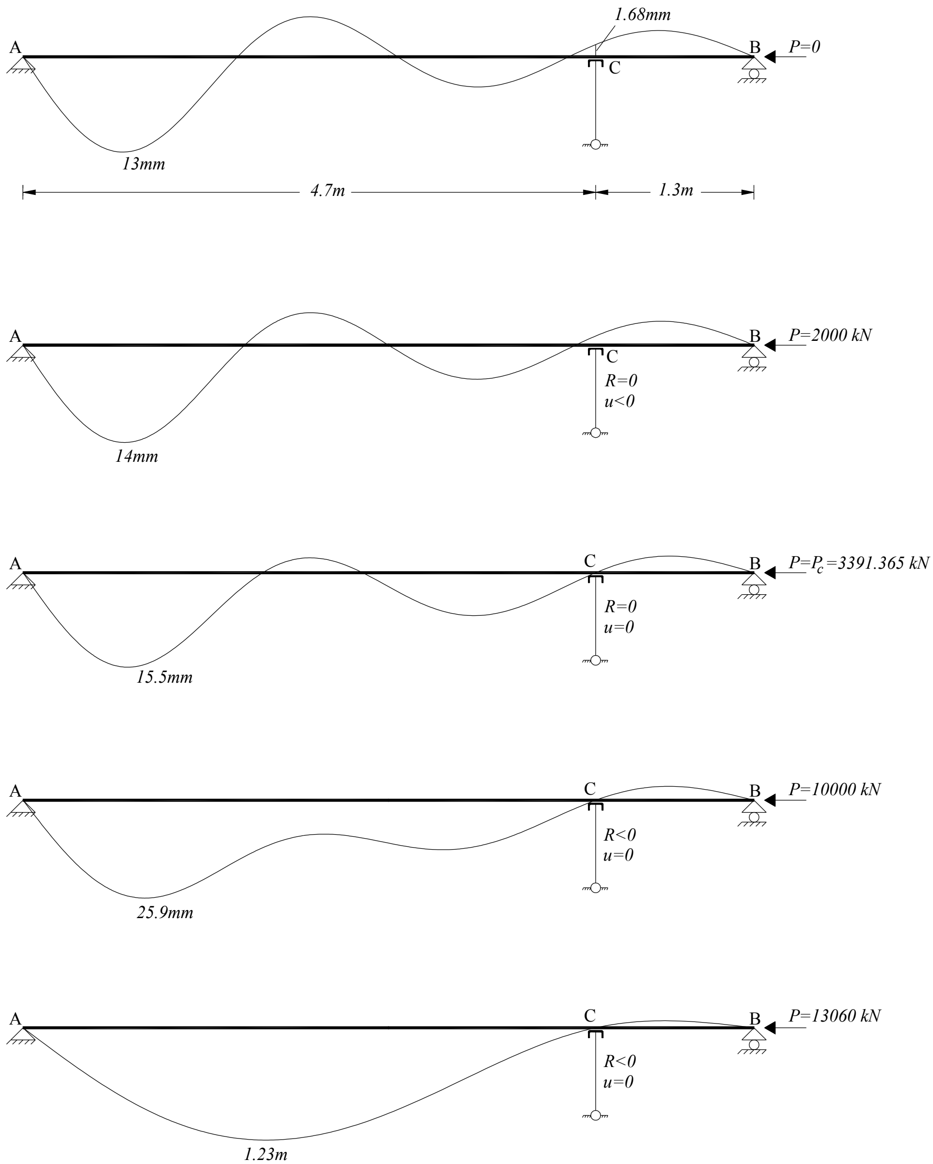


Fig. 7.2 Progressive deflection of the beam of Example 1 of Section 7.3.1 till instability.

It has to be noticed that the type of instability which arises in the studied example (i.e. extremely large deflections as the applied load approaches the critical eigenvalue $P_{cr,eig}^{(2)} = 13108.87\, kN$), results from the validity of the fundamental Theorem 3.4. More specifically, for the case treated here the right hand side of the fundamental differential equations (4.55) and (4.56) take the form:

$$f_1(x) = \sum_{r=1}^{5} b_r \left(\frac{r\pi}{6}\right)^4 \sin\left(\frac{r\pi x_1}{6}\right), \;\; x_1 \in [0, 4.7]. \tag{7.8}$$

$$f_2(x) = -\sum_{r=1}^{5} b_r \left(\frac{r\pi}{6}\right)^4 \sin\left(\frac{r\pi x_2}{6}\right)(-1)^r, \;\; x_2 \in [0, 1.3]. \tag{7.9}$$

For the critical eigenvalue $P_{cr,eig}^{(2)} = 13108.87\, kN$, the corresponding eigenmode is written as:

$$\varphi_1^{(2)}(x) = B_1 \sin kx_1 + C_1 x_1, \;\; x_1 \in [0, 4.7]. \tag{7.10}$$

$$\varphi_2^{(2)}(x_2) = B_2 \sin kx_2 + C_2 x_2, \;\; x_2 \in [0, 1.3]. \tag{7.11}$$

For:

$$k_2 = \sqrt{\frac{P_{cr,eig}^{(2)}}{EI}} = \sqrt{\frac{13108.87}{16989}} = 0.878413 \tag{7.12}$$

and using relations (4.35)-(4.37) the unknown coefficients B_1, C_1, C_2 are determined as a function of the coefficient $B_2 < 0$. Therefore, equations (7.10), (7.11) take the following form:

$$\begin{aligned}\varphi_1(x) &= \left(-\frac{1}{0.91743}\right) B_2 \sin kx_1 - \frac{0.83438}{0.91743} B_2 x_1 = \\ &= -B_2 \left[\left(-\frac{1}{0.91743}\right)\sin kx_1 + \left(\frac{0.83438}{0.91743}\right)x_1\right], \;\; x_1 \in [0, 4.7]\end{aligned} \tag{7.13}$$

$$\varphi_2(x) = B_2 \sin kx_2 - \left(\frac{3.01649}{0.91743}\right) B_2 x_2 = B_2 \left[\sin kx_2 - \left(\frac{3.01649}{0.91743}\right) x_2\right], \;\; x_2 \in [0, 1.3]. \tag{7.14}$$

Taking the scalar product of the functions $f(x)=\begin{cases} f_1(x_1), \; x_1 \in [0,4.7] \\ f_2(x_2), \; x_2 \in [0,1.3] \end{cases}$ and $\varphi^{(2)}(x)=\begin{cases} \varphi_1^{(2)}(x_1), \; x_1 \in [0,4.7] \\ \varphi_2^{(2)}(x_2), \; x_2 \in [0,1.3] \end{cases}$ the following equations are derived:

$$\left(f_1,\varphi_1^{(2)}\right)=\int_0^{4.7}\left(f_1(x_1)\cdot\varphi_1^{(2)}(x_1)\right)dx_1=-0.0852B_2\neq 0 \tag{7.15}$$

$$\left(f_2,\varphi_2^{(2)}\right)=\int_0^{1.3}\left(f_2(x_2)\cdot\varphi_2^{(2)}(x_2)\right)dx_2=0.1095B_2\neq 0 \; . \tag{7.16}$$

Therefore, for every eigenmode $\varphi^{(2)}(x)$ (with $B_2<0$), which corresponds to the eigenvalue (7.12), the function $f(x)$ is not orthogonal to that eigenmode. Consequently, the initial constrained BVP is unsolvable for the eigenvalue (7.12). As it was stated in Chapter 3, this result indicates instability as the applied load approaches the critical eigenvalue for which the problem is unsolvable. It has to be mentioned that in case for which the integrals (7.15) and (7.16) are equal to zero, the eigenmode $\varphi^{(2)}(x)$ is orthogonal to the function $f(x)$ leading to infinite solutions. Again, the critical eigenvalue constitutes singular point for the solution of the initial BVP, but in that case a different type of instability arises (see Example 3.4).

7.3.2 Example 2

Let us consider a beam with a length of 6m and the same features as considered in the previous example. In present example a new initial geometric imperfection is considered, which is described by a Fourier sine series of twenty terms (Table 7.3). This initial imperfection is presented schematically in Fig. 7.3.

$$w_0(x)=\sum_{r=1}^{20} b_r \sin\frac{r\pi x}{6}, \quad x\in[0,6] \tag{7.17}$$

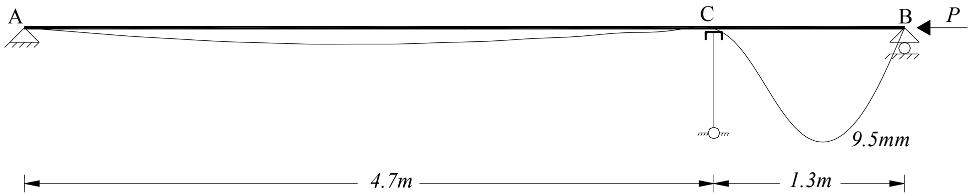


Fig. 7.3 The imperfect beam of Example 2.

Fourier coefficients for the imperfection function			
$b_1 = 0.0018056$	$b_6 = -0.0016562$	$b_{11} = -0.0000502$	$b_{16} = -0.0000419$
$b_2 = -0.0008027$	$b_7 = 0.0012820$	$b_{12} = 0.0001333$	$b_{17} = 0.0000632$
$b_3 = 0.0016436$	$b_8 = -0.0008533$	$b_{13} = -0.0001331$	$b_{18} = -0.0000586$
$b_4 = -0.0019047$	$b_9 = 0.0004570$	$b_{14} = 0.0000769$	$b_{19} = 0.0000327$
$b_5 = 0.0018976$	$b_{10} = -0.0001468$	$b_{15} = -0.000012$	$b_{20} = -0.0000015$

Table. 7.3 The Fourier coefficients for the function of the geometric initial imperfection of Example 2 of Section 7.3.2.

Due to the fact that the initial imperfection satisfies the equality condition of the restriction (4.50), i.e. $w_0(4.7)=0$, the beam is initially in contact with the unilateral support at point C. The bending behaviour of the above problem is described through equations (4.55) and (4.56), the boundary conditions (4.61)-(4.68) and the restrictions (4.11)-(4.13). Therefore, in order to determine the instability load of the beam, the following steps are followed according to the calculation procedure of Section 4.3.4.

- Application of Steps 1 and 3

Because the beam is initially in contact with the unilateral support, it is checked whether a valid value P_s exists, yielded by relation (4.104) when it holds as equality (i.e. $R=0$). The calculation for the specific problem prove that, when the load takes the value $P_s \cong 11740.587\ kN$ the reaction force of the constraint becomes zero and the beam develops the tendency to be separated from the unilateral support. For values of the load $P < P_s$ the reaction force of the unilateral support is active and therefore the deflection curve of the beam can be described by the following relations:

$$w_1(x_1) = B_1 \sin kx_1 + C_1 x_1 + \sum_{r=1}^{20} b_r F_r \sin\left(\frac{r\pi x_1}{6}\right), \quad x_1 \in [0, 4.7] \tag{7.18}$$

$$w_2(x_2) = B_2 \sin kx_2 + C_2 x_2 - \sum_{r=1}^{20} b_r F_r \sin\left(\frac{r\pi x_2}{6}\right)(-1)^r, \quad x_2 \in [0, 1.3], \tag{7.19}$$

where the coefficients B_1, B_2, C_1, C_2 are calculated through equations (4.98), (4.99), (4.100) and (4.101). It is essential to notice that the above solution is valid for all the values of load P which lie inside the interval $(0, 11740.587]$ because no one of the eigenvalues of the corresponding bifurcation buckling problem belongs to that interval (see Example 1 of section 6.2.1).

- Application of Step 5

For $P > P_s$, the contact status will change. Due to the fact that, the first eigenvalue of the corresponding BVP (case of the inactive constraint), is smaller than P_s (see Table 7.2 of the previous example), the beam cannot sustain more loading and will suddenly buckle. The arising deflections of the beam are accompanied by an abrust decrease in the values of loading. In that case the buckling response of the beam is similar to the one appearing in simply supported beams, i.e. the beam buckles for $P \rightarrow P_{cr,eig}^{(1)} = 4657.65\ kN$, thus the instability load is equal to:

$$P_i = P_{cr,eig}^{(1)} = 4657.65\ kN\ . \tag{7.20}$$

The progressive deflection of the beam till the instability is presented in Fig. 7.4.

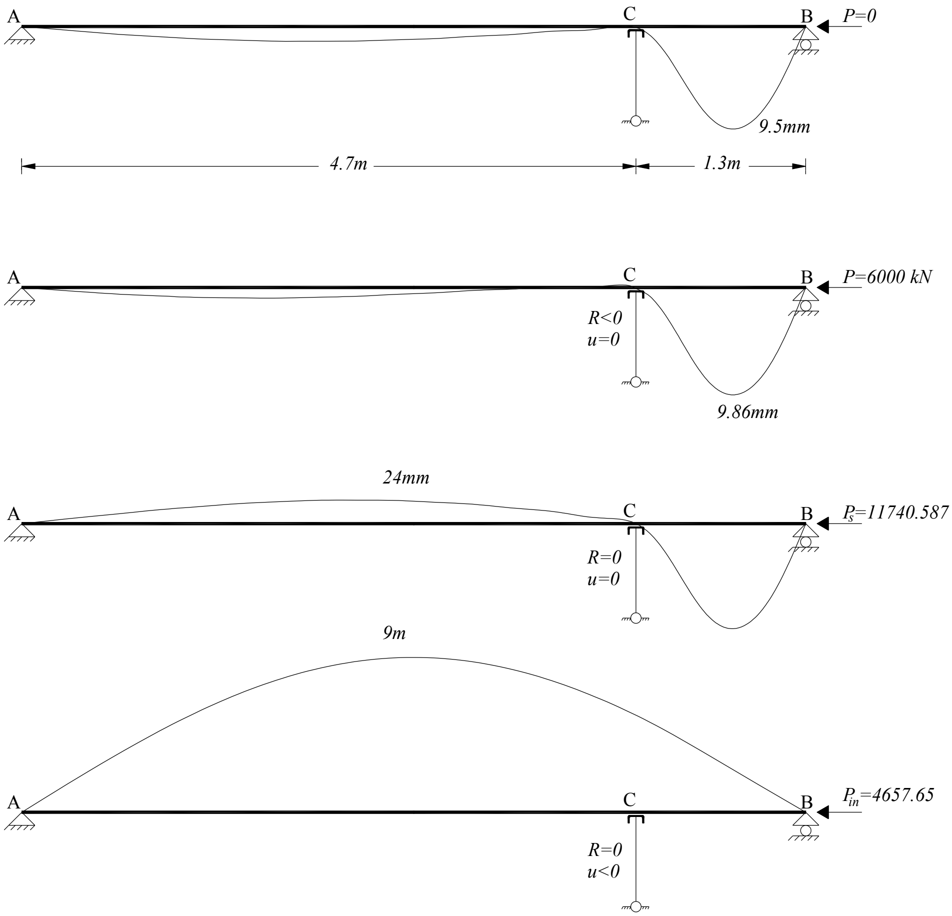


Fig. 7.4 Progressive deflection of the beam of Example 2 of Section 7.3.2.

7.3.3 Example 3

In the present example the same continuous beam is again considered but with an initial imperfection which has the shape of the second eigenmode of the corresponding homogeneous BVP (see Section 6.3, Fig. 6.2). For this eigenmode the unilateral support is active. The description of this imperfection by means of a Fourier sine series, is achieved through the Discrete Fourier Transformation[16] (DFT) method taking into account twenty Fourier terms.

[16] Details concerning the transformation of a function into a Fourier series and especially for the application of the DFT method, are given in Appendix C.

$$w_0(x) = \sum_{r=1}^{20} b_r \sin\frac{r\pi x}{6}, \quad x \in [0,6]. \tag{7.21}$$

The imperfection is presented in Fig. 7.5, while the Fourier coefficients are given in Table 7.4.

Fourier coefficients for the imperfection function			
$b_1 = 0.0039782$	$b_6 = -0.0000367$	$b_{11} = -0.0000031$	$b_{16} = -0.0000007$
$b_2 = -0.0026521$	$b_7 = 0.0000227$	$b_{12} = 0.0000022$	$b_{17} = 0.0000005$
$b_3 = -0.005568$	$b_8 = -0.0000095$	$b_{13} = -0.0000009$	$b_{18} = -0.0000001$
$b_4 = 0.0001480$	$b_9 = 0.0000012$	$b_{14} = -0.0000001$	$b_{19} = -0.0000001$
$b_5 = 0.0000280$	$b_{10} = 0.0000025$	$b_{15} = 0.0000007$	$b_{20} = 0.0000002$

Table. 7.4 The Fourier coefficients for the function of the geometric initial imperfection of Example 3 of Section 7.3.3.

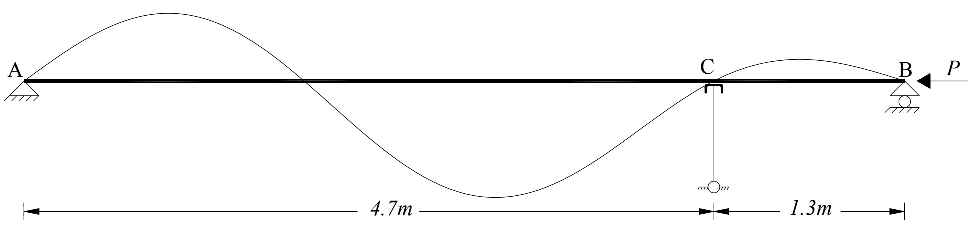


Fig. 7.5 The imperfect beam of Example 3 of Section 7.3.3.

The initial imperfection satisfies the equality condition of the restriction (4.50), i.e. $w_0(4.7) = 0$, thus the beam is initially in contact with the unilateral support at point C. The bending behaviour of the above problem can be described through equations (4.55) and (4.56), the boundary conditions (4.61)-(4.68) and the restrictions (4.11)-(4.13). The determination of the instability load of the beam of Fig. 7.5 is accomplished by applying the calculation procedure of Section 4.3.4. More specifically,

- Application of Steps 1 and 3

Due to the fact that the beam is initially in contact with the unilateral support, it has to be checked if a value of the applied load P exists, such that the restriction (4.102) is not fulfilled. The inequality restriction (4.102), when it holds as equality, yields the value $P_s = 13108.306\,kN$. It is noticed that the reaction force of the unilateral constraint increases together with the loading. However, when the applied load takes values sufficiently close to P_s, then the reaction force is starting to decrease. For $P = P_s = 13108.306\,kN$, the reaction force of the unilateral support becomes zero and the beam develops the tendency to be separated from the unilateral support.

For $P < P_s$ the deflection curve of the beam is described by the following relations:

$$w_1(x_1) = B_1 \sin kx_1 + C_1 x_1 + \sum_{r=1}^{20} b_r F_r \sin\left(\frac{r\pi x_1}{6}\right), \quad x_1 \in [0, 4.7] \tag{7.22}$$

$$w_2(x_2) = B_2 \sin kx_2 + C_2 x_2 - \sum_{r=1}^{20} b_r F_r \sin\left(\frac{r\pi x_2}{6}\right)(-1)^r, \quad x_2 \in [0, 1.3]\,. \tag{7.23}$$

In the above, the coefficients B_1, B_2, C_1, C_2 are calculated through equations (4.98)-(4.101).

- Application of Step 5

Due to the fact that the critical eigenvalue which corresponds to the contact case of the inactive constraint is equal to the Euler load:

$$P_{cr,eig}^{(1)} = 4657.63\ kN\,, \tag{7.24}$$

which is obtained from Table 7.2, the beam cannot sustain more loading and will suddenly buckle (notice that the homogenous BVP is the same for all the studied examples). It is interesting to notice in this specific example that the critical eigenvalue of the buckling problem of the geometrically perfect beam when the unilateral constraint is active, is equal to (see Table 7.2):

$$P_{cr,eig}^{(2)} = 13108.87\ kN\,. \tag{7.25}$$

Practically, the beam has the tendency to buckle with respect to the buckling mode which corresponds to the above critical load. This actually happens when the support at point C is considered as bilateral (Fig.7.6). In this case, reaction force at point C changes sign for $P = P_s = 13108.306\,kN$. As a result the beam buckles when the load approaches the value $P^{(2)}_{cr,eig} = 13108.87\ kN$, developing a buckling mode similar to the one in Fig.7.6.

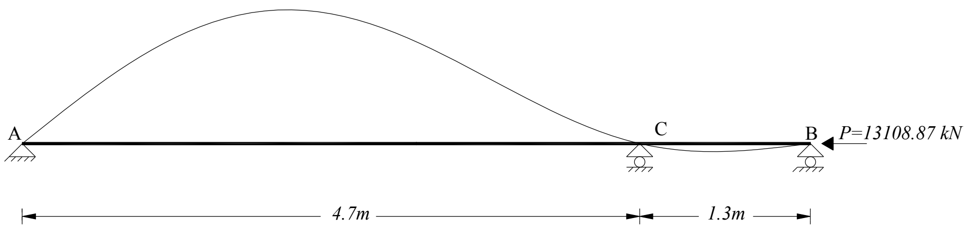


Fig. 7.6 The buckling mode of the bilateral problem.

However, in the studied case, the nature of the constraint does not offer to the beam the potential of buckling according to that way, due to the fact that the support at point C is unilateral. Therefore, for $P = P_s = 13108.306\ kN$ the beam is separated from the unilateral support. The moment that the beam is separated from the unilateral support at point C, the elastic energy which has been stored in the beam corresponds to the applied load P_s. Obviously, the beam cannot be in equilibrium for this value of load, due to the fact that for the certain contact status (inactive constraint) the maximum load is equal to $P^{(1)}_{cr,eig} = 4657.65\ kN$. Therefore, the arising deflections of the beam (which happen instantaneously) are accompanied by a "violent" jump of the applied load to lower values which approach the load $P^{(1)}_{cr,eig}$. Thus, the instability load of the beam is equal to:

$$P_i = P^{(1)}_{cr,eig} = 4657.65\ kN \tag{7.26}$$

The above results are depicted schematically in Fig. 7.7.

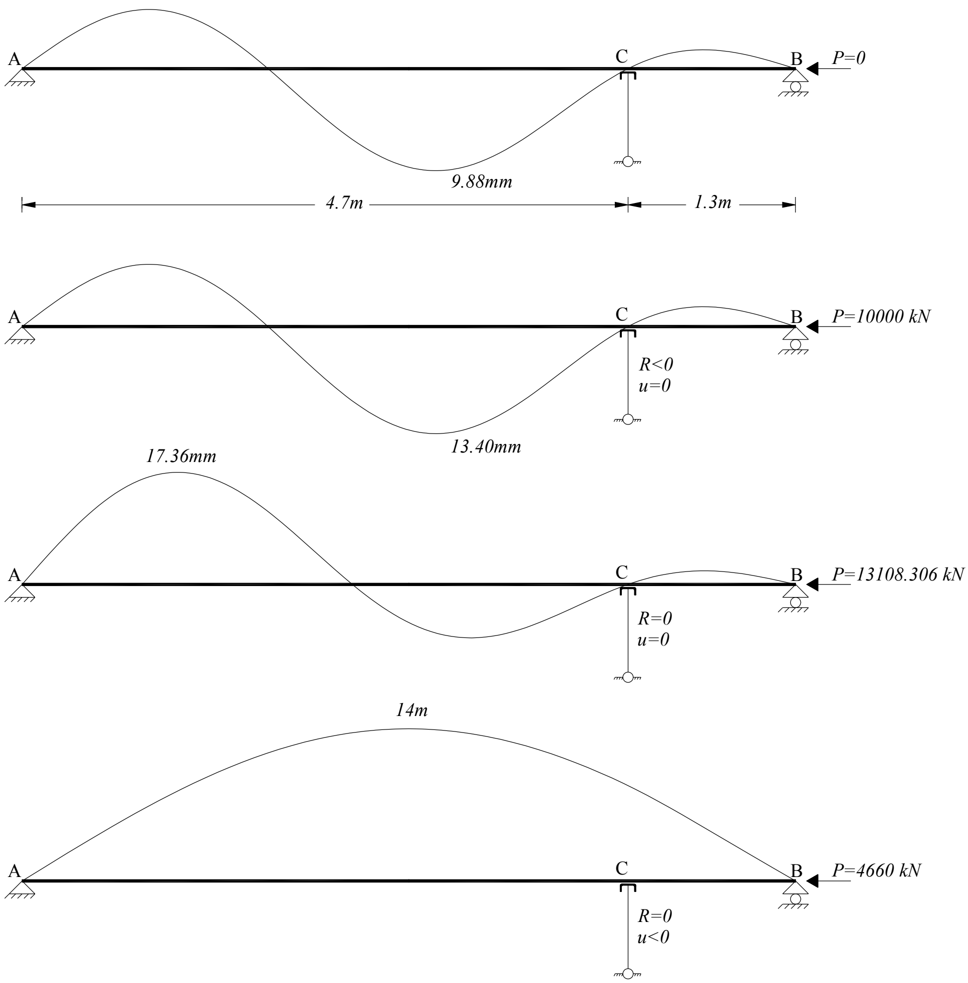


Fig. 7.7 Progressive deflection of the beam of Example 3 of Section 7.3.3.

7.3.4 Example 4

Let us consider now the geometrically imperfect beam of the first example (Fig 7.1). The stiffness rigidity of the beam is equal to $EI = 16989\ kNm^2$. This value corresponds to a HEB 220 steel section. It is supposed that the quality of the steel used is S460N with a yield stress equal to $f_y = 460\ Mpa$. For the certain initial geometric imperfection, it is possible that the instability load which was calculated in Example 1, is not the ultimate load that the beam is able to sustain. Due to the fact that the development of large deformations causes stresses that may exceed the capacity of the cross-section, the ultimate load is limited by the actual strength of the cross-section. Therefore, in order to determine the ultimate load of the beam, strength criteria should be employed in the calculation procedure. For the determination of the actual strength of the used cross-section, the provision of Eurocode 3 (EN 1993.01.01. (2005)) are considered. According to the latter, the bending strength of an –H cross-section for bending about the y-y axis, is given by the following relations:

• If the design axial load satisfies simultaneously the following inequalities:

$$N_{Ed} \leq 0.25 N_{pl.Rd} \tag{7.27}$$

$$N_{Ed} \leq \frac{0.5 h_w t_w f_y}{\gamma_{M_0}} \tag{7.28}$$

$$\gamma_{M_0} = 1 \tag{7.29}$$

then, allowance need not be made for the effect of the axial force on the plastic resistance moment, thus:

$$M_{Rd}^{y-y} = M_{pl.Rd} \tag{7.30}$$

• If the design axial load does not satisfy both the inequalities (7.18) and (7.19), then the bending strength of the cross-section is calculated approximately through the following equation:

$$M_{Rd}^{y-y} = M_{N,y,Rd} = M_{pl,y,Rd} \frac{1-n}{1-0.5a} \leq M_{pl,y,Rd} \tag{7.31}$$

where:

$$n = \frac{N_{Ed}}{N_{pl.Rd}} \tag{7.32}$$

$$a = \frac{A - 2bt_f}{A} \leq 0.5 \tag{7.33}$$

In all the above relations, A denotes the area of the cross-section, b denotes the width of the cross-section, t_f denotes the thickness of the flanges while h_w and t_w denote the height and the thickness of the web, respectively. The term γ_{M_0} constitutes the safety factor for the material strength and is taken unity. It is noted that, the design resistance $N_{pl.Rd}$ of a cross-section in uniform compression is determined as follows:

$$N_{pl.Rd} = \frac{Af_y}{\gamma_{M_0}}. \tag{7.34}$$

Following the above calculation procedure, the axial force-bending moment interaction diagram is obtained for the HEB 220 cross-section. The latter is depicted in Fig. 7.8.

For the determination of the design second order bending moment, the calculation procedure of Section 4.4 is followed. More specifically, the bending moments along the beam for the imperfect beam of Example 1 are given by the following equations:

• For $P \leq P_c = 3391.365\,kN$ the function of the bending moment is given by applying equations (4.113) and (4.114). For the case treated here, these equations take the form:

$$M_{Ed}(x_1) = -EI\left[\sum_{r=1}^{5} b_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right] \quad x_1 \in [0,4.7] \tag{7.35}$$

$$M_{Ed}(x_2) = -EI\left[\sum_{r=1}^{5} b_r \left(\frac{r\pi}{L}\right)^2 (F_r-1)\sin\left(\frac{r\pi x_2}{L}\right)(-1)^r\right] \quad x_2 \in [0,1.3]. \tag{7.36}$$

• For $P > P_c$ the function of the bending moment is given by applying equations (4.115) and (4.116), therefore:

$$M_{Ed}(x_1) = -EI\left[-B_1 k^2 \sin(kx_1) + \sum_{r=1}^{5} b_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right], \quad x_1 \in [0,4.7] \tag{7.37}$$

$$M_{Ed}(x_2) = -EI\left[-B_2 k^2 \sin(kx_2) + \sum_{r=1}^{5} b_r \left(\frac{r\pi}{L^2}\right)(F_r - 1)\sin\left(\frac{r\pi x_2}{L}\right)(-1)^r\right],$$
$$x_2 \in [0,(1-a)L]. \quad (7.38)$$

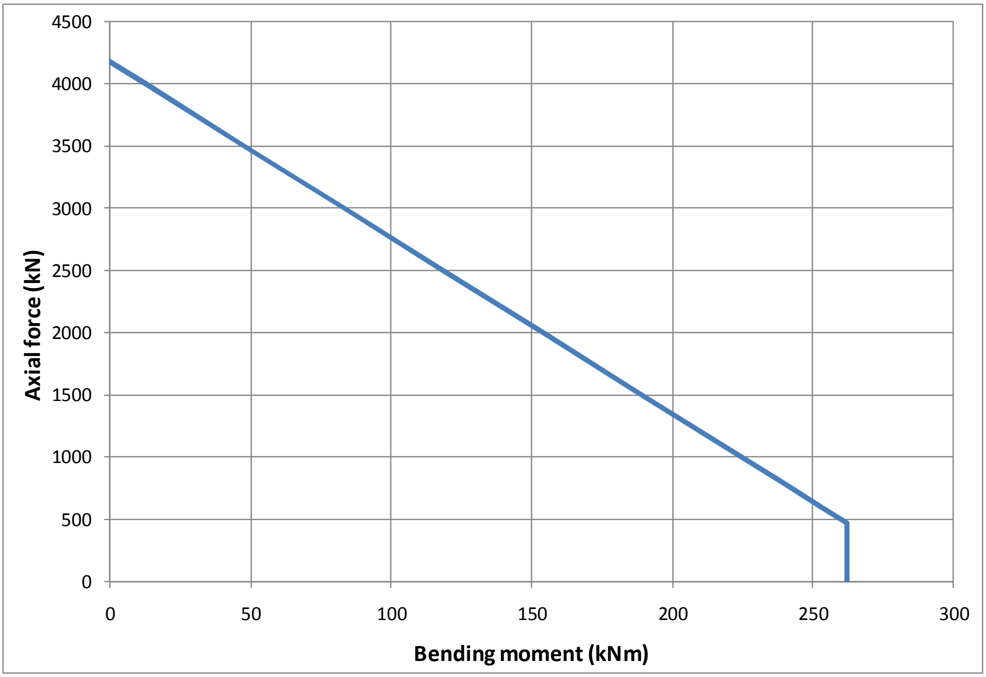


Fig. 7.8 The axial force-bending moment interaction diagram for the HEB 220 steel cross-section according to the provisions of Eurocode 3 (EN 1993.01.01 (2005)).

Then, for each pair of values (P, M_{Ed}) it is examined if the strength criterion:

$$M_{Ed} \leq M_{Rd}^{y-y} \quad (7.39)$$

is satisfied. If a value of the applied load P exists such that the above criterion holds as equality and, simultaneously, is smaller than the instability load P_i, then this load represents the ultimate load of the beam and is termed as P_{ult}. In the case that such load does not exist, the instability load is also the ultimate load of the beam.

Fig. 7.9 presents the values of the second order bending moments for various characteristic values of the applied load P up to failure, which occurs for

$P_{ult} = 3431.07\ kN$. For this value of the axial force, the maximum value of the bending moment at the beam is equal to $M = 53.32\ kNm$. This pair of axial force and bending moment lies on the boundary of the interaction diagram signalling the failure of the beam. Therefore, the ultimate load capacity of the specific geometrically imperfect beam is far away from the theoretical instability load which was calculated earlier, in Example 1 ($P_{in} = 13108.87\ kN$). This means that in the certain studied case the maximum loading is defined by the failure of the material rather than from instability.

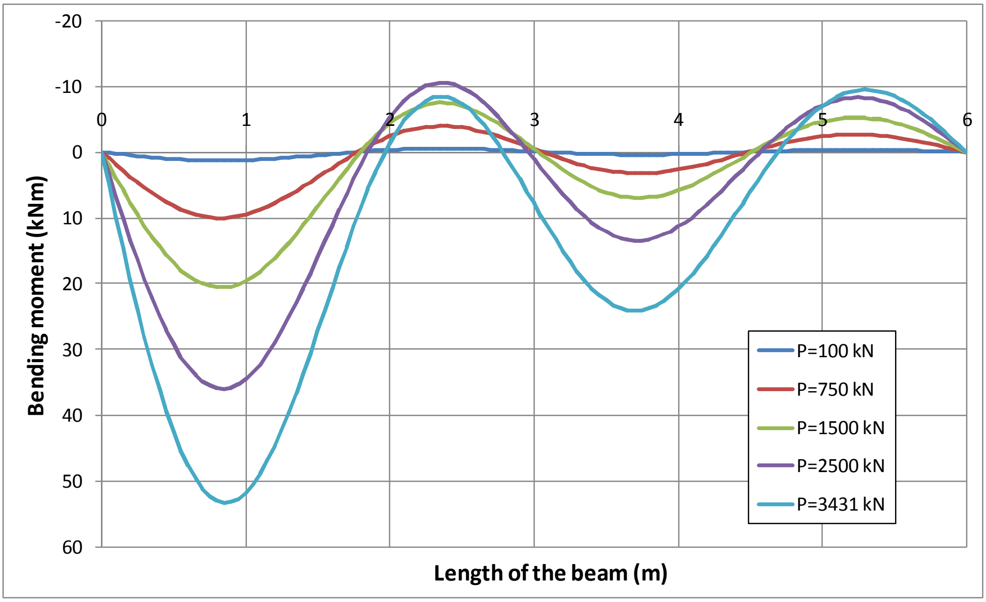


Fig. 7.9 Variation of the second order bending moment of the beam of Example 1 for various values of the loading until failure.

Finally, Fig. 7.10 presents the results of a parametric study in which the imperfection amplitude increases, while keeping the shape of the imperfection curve the same. All the imperfection shapes applied herein do not violate the unilateral contact support. As it was expected, the failure load decreases as the amplitude of the imperfection curve increases.

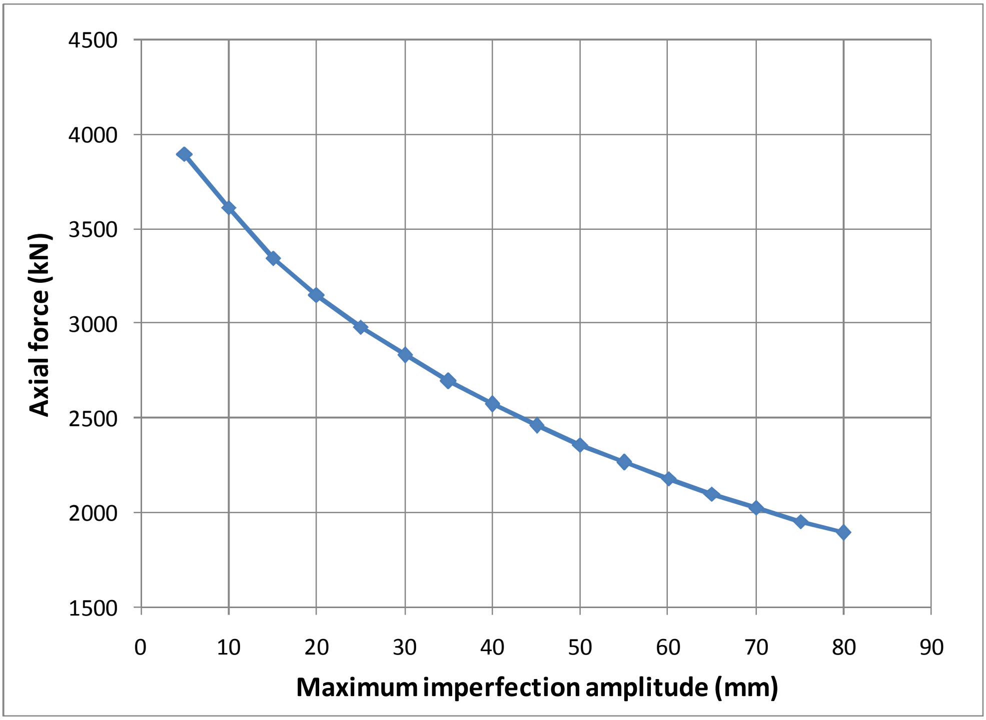


Fig. 7.10 Axial failure load vs. maximum imperfection amplitude for the beam of Example 1 of Section 7.3.1

7.4 Geometrically imperfect beams with two opposite functioning unilateral supports

7.4.1 Example 1

Let us consider the beam of Fig. 7.11 with a length of 6m and stiffness rigidity equal to $EI = 6216 kNm^2$. The beam is divided into three unequal spans by two unilateral supports functioning in opposite direction. The first one is placed at a distance of 1.5 m from the left end, while the other is placed 1.5m from the right end of the beam. The initial imperfection of the beam is supposed to be described by a Fourier sine series of seven terms (Fig.7.11). The function of the imperfection is given by the following relation

$$w_0(x) = \sum_{r=1}^{7} g_r \sin\frac{r\pi x}{6}, \quad x \in [0,6] \ , \tag{7.40}$$

while the Fourier coefficients are given in Table 7.5.

Fourier coefficients for the imperfection function
$g_1 = 0.00125$
$g_2 = 0.0025$
$g_3 = 0.0033$
$g_4 = 0.0065$
$g_5 = 0.00125$
$g_6 = 0.000015$
$g_7 = 0.0015$

Table. 7.5 The Fourier coefficients for the function of the geometric initial imperfection of Example 1 of Section 7.4.1.

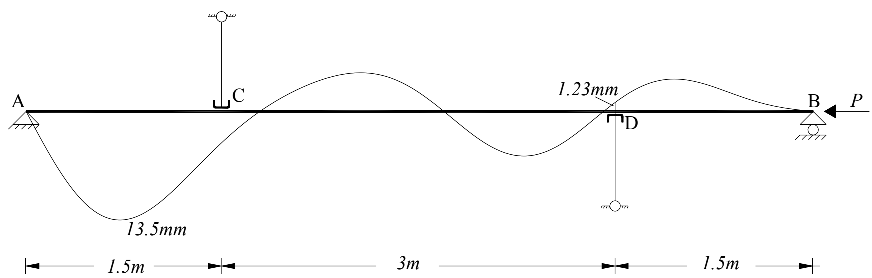


Fig. 7.11 The imperfect beam of Example 1 of Section 7.4.1.

The initial imperfection (7.40) is compatible with the unilateral constraints due to the fact that satisfies the required inequalities (5.140) and (5.141). More specifically:

$$w_0(aL) = w_0(1.5) = 0.00377 > 0 \tag{7.41}$$

$$w_0((a+b)L) = w_0(4.5) = -0.00123 < 0\,. \tag{7.42}$$

Obviously, the beam is not in contact with any of the two unilateral supports. The gap at point C is equal to 3.77mm while the gap at point D is equal to 1.23mm. The bending behaviour of the above problem is described through the equations (5.148)-(5.150), the boundary conditions (5.4)-(5.15) and the restrictions (5.16)-(5.18). The solution of a non-homogeneous BVP depends on the eigenvalues of the corresponding homogeneous problem. Table 7.6 summarizes the critical eigenvalues of the different contact cases which can occur during the bending deformation. The required calculation for the determination of the eigenvalues of Table 7.6 were given in Example 1 of Section 6.3.

Contact Case	Contact Status	Accepted Eigenvalue	Critical load $P_{cr,eig}$
CC3	A-I	$k_1 = 0.9022$	$P^{(1)}_{cr,eig} = 5059.6\ kN$
CC4	I-A	$k_2 = 0.9022$	$P^{(2)}_{cr,eig} = 5059.6\ kN$
CC2	I-I	$k_3 = 1.0472$	$P^{(3)}_{cr,eig} = 6816.6\ kN$
CC9	N-N	$k_4 = 2.0944$	$P^{(4)}_{cr,eig} = 27266.5\ kN$
CC1	A-A	$k_5 = 2.9956$	$P^{(5)}_{cr,eig} = 55780\ kN$

Table. 7.6 The critical eigenvalues of the corresponding bifurcation problem.

For the determination of the instability load of the beam the procedure of Section 5.3.3 is followed:

- Due to the fact that the beam is not in contact with any of the unilateral constraints (I-I contact status), the CC2 contact case is valid. Therefore, at the beginning of loading the equations of Section 5.3.2.2 hold and the deflection curve of the beam is given by the following equations:

$$w_1(x_1) = \sum_{r=1}^{7} g_r F_r \sin(\frac{r\pi x_1}{6}) \quad x_1 \in [0,1.5] \tag{7.43}$$

$$w_2(x_2) = \sum_{r=1}^{7} g_r F_k \sin(\frac{r\pi (x_2 + 1.5)}{6}) \quad x_2 \in [0,4.5] \tag{7.44}$$

$$w_3(x_3) = -\sum_{r=1}^{7} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r \quad x_3 \in [0,1.5]. \tag{7.45}$$

These equations are valid only for values of load which satisfy the required restrictions for the specific contact case. Additionally, these values of load should not constitute eigenvalues of the corresponding homogeneous BVP.

Therefore, it has to be examined if a value of the applied load P exists, for which at least one of the inequality restrictions (5.199) and (5.201) is not fulfilled.

If such a value exists, then, for greater values of load, the contact situation will change. For the example studied herein, the inequality criterion (5.201) holds as equality when the load takes the value $P = P_c^D = 1088kN$. For this value of load the beam comes in contact with the unilateral support at point D, while at the same time the criterion (5.199) is satisfied (i.e. the constrained at point C remains inactive). For $P > P_c^D$ the bending behaviour of the beam is determined by the equations which correspond to the CC4 contact case (I-A contact status, see Section 5.3.2.4). More specifically, the deflection curve is calculated through the following equations:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{7} g_r F_r \sin(\frac{r\pi x_1}{6}) \quad x_1 \in [0,1.5], \qquad (7.46)$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{7} g_r F_r \sin(\frac{r\pi (x_2 + 1.5)}{6}) \quad x_2 \in [0,4.5] \qquad (7.47)$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 - \\ - \sum_{r=1}^{7} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r \quad x_3 \in [0,1.5] \qquad (7.48)$$

where the unknown coefficients are determined by equations (5.219)-(5.226). It is then noticed that when the load approaches the value $P_{cr,eig}^{(2)} = 5059.6\ kN$, the beam develops extremely large deflections. Furthermore, the inequality restrictions (5.215) and (5.218) of the contact case CC4 (contatct status I-A) are fulfilled for every $P \in (P_c^D, 5059.437)$ i.e. the constraint at point D is active while the constraint at point C is inactive for each value inside that interval. Therefore, the instability load of the beam is equal to:

$$P_{in} = P_{cr,eig}^{(2)} = 5059.6\ kN\,. \qquad (7.49)$$

Fig. 7.12 presents the progressive deflection of the beam up to the instability load.

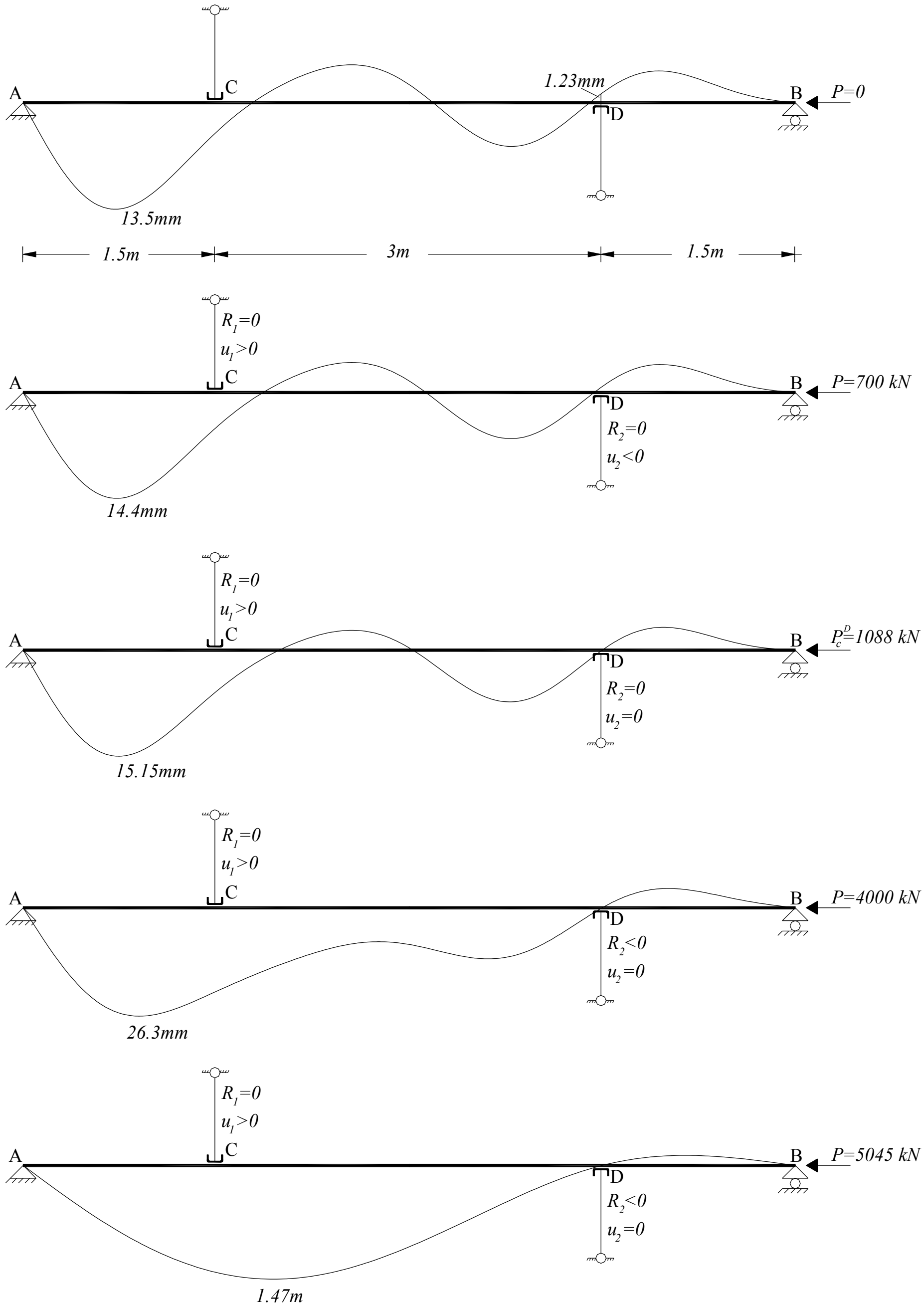


Fig. 7.12 Progressive deflection of the beam of Example 1 of Section 7.4.1.

7.4.2 Example 2

The beam of Fig. 7.13 with a length of 6m and stiffness rigidity equal to $EI = 6216kNm^2$ is considered, which is divided into three unequal spans by two unilateral supports functioning in opposite direction. The first one is placed in a distance of 1.5 m from the left end, while the other is placed 4m from the right end of the beam. The initial imperfection of the beam is supposed to be described by a Fourier sine series of ten terms (Fig.7.13). The function of the imperfection is given by the following relation

$$w_0(x) = \sum_{r=1}^{10} g_r \sin\frac{r\pi x}{6}, \quad x \in [0,6], \tag{7.50}$$

while the Fourier coefficients are given in Table 7.7

Fourier coefficients for the imperfection function	
$g_1 = 0.001912$	$g_6 = -5.7\times10^{-5}$
$g_2 = -0.00231$	$g_7 = 1.1803\times10^{-5}$
$g_3 = 0.001252$	$g_8 = 1.34\times10^{-5}$
$g_4 = 0.000279$	$g_9 = -2.48\times10^{-6}$
$g_5 = -2.714\times10^{-5}$	$g_{10} = -5.73\times10^{-6}$

Table. 7.7 The Fourier coefficients for the function of the geometric initial imperfection of Example 2 of Section 7.4.2.

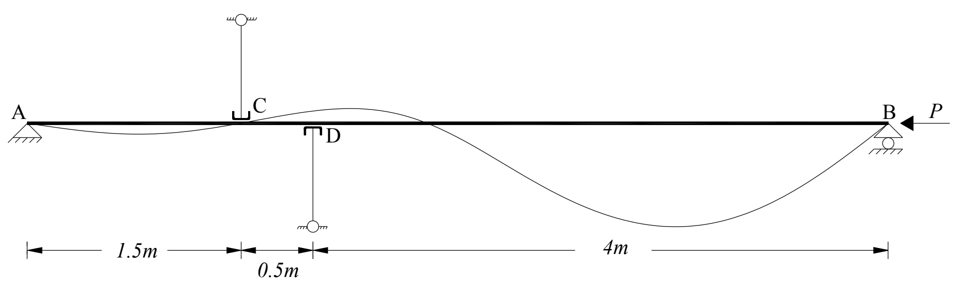


Fig. 7.13 The imperfect continuous beam of Example 2 of Section 7.4.2.

The initial imperfection (7.50) is compatible with the unilateral constraints, due to the fact that satisfies the required inequalities (5.140) and (5.141). More specifically:

$$w_0(aL) = w_0(1.5) = 0 \tag{7.51}$$

$$w_0((a+b)L) = w_0(2) = -0.00054 < 0\ . \tag{7.52}$$

The bending behaviour of the above problem is described through the equations (5.148)-(5.150), the boundary conditions (5.4)-(5.15) and the restrictions (5.16)-(5.18). The solution of the problem is associated with the critical eigenvalues of the corresponding homogeneous problem, which can be calculated applying the procedures introduced in Chapter 4. Table 7.8 summarizes presents the critical eigenvalues for the contact cases which occur.

Contact Case	Contact Status	Accepted Eigenvalue	Critical load P_{cr_eigen}
CC3	A-I	$k_1 = 0.9022$	$P^{(1)}_{cr,eig} = 5059.6\ kN$
CC4	I-A	$k_2 = 0.9642$	$P^{(2)}_{cr,eig} = 5778.9\ kN$
CC1	A-A	$k_3 = 1.082$	$P^{(3)}_{cr,eig} = 7277.2\ kN$

Table. 7.8 The critical eigenvalues of the corresponding homogeneous BVP, for all the contact cases occur in Example 2 of Section 7.4.2.

Then, for the determination of the instability load of the beam the procedure of Section 5.3.2 should be followed:

- Initially, the contact at the beginning of loading is examined. For the case treated here, the CC3 contact case is valid (contact status A-I). This means that initially the equations of Section 5.3.2.3 hold. Therefore, the deflection curve of the beam is given by the following equations:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_1}{6})\ , \quad x_1 \in\ [0,1.5] \tag{7.53}$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{10} g_r F_k \sin(\frac{r\pi (x_2+1.5)}{6})\ , \quad x_2 \in\ [0,0.5] \tag{7.54}$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 -$$

$$-\sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r \ , \quad x_3 \in [0,4]. \qquad (7.55)$$

where the unknown coefficients are determined by the utilization of equations (5.207)-(5.214). These equations are valid only for values of load which satisfy the required restrictions for the specific contact case. Additionally, these values of load should not constitute eigenvalues of the corresponding homogeneous BVP.

Therefore, it has to be examined if a value of the applied load P exists, for which at least one of the inequality restrictions (5.204) and (5.205) is not fulfilled. If such a value exists, then the contact situation will change for greater load values. For the under study example it is found that when the load takes the value of $P = P_s^C = 32.26kN$, the criterion (5.204) is satisfied as equality. The beam appears the tendency to be separated from the unilateral support at point C while at the same time, the criterion (5.205) is satisfied, i.e. the beam has not touched the unilateral support at point D.

In the sequel, it has to be checked if any of the eigenvalues of the corresponding bifurcation problem lies on the interval $[0, P_s^C]$. The smallest eigenvalue of the buckling problem of the geometrically perfect beam of Fig. 7.13, is obtained from Table 7.8 when the case CC3 holds (contact status A-I) and it is equal to $P_{cr,eig}^{(1)} = 5059.6\ kN$.

Obviously, none of the eigenvalues lie inside the interval, $[0, P_s^C]$ therefore, the beam is able to sustain more loading. For values of load slightly greater than the load $P = P_s^C = 32.26kN$ the beam is not in contact with any of the unilateral supports (i.e. inactive constraints, contact case CC2 (contact status I-I)). For this contact case the equations of Section 5.3.2.2 hold. The description of the deflection curve is then given by the following equations:

$$w_1(x_1) = \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_1}{6}) \ , \quad x_1 \in [0,1.5] \qquad (7.56)$$

$$w_2(x_2) = \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi (x_2 + 1.5)}{6}) \ , \quad x_2 \in [0,0.5] \qquad (7.57)$$

$$w_3(x_3) = \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_3}{6}) \ , \quad x_3 \in [0,4] \qquad (7.58)$$

The above equations describe the deflection curve of the beam for the specific contact case, where the two unilateral supports are inactive. These equations are, of course, valid only for values of load which are greater than P_s^C and also satisfy the required restrictions (5.199) and (5.201). Additionally, these load values should not constitute eigenvalues of the corresponding homogeneous BVP (i.e. the eigenvalues of the corresponding perfect beam for the CC2 contact case).

Therefore, it has to be examined for the specific contact case, if a value of the load P exists, for which at least one of the restrictions (5.199) and (5.201) is not satisfied. From the criterion (5.201) it is derived that for the value $P = P_c^D = 509.42\ kN$ the beam will come in contact with the unilateral support at point D, while simultaneously, the beam will not be in contact with the unilateral support at point C. This contact situation is termed as CC4 (contact status I-A, see Table 5.1).

Additionally, there does not exist an eigenvalue which satisfies the restrictions of the contact case CC2 and at the same time lies inside the interval $[P_s^C, P_c^D]$. This means that equations (7.56)-(7.58) describe the deflection of the beam for values of $P \in [P_s^C, P_c^D]$. For $P > P_c^D$ the following equations hold for the description of the deflection curve of the beam:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_1}{6}) , \quad x_1 \in [0,1.5] \quad (7.59)$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{10} g_r F_k \sin(\frac{r\pi (x_2 + 1.5)}{6}) , \quad x_2 \in [0, 0.5] \quad (7.60)$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 - \\ - \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r , \quad x_3 \in [0,4] . \quad (7.61)$$

In the above equations, the determination of the unknown coefficients is achieved through the utilization of the relations of Section 5.3.2.4. Equations (7.59)-(7.61) do not hold for every $P > P_c^D$. If a value of the load exists, for which the inequality restrictions (5.215) and (5.218) are not fulfilled, then the above equations do not hold. Additionaly, the latter hold under the prerequisite that the values of the applied load do not constitute eigenvalues of the corresponding homogeneous BVP.

Therefore, it has to be examined if a value of load P exists, such that at least one of the restrictions (5.215) and (5.218) is violated. For the case treated herein, the value $P = P_c^C = 2916.71\,kN$ leads the deflection curve of the beam to come in contact, for a second time, with the unilateral support at point C. Additionally, the critical eigenvalue of the corresponding buckling problem of the geometrically perfect beam when the case CC4 holds (contact status I-A) is obtained from Table 7.8 and is equal to $P_{cr,eig}^{(2)} = 5778.9 kN$.

Consequently, none of the admissible eigenvalues lies inside the interval $[P_c^D, P_c^C]$, thus equations (7.59)-(7.61) hold for every $P \in [P_c^D, P_c^C]$. Therefore, the beam has the ability to be loaded with greater load values. For these values, both the unilateral constraints are active (contact case CC1) and, therefore, the equations of Section 5.3.2.1 hold.

It is then noticed, that when the load approaches the value $P_{cr,eig}^{(3)} = 7277.2\ kN$ the deflections of the beam take disproportionate large values. This specific value is the first eigenvalue of the corresponding bifurcation problem and can be derived by the buckling equation (5.49). It is also noticed that for every $P \in \left[P_c^C, 7277.2\right]$ the restrictions (5.183) and (5.185) which refer to the specific contact case CC1 are fulfilled. Thus, instability occurs when both of the constraints are active and the load approaches the following instability load:

$$P_i = P_{cr,eig}^{(3)} = 7277.2\ kN \tag{7.62}$$

The progressive deflection of the beam for characteristic values of the axial load P is demonstrated in Fig.7.14 .

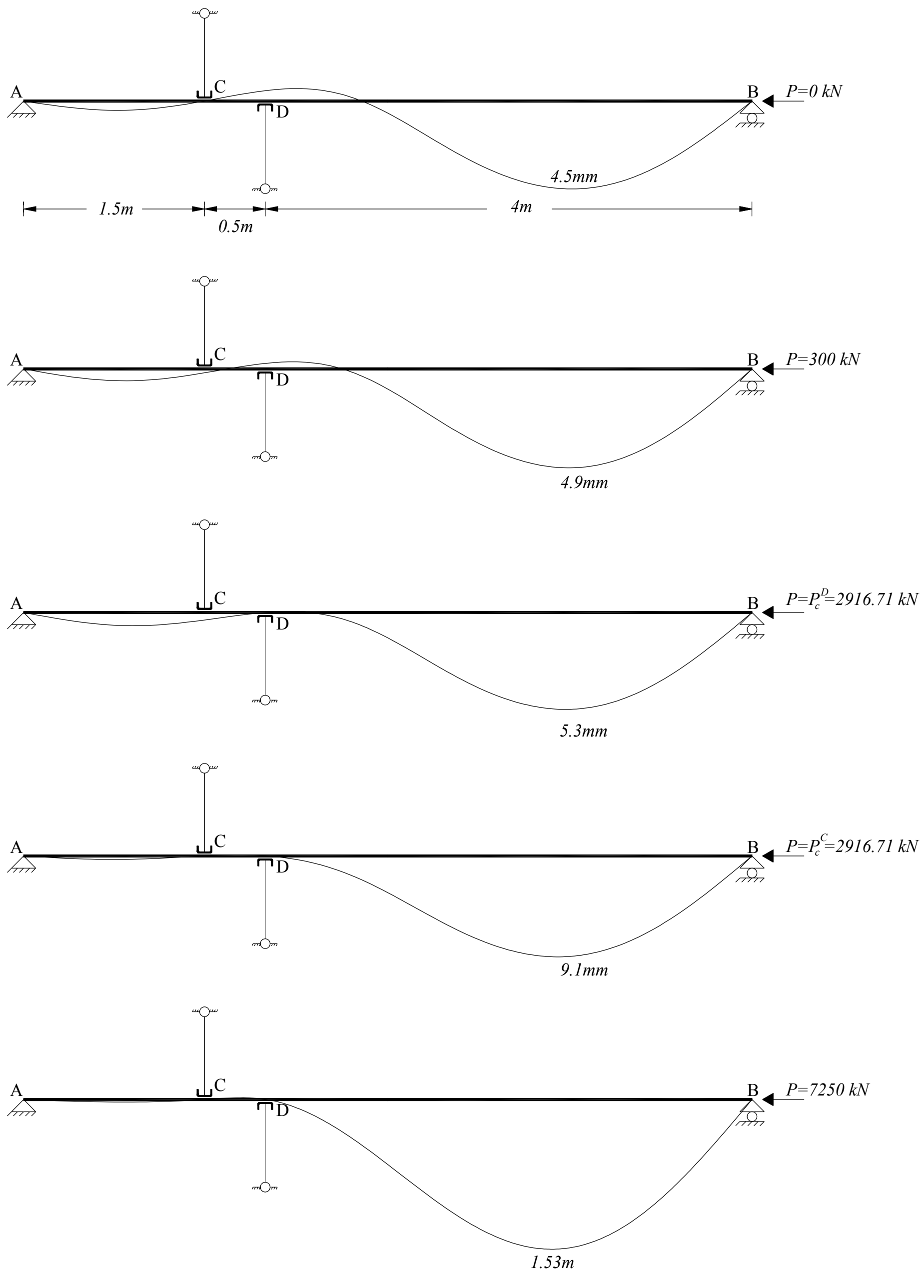


Fig. 7.14 Progressive deflection of the beam of Example 2 of Section 7.4.2.

7.4.3 Example 3

In the present example the geometrically imperfect beam of Fig. 7.15 with a length of 6m and stiffness rigidity equal to $EI = 16989\ kNm^2$ is considered. The beam is supported by two unilateral constraints functioning in opposite directions. The first one is placed at a distance of 2.6 m from the left end, while the other is placed 1.3m from the roll support of the beam. The imperfect shape of the beam is described by a Fourier sine series of ten terms (Fig.7.15). The function of the imperfection is given by the following relation:

$$w_0(x) = \sum_{r=1}^{10} g_r \sin\frac{r\pi x}{6}, \quad x \in [0,6], \tag{7.63}$$

while the Fourier coefficients are given in Table 7.5

Fourier coefficients for the imperfection function	
$g_1 = 0.0039789$	$g_6 = -0.0000367$
$g_2 = -0.0026521$	$g_7 = 0.0000227$
$g_3 = -0.0055680$	$g_8 = -0.0000095$
$g_4 = 0.0001480$	$g_9 = 0.0000012$
$g_5 = 0.0000278$	$g_{10} = 0.0000025$

Table. 7.9 The Fourier coefficients for the function of the geometric initial imperfection of Example 3 of Section 3.

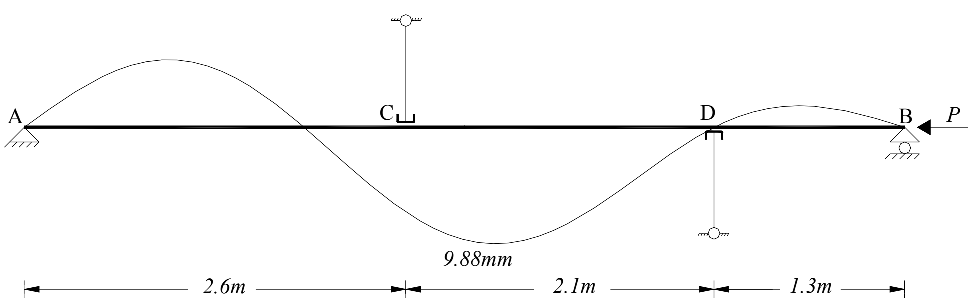


Fig. 7.15 The imperfect beam of Example 3.

The initial imperfection is compatible with the unilateral constraints due to the fact that satisfies the required inequalities (5.140) and (5.141). More specifically:

$$w_0(aL) = w_0(2.6) = 0.007195m \tag{7.64}$$

$$w_0((a+b)L) = w_0(4.7) = 0\ . \tag{7.65}$$

Therefore, the bending behaviour of the buckling problem of Fig. 7.15 is described through the equations (5.148)-(5.150), the boundary conditions (5.4)-(5.15) and the restrictions (5.16)-(5.18). For the calculation of the instability load of the beam the procedure of 5.3.3 is followed. For the implementation of that procedure the calculation of the eigenvalues of the corresponding homogeneous constrained BVP is required. Table 7.6 summarizes the critical eigenvalues which occur in the presented example. The calculation of these values is based on the theory developed in Section 5.2 and has presented thoroughly in several examples in Chapter 6. For the sake of brevity the required calculations are not presented.

Contact Case	Contact Status	Accepted Eigenvalue	Critical load P_{cr_eigen}
CC4	I-A	$k_1 = 0.87841$	$P^{(1)}_{cr,eig} = 13108.87\ kN$
CC3	A-I	$k_3 = 1.02985$	$P^{(2)}_{cr,eig} = 18018.38\ kN$
CC2	I-I	$k_4 = 1.04720$	$P^{(3)}_{cr,eig} = 18630.61\ kN$
CC1	A-A	$k_5 = 1.39471$	$P^{(4)}_{cr,eig} = 33047.27\ kN$

Table. 7.10 The critical eigenvalues of the bifurcation problem corresponding to Example 3 of Section 7.4.3 .

According to relations (7.64) and (7.65), the beam is initially in contact with the unilateral support at point D, while it is not in contact with the unilateral support at point C. The existing gap at point C is equal to 7.195mm. According to the calculation procedure 5.3.3 the following steps are considered:

- Firts, it is examined which of the contact cases presented in Table 5.1 is satisfied by the initial imperfection. For the case treated here, the CC4 contact case is valid (contact status I-A). Therefore, at the beginning of the loading, the equations of Section 5.3.2.4 hold. In this case the deflection curve of the beam is given by the following equations:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_1}{6}) \,, \quad x_1 \in [0, 2.6] \quad (7.66)$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{10} g_r F_k \sin(\frac{r\pi(x_2 + 2.6)}{6}) \,, \quad x_2 \in [0, 2.1] \quad (7.67)$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 - \\ - \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r \,, \quad x_3 \in [0, 1.3] \,. \quad (7.68)$$

where the unknown coefficients are determined by the utilization of equations (5.219)-(5.226). These equations are valid only for values of load which satisfy the required restrictions for the specific contact case. Additionally, these values of load should not constitute eigenvalues of the corresponding homogeneous BVP.

For this reason, it has to be examined if a value of the applied load P exists, for which at least one of the inequality restrictions (5.215) and (5.218) is not fulfilled. The existence of such a value will lead the beam to a different contact status. For the problem tretated herein, it is found that when the load takes the value of $P = P_s^D = 13103.272 \; kN$, the criterion (5.218) is satisfied as equality. The beam develops the tendency to be separated from the unilateral support at point D while at the same time, the criterion (5.215) is satisfied, i.e. the beam is not in touch with the unilateral support at point C.

Moreover, it has to be checked if any of the eigenvalues of the corresponding bifurcation problem (i.e. when the contact case CC4 is considered) lies on the interval $[0, P_s^D]$. According to Table 7.10, such an eigenvalue does not exist (the smallest eigenvalue of the buckling problem of the geometrically perfect beam which corresponds to the contact case CC4 is equal to $P_{cr,eig}^{(1)} = 13108.87 \; kN$. Therefore, the beam can sustain more axial loading. For values of load slightly greater than $P = P_s^D = 13103.272 \; kN$ the beam is not in contact with none of the unilateral supports (the contact case CC2 is valid, contact status I-I). However, the beam cannot be in equilibrium for $P > P_s^D$ due to the fact that the instability load of the simply supported beam is equal to:

$$P_{in} = \frac{\pi^2 EI}{L^2} = 4657.63 \; kN \,. \quad (7.69)$$

The structure develops the tendency to buckle according to the first eigenmode of the simply supported beam, therefore, the deflections of the beam are accompanied by a decrease of the applied load to lower values. For values of load $P < P_s^D$ the contact case CC2 is valid thus, equations which refer to that condition hold (see Section 5.3.2.2). The existence of the unilateral support at point C constrains the development of the buckling mode. Thus, the decrease of the load has a limit which is equal to the value:

$$P_c^C = 9902.35\ kN, \tag{7.70}$$

for which the beam will come in contact with the unilateral support at point C. The P_c^C load is derived through the restriction (5.199) when it holds as equality.

When the beam comes in contact with the unilateral support at point D, the contact case CC3 is activated (contact status A-I). Due to the fact that the critical eigenvalue of this contact situation is equal to $P_{cr,eig}^{(2)} = 18018.38\ kN$ (see Table 7.10) the beam is able to sustain more loading. Therefore, for $P > P_c^C$ the deflection curve of the beam is given by the following equations:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_1}{6}), \quad x_1 \in [0,2.6] \tag{7.71}$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{10} g_r F_k \sin(\frac{r\pi(x_2+2.6)}{6}), \quad x_2 \in [0,2.1] \tag{7.72}$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 - \\ - \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r, \quad x_3 \in [0,1.3]. \tag{7.73}$$

In the above equations the determination of the unknown coefficients is based on relations (5.207)-(5.214). Equations (7.71)-(7.72) do not hold for every $P > P_c^C$. If a value of the applied load exists, for which the inequality restrictions (5.204) or (5.205) are not fulfilled, then the above equations do not hold.

Therefore, it has to be examined for the specific contact case, if a value of the load P exists, for which at least one of the restrictions (5.204) and (5.205) is not satisfied. From the criterion (5.205) it is derived that for the value $P = P_c^D = 13429.9\ kN$ the beam will come in contact with the unilateral support at point D, while simultaneously, the beam will remain in contact with the unilateral support at point C. This contact case is termed as CC1 (contact status A-A, see Table 5.1).

For $P > P_c^D$ equations (7.71)-(7.72) hold for the description of the deflection curve of the beam, where now the determination of the unknown coefficients is achieved through the utilization of relations (5.187)-(5.198d) of Section 5.3.2.1. It is then noted, that when the load approaches the value $P_{cr,eig}^{(3)} = 33047.27\ kN$, the deflections of the beam tend to infinity. This specific value is the first eigenvalue of the corresponding bifurcation problem and can be derived by the buckling equation (5.49). Additionally, for every $P \in \left[P_c^D, 33047.27\right)$ the restrictions (5.183) and (5.185) which refer to the specific contact case CC1 are fulfilled. Thus, instability occurs when both of the constraints are active and the load approaches the following instability load:

$$P_{cr,eig}^{(4)} = 33047.27\ kN\ . \tag{7.74}$$

In Fig. 7.16 the progressive deflection of the beam for characteristic values of the axial load P is depicted. Notice that when the beam tends to be separated from the unilateral support at point D, then the deflection curve takes instantaneously the shape which corresponds to the load $P_c^C = 9902.35\ kN$.

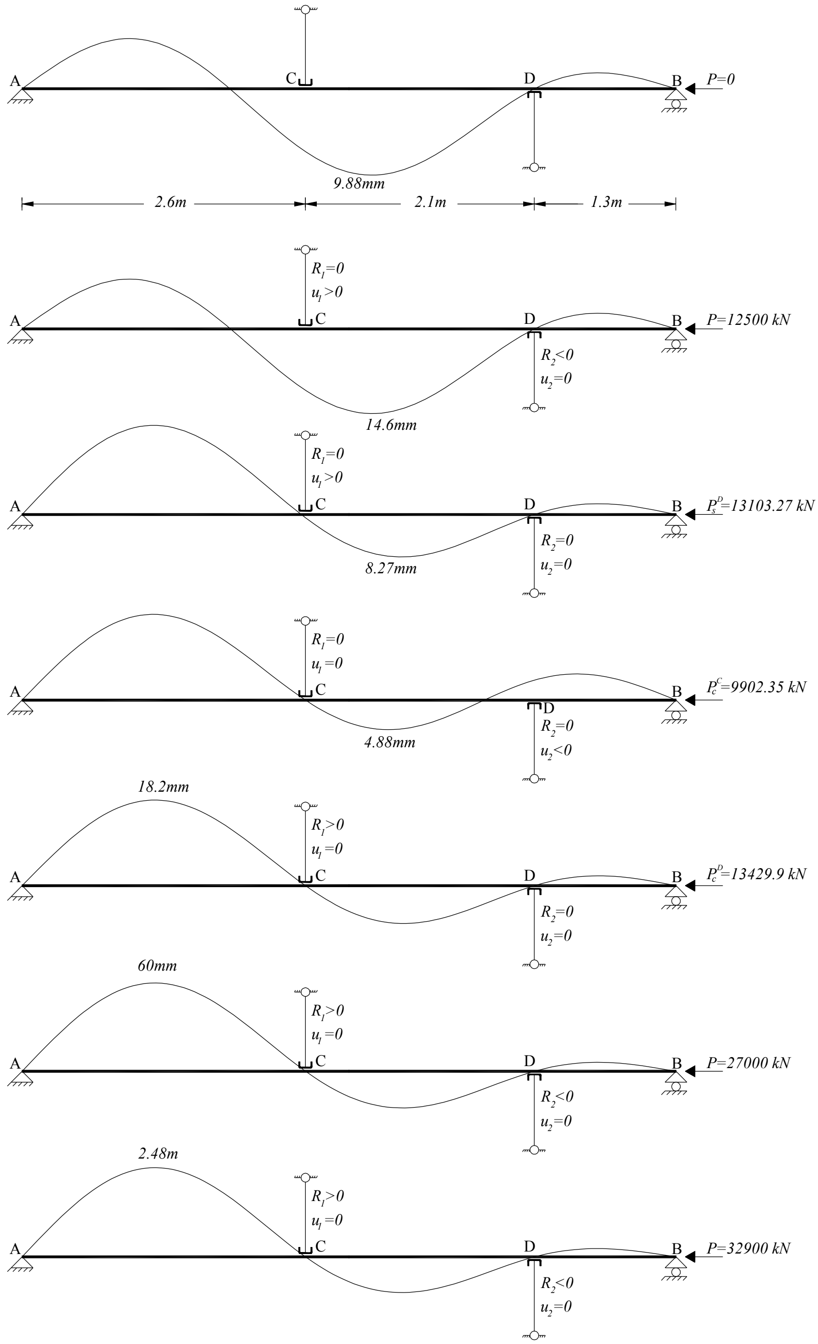


Fig. 7.16 Progressive deflection of the beam of Example 3 of Section 7.4.3.

7.4.4 Example 4

Let us consider a geometrically imperfect beam with a length of 6m and stiffness rigidity equal to $EI = 6216\, kNm^2$ (Fig.7.17). Two unilateral supports, functioning in opposite directions, are placed in the beam. The first one is placed at a distance of 1.5m from the left end, while the other is placed 4.5m from the right end of the beam (Fig.7.17). The initial imperfection is supposed to be described by a Fourier sine series of ten terms. The function of the imperfection is given by the following relation:

$$w_0(x) = \sum_{r=1}^{10} g_r \sin\frac{r\pi x}{6}, \quad x \in [0,6], \tag{7.75}$$

while the Fourier coefficients are given in Table 7.11

Fourier coefficients for the imperfection function	
$g_1 = 0.0203836$	$g_6 = -0.0048096$
$g_2 = -0.0334536$	$g_7 = 0.0003248$
$g_3 = -0.0327362$	$g_8 = 0.0000000$
$g_4 = 0.0000000$	$g_9 = -0.0000898$
$g_5 = -0.0521351$	$g_{10} = 0.0003099$

Table. 7.11 The Fourier coefficients for the function of the geometric initial imperfection of Example 4 of Section 7.4.4.

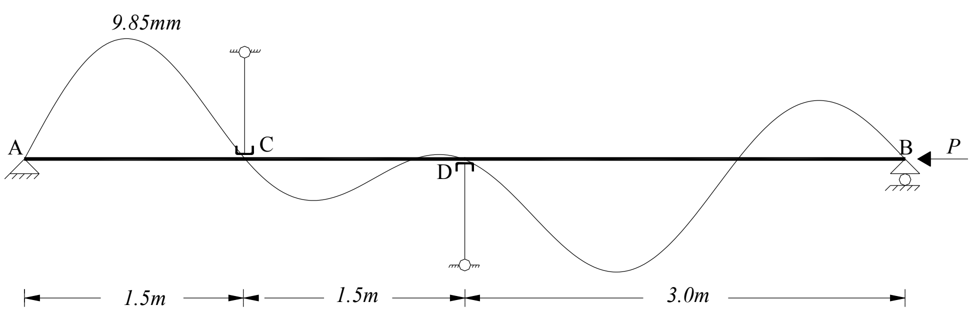


Fig. 7.17 The imperfect beam of Example 4 of Section 7.4.4.

The initial geometric imperfection is compatible with the unilateral constraints because it satisfies the required inequalities (5.140) and (5.141). More specifically:

$$w_0(aL) = w_0(1.5) = 0 \tag{7.76}$$

$$w_0((a+b)L) = w_0(3) = 0\ . \tag{7.77}$$

From the above relations, it is clearly concluded that the beam is initially in contact with both the unilateral supports. The bending behaviour of the problem of Fig. 7.17, can be described mathematically by a non homogeneous constrained BVP which is formulated through the equations (5.148)-(5.150), the boundary conditions (5.4)-(5.15) and the restrictions (5.16)-(5.18). For this reason, for the determination of the instability load of the beam, the calculation procedure of Section 5.3.3 should be followed.

The solution of the studied example is strongly connected with the eigenvalues of the corresponding homogeneous BVP. For this reason, Table 7.12 summarizes the critical eigenvalues of that problem for all the contact cases which occur in the considered example .

Contact Case	Contact Status	Accepted Eigenvalue	Critical load P_{cr_eigen}
CC3	A-I	$k_1 = 0.9022$	$P^{(1)}_{cr,eig} = 5059.6\ kN$
CC6	I-N	$k_2 = 1.0472$	$P^{(2)}_{cr,eig} = 6816.6\ kN$
CC1	A-A	$k_3 = 1.3090$	$P^{(3)}_{cr,eig} = 10650.2\ kN$
CC4	I-A	$k_4 = 1.4978$	$P^{(4)}_{cr,eig} = 13945.0\ kN$
CC2	I-A	$k_5 = 1.5708$	$P^{(5)}_{cr,eig} = 14540.0\ kN$

Table. 7.12 The critical eigenvalues of the corresponding, to Example 4, homogeneous constrained BVP.

- Considering the fact that the beam is in contact with the unilateral supports, the CC1 contact case is valid. This means that at the beginning of the loading the equations of Section 5.3.2.1 hold. Therefore, the deflection curve of the beam can be described by the following equations:

$$w_1(x_1) = A_1 \cos kx_1 + B_1 \sin kx_1 + C_1 x_1 + D_1 + \sum_{r=1}^{10} g_r F_r \sin\left(\frac{r\pi x_1}{6}\right)\ , \quad x_1 \in [0,1.5] \tag{7.78}$$

$$w_2(x_2) = A_2 \cos kx_2 + B_2 \sin kx_2 + C_2 x_2 + D_2 + \\ + \sum_{r=1}^{10} g_r F_k \sin(\frac{r\pi(x_2+1.5)}{6}) \, , \quad x_2 \in [0,1.5] \qquad (7.79)$$

$$w_3(x_3) = A_3 \cos kx_3 + B_3 \sin kx_3 + C_3 x_3 + D_3 - \\ - \sum_{r=1}^{10} g_r F_r \sin(\frac{r\pi x_3}{6})(-1)^r \, , \quad x_3 \in [0,3] \, . \qquad (7.80)$$

where the unknown coefficients are determined by the utilization of equations (5.187)-(5.198d). These equations are valid only for values of load which satisfy the required restrictions for the specific contact case. Additionally, these values of load should not constitute eigenvalues of the corresponding homogeneous BVP.

Therefore, it has to be examined if a value of the applied load P exists, for which at least one of the inequality restrictions (5.183) and (5.185) is not fulfilled. If such a value exists, then the beam will change contact status for greater values of the applied load. For the specific example it is found that when the load takes the value of $P = P_s^D = 10624.698 kN$, the criterion (5.185) is satisfied as equality. This means that the reaction force of the unilateral support at point D becomes zero and the beam tends to be separated from that support. At the same time the restriction (5.183) is satisfied, thus the beam remains in contact with the unilateral support at point C. Furthermore, it is noticed that the smallest eigenvalue of the homogeneous BVP corresponding to that contact case (i.e. both constraints are active, CC1 contact case) is equal to $P_{cr,eig}^{(3)} = 10650.2 \ kN$ (see Table 7.12). Therefore, equations (7.78)-(7.80) are valid for every $P \in [0, P_s^D]$. For values of load slightly greater than P_s^D, the beam is in contact with the unilateral support at point C while it is not in contact with the unilateral support at point D. The response of the beam for the present contact case (CC3) can be described by the equations of Section 5.3.2.3.

However, the beam cannot be in equilibrium for $P > P_s^D$ due to the fact that the instability load which corresponds to the contact case CC3 is smaller than P_s^D (see Table 7.12), i.e.:

$$P_{cr,eig}^{(1)} = 5059.6 \ kN < P_s^D \, . \qquad (7.81)$$

The moment that the beam is separated from the unilateral support at point D, the elastic energy which has been stored in the beam corresponds to to the applied load $P_s^D = 10624.198 \, kN$. Obviously, the beam cannot be in equilibrium for this value of load due to the fact that for the certain contact case (i.e. the CC3), the maximum load is equal to $P^{(1)}_{cr,eig} = 5059.6\ kN$. Therefore, the deflections of the beam are accompanied by a "violent" decrease of the applied load to lower values which approach the load $P^{(1)}_{cr,eig}$ (Fig.7.18). Thus, the instability load of the beam is equal to:

$$P_{in} = P^{(1)}_{cr,eig} = 5059.6\ kN \tag{7.82}$$

It is noticed that for all the values of the applied load $P \in [P^D_s\ 5059.6)$ the equilibrium of the beam is unstable.

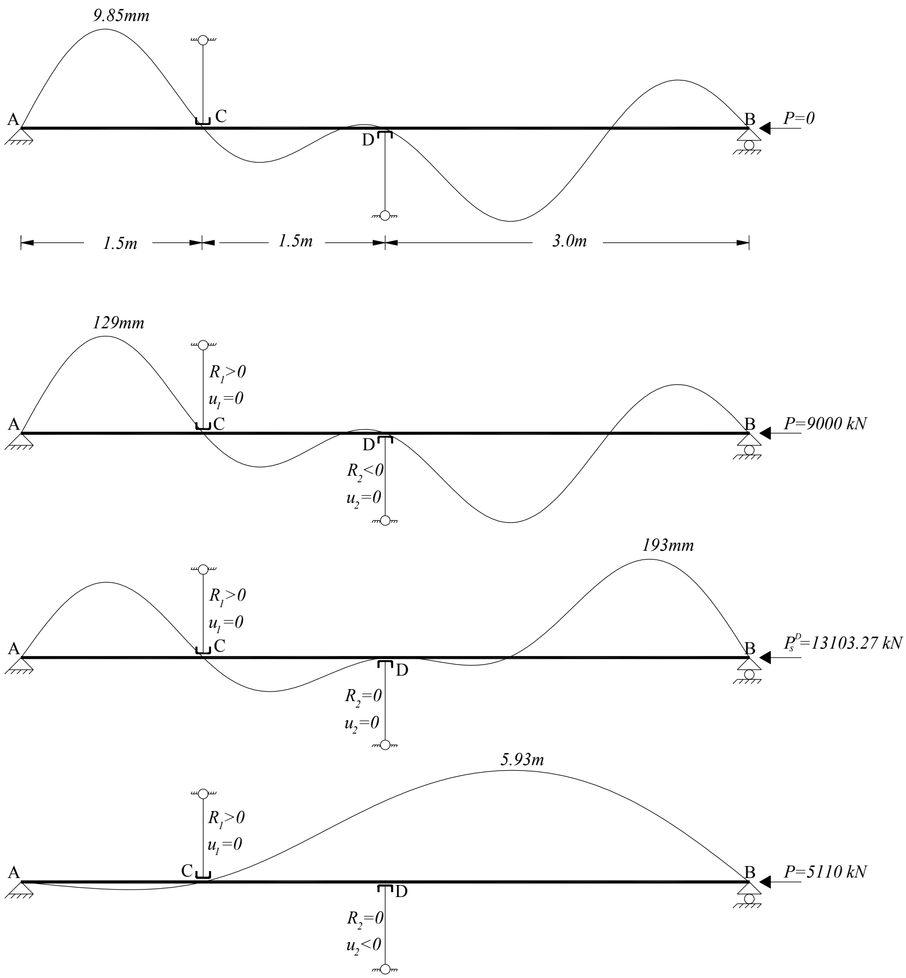


Fig. 7.18 Progressive deflection of the beam of Example 4 due to loading.

7.4.5 Example 5

Let us consider herein the geometrically imperfect beam of the second example of this section (Fig 7.13). The stiffness rigidity of the beam is equal to $EI = 6216\ kNm^2$. This value corresponds to the rectangular hollow QHS 200x6.3 steel section. The quality of the steel used is S355 with a yield stress equal to $f_y = 355\ Mpa$. In order to calculate the ultimate load of the beam, strength criteria should be taken itno account in the calculation procedure. The actual strength of the used cross-section is determined through the provisions of Eurocode 3 (EN 1993.01.01. (2005)).

According to the latter, the bending strength of a rectangular hollow cross-section for bending about the y-y axis, is given by the following relations:
where:

$$M_{Rd}^{y-y} = M_{N,y,Rd} = M_{pl,y,Rd} \frac{1-n}{1-0.5a_w} \leq M_{pl,y,Rd} \tag{7.83}$$

$$n = \frac{N_{Ed}}{N_{pl.Rd}} \tag{7.84}$$

$$a_w = \frac{A-2b_t}{A} \leq 0.5 \tag{7.85}$$

$$N_{pl.Rd} = \frac{Af_y}{\gamma_{M_0}}. \tag{7.86}$$

$$M_{pl,y,Rd} = \frac{W_{pl}^{y-y} f_y}{\gamma_{M_0}}. \tag{7.87}$$

$$\gamma_{M_0} = 1. \tag{7.88}$$

In all the above relations, A denotes the area of the cross-section, W_{pl}^{y-y} denotes the plastic moduli about the y-y axis, while b and t denote the width of the cross-section and the thickness of the flanges respectively.
Following the above calculation procedure, the axial force-bending moment interaction diagram is obtained for the QHS 200x6.3 cross-section. The latter is depicted in Fig. 7.19.

For the determination of the design second order bending moment, the calculation procedure of Section 5.4 is followed. More specifically, the bending moments for the imperfect beam of Example 1 are given by the following equations:

• For $P \le P_s^c = 32.26\, kN$ the beam is in contact with the unilateral constraint at point C, thus the function of the bending moment is given by applying equations (5.267)-(5.269). For the case treated here, these equations take the form:

$$M_{Ed}(x_1) = -EI\left[-B_1 k^2 \sin(kx_1) + \sum_{r=1}^{10} g_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right], \quad x_1 \in [0, 2.6] \quad (7.89)$$

$$M_{Ed}(x_2) = -EI\left[\begin{array}{l} -A_2 k^2 \cos(kx_2) - B_2 k^2 \sin(kx_2) + \\ +\sum_{r=1}^{10} g_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi(x_2+aL)}{L}\right)\end{array}\right], \quad x_1 \in [0, 2.1] \quad (7.90)$$

$$M_{Ed}(x_3) = -EI\left[-B_3 k^2 \sin(kx_3) + \sum_{r=1}^{10} g_r \left(\frac{r\pi}{L}\right)^2 (F_r - 1)\sin\left(\frac{r\pi x_3}{L}\right)(-1)^r\right], \quad x_3 \in [0, 1.3]$$

(7.91)

As it is depicted in Fig. 7.20, for $P = P_s^c = 32.26\, kN$ the bending moments along the beam are very small, thus the beam is able to sustain more loading.

• For $P \in [P_s^c, P_c^D)$, the beam is not in contact with none of the unilateral supports, thus the function of the bending moment is given by the same set of equations (5.267)-(5.269) setting $B_1 = A_2 = B_2 = B_3 = 0$. More specifically, the following equations hold for the determination of the bending moment when the constraints are inactive:

$$M_{Ed}(x_1) = -EI\left[\sum_{r=1}^{10} g_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi x_1}{L}\right)\right], \quad x_1 \in [0, 2.6] \quad (7.92)$$

$$M_{Ed}(x_2) = -EI\left[\sum_{r=1}^{10} g_r \left(\frac{r\pi}{L}\right)^2 (1-F_r)\sin\left(\frac{r\pi(x_2+aL)}{L}\right)\right], \quad x_1 \in [0, 2.1] \quad (7.93)$$

$$M_{Ed}(x_3) = -EI\left[\sum_{r=1}^{10} g_r \left(\frac{r\pi}{L}\right)^2 (F_r - 1)\sin\left(\frac{r\pi x_3}{L}\right)(-1)^r\right], \quad x_3 \in [0,1.3]. \tag{7.94}$$

It is then noticed that when the load takes the value $P = 1603.5\ kN$, which lies inside the interval $[P_s^c, P_c^D)$, the maximum value of the bending moment is equal to $M_{Ed}^{\max} = 9.077\ kNm$. The pair of values $(1603.5,\ 9.077)$ lies on the boundary of the interaction diagram (Fig. 7.19), signalling the failure of the beam. Therefore, the ultimate load that the beam is able to sustain is:

$$P_{ult} = 1603.5\ kN. \tag{7.95}$$

Fig. 7.20 presents the values of the second order bending moments for various characteristic values of the applied load P up to failure, which occurs for $P_{ult} = 1603.5\ kN$.

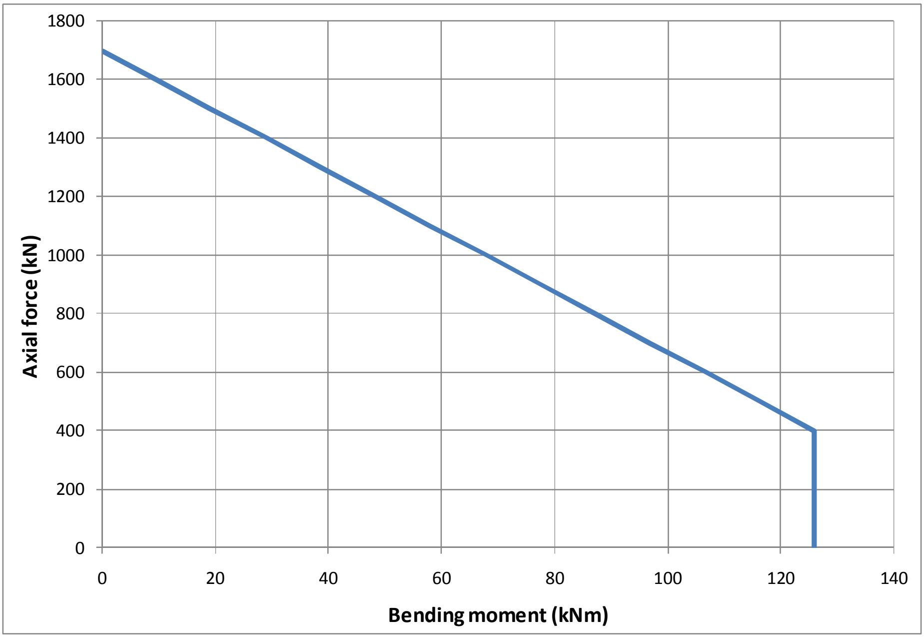


Fig. 7.19 The axial force-bending moment interaction diagram for the HEB 220 steel cross-section according to the provisions of Eurocode 3 (EN 1993.01.01 (2005)).

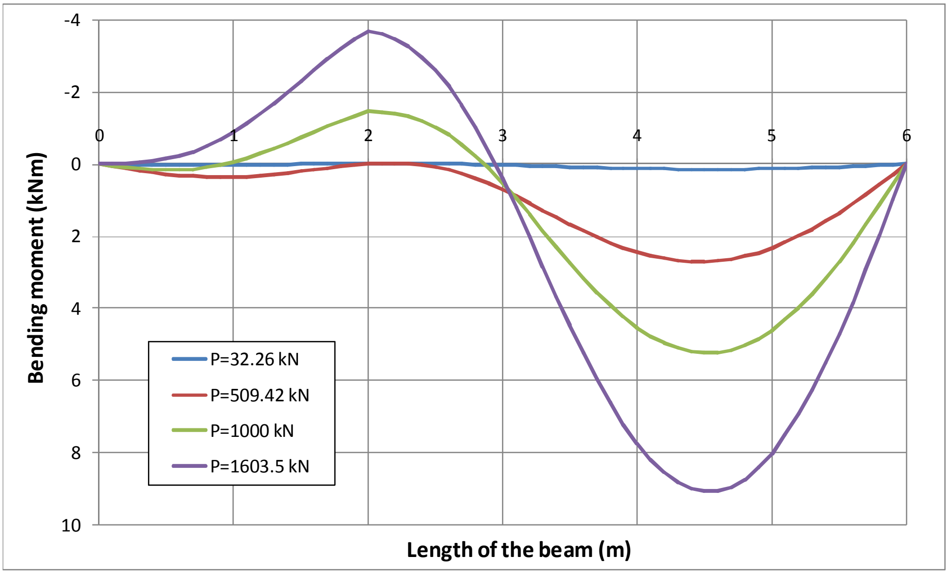


Fig. 7.20 Variation of the second order bending moment of the beam of Example 2 for various loading until failure.

8 Summary and Conclusions

In the present dissertation the contact buckling problem of beams in the presence of one or two intermediate unilateral supports was investigated. The study uses tools from the theory of elastic stability in order to formulate a constraint BVP which is solved analytically. Generally, the contact buckling problem in steel and composite structures has been of interest for many researchers until today, due to the existence of a variety of applications where instability phenomena involve unilateral contact conditions. Most of the reported works dealt with methods and techniques based on the computational and variational approach. Theoretical and analytical research is also important, but the published works are rather few in that case. This results from the fact that contact buckling problems exhibit highly nonlinear response leading to complex mathematical formulations which are difficult to be solved analytically. However, significant work has been developed in this field for individual structural members which can be modeled as plates or beams.

Concerning the buckling problem of beams, special attention has been given to the calculation of the crtitical load. In the most of the related works the proposed methods aim to determine the critical buckling load for beams resting on an elastic foundation or for beams where their displacement is restrained by rigid obstacles.

The major innovative point of the present research in that field is that it considers geometrically imperfect beams (i.e. beams with arbitrary initial geometric imperfections). This consideration leads to a non-homogeneous constrained BVP. The solution of that problem is obtained after the separation of the initial BVP into specific constained subproblems. The separation is based on the different contact cases which are possible to occur during the bending deformation of the beam (for example one of the constaints may be active and the other inactive, or both of the constaints may be active etc.), however, the inequality character of the problem is kept. Within this solution procedure, analytical relations are derived for each contact case which, of course, are valid under certain restrictions introduced by the unilateral supports. The main problem in the case of imperfect beams is that during the bending deformation the contact conditions may change. In order to take that into account, all the obtained solutions are unified in a special algorithm which is able to treat various problems with different contact conditions. In that way, the instability load and the deflection curve of the beam can be determined.

Due to the fact that the solution of the constrained non-homogeneous BVP is strongly connected with the solution of the corresponding homogenous BVP, the latter is also investigated producing eigenvalues and eigenmodes for each contact case. As it is well known from the mathematical field of ordinary differential equations and BVPs, the eigenvalues of a homogeneous BVP constitute singular points in the solution of a respective non-homogeneous BVP, due to the fact that for these eigenvalues the non-homogeneous BVP may be either unsolvable or solvable but not uniquely. Obviously, from the mechanical point of view, in the case of buckling of beams, the existence of singular points indicates instability phenomena. It has to be noticed that these singular points cannot be detected easily using common finite element software. Fisrt of all, advanced geometrically nonlinear finite element analysis codes should be employed which are able to take into account unilateral contact conditions. Even in that case, the correct solution is not guaranteed. Special attention should be given to the incrementation of the applied load because the analysis procedure may overclme the critical points which are not always obvious. In the present study, this target is achieved quickly in a simple manner without the utilization of any kind of load-incrementation schemes.

Among the others, the proposed methodology offers the potential of treating problems with various initial geometric imperfections and initial contact conditions. The initial contact conditions are defined by the position of the unilateral supports. In that way, particular cases can be examined and several conclusions concerning the influence of the position of the unilateral supports on the value of the critical load and the buckling mode can be extracted.

Due to the fact that in real world applications the failure of a structural system is influenced by material nonlinearity, strength criteria which take into account the strength of the cross-section of the beam are employed in the proposed methodology. Within that, the ultimate load that the beam is able to sustain can be determined considering the interaction between bending moment and axial force in the stress analysis.

By the implementation of the proposed methodology in various cases, a series of topics with theoretical interest were treated. More specifically, issues concerning the passage from one contact case to another during the bending deformation were revealed. These interesting points were studied through the demonstration of several examples, for a class of different cases. However, further research, mainly from the mathematical point of view, can be a challenge. Similar topics were studied numerically by many researchers (e.g. Stein and Wriggers, 1984 ; Simo et al, 1986; Wriggers, 2006 etc.) and, therefore, the future research can be supported in some way. Furthermore, the consideration in the formulation of the problem, of a bending theory different from that of the Euler-Bernoulli, may give the potential of treating problems with large deformations. In that way, the postcritical behaviour of the beam, after buckling, can be studied as well.

Appendix A - The space $L_2(a,b)$

A.1 Introduction

The mathematical formulation of the buckling problem of beams in the framework of the second-order bending theory, consists of fourth order ordinary differential equations together with the appropriate boundary conditions (see Chapter 3). In order to solve such problems the functions which constitute the solutions of the ordinary BVP should belong to a specific space of functions where certain properties hold. For the studied homogeneous and non-homogeneous BVPs of the present dissertation the appropriate space in which the mathematical problem should be formulated , is the space of the square-integrable functions, the so-called $L_2(a,b)$ space. This space of functions is actually one of the most important examples of Hilbert spaces. In general, Hilbert spaces have a rich geometric structure because they are endowed with an inner (or scalar) product that allows the introduction of the concept of orrthogonality of functions. That concept is very significant for the solution of problems involving differential equations because it gives the potential to approximate functions through a series of orthonormal functions. Classical example of this potential is the Fourier series expansion, a mathematical tool which is fundamental in the study of differential equations. As it can be proved (Debnath and Mikusinski, 2005; Rektorys, 1980) in order to represent a fuction f through the sum of other orthonormal functions, the following term should be introduced:

$$\|f\|^2 = \int_a^b [f(x)]^2 dx \,. \tag{A.1}$$

The above term is the inner product of function f with itself and represents the norm of the $L_2(a,b)$ space. This, in turn, means that the function f should be restricted to the class of square-integrable functions. In the following paragraphs the basic concept of the space $L_2(a,b)$ is displayed and the appropriate definitions and properties are given.

A.2 Square integrable functions

The space $L_2(a,b)$ is a linear space, the elements of which are square integrable functions in the bounded interval $[a,b]$. This means, that the following integrals exist and are finite.

$$\int_a^b f(x)dx \tag{A.2}$$

$$\int_a^b f^2(x)dx \,. \tag{A.3}$$

The above integrals are considered in the Lebesgue sence. The Lebesgue integral is an extension of the Riemman integral and it is essential from the point of view of mathematics rather than from the point of view of mechanics. For the needs of the most of applications no difference exist in the calculation of an integral through the Lebesgue and the Riemman sences. For this reason further details are not presented.

A.3 Inner product, norm, distance and orthogonality in *$L_2(a,b)$* space

The inner (or scalar) product of two functions f, g in the $L_2(a,b)$ space is defined as the integral:

$$(f,g) = \int_a^b f(x)g(x)dx \,. \tag{A.4}$$

The norm and the distance (or metrics) are defined by the following relations respectively:

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_a^b f(x)f(x)dx} \,. \tag{A.5}$$

$$\rho(f,g) = \|f-g\| = \sqrt{\int_a^b \big(f(x)-g(x)\big)\big(f(x)-g(x)\big)dx} \tag{A.6}$$

It is noted that the concept of norm is an abstract generalization of the length of a vector. Any real valued function which satisfies certain conditions (see Debnath and Mikusinski, 2005) is called a norm. A function f is called normed in the space $L_2(a,b)$ if its norm is equal to unity.

A.4 Orthogonality and ortonormal functions

Two functions f, g are called orthogonal in $L_2(a,b)$ if their inner product is equal to zero, i.e.:

$$(f,g) = \int_a^b f(x)g(x)dx = 0\,. \tag{A.7}$$

A system of functions $f_1, f_2, f_3 \ldots.$ is called orthogonal in the space $L_2(a,b)$, if every two functions f_i, f_j of this system, with $i \neq j$, are orthogonal in that space. Furthermore, if every function of that system is normed, then the system is called orthonormalized or orthonormal in that space. An orthonormal system constitutes a base in the vector space $L_2(a,b)$.

A.5 Complete systems – Fourier expansion

An orthonormal basis (having elements the functions f_n) is called complete in the space $L_2(a,b)$, if for any square integrable function $f \in L_2(a,b)$ and any $\varepsilon > 0$ there is a finite linear combination:

$$\sum_{i=1}^{n} b_n f_n(x) \tag{A.8}$$

such that,

$$\left\| \sum_{i=1}^{n} b_n f_n(x) - f(x) \right\| < \varepsilon\,. \tag{A.9}$$

This means that any function $f(x)$ can be expressed approximately as a linear combination of the elements of the orthonormal basis f_n, i.e:

$$f(x) = b_1 f_1(x) + b_2 f_2(x) + b_3 f_3(x) + \ldots = \sum_{i=1}^{n} b_n f_n(x)\,. \tag{A.10}$$

The above series is called the Fourier series or Fourier expansion of the function f with respect to the orthonormal basis. The coefficients b_n are called Fourier coefficients and are defined by the following relation:

$$b_n = (f, f_n) = \int_a^b f(x) f_n(x) dx , \tag{A.11}$$

where $n = 1, 2, 3 \ldots$.

A.6 Orthogonality of Eigenfunctions in the $L_2(a,b)$

As it was stated in Chapter 3 a BVP in the form:

$$M(f) - \lambda N(f) = g(x) \tag{A.12}$$

$$a_{i0} w(a) + b_{i0} w(b) + a_{i1} w'(a) + b_{i1} w'(b) + \ldots. + a_{i,2m-1} w^{(2m-1)}(a) + b_{i,2m-1} w^{(2m-1)}(b) = 0 ,$$

$$i = 1, 2, 3 \ldots., 2m . \tag{A.13}$$

where (A.12) represents the differential equation and (A.13) represent the boundary conditions of the problem, has apart from the zero solution, nonzero solutions, which are called eigenenfunctions. The values of the parameter λ which produce the nonzero solutions are called eigenvalues. For the eigenvalues of a specific BVP the following theorem holds.

Theorem A.1

If λ_i, λ_j with $i \neq j$ are two eigenvalues of the BVP (A.10) and (A.11) and f_i, f_j are the corresponding eigenfunctions, then the functions λ_i and λ_j are orthogonal in the space $L_2(a,b)$, i.e.:

$$\lambda_i \neq \lambda_j \Rightarrow (f_i, f_j) = \int_a^b f_i(x) f_j(x) dx = 0 . \tag{A.14}$$

From (A.12) results the fact that eigenfunctions corresponding to different eigenvalues are orthogonal in the space $L_2(a,b)$. Obviously, the system of normed eigenfunctions constitutes an orthonormal basis in $L_2(a,b)$.

Appendix B - Fourier series and the Discrete Fourier Transform method

B.1 Sine Fourier series

Fourier series are complete sets of functions (see paragraph A.5 in Appendix A) which satisfy some certain identities and are used to approximate any square integrable function in the space $L_2(a,b)$. Basically, three different set of Fourier series exist. The full Fourier series expansion, the sine Fourier series and the cosine Fourier series. Depending on the type of the boundary conditions for a given BVP and the type of the function to be approximated, the appropriate Fourier series should be used.

In the present dissertation the sine Fourier series are used in order to express the particular solution of the non-homogeneous BVP which describes the bending behavior of a beam in the framework of the second-order bending theory. This set of Fourier series is the appropriate set for the given BVP due to the fact that it is well adapted to functions which are zero to $x=a$ and $x=b$. Except that, the function of the deflection curve described in the interval $[a,b]$, is an odd summetric function, i.e.:

$$f(-x)=-f(x), \quad \forall x\in[a,b] \tag{C.1}$$

and therefore sine Fourier series is the appropriate tool in order to approximate functions with that property.

A sine Fourier series represents a function as:

$$f(x)=\sum_{i=1}^{n} b_n \sin\frac{n\pi x}{L}, \tag{C.2}$$

where with the term L is denoted the length of the interval $[a,b]$. The formula for the determination of the Fourier coefficients is defined by the following relation:

$$b_n=\frac{2}{L}\int_a^b f(x)\sin\frac{n\pi x}{L}. \tag{C.3}$$

For the sine Fourier series the following fundamental theorem holds:

Theorem B.1

If the function $f(x)$ is square integrable in an interval $[a,b]$, then:

- all the coefficients b_n of equation (C.3) are well defined,

- all the coefficients b_n are uniquely determined by $f(x)$ and depend linearly on $f(x)$,

- the series (C.2) converges to the function $f(x)$ in the root mean square sense, i.e.:

$$\left\|\sum_{i=1}^{n} b_n f_n(x) - f(x)\right\| \to 0, \tag{C.4}$$

- the following relation holds:

$$\|f(x)\|^2 = \frac{L}{2}\sum_{i=1}^{n}|b_n|^2, \tag{C.5}$$

where the right hand side part always converges.

Conversely, given a square summable sequence b_n such that the term

$$\sum_{i=1}^{n}|b_n|^2 \tag{C.6}$$

is finite, a square integrable function can be determined uniquely, such that all the above statements hold.

B.2 The Discrete Fourier Transform

In order to approximate a given function $f(x)$ through a sine Fourier series, the integrals (C.3) have to be calculated. When the function $f(x)$ is known and the integral can be determined analytically, no difficulty exists in the approximation procedure. In cases where the function $f(x)$ is unknown, for instance, when function values are given only at discerte values (as with physical measurements) or the integral (C.3) cannot be calculated in an analytical way, the Discrete Fourier

Transform (DFT) method is applied. The DFT method is actually the evaluation of the Fourier Transform by numerical computing. The integral (C.3) is determined considering a numerical scheme for the integration. In the framework of the present dissertation the Simpson's rule was adopted for that purpose.

Appendix C - Solution of nonlinear algebraic equations

C.1 The Newton's method

Suppose we want to find the value of the variable x that satisfies the nonlinear algebraic equation:

$$f(x)=0 \tag{A.1}$$

where $f(x)$ can be a polynomial or a transcendental function. Transcedental are called the functions which cannot be expressed as the sum, the difference, the product, the ratio of two polynomials, or the the root of one polynomial. A classic example of a transedental function is the following:

$$f(x)=-x+\sin x\,. \tag{A.2}$$

The problem of finding the solution to equation (A.1) is not necessarily a simple issue. The solution of such equations is achieved ny means of numerical methods. Such methods are the well konown bracketing methods (e.g. the Bisection method), the one-point iterations methods etc. A very powerful method for obtaining solutions to nonlinear algebraic equations is the Newton's method. In the present dissertation all the encountered transcendental equations are solved by the Newton's method with the utilization of the Mathematica software. In the following, a brief description of the Newton's method is presented.

Let us consider a function $f:\mathbb{R}\rightarrow\mathbb{R}$ which is infinitely differentiable in a neighborhoud of a real number $x^{(k)}$. Then, the function f can be represented as a Taylor series expansion which is an infinite sum of terms calculated from the values of its derivatives at point $x^{(k)}$, as follows:

$$f(x)=f(x^{(k)})+f'(x^{(k)})(x-x^{(k)})+\frac{f''(x^{(k)})}{2!}(x-x^{(k)})^2+\frac{f'''(x^{(k)})}{3!}(x-x^{(k)})^3+.. \tag{A.3}$$

If we retain only the first two terms of the above sum, the linear approximate representation of the function f in the neighbourhood of $x^{(k)}$ becomes:

$$f(x)=f(x^{(k)})+f'(x^{(k)})(x-x^{(k)})\,. \tag{A.4}$$

The right-hand side of the above equation represents a straight line that is tangent to the curve $f(x)$ and passes through the point $x^{(k)}$. The second term on the right-hand side is the approximate change in the function $f(x)$ with respect to a movement to the left or the right of the point $x^{(k)}$. That approximate change of the function $f(x)$ becomes exact, as the variable x approaches $x^{(k)}$.

Recall now equation (A.1) and suppose that the function $f(x)$ of this equation is approximated through the previous linearization procedure about an arbitrary point x^k. Then, equation (A.1) becomes:

$$f(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) = 0. \tag{A.5}$$

The point $x^{(k)}$ is called an initial guess. Obviously, the solution of equation (A.5) will identify the point $x^{(k+1)}$ which is defined by the intersection of the x axis with the tangential line (Fig. A.1), i.e.:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})} . \tag{A.6}$$

The above equation produces a sequence of approximations based on the iteration function:

$$g(x) = x - \frac{f(x)}{f'(x)} \tag{A.7}$$

Once $x^{(k+1)}$ is obtained, the value $f(x^{(k+1)})$ can be determined and checked how much this value deviates from zero. This difference represents the error associated with the initial guess $x^{(k)}$ and is reffered as the residual. If the residual is larger than some user-defined tolerance, then the next guess $x^{(k+2)}$ is computed and checked against the tolerance. The process is repeated until the residual is small compared with the defined tolerance.

Generally, the sequence (A.6) converges when the iteration function $g(x)$ fulfils the iequality:

$$|g'(X)| < 1, \tag{A.8}$$

where X constitutes root of the nonlinear equation (A.1). It can be proved (see e.g. C. Pozrikidis, 1998; Ostrowski, 1966; Rabinowitz, 1970) that the iteration function (A.7) satisfies inequality (A.8) .

It has to be noticed that in the practical implementation of Newton's method, the derivative f' is often computed by numerical differentiation.

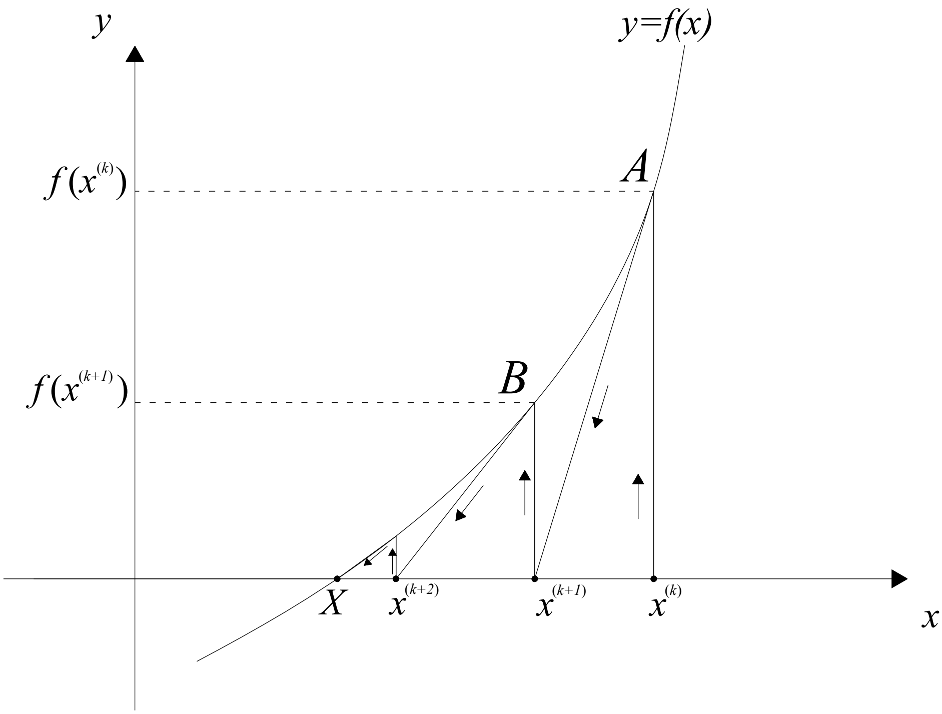


Fig. C.1 Schematical representation of Newton's method for solving nonlinear equations.

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