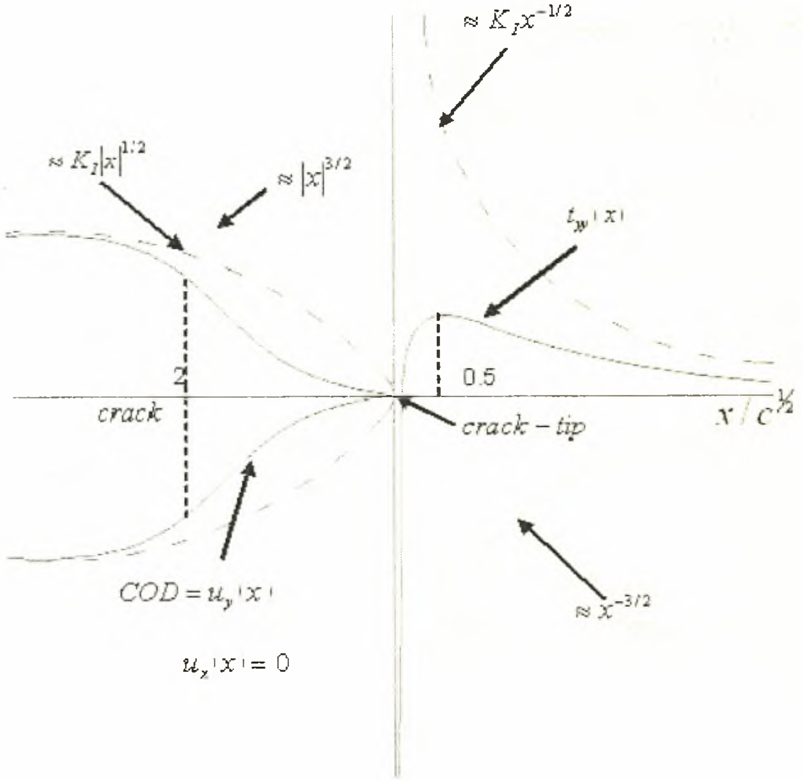
UNIVERSITY OF THESSALY
SCHOOL OF ENGINEERING
DEPARTMENT OF CIVIL ENGINEERING

THESIS

FOR THE FULFILMENT OF THE MASTER'S OF SCIENCE DEGREE

CONNECTING FAR-FIELD LOADING WITH INTENSITY OF TRACTIONS AND DISPLACEMENTS AT THE MODE I CRACK-TIP FOR STRAIN-GRADIENT ELASTIC MATERIALS



GAVARDINAS IOANNIS

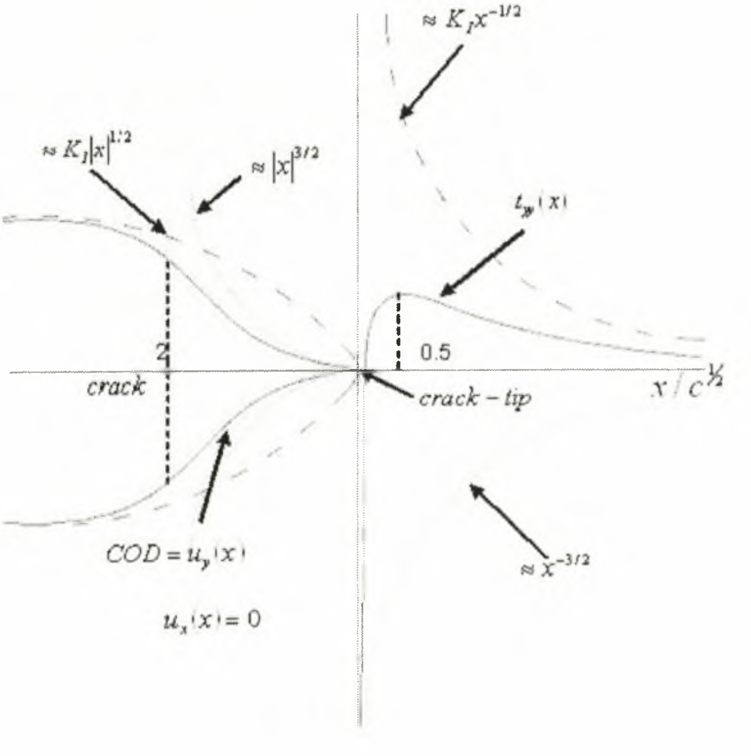
MEMBERS OF THESIS COMMITTEE:
PROFESSOR GIANNAKOPOULOS A., (ADVISOR)
PROFESSOR ARAVAS N.
PROFESSOR PERDIKARIS PH.

32

UNIVERSITY OF THESSALY
SCHOOL OF ENGINEERING
DEPARTMENT OF CIVIL ENGINEERING

THESIS
FOR THE FULFILMENT OF THE MASTER'S OF SCIENCE DEGREE

CONNECTING FAR-FIELD LOADING WITH INTENSITY OF TRACTIONS AND DISPLACEMENTS AT THE MODE I CRACK-TIP FOR STRAIN-GRADIENT ELASTIC MATERIALS



GAVARDINAS IOANNIS

MEMBERS OF THESIS COMMITTEE:
PROFESSOR GIANNAKOPOULOS A., (ADVISOR)
PROFESSOR ARAVAS N.
PROFESSOR PERDIKARIS PH.

Ἐὰν ᾖς φιλομαθής, ἔσει πολυμαθής. Ἃ μὲν ἐπίστασαι, ταῦτα διαφύλαττε ταῖς μελέταις, ἃ δὲ μὴ μεμάθηκας, προσλάμβανε ταῖς ἐπιστήμαις· ὁμοίως γὰρ αἰσχρὸν ἀκούσαντα χρήσιμον λόγον μὴ μαθεῖν καὶ διδόμενόν τι ἀγαθὸν παρὰ τῶν φίλων μὴ λαβεῖν. Καταναλισκε τὴν ἐν τῷ βίῳ σχολὴν εἰς τὴν τῶν λόγων φιληκοΐαν· οὕτω γὰρ τα τοῖς ἄλλοις χαλεπῶς εὑρημένα συμβήσεταί σοι ῥᾳδίως μανθάνειν.

Ἡγοῦ τῶν ἀκουσμάτων πολλὰ πολλῶν εἶναι χρημάτων κρείττω· τὰ μὲν γὰρ ταχέως ἀπολείπει, τὰ δὲ πάντα τὸν χρόνον παραμένει· σοφία γὰρ μόνον τῶν κτημάτων ἀθάνατον.

Isocrates, Ad Demonicus, 18.1 - 19.1

ACKNOWLEDGEMENTS

With the completion of this thesis, I consider it necessary to thank those who, in any way, assisted me in finishing this task.

I thank my friends Vaggelis Karaiskos, Fedon Giannopoulos, Panagiotis Gatsonis, Mihalis Kavouras, Kostas Tzaros, Lambros Tzaros, Vassilis Evaggelou, for all they offered in many ways, and for sharing common experiences during my years in the University. I also thank my family for their vast support, in the past and still continuing.

I express my gratitudes to Professors Aravas N. and Perdikaris Ph. for accepting to participate in the supervising committee, despite their many other ongoing obligations.

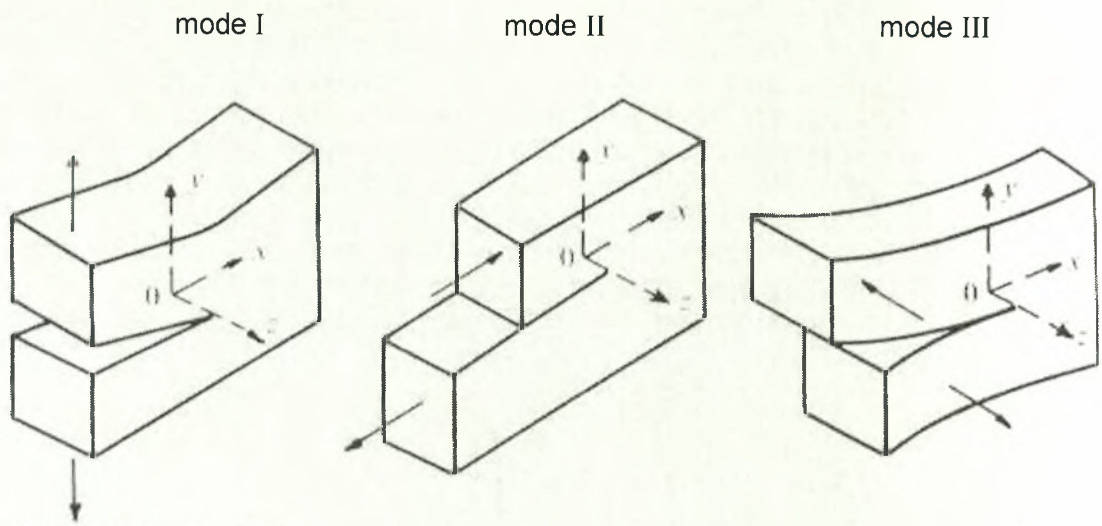
Finally, I express my gratitudes to my supervisor, Professor Giannakopoulos A., not only for sharing knowledge with me, but to his professional ethos, and for entrusting me with the burden and opportunity to deal with scientifically important topics of fracture mechanics.

I.D. Gavardinas

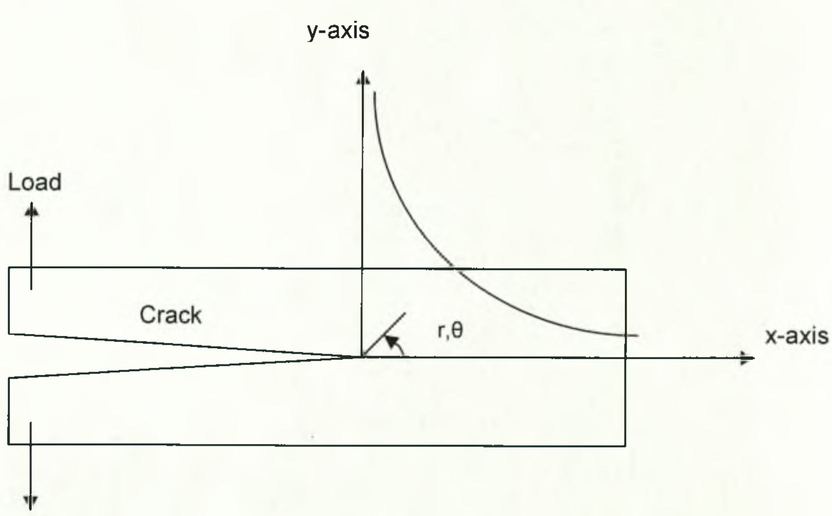
PREFACE

The present work deals with the evaluation of the J-Integral, as well as the other integrals that appear and are useful in fracture mechanics of gradient elastic materials (J_2, L-Integrals). The analysis pertains to simple linear elastic strain gradient materials. We succeeded in expressing the J-Integral, as the energy release rate of a quasi-statically advancing crack-tip. The importance of results of this work is the close form relations between the J-Integral and the amplitudes of the expansions of the local deformation and stress fields that are closest to the crack-tip. In this way, it is possible to connect the macroscopic far-field loading, geometry and material properties with the strength of the material, as it manifests with the amplitudes of the singular terms of stress and deformation.

INTRODUCTION



mode I (tensile opening)
mode II (in-plane sliding)
mode III (anti-plane shear or tearing)



Mode I, plane strain, asymptotic behaviour of t_{yy} near the crack-tip (singularity of t_{yy} traction)

For the plane problem, the leading terms for mode I stresses in Cartesian coordinates are [Williams, M.L., (1957)]:

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{Bmatrix}=\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\begin{Bmatrix}1-\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\\ 1+\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)\end{Bmatrix} \tag{1}$$

When written in cylindrical coordinates, the stresses for mode I have the following leading terms:

$$\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{Bmatrix}=\frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\begin{Bmatrix}1+\sin^2\left(\frac{\theta}{2}\right)\\ \cos^2\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\end{Bmatrix} \tag{2}$$

$$\sigma_{zz}=\nu_1(\sigma_{xx}+\sigma_{yy}) = \nu_1(\sigma_{rr}+\sigma_{\theta\theta}) \tag{3}$$

$$\sigma_{xz}=\sigma_{yz}=\sigma_{rz}=\sigma_{\theta z}=0 \tag{4}$$

The corresponding displacements are [Williams, M.L., (1957)]:

$$\begin{Bmatrix}u_x\\ u_y\end{Bmatrix}=\frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\begin{Bmatrix}(1+\mathrm{v})\left[(2\kappa-1)\cos\left(\frac{\theta}{2}\right)-\cos\left(\frac{3\theta}{2}\right)\right]\\ (1+\mathrm{v})\left[(2\kappa+1)\sin\left(\frac{\theta}{2}\right)-\sin\left(\frac{3\theta}{2}\right)\right]\end{Bmatrix} \tag{5}$$

$$\begin{Bmatrix}u_r\\ u_\theta\end{Bmatrix}=\frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\begin{Bmatrix}(1+\mathrm{v})\left[(2\kappa-1)\cos\left(\frac{\theta}{2}\right)-\cos\left(\frac{3\theta}{2}\right)\right]\\ (1+\mathrm{v})\left[-(2\kappa-1)\sin\left(\frac{\theta}{2}\right)+\sin\left(\frac{3\theta}{2}\right)\right]\end{Bmatrix} \tag{6}$$

$$u_z=-\left(\frac{\nu_2 z}{E}\right)\left(\sigma_{xx}+\sigma_{yy}\right)=-\left(\frac{\nu_2 z}{E}\right)\left(\sigma_{rr}+\sigma_{\theta\theta}\right) \tag{7}$$

For plane stress,

$$\kappa=\left(\frac{3-\nu}{1+\nu}\right),\nu_1=0,\nu_2=\nu \tag{8}$$

and, for plane strain,

$$\kappa = (3-4\nu),\ \nu_1 = 0,\ \nu_2 = \nu \qquad (9)$$

The term K_I is the so-called *stress intensity factor*, which incorporates the boundary conditions of the cracked body and is a function of loading, crack length and geometry. For plane problems, it is independent of the elastic constants (E, elastic modulus, v, Poisson's ratio).

For the plane strain problem, the displacement u_y on the left of the crack-tip can be simplified using equations (5), (9), taking into account that θ=π. The following expression is obtained: [sin(π /2)=1, sin(3π /2)=-1]

$$u_y = \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\left\{ (1+\mathrm{v})\left[\left(2(3-4\nu)+1\right)\times 1-(-1)\right] \right\} =$$
$$= \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\left\{ (1+\mathrm{v})\left[6-8\nu+1+1\right] \right\} =$$
$$= \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\left\{ (1+\mathrm{v})\left[8-8\nu\right] \right\} =$$
$$= \frac{K_I}{2E}\sqrt{\frac{r}{2\pi}}\ (1+\mathrm{v})\ 8(1-\mathrm{v}) \Rightarrow$$

$$\Rightarrow \boxed{u_y = \frac{4}{\sqrt{2\pi}}\ \frac{K_I\left(1-\mathrm{v}^2\right)}{E}\sqrt{\mathrm{r}}}$$

Since $\boxed{\mathcal{G} = \frac{1-\nu^2}{\mathrm{E}} K_i^2}$ for plane strain, where $\mathcal{G}$ is the energy release rate (not the shear modulus)

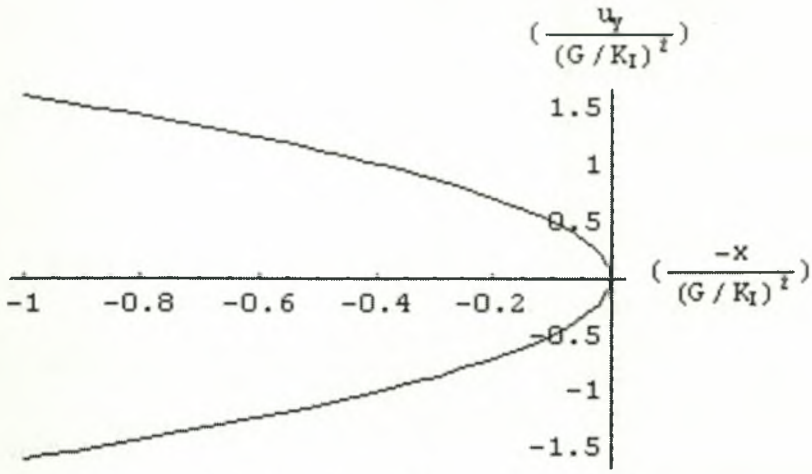
$$\Rightarrow u_y = \frac{4}{\sqrt{2\pi}}\ \frac{\mathcal{G}}{K_I}\ \sqrt{\mathrm{r}} \Rightarrow \frac{u_y}{\left(\frac{\mathcal{G}}{K_I}\right)^2} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{\mathrm{r}}{\left(\frac{\mathcal{G}}{K_I}\right)^2}} \Rightarrow$$

$$\Rightarrow \frac{u_y}{\left(\frac{\mathcal{G}}{K_I}\right)^2} = \frac{4}{\sqrt{2\pi}}\sqrt{\frac{-\mathrm{x}}{\left(\frac{\mathcal{G}}{K_I}\right)^2}}$$

because r=-x, x<0 (the negative semi-axis x is situated on the left of the crack-tip). The above expression for u_y is of the form $Q = \frac{4}{\sqrt{2\pi}} \cdot \sqrt{P}$, where

$$\mathrm{Q} = \frac{u_y}{\left(\frac{\mathcal{G}}{K_I}\right)^2},\ \mathrm{P} = \sqrt{\frac{-\mathrm{x}}{\left(\frac{\mathcal{G}}{K_I}\right)^2}}$$

and is plotted as illustrated below:



Plot of $-x/(\mathcal{G}/K_I)^2$ vs $u_y/(\mathcal{G}/K_I)^2$

It should be noted that both metrics are dimensionless.

In the same fashion, from eq.(1), σ_{yy} is, for θ=0 (that is ahead of the crack-tip)

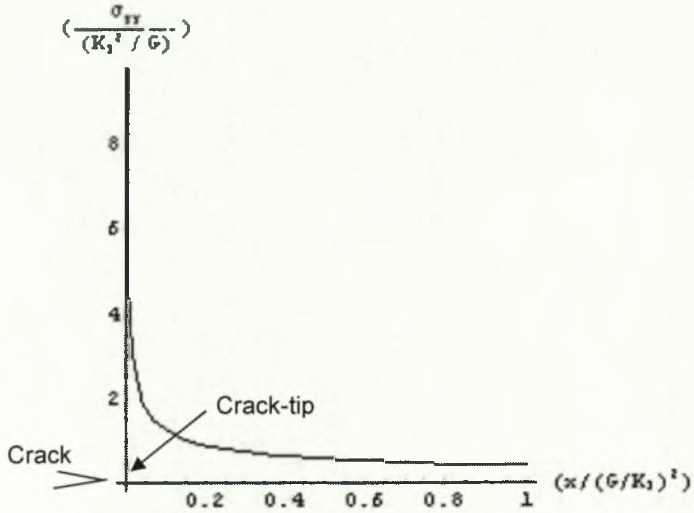
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos 0\times(1+0) = \frac{K_I}{\sqrt{2\pi r}} \Rightarrow$$

$$\Rightarrow \sigma_{yy} = \frac{\dfrac{K_I}{(\mathcal{G}/K_I)}}{\sqrt{\dfrac{2\pi r}{(\mathcal{G}/K_I)^2}}} = \frac{(K_I^2/\mathcal{G})}{\sqrt{2\pi}\sqrt{\dfrac{r}{(\mathcal{G}/K_I)^2}}} \Rightarrow$$

$$\Rightarrow \boxed{\frac{\sigma_{yy}}{(K_I^2/\mathcal{G})} = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{1}{\dfrac{+x}{(\mathcal{G}/K_I)^2}}}}$$

because r=+x, x>0.

The metrics are again non-dimensional.



Plot of $x/(\mathcal{G}/K_I)^2$ vs $\sigma_{yy}/(K_I^2/\mathcal{G})$

As we will show later [Sifnaiou M., (2006)], the new, non classical solution for the displacement u_y is given by:

$$u_y = -(15/2)r^{3/2}\left[(17A - 83B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right]$$

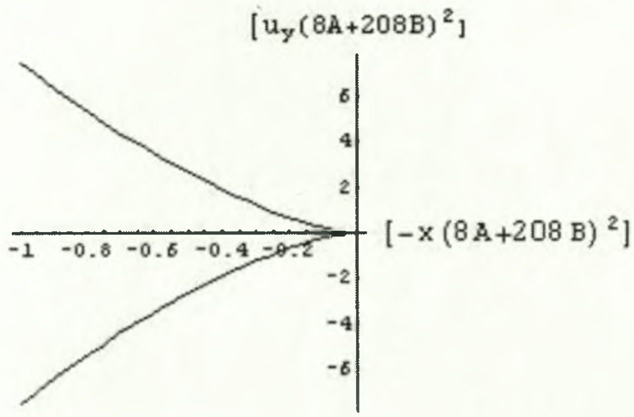
For θ=π, r= -x :

$$u_y = -(15/2)r^{3/2}\left[(17A - 83B)(-1) + 25(A+5B)(1)\right] \Rightarrow$$

$$u_y = -(15/2)r^{3/2}(8A+208B) \Rightarrow u_y(8A+208B)^2 = -(15/2)r^{3/2}(8A+208B)^3 \Rightarrow$$

$$u_y(8A+208B)^2 = -(15/2)\left[r\left((8A+208B)^2\right)\right]^{3/2} \Rightarrow$$

$$\Rightarrow \boxed{u_y(8A+208B)^2 = -(15/2)\left[-x\left((8A+208B)^2\right)\right]^{3/2}}$$



Plot of $u_y(8A+208B)^2$ vs $(-x(8A+208B)^2)^{3/2}$

The total traction t_{yy} ahead of the crack-tip, and more specifically along the straight line *y=0* which bisects the angle of the crack, has an asymptotic behaviour, given by the following equation [Sifnaiou M., (2006)]:

$$t_{yy}(x>0, y=0) = t_{yy}(r, \theta=0) = 2c\mu r^{\omega-3}\,\omega(\omega^2-1)(\omega-2)(A_1 + A_2 + B_1 + B_2)$$

For plane strain, the coefficients A_1, A_2, B_1, B_2 are related as given below:

$$A_2 = -125A_1,\ B_2 = -625B_1,\ A_1 = A,\ B_1 = B$$

Note that the gradient elasticity requires two constants (A, B) to describe the local fields, contrary to the classic case that requires one.
The coefficient ω emerges from the boundary conditions for mode I, which lead to an eigenvalue problem. The value that produces the greatest singularity of the t_{yy} traction is **ω=3/2**. Denoting by $\mu = \dfrac{E}{2(1+\nu)}$ the shear modulus and by $c^{1/2}$ the characteristic length of the body (e.g. grainsize etc), we have [Sifnaiou M., (2006)]:

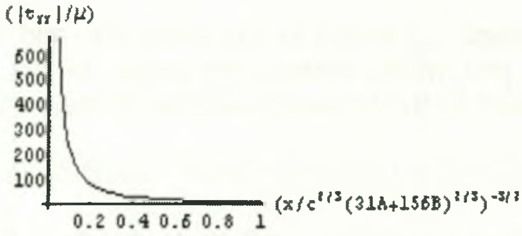
$$\begin{aligned}
t_{yy} &= 2c\mu r^{\omega-3}\omega(\omega^2-1)(\omega-2)(A_1+A_2+B_1+B_2) = \\
&= 2c\mu r^{\frac{3}{2}-3}\frac{3}{2}\left(\left(\frac{3}{2}\right)^2-1\right)\left(\frac{3}{2}-2\right)(A-125A+B-625B) = \\
&= \not{2}c\mu r^{-3/2}\frac{3}{\not{2}}\left(\frac{5}{4}\right)\left(-\frac{1}{2}\right)(-124A-624B) = \\
&= \left(-\frac{15}{8}\right)c\mu r^{-3/2}(-124A-624B) \Rightarrow
\end{aligned}$$

$$\Rightarrow \boxed{t_{yy} = \left(\frac{15}{2}\right)c\mu r^{-3/2}(31A+156B)}$$

From this representation, one can get the dimensionless metric:

$$\begin{aligned}
&t_{yy} = \left(\frac{15}{2}\right)c\mu r^{-3/2}(31A+156B) \Rightarrow \\
&\Rightarrow \frac{|t_{yy}|}{\mu} = \left(\frac{15}{2}\right)r^{-3/2}\left[c^{2/3}(31A+156B)^{2/3}\right]^{3/2} \Rightarrow
\end{aligned}$$

$$\Rightarrow \boxed{\frac{|t_{yy}|}{\mu} = \left(\frac{15}{2}\right)\left|\frac{c^{2/3}(31A+156B)^{2/3}}{x}\right|^{3/2}} \quad (r=+x)$$



Plot of $(|t_{yy}|/\mu)$ vs $[x/c^{2/3}(31A+156B)^{2/3}]^{-3/2}$

The asymptotic displacement fields are then calculated to be:

$$u_x = -(5/2)r^{3/2}\left[(49A - 251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right]$$

$$u_y = -(15/2)r^{3/2}\left[(17A - 83B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right]$$

The existence of a crack presupposes that the maximum principal strain is tensile, i.e. $\boxed{\varepsilon_{yy}>0}$ ahead of the crack-tip, and additionally: $\boxed{u_y>0}$.
The expressions for ε_{yy} and u_y are, respectively:

$$\varepsilon_{yy} = (+15/4)\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) - (101A+B)\cos(\theta/2)\right]$$

$$u_y = -(15/2)r^{3/2}\left[(17A - 83B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right]$$

The first expression for θ=0 gives:

$$\begin{aligned}\varepsilon_{yy} &= (+15/4)\, r^{1/2}\left[25(A+5B)\cos 0 - (101A+B)\cos 0\right] = \\ &= (+15/4)\, r^{1/2}\left[25(A+5B) - (101A+B)\right] = \\ &= (+15/4)\, r^{1/2}\left[25A+125B - 101A - B\right] = \\ &= (+15/4)\, r^{1/2}\left[-76A+124B\right] = \\ &= (+15)\, r^{1/2}\left(-19A+31B\right)\end{aligned}$$

and:

$$\varepsilon_{yy} = (+15)\, r^{1/2}\left(-19A+31B\right) > 0 \Rightarrow$$
$$\Rightarrow \boxed{-19A+31B>0}$$

The second expression for θ=π gives:

$$\begin{aligned}u_y &= -(15/2)r^{3/2}\left[(17A - 83B)\sin(3\pi/2) + 25(A+5B)\sin(\pi/2)\right] = \\ &= -(15/2)r^{3/2}\left[(17A - 83B)(-1) + 25(A+5B)(+1)\right] = \\ &= -(15/2)r^{3/2}\left[-17A + 83B + 25A+125B\right] = \\ &= -(15/2)r^{3/2}\left[8A + 208B\right]\end{aligned}$$

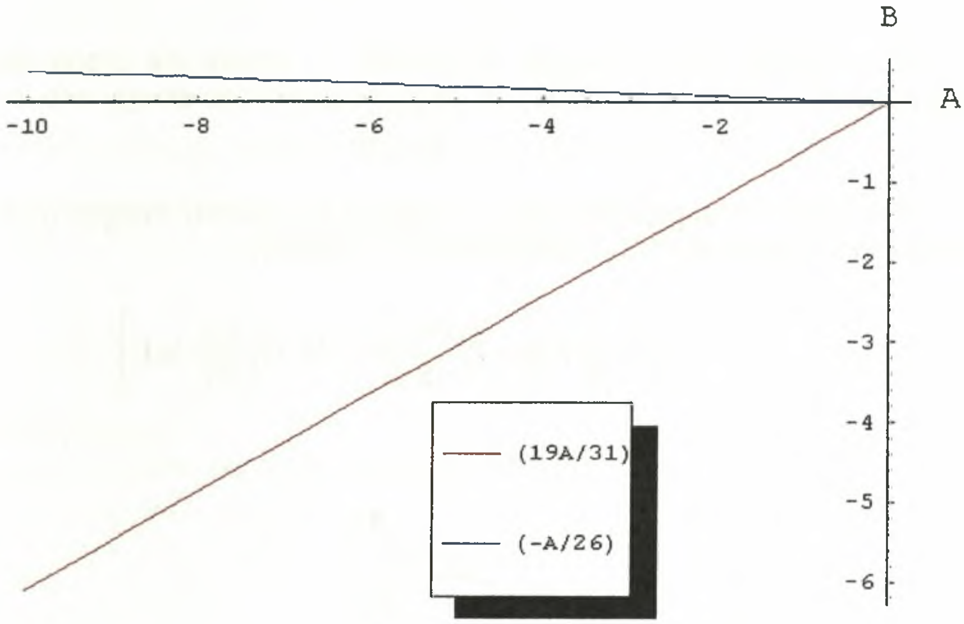
$$u_y = -(15/2)r^{3/2}\left[8A + 208B\right] > 0 \Rightarrow$$
$$\Rightarrow 8A + 208B < 0 \Rightarrow$$

$$\Rightarrow \boxed{A + 26B < 0}$$

By solving the resulting system of inequalities, the following are obtained:

A < 0, 19A/31 < B < -A/26

This can be plotted in the A-B plane (2-D) as shown bellow. The restriction A<0 gives to A negative values, thus values situated on the left of the vertical axis. By the second restriction, we obtain for B values between the lines 19A/31 < B < -A/26.



Since: 19A/31 < B < -A/26, then:

$$\frac{19A}{31} < B < -\frac{A}{26} \Rightarrow$$

$$156\frac{19A}{31} < 156B < -156\frac{A}{26} \Rightarrow$$

$$156\frac{19A}{31} + 31A < 31A + 156B < -156\frac{A}{26} + 31A \Rightarrow$$

$$31A + 156B < \frac{-156 + 31\times 26}{26}A \Rightarrow$$

$$31A + 156B < 25A < 0$$

(since A<0)

$$\Rightarrow \left(\frac{15}{2}\right) c\mu r^{-3/2}(31A + 156B) < 0 \Rightarrow$$

$$\Rightarrow \boxed{t_{yy} < 0}$$

The last inequality means that the traction t_{yy} is always compressive!

This result is in agreement with finite element calculations [Amanatidou, E. and Aravas, N. (2002)], [Wei, Y., (2006)]

CHAPTER 1

THE J_1-INTEGRAL

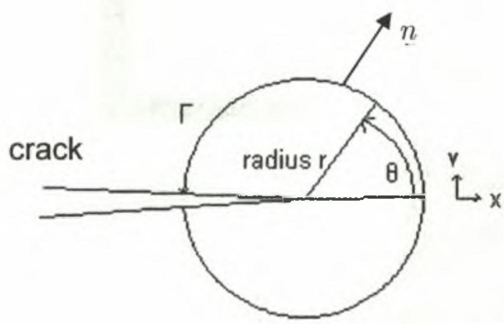
Presentation of the J-Integral

J is actually the J_1-Integral. Physically, J_1 shows the propensity of the crack-tip to advance forward, thus it is exactly equal to the energy release rate:

$$\boxed{J_1 = \mathcal{G}}.$$

For plane-strain problems, the J_1-Integral is a contour integral given by the expression [Georgiadis H.G., Grentzelou C.G., (2006)]:

$$J_1 = \int_\Gamma \left[W dy - P_q \frac{\partial u_q}{\partial x} d\Gamma - R_z D\left(\frac{\partial u_z}{\partial x}\right) d\Gamma \right]$$



$d\Gamma = r d\theta$

where Γ is a closed integration curve lying in the Cartesian $(x,y)=(x_1,x_2)$ plane and encircling the crack-tip.

The J_1-Integral is path independent [Georgiadis H.G., Grentzelou C.G., (2006)]. Therefore, any closed curve can be chosen as an integration curve. As it can be observed, all terms are functions of r and θ. Consequently, those terms whose order of r will be greater than zero, after the integration, then in the limit $\mathbf{r \to 0}$, they will actually be zero (terms that vanish at crack-tip)!

W is the strain energy density, u_q are the components of the displacement field (u_x, u_y), P_q is the total stress traction and R_z the dipolar stress traction.

The aforementioned metrics are defined as follows:

W is defined as:

$$W=(\lambda/2)\left(\varepsilon_{xx}+\varepsilon_{yy}\right)^2+\mu\left(\varepsilon_{xx}^2+2\varepsilon_{xy}+\varepsilon_{yy}^2\right)+(\lambda c/2)\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]+$$

$$\mu c\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2+2\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2+2\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2+\left(\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]$$

$(\lambda/2)\left(\varepsilon_{xx}+\varepsilon_{yy}\right)^2$ is the hydrostatic strain energy density,

$\mu\left(\varepsilon_{xx}^2+2\varepsilon_{xy}+\varepsilon_{yy}^2\right)$ is the shear strain energy density

$(\lambda c/2)\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]$ is the gradient hydrostatic strain energy density,

λ,*μ* are the Lamé constants, satisfying the relations:

$$\frac{\lambda}{\mu}=\frac{2\nu}{1\text{-}2\nu},\ \mu=\frac{E}{2(1+\nu)},$$

with E: elastic modulus, v the Poisson ratio, *μ* the shear modulus, $c^{1/2}$ [m] is a characteristic material length

and

$\mu c\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2+2\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2+2\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2+\left(\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]$ is

the gradient strain energy density.

Along Γ, where Γ is any internal or external boundary of the body,

$$P_q=\mathrm{n}_p\left(\tau_{pq}-\partial_r m_{rpq}\right)-\mathrm{D}_p\left(\mathrm{n}_r m_{rpq}\right)+\left(\mathrm{D}_j\mathrm{n}_j\right)\mathrm{n}_r\mathrm{n}_p m_{rpq}$$

τ_{pq}, m_{rpq} are the Cauchy and dipolar stresses, respectively.

$\mathrm{R}_z=\mathrm{n}_r\mathrm{n}_p m_{rpz}$ along Γ,

$D_p(\ )\equiv\partial_r(\ )-n_p D(\ )$ is the surface gradient operator, ($\partial_r(\ )=\partial(\ )/\partial x_r$; r=1,2)

$D(\ )\equiv n_r\,\partial_r(\ )$ is the normal gradient operator,

n_p is the unit outward normal vector of Γ,

$d\Gamma = r\, d\theta$.

Repeated indices imply summation from 1 to 2.

We will use the asymptotic results that are summarized in Appendix 1. Intermediate results are shown in Appendices 2 to 3.

1.1 Terms W in J_1

Calculation of terms W

W is:

$$W = (\lambda/2)\left(\varepsilon_{xx} + \varepsilon_{yy}\right)^2 + \mu\left(\varepsilon_{xx}^2 + 2\varepsilon_{xy} + \varepsilon_{yy}^2\right) + (\lambda c/2)\left[\left(\frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon_{xx}}{\partial y} + \frac{\partial \varepsilon_{yy}}{\partial y}\right)^2\right] +$$

$$\mu c\left[\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)^2 + 2\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2 + 2\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2 + \left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2\right]$$

where the strains are:

$$\varepsilon_{xx} = (-15/4)\, r^{1/2} \left[\, 25(A + 5B)\cos(3\theta/2) \,+\, (99A - B)\cos(\theta/2)\right]$$

$$\varepsilon_{xy} = (-15/4)\, r^{1/2} \left[\, 25(A + 5B)\sin(3\theta/2) \,+\, (A + B)\sin(\theta/2)\right]$$

$$\varepsilon_{yy} = (+15/4)\, r^{1/2} \left[\, 25(A + 5B)\cos(3\theta/2) \,-\, (101A + B)\cos(\theta/2)\right]$$

For practical reasons we name the terms consisting *W* as seen below, following their order of appearance in the above expression.

- Term *W*1: the term whose multiplying factor is (λ/2) (Note that the respective multiplying factors are not included. All results will be then multiplied by their appropriate coefficient.)
- Term *W*2: the term whose multiplying factor is *μ*
- Term *W*3: the term whose multiplying factor is (λ*c*/2)
- Term *W*4: the term whose multiplying factor is *μc.*

Further, in the same manner, term *W*3, consists of the terms:

Term *W*3(a): $\left(\frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$

Term $W3$(b): $\left(\frac{\partial \varepsilon_{xx}}{\partial y} + \frac{\partial \varepsilon_{yy}}{\partial y}\right)^2$

while term $W4$ of these:

Term $W4$(a): $\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)^2$

Term $W4$(b): $\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2$ (without the coefficient 2)

Term $W4$(c): $\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$

Term $W4$(d): $\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2$

Term $W4$(e): $\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2$ (without the coefficient 2)

Term W4(f): $\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2$

In the expressions of the strains $\varepsilon_{xx}, \varepsilon_{yy}$, r is of order ½. Consequently, the expressions $(\lambda/2)\left(\varepsilon_{xx} + \varepsilon_{yy}\right)^2$ and $\mu\left(\varepsilon_{xx}^2 + 2\varepsilon_{xy} + \varepsilon_{yy}^2\right)$ (terms $W1$ and $W2$, respectively) are of order 1. When multiplied by: $dy = n_1\, d\Gamma = \cos\theta\, d\Gamma = \cos\theta\, r\, d\theta = r\cos\theta\, d\theta$, r should become of order 2. So integrated along a curve Γ, which is chosen to be a circle of radius r, then in the limit r→0, the integral itself will tend to 0!

Each of the rest terms will be calculated separately and then integrated (so as to form the integral $\int W dy$)

The transformation of the derivative from (x,y) Cartesian coordinates to (r,θ) polar coordinates follows the rule:

$$\frac{\partial}{\partial x}(\ ) = \cos\theta\ \frac{\partial}{\partial r}(\ ) - \frac{\sin\theta}{r}\ \frac{\partial}{\partial \theta}(\ )$$

$$\frac{\partial}{\partial y}(\ ) = \sin\theta\ \frac{\partial}{\partial r}(\ ) + \frac{\cos\theta}{r}\ \frac{\partial}{\partial \theta}(\ )$$

We finally get:

Term *W3*

Calculation of Term *W3*(a) $\left(\frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$

Instead of calculating $\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$, it is easier to calculate $\left(\frac{\partial}{\partial x}(\varepsilon_{xx}+\varepsilon_{yy})\right)^2$.

As already said, the transformation of the derivative from (x,y) Cartesian coordinates to (r,θ) polar coordinates follows the rule:

$$\frac{\partial}{\partial x}(\ ) = \cos\theta\ \frac{\partial}{\partial r}(\ ) - \frac{\sin\theta}{\mathrm{r}}\ \frac{\partial}{\partial \theta}(\ )$$

$$\frac{\partial}{\partial y}(\ ) = \sin\theta\ \frac{\partial}{\partial r}(\ ) + \frac{\cos\theta}{\mathrm{r}}\ \frac{\partial}{\partial \theta}(\ )$$

It is calculated that

$$\boxed{\left(\varepsilon_{xx}+\varepsilon_{yy}\right) = (\text{-}750\mathrm{A})\ \mathrm{r}^{1/2}\ \cos(\theta/2)}$$

and

$$\boxed{\frac{\partial}{\partial x}(\varepsilon_{xx}+\varepsilon_{yy}) = (\text{-}375\mathrm{A})\ \mathrm{r}^{-\frac{1}{2}}\left[\ \cos(\theta/2)\right]}$$

By taking the square of it:

$$\Rightarrow \boxed{\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 = 140625\mathrm{A}^2\ \mathrm{r}^{-1}\ \cos^2(\theta/2)}$$

<u>Calculation of Term *W*3(b)</u> $\left(\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy})\right)^2$

This term will be calculated as the previous one.

$$\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy}) = (\text{-}375\mathrm{A})\ \mathrm{r}^{-\frac{1}{2}}\ \sin(\theta/2)\ ,$$

the square of which is:

$$\boxed{\left(\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy})\right)^2 = 140625\ \mathrm{A}^2\ \mathrm{r}^{-1}\ \sin^2(\theta/2)}$$

It is remarkable to notice that the term *W*3 converts into:

$$\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_{xx}}{\partial y}+\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 = 140625\mathrm{A}^2\ \mathrm{r}^{-1}\ \cos^2(\theta/2) + 140625\ \mathrm{A}^2\ \mathrm{r}^{-1}\sin^2(\theta/2) =$$

$$= 140625\mathrm{A}^2\ \mathrm{r}^{-1}\ \left(\cos^2(\theta/2)+\sin^2(\theta/2)\right) = 140625\mathrm{A}^2\ \mathrm{r}^{-1}\bullet\ 1 \Rightarrow$$

$$\Rightarrow \left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 + \left(\frac{\partial \varepsilon_{xx}}{\partial y}+\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 = 140625\mathrm{A}^2\ \mathrm{r}^{-1}$$

Term *W4*

Calculation of Term *W4*(a): $\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)^2$

$$\frac{\partial \varepsilon_{xx}}{\partial x} = (-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\left[50(\mathrm{A}+5\mathrm{B})(\cos\theta-\cos(2\theta))+124(\mathrm{A}+\mathrm{B})\right]$$

Calculation of Term *W4*(b): $\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2$

The simplified expression for $\frac{\partial \varepsilon_{xy}}{\partial x}$ is:

$$\frac{\partial \varepsilon_{xy}}{\partial x} = \left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(24\mathrm{A}+124\mathrm{B})\right]$$

Calculation of Term *W4*(c): $\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$

$$\frac{\partial \varepsilon_{yy}}{\partial x} = \left(+\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left[50(\mathrm{A}+5\mathrm{B})(\cos\theta-\cos(2\theta))+(-76\mathrm{A}+124\mathrm{B})\right]$$

Calculation of Term *W4*(d): $\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2$

$$\frac{\partial \varepsilon_{xx}}{\partial y} = \left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(24\mathrm{A}-376\mathrm{B})\right]$$

Calculation of Term *W4*(e): $\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2$

$$\frac{\partial \varepsilon_{xy}}{\partial y} = (-\frac{15}{8}) r^{-1/2} \cos(\theta/2)\left[50(A+5B)(-\cos\theta+\cos(2\theta))+(76A+376\ B)\right]$$

Calculation of Term *W*4(f): $\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2$

$$\frac{\partial \varepsilon_{yy}}{\partial y} = \left(\frac{15}{8}\right) r^{-1/2} \sin(\theta/2)\left[-50(A+5B)(\cos\theta+\cos(2\theta))+(-176A-376B)\right]$$

1.2 Terms R_q in J_1

Calculation of terms $R_q D\left(\frac{\partial u_q}{\partial x}\right)$

The term $R_q D\left(\frac{\partial u_q}{\partial x}\right)$ can be expanded in the following way:

$$R_q D\left(\frac{\partial u_q}{\partial x}\right) = R_x D\left(\frac{\partial u_x}{\partial x}\right) + R_y D\left(\frac{\partial u_y}{\partial x}\right).$$

Consequently, these are the terms that should be calculated:

$$R_x, D\left(\frac{\partial u_x}{\partial x}\right), R_y, D\left(\frac{\partial u_y}{\partial x}\right).$$

These are their definitions:

$$\begin{aligned} R_x &= n_r n_p m_{rpx} = \\ &= n_x n_p m_{xpx} + n_y n_p m_{ypx} = \\ &= n_x n_x m_{xxx} + n_x n_y m_{xyx} + n_y n_x m_{yxx} + n_y n_y m_{yyx} \end{aligned}$$

$$\begin{aligned} D\left(\frac{\partial u_x}{\partial x}\right) &= n_r \partial_r\left(\frac{\partial u_x}{\partial x}\right) = n_x \partial_x\left(\frac{\partial u_x}{\partial x}\right) + n_y \partial_y\left(\frac{\partial u_x}{\partial x}\right) = \\ &= n_x \frac{\partial^2 u_x}{\partial x^2} + n_y \frac{\partial^2 u_x}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_x}{\partial x^2} + \sin\theta \frac{\partial^2 u_x}{\partial x \partial y} \end{aligned}$$

(given that $n_x = \cos\theta$ and $n_y = \sin\theta$)

and

$$\mathrm{R}_y = n_r n_p m_{rpy} =$$
$$= \quad n_x n_p m_{xpy} \quad + \quad n_y n_p m_{ypy} \quad =$$
$$= n_x n_x m_{xxy} + n_x n_y m_{xyy} \; + \; n_y n_x m_{yxy} + n_y n_y m_{yyy}$$

$$D\left(\frac{\partial u_y}{\partial x}\right) = \mathrm{n}_r\, \partial_r \left(\frac{\partial u_y}{\partial x}\right) = \mathrm{n}_x\, \partial_x \left(\frac{\partial u_y}{\partial x}\right) + \mathrm{n}_y\, \partial_y \left(\frac{\partial u_y}{\partial x}\right) =$$
$$= \mathrm{n}_x \frac{\partial^2 u_y}{\partial x^2} + \mathrm{n}_y \frac{\partial^2 u_y}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_y}{\partial x^2} + \sin\theta \frac{\partial^2 u_y}{\partial x \partial y}$$

The dipolar stresses, needed to form the terms $\mathrm{R}_x, \mathrm{R}_y$ are defined as [Sifnaiou M., (2006)]:

$$m_{xxx} = (+15/4)\ \mathrm{c}\ \mathrm{r}^{-1/2} \left[\ 25\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)\ -\ (100\mathrm{A}\lambda + 149\mathrm{A}\mu + 249\mathrm{B}\mu)\cos(\theta/2)\right]$$

$$m_{xxy} = m_{xyx} = (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2} \left[\ 25(\mathrm{A}+5\mathrm{B})\sin(5\theta/2)\ -\ (49\mathrm{A}+249\mathrm{B})\sin(\theta/2)\right]$$

$$m_{xyy} = (-15/4)\ \mathrm{c}\mathrm{r}^{-1/2} \left[\ 25\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)\ +\ (100\mathrm{A}\lambda + 51\mathrm{A}\mu - 249\mathrm{B}\mu)\cos(\theta/2)\right]$$

$$m_{yxx} = (+15/4)\ \mathrm{c}\mathrm{r}^{-1/2} \left[\ 25\mu(\mathrm{A}+5\mathrm{B})\sin(5\theta/2)\ -\ (100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)\sin(\theta/2)\right]$$

$$m_{yyx} = m_{yxy} = (-15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2} \left[\ 25(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)\ +\ (51\mathrm{A}+251\mathrm{B})\cos(\theta/2)\right]$$

$$m_{yyy} = (-15/4)\ \mathrm{c}\mathrm{r}^{-1/2} \left[\ 25\mu(\mathrm{A}+5\mathrm{B})\sin(5\theta/2)\ +\ (100\mathrm{A}\lambda + 151\mathrm{A}\mu + 251\mathrm{B}\mu)\sin(\theta/2)\right]$$

<u>Calculation of the subterms that compose the term</u> $\mathrm{R}_q\, D\left(\frac{\partial u_q}{\partial x}\right)$

<u>CALCULATION OF</u> R_x

$$\mathrm{R}_x = n_x n_x m_{xxx} + n_x n_y m_{xyx} \quad + \quad n_y n_x m_{yxx} \quad + \; n_y n_y m_{yyx} =$$
$$= \cos^2\theta\, m_{xxx} + \cos\theta \sin\theta\, m_{xyx} + \sin\theta \cos\theta\, m_{yxx} + \sin^2\theta\, m_{yyx}$$

Finally, R_x is:

$$R_x = +(15/4)\ c\ r^{-1/2}\cos(\theta/2)(-2)\left[(13A+63B)\mu + (50A\lambda + 49A\mu - B\mu)\cos\theta\right] \Rightarrow$$

$$\Rightarrow \boxed{R_x = (-15/2)\ c\ r^{-1/2}\cos(\theta/2)\left[(13A+63B)\mu + (50A\lambda + 49A\mu - B\mu)\cos\theta\right]}$$

CALCULATION OF R_y

$$\begin{aligned} R_y &= n_x n_x m_{xxy} + n_x n_y m_{xyy} + n_y n_x m_{yxy} + n_y n_y m_{yyy} = \\ &= \cos^2\theta\, m_{xxy} + \cos\theta\sin\theta\, m_{xyy} + \sin\theta\cos\theta\, m_{yxy} + \sin^2\theta\, m_{yyy} = \\ &= \cos^2\theta\, m_{xxy} + \cos\theta\sin\theta\, \left(m_{xyy} + m_{yxy}\right) + \sin^2\theta\, m_{yyy} \end{aligned}$$

The final form for R_y is:

$$\Rightarrow \boxed{R_y = (-15/2) c\, r^{-1/2}\sin(\theta/2)\left\{50A\lambda + 63(A\mu + B\mu) + (50A\lambda + 51A\mu + B\mu)\cos\theta\right\}}$$

Calculation of the partial derivatives of the displacements

Before continuing, it is presupposed to determine the following partial derivatives: $\frac{\partial u_x}{\partial x}, \frac{\partial^2 u_x}{\partial x^2}, \frac{\partial^2 u_x}{\partial x \partial y}$ and $\frac{\partial u_y}{\partial x}, \frac{\partial^2 u_y}{\partial x^2}, \frac{\partial^2 u_y}{\partial x \partial y}$

CALCULATION OF THE PARTIAL DERIVATIVES

$\frac{\partial u_x}{\partial x}, \frac{\partial^2 u_x}{\partial x^2}, \frac{\partial^2 u_x}{\partial x \partial y}$ and $\frac{\partial u_y}{\partial x}, \frac{\partial^2 u_y}{\partial x^2}, \frac{\partial^2 u_y}{\partial x \partial y}$

DEFINITIONS OF u_x and u_y

$$u_x = (-5/2) r^{3/2}\left[(49A-251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right]$$

$$u_y = (-15/2) r^{3/2}\left[(17A-83B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right]$$

1. $\frac{\partial u_x}{\partial x}$

$$\boxed{\frac{\partial u_x}{\partial x} = (-\frac{15}{4}) r^{1/2}\cos(\theta/2)\left[(49A-251B) + 25(A+5B)(2\cos\theta + 1)\right]}$$

2. $\frac{\partial^2 u_x}{\partial x^2}$

$$\frac{\partial^2 u_x}{\partial x^2} = (-\frac{15}{8}) r^{-1/2} \cos(\theta/2) \left\{ (49A\text{-}251B) + 25(A+5B)\left[3 + 2\cos\theta - 2\cos(2\theta)\right] \right\}$$

3. $\frac{\partial^2 u_x}{\partial x \partial y}$

$$\frac{\partial^2 u_x}{\partial x \partial y} = (-\frac{15}{8}) r^{-1/2} \sin(\theta/2) \left\{ (49A\text{-}251B) - 25(A+5B)\left[1 + 2\cos\theta + 2\cos(2\theta)\right] \right\}$$

4. $\frac{\partial u_y}{\partial x}$

$$\frac{\partial u_y}{\partial x} = (-\frac{15}{4}) r^{1/2} \sin(\theta/2) \left[3(17A\text{-}83B) + 25(A+5B)\left(2\cos\theta\text{-}1\right) \right]$$

5. $\frac{\partial^2 u_y}{\partial x^2}$

$$\frac{\partial^2 u_y}{\partial x^2} = \left(+\frac{15}{8}\right) r^{-1/2} \sin(\theta/2) \left[3(17A\text{-}83B) + 25(A+5B)\left(-3 + 2\cos\theta + 2\cos(2\theta)\right) \right]$$

6. $\frac{\partial^2 u_y}{\partial x \partial y}$

$$\frac{\partial^2 u_y}{\partial x \partial y} = \left(-\frac{15}{8}\right) r^{-1/2} \cos(\theta/2) \left\{ 3(17A\text{-}83B) + 25(A+5B)\left[1 - 2\cos\theta + 2\cos(2\theta)\right] \right\}$$

<u>Calculation of</u> $D\left(\frac{\partial u_x}{\partial x}\right) = n_x \frac{\partial^2 u_x}{\partial x^2} + n_y \frac{\partial^2 u_x}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_x}{\partial x^2} + \sin\theta \frac{\partial^2 u_x}{\partial x \partial y}$

$$\boxed{D\left(\frac{\partial u_x}{\partial x}\right) = (-\frac{15}{4})r^{-1/2}\cos(\theta/2)\{(37A\text{-}63B) + 25(A+5B)\cos\theta\}}$$

<u>Calculation of:</u> $D\left(\frac{\partial u_y}{\partial x}\right) = n_x\frac{\partial^2 u_y}{\partial x^2} + n_y\frac{\partial^2 u_y}{\partial x\partial y} = \cos\theta\frac{\partial^2 u_y}{\partial x^2} + \sin\theta\frac{\partial^2 u_y}{\partial x\partial y}$

$$D\left(\frac{\partial u_y}{\partial x}\right) = \cos\theta\frac{\partial^2 u_y}{\partial x^2} + \sin\theta\frac{\partial^2 u_y}{\partial x\partial y} \Rightarrow$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_y}{\partial x}\right) = (-\frac{15}{4})r^{-1/2}\sin(\theta/2)\{(13A\text{-}187B)+25(A+5B)\cos\theta\}}$$

At this point, each one of the tems $R_x, D\left(\frac{\partial u_x}{\partial x}\right), R_y, D\left(\frac{\partial u_y}{\partial x}\right)$ has been calculated. The next step is to estimate the products:

$$R_x\, D\left(\frac{\partial u_x}{\partial x}\right), R_y\; D\left(\frac{\partial u_y}{\partial x}\right)$$

<u>Calculation of the product:</u> $R_x\, D\left(\frac{\partial u_x}{\partial x}\right)$

$$\boxed{\begin{aligned} &R_x\, D\left(\frac{\partial u_x}{\partial x}\right) = \\ &= (\frac{225}{8})cr^{-1}\cos^2(\theta/2)\left\{\begin{aligned} &\left(481A^2 + 1512AB\text{-}3969B^2\right)\mu + \\ &+\left[\left(1850A^2 - 3150AB\right)\lambda + \left(2138A^2 + 76AB + 7938B^2\right)\mu\right]\cos\theta + \\ &+25\left[50\left(A^2+5AB\right)\lambda + \left(49A^2 + 244AB - 5B^2\right)\mu\right]\cos^2\theta \end{aligned}\right\}\end{aligned}}$$

<u>Calculation of the product:</u> $R_y\; D\left(\frac{\partial u_y}{\partial x}\right)$

$$R_y\, D\left(\frac{\partial u_y}{\partial x}\right) =$$

$$= \left(\frac{225}{8}\right) c\, r^{-1} \sin^2(\theta/2) \begin{Bmatrix} (650A^2 - 9350AB)\lambda + \left(819A^2 - 10962AB - 11781B^2\right)\mu + \\ + \left[\left(1900A^2 - 3100AB\right)\lambda + \left(2238A^2 - 74AB + 7688B^2\right)\mu\right]\cos\theta + \\ + \left[\left(1250A^2 + 6250AB\right)\lambda + \left(1275A^2 + 6400AB + 125B^2\right)\mu\right]\cos^2\theta \end{Bmatrix}$$

1.3 Terms P_z in J_1

Calculation of terms $P_z\left(\frac{\partial u_z}{\partial x}\right)$

The term $P_z\left(\frac{\partial u_z}{\partial x}\right)$ is expanded this way, following the summation convention:

$$P_z\left(\frac{\partial u_x}{\partial x}\right) = P_x\left(\frac{\partial u_x}{\partial x}\right) + P_y\left(\frac{\partial u_y}{\partial x}\right)$$

Its definition is:

$$P_z = n_p\left(\tau_{pz} - \partial_r m_{rpz}\right) - D_p\left(n_r m_{rpz}\right) + \left(D_j n_j\right) n_r n_p m_{rpz} =$$
$$= n_p \tau_{pz} - n_p \partial_r m_{rpz} - D_p\left(n_r m_{rpz}\right) + \left(D_j n_j\right) n_r n_p m_{rpz}$$

CALCULATION OF P_x

$$P_x = n_p \tau_{px} - n_p \partial_r m_{rpx} - D_p\left(n_r m_{rpx}\right) + \left(D_j n_j\right) n_r n_p m_{rpx} =$$
$$= n_p \tau_{px} - n_p \partial_r m_{rpx} - D_p\left(n_r m_{rpx}\right) + \left(D_j n_j\right) R_x =$$
$$= n_x \tau_{xx} - n_x \partial_r m_{rxx} - D_x\left(n_r m_{rxx}\right) + \left(D_j n_j\right) R_x +$$
$$n_y \tau_{yx} - n_y \partial_r m_{ryx} - D_y\left(n_r m_{ryx}\right) =$$
$$= \left(n_x \tau_{xx} + n_y \tau_{yx}\right) - \left(n_x \partial_r m_{rxx} + n_y \partial_r m_{ryx}\right) - \left[D_x\left(n_r m_{rxx}\right) + D_y\left(n_r m_{ryx}\right)\right] + \left(D_j n_j\right) R_x =$$

$$=\left(n_x\tau_{xx}+n_y\tau_{yx}\right)-\left(n_x\partial_x m_{xxx}+n_y\partial_x m_{xyx}\right)-\left[D_x\left(n_x m_{xxx}\right)+D_y\left(n_x m_{xyx}\right)\right]+\left(D_j n_j\right)\mathrm{R}_x-$$

$$-\left(n_x\partial_y m_{yxx}+n_y\partial_y m_{yyx}\right)-\left[D_x\left(n_y m_{yxx}\right)+D_y\left(n_y m_{yyx}\right)\right]\Rightarrow$$

$$\boxed{\Rightarrow \mathrm{P}_x=\left(n_x\tau_{xx}+n_y\tau_{yx}\right)-\begin{bmatrix}n_x\partial_x m_{xxx}+n_y\partial_x m_{xyx}+\\+n_x\partial_y m_{yxx}+n_y\partial_y m_{yyx}\end{bmatrix}-\begin{bmatrix}D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)+\\+D_y\left(n_x m_{xyx}+n_y m_{yyx}\right)\end{bmatrix}+\left(D_j n_j\right)\mathrm{R}_x}$$

For practical reasons, the first set of terms wil be referred to as Subterm A of P_x, the second as Subterm B etc, following their order of appearance in the above expression.

The full term P_x, consisting of the Subterms A, B, C, D, is going to be multiplied by the derivative of the appropriate displacement (either by $\frac{\partial u_x}{\partial x}$ for the J_1-Integral or by $\frac{\partial u_x}{\partial y}$ for the J_2-Integral).

Its component $\left(n_x\tau_{xx}+n_y\tau_{yx}\right)$ (Subterm A) is ~ $r^{1/2}$.

The product $\left(n_x\tau_{xx}+n_y\tau_{yx}\right)\frac{\partial u_x}{\partial x}d\Gamma \sim r^{1/2}\,r^{1/2}\,r\,d\theta \sim r^2 \to 0$ for $r\to 0$.

Consequently, this Subterm's integral will be 0!

<u>CALCULATION OF SUBTERM B:</u> $\begin{bmatrix}n_x\partial_x m_{xxx}+n_y\partial_x m_{xyx}+\\+n_x\partial_y m_{yxx}+n_y\partial_y m_{yyx}\end{bmatrix}$

$$n_x\partial_x m_{xxx}+n_y\partial_x m_{xyx}+n_x\partial_y m_{yxx}+n_y\partial_y m_{yyx}=$$

$$=\cos\theta\left[\cos\theta\frac{\partial}{\partial r}m_{xxx}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}m_{xxx}\right]+\sin\theta\left[\cos\theta\frac{\partial}{\partial r}m_{xyx}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}m_{xyx}\right]+$$

$$+\cos\theta\left[\sin\theta\frac{\partial}{\partial r}m_{yxx}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}m_{yxx}\right]+\sin\theta\left[\sin\theta\frac{\partial}{\partial r}m_{yyx}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}m_{yyx}\right]=$$

$$= \cos^2\theta \frac{\partial}{\partial r} m_{xxx} - \frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta} m_{xxx} + \sin\theta\cos\theta\frac{\partial}{\partial r} m_{xyx} - \frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta} m_{xyx} +$$

$$+\cos\theta\sin\theta\frac{\partial}{\partial r} m_{yxx} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta} m_{yxx} + \sin^2\theta\frac{\partial}{\partial r} m_{yyx} + \frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta} m_{yyx} =$$

$$= \cos^2\theta \frac{\partial}{\partial r} m_{xxx} + \sin\theta\cos\theta\left[\frac{\partial}{\partial r} m_{xyx} + \frac{\partial}{\partial r} m_{yxx}\right] - \frac{\cos\theta\sin\theta}{r}\left[\frac{\partial}{\partial\theta} m_{xxx} - \frac{\partial}{\partial\theta} m_{yyx}\right] -$$

$$-\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta} m_{xyx} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta} m_{yxx} + \sin^2\theta\frac{\partial}{\partial r} m_{yyx}$$

Finally,

$$\boxed{\text{Subterm B of } P_x = (+15/8)c\,\mu\, r^{-3/2}\cos(\theta/2)\left[200(A+5B)\right]}$$

<u>**CALCULATION OF SUBTERM C:**</u> $\left[\begin{array}{l} D_x\left(n_x m_{xxx} + n_y m_{yxx}\right) + \\ + D_y\left(n_x m_{xyx} + n_y m_{yyx}\right)\end{array}\right]$

Subterm C can be decomposed in two parts : $D_x\left(n_x m_{xxx} + n_y m_{yxx}\right)$ and $D_y\left(n_x m_{xyx} + n_y m_{yyx}\right)$.

<u>First part:</u> $D_x\left(n_x m_{xxx} + n_y m_{yxx}\right)$

$$\begin{aligned}
&D_x\left(n_x m_{xxx} + n_y m_{yxx}\right) = \\
&= \partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - n_x D\left(n_x m_{xxx} + n_y m_{yxx}\right) = \\
&= \partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - n_x\, n_r\, \partial_r\left(n_x m_{xxx} + n_y m_{yxx}\right) = \\
&= \partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - n_x\left[n_x\,\partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) + n_y\,\partial_y\left(n_x m_{xxx} + n_y m_{yxx}\right)\right] = \\
&= \partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - n_x^{\,2}\partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - n_x n_y \partial_y\left(n_x m_{xxx} + n_y m_{yxx}\right) =
\end{aligned}$$

$$\begin{aligned}
&= \left(1 - n_x^{\,2}\right)\partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - n_x n_y \partial_y\left(n_x m_{xxx} + n_y m_{yxx}\right) = \\
&= \left(1 - \cos^2\theta\right)\partial_x\left(n_x m_{xxx} + n_y m_{yxx}\right) - \cos\theta\,\sin\theta\,\partial_y\left(n_x m_{xxx} + n_y m_{yxx}\right) \Rightarrow
\end{aligned}$$

$$\Rightarrow D_x\left(n_x m_{xxx} + n_y m_{yxx}\right) =$$
$$= \sin^2\theta\,\partial_x\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) - \cos\theta\,\sin\theta\,\partial_y\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right)$$

Second part: $D_y\left(n_x m_{xyx} + n_y m_{yyx}\right)$

$$D_y\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right) - n_y D\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right) - n_y n_r\,\partial_r\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right) - n_y\left[n_x\,\partial_x\left(n_x m_{xyx} + n_y m_{yyx}\right) + n_y\,\partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right)\right] =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right) - n_y\, n_x\,\partial_x\left(n_x m_{xyx} + n_y m_{yyx}\right) - n_y^{\,2}\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$

$$= \left(1 - n_y^{\,2}\right)\partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right) - n_y\, n_x\,\partial_x\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$
$$= \left(1 - \sin^2\theta\right)\partial_y\left(n_x m_{xyx} + n_y m_{yyx}\right) - \sin\theta\cos\theta\,\partial_x\left(n_x m_{xyx} + n_y m_{yyx}\right) \Rightarrow$$

$$\Rightarrow D_y\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$
$$= \cos^2\theta\,\partial_y\left(\cos\theta\, m_{xyx} + \sin\theta\, m_{yyx}\right) - \sin\theta\cos\theta\,\partial_x\left(\cos\theta\, m_{xyx} + \sin\theta\, m_{yyx}\right)$$

Finally, Subterm C of P_x in its most compact form becomes:

$$\text{Subterm C of } \mathrm{P}_x = (+15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix} 100\mathrm{A}\lambda(-1+\cos\theta)+ \\ +\mathrm{A}\mu(-174+98\cos\theta)+ \\ +\mathrm{B}\mu(-374-2\cos\theta) \end{Bmatrix}$$

CALCULATION OF SUBTERM D: $\left(D_j n_j\right)\mathrm{R}_x$

$$\left(D_j n_j\right)\mathrm{R}_x = \left(D_x n_x + D_y n_y\right)\mathrm{R}_x$$

Each one of $D_x n_x, D_y n_y$ will be estimated separately.

$$D_x n_x = \partial_x(n_x) - n_x D(n_x) = \partial_x(n_x) - n_x n_r \partial_r(n_x) =$$
$$= \partial_x(n_x) - n_x\left[n_x \partial_x(n_x) + n_y \partial_y(n_x)\right] =$$
$$= \partial_x(n_x) - n_x{}^2 \partial_x(n_x) - n_x n_y \partial_y(n_x) =$$
$$= \left(1 - n_x{}^2\right)\partial_x(n_x) - n_x n_y \partial_y(n_x) =$$
$$= \left(1 - \cos^2\theta\right)\partial_x(\cos\theta) - \cos\theta \sin\theta\, \partial_y(\cos\theta) \Rightarrow$$

$$\Rightarrow D_x n_x = \sin^2\theta\ \partial_x(\cos\theta) - \cos\theta\sin\theta\,\partial_y(\cos\theta)$$

$$\Rightarrow\ D_x n_x = \frac{\sin^2\theta}{r}$$

In the same manner,

$$D_y n_y = \partial_y(n_y) - n_y D(n_y) = \partial_y(n_y) - n_y n_r \partial_r(n_y) =$$
$$= \partial_y(n_y) - n_y\left[n_x \partial_x(n_y) + n_y \partial_y(n_y)\right] =$$
$$= \partial_y(n_y) - n_y n_x \partial_x(n_y) - n_y{}^2 \partial_y(n_y) =$$
$$= \left(1 - n_y{}^2\right)\partial_y(n_y) - n_y n_x \partial_x(n_y) =$$
$$= \left(1 - \sin^2\theta\right)\partial_y(\sin\theta) - \sin\theta\cos\theta\,\partial_x(\sin\theta) \Rightarrow$$

$$\Rightarrow D_y n_y = \cos^2\theta\ \partial_y(\sin\theta) - \sin\theta\cos\theta\,\partial_x(\sin\theta)$$

$$\Rightarrow D_y n_y = \frac{\cos^2\theta}{r}$$

The sum $\left(D_j n_j\right) = D_x n_x + D_y n_y$ leads to:

$$\left(D_j n_j\right) = D_x n_x + D_y n_y =$$
$$= \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} =$$
$$= \frac{1}{r} \Rightarrow$$

$$\Rightarrow\ \left(D_j n_j\right) = \frac{1}{r}$$

Eventually, Subterm D of P_x takes the form:

$$\left(D_j n_j\right)\mathrm{R}_x = \left(D_x n_x + D_y n_y\right)\mathrm{R}_x \Rightarrow$$

$$\Rightarrow \boxed{\left(D_j n_j\right)\mathrm{R}_x = \frac{\mathrm{R}_x}{r}}$$

Having estimated all Subterms of P_x, its full form can be acquired. First, the sum: Subterm B+ Subterm C will be calculated.
Then, P_x= -(Subterm B+ Subterm C) + Subterm D.

<u>CALCULATION OF SUM: SUBTERM B+ SUBTERM C</u>

Subterm B+ Subterm C=

$$= (+15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\cos(\theta/2)\begin{Bmatrix} 100\mathrm{A}\lambda\left(-1+\cos\theta\right)+ \\ +\mathrm{A}\mu\left(26+98\cos\theta\right)+ \\ +\mathrm{B}\mu\left(626-2\cos\theta\right) \end{Bmatrix}$$

<u>CALCULATION OF SUM: -(SUBTERM B+ SUBTERM C)+ SUBTERM D</u>

Now,
-(Subterm B of P_x+ Subterm C of P_x) + Subterm D of P_x=

$\Rightarrow$-(Subterm B of P_x+ Subterm C of P_x) + Subterm D of P_x=

$$=(-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\cos(\theta/2)\begin{Bmatrix} -100\mathrm{A}\lambda+78\mathrm{A}\mu+878\mathrm{B}\mu+ \\ +\left(300\mathrm{A}\lambda+294\mathrm{A}\mu-6\mathrm{B}\mu\right)\cos\theta \end{Bmatrix}$$

It should be reminded that P_x also contains the term $\left(n_x\tau_{xx}+n_y\tau_{yx}\right)$. The full form for P_x is:

$$\mathrm{P}_x = \left(n_x\tau_{xx}+n_y\tau_{yx}\right)+$$
$$+(-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\cos(\theta/2)\begin{Bmatrix} -100\mathrm{A}\lambda+78\mathrm{A}\mu+678\mathrm{B}\mu+ \\ +\left(300\mathrm{A}\lambda+294\mathrm{A}\mu-6\mathrm{B}\mu\right)\cos\theta \end{Bmatrix}$$

The product $\mathrm{P}_x\left(\dfrac{\partial u_x}{\partial x}\right)$ should now be estimated. The component $\left(n_x\tau_{xx}+n_y\tau_{yx}\right)$ of P_x will not be included in this estimation of the product (since this term, after being integrated, vanishes for r→0).

Calculation of the product: $P_x\left(\frac{\partial u_x}{\partial x}\right)$

$$P_x\left(\frac{\partial u_x}{\partial x}\right)=$$

$$(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+78A\mu+878B\mu+\\+(300A\lambda+294A\mu-6B\mu)\cos\theta\end{Bmatrix}\cdot$$

$$\cdot(-\frac{15}{4})r^{1/2}\cos(\theta/2)\left[(49A-251B)+25(A+5B)(2\cos\theta+1)\right]\Rightarrow$$

$$\Rightarrow P_x\left(\frac{\partial u_x}{\partial x}\right)=$$
$$=(+225/32)c\ r^{-1}\cos^2(\theta/2)\cdot$$
$$\cdot\begin{Bmatrix}\left(-7400A^2+12600AB\right)\lambda+\left(5772A^2+55144AB-110628B^2\right)\mu+\\+\left[\left(17200A^2-62800AB\right)\lambda+\left(25656A^2+25912AB+220256B^2\right)\mu\right]\cos\theta+\\+\left[15000\left(A^2+5B\right)\lambda+\left(14700A^2+73200AB\text{-}1500B^2\right)\mu\right]\cos^2\theta\end{Bmatrix}$$

CALCULATION OF P_y

$$P_y = n_p\tau_{py}-n_p\partial_r m_{rpy} - D_p\left(n_r m_{rpy}\right) + \left(D_j n_j\right) n_r n_p m_{rpy} =$$
$$= n_p\tau_{py}-n_p\partial_r m_{rpy} - D_p\left(n_r m_{rpy}\right) + \left(D_j n_j\right) R_y =$$
$$= n_x\tau_{xy}-n_x\partial_r m_{rxy} - D_x\left(n_r m_{rxy}\right) + \left(D_j n_j\right) R_y +$$
$$n_y\tau_{yy}-n_y\partial_r m_{ryy} - D_y\left(n_r m_{ryy}\right) =$$
$$=\left(n_x\tau_{xy}+n_y\tau_{yy}\right)-\left(n_x\partial_r m_{rxy}+n_y\partial_r m_{ryy}\right)-\left[D_x\left(n_r m_{rxy}\right)+D_y\left(n_r m_{ryy}\right)\right]+\left(D_j n_j\right) R_y =$$

$$=\left(n_x\tau_{xy}+n_y\tau_{yy}\right)-\left(n_x\partial_x m_{xxy}+n_y\partial_x m_{xyy}\right)-\left[D_x\left(n_x m_{xxy}\right)+D_y\left(n_x m_{xyy}\right)\right]+\left(D_j n_j\right) R_y -$$

$$-\left(n_x\partial_y m_{yxy}+n_y\partial_y m_{yyy}\right)-\left[D_x\left(n_y m_{yxy}\right)+D_y\left(n_y m_{yyy}\right)\right]\Rightarrow$$

$$\Rightarrow P_y=\left(n_x\tau_{xy}+n_y\tau_{yy}\right)-\begin{bmatrix}n_x\partial_x m_{xxy}+n_y\partial_x m_{xyy}+\\+n_x\partial_y m_{yxy}+n_y\partial_y m_{yyy}\end{bmatrix}-\begin{bmatrix}D_x\left(n_x m_{xxy}+n_y m_{yxy}\right)+\\+D_y\left(n_x m_{xyy}+n_y m_{yyy}\right)\end{bmatrix}+$$
$$+\left(D_j n_j\right)R_y$$

The process followed for the calculation of P_x will be performed for the calculation of P_y, too.

The term $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$ (Subterm A') is ~ $r^{1/2}$.

The product $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)\frac{\partial u_y}{\partial x}d\Gamma \sim r^{1/2}\,r^{1/2}\,r\,d\theta \sim r^2 \to 0$ for $r\to 0$.

Consequently, this Subterm does not contribute to the final result.

<u>CALCULATION OF SUBTERM B'</u>: $\begin{bmatrix} n_x\partial_x m_{xxy}+n_y\partial_x m_{xyy}+ \\ +n_x\partial_y m_{yxy}+n_y\partial_y m_{yyy} \end{bmatrix}$

$$n_x\partial_x m_{xxy}+n_y\partial_x m_{xyy}+n_x\partial_y m_{yxy}+n_y\partial_y m_{yyy}=$$

$$=\cos\theta\left[\cos\theta\frac{\partial}{\partial r}m_{xxy}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}m_{xxy}\right]+\sin\theta\left[\cos\theta\frac{\partial}{\partial r}m_{xyy}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}m_{xyy}\right]+$$

$$+\cos\theta\left[\sin\theta\frac{\partial}{\partial r}m_{yxy}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}m_{yxy}\right]+\sin\theta\left[\sin\theta\frac{\partial}{\partial r}m_{yyy}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}m_{yyy}\right]=$$

$$=\cos^2\theta\frac{\partial}{\partial r}m_{xxy}-\frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta}m_{xxy}+\sin\theta\cos\theta\frac{\partial}{\partial r}m_{xyy}-\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}m_{xyy}+$$

$$+\cos\theta\sin\theta\frac{\partial}{\partial r}m_{yxy}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta}m_{yxy}+\sin^2\theta\frac{\partial}{\partial r}m_{yyy}+\frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta}m_{yyy}\Rightarrow$$

$$\Rightarrow n_x\partial_x m_{xxy}+n_y\partial_x m_{xyy}+n_x\partial_y m_{yxy}+n_y\partial_y m_{yyy}=$$

$$=\cos^2\theta\frac{\partial}{\partial r}m_{xxy}+\sin\theta\cos\theta\left[\frac{\partial}{\partial r}m_{xyy}+\frac{\partial}{\partial r}m_{yxy}\right]-\frac{\cos\theta\sin\theta}{r}\left[\frac{\partial}{\partial\theta}m_{xxy}-\frac{\partial}{\partial\theta}m_{yyy}\right]-$$

$$-\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}m_{xyy}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta}m_{yxy}+\sin^2\theta\frac{\partial}{\partial r}m_{yyy}\Rightarrow$$

$$\Rightarrow \boxed{\text{Subterm B' of } P_y=(+15/8)c\mu r^{3/2}\sin(\theta/2)\left[200(A+5B)\right]}$$

CALCULATION OF SUBTERM C': $\begin{bmatrix} D_x\left(n_x m_{xxy} + n_y m_{yxy}\right) + \\ + D_y\left(n_x m_{xyy} + n_y m_{yyy}\right) \end{bmatrix}$

The decomposition of this Subterm in 2 parts gives: $D_x\left(n_x m_{xxy} + n_y m_{yxy}\right)$ and $D_y\left(n_x m_{xyy} + n_y m_{yyy}\right)$.

First part: $D_x\left(n_x m_{xxy} + n_y m_{yxy}\right)$

$$
\begin{aligned}
&D_x\left(n_x m_{xxy} + n_y m_{yxy}\right) = \\
&= \partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - n_x D\left(n_x m_{xxy} + n_y m_{yxy}\right) = \\
&= \partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - n_x\, n_r\, \partial_r\left(n_x m_{xxy} + n_y m_{yxy}\right) = \\
&= \partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - n_x\left[n_x\, \partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) + n_y\, \partial_y\left(n_x m_{xxy} + n_y m_{yxy}\right)\right] = \\
&= \partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - n_x^{\,2}\partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - n_x n_y \partial_y\left(n_x m_{xxy} + n_y m_{yxy}\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - n_x^{\,2}\right)\partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - n_x n_y \partial_y\left(n_x m_{xxy} + n_y m_{yxy}\right) = \\
&= \left(1 - \cos^2\theta\right)\partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - \cos\theta\, \sin\theta\, \partial_y\left(n_x m_{xxy} + n_y m_{yxy}\right) \Rightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow D_x\left(n_x m_{xxy} + n_y m_{yxy}\right) = \\
&= \sin^2\theta\, \partial_x\left(n_x m_{xxy} + n_y m_{yxy}\right) - \cos\theta\, \sin\theta\, \partial_y\left(n_x m_{xxy} + n_y m_{yxy}\right)
\end{aligned}
$$

Second part: $D_y\left(n_x m_{xyy} + n_y m_{yyy}\right)$

$$
\begin{aligned}
&D_y\left(n_x m_{xyy} + n_y m_{yyy}\right) = \\
&= \partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - n_y D\left(n_x m_{xyy} + n_y m_{yyy}\right) = \\
&= \partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - n_y n_r\, \partial_r\left(n_x m_{xyy} + n_y m_{yyy}\right) = \\
&= \partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - n_y\left[n_x\, \partial_x\left(n_x m_{xyy} + n_y m_{yyy}\right) + n_y\, \partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right)\right] = \\
&= \partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - n_y\, n_x\, \partial_x\left(n_x m_{xyy} + n_y m_{yyy}\right) - n_y^{\,2}\left(n_x m_{xyy} + n_y m_{yyy}\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= \left(1 - n_y^{\,2}\right)\partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - n_y\, n_x\, \partial_x\left(n_x m_{xyy} + n_y m_{yyy}\right) = \\
&= \left(1 - \sin^2\theta\right)\partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - \sin\theta\cos\theta\, \partial_x\left(n_x m_{xyy} + n_y m_{yyy}\right) \Rightarrow
\end{aligned}
$$

$$\Rightarrow D_y\left(n_x m_{xyy} + n_y m_{yyy}\right) =$$
$$= \cos^2\theta\ \partial_y\left(n_x m_{xyy} + n_y m_{yyy}\right) - \sin\theta\cos\theta\ \partial_x\left(n_x m_{xyy} + n_y m_{yyy}\right)$$

Having calculated the partial derivatives of sum1' and sum 2' with respect to x and y, the full form of Subterm C' of P_y can be reached.

Finally, Subterm C' of P_y in its most compact form is:

$$\text{Subterm C' of } P_y = (+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix} 100A\lambda\cos\theta + \\ +(-74+102\cos\theta)A\mu + \\ +(-374+2\cos\theta)B\mu \end{bmatrix}$$

It should also be reminded that P_y has a component $(n_x\tau_{xy} + n_y\tau_{yy})$. This component's order of r, after the integration, is greater than zero, which means that this term does not contribute to the final result for the J_1-Integral.

CALCULATION OF SUBTERM D: $(D_j n_j)R_y$

It has already been shown, while calculating the Subterm D of P_x that the operator $(D_j n_j)$ is:

$$(D_j n_j) = D_x n_x + D_y n_y =$$
$$= \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} = \frac{1}{r} \Rightarrow$$
$$\Rightarrow (D_j n_j) = \frac{1}{r}$$

In virtue of this, Subterm D' of P_y is:

$$\Rightarrow \text{Subterm D' } (D_j n_j)R_y = \frac{R_y}{r}$$

With all 4 Subterms of P_y estimated, the full form of P_y can be acquired. First, the Subterms B' and C' will be summed. Then, P_y=-(Subterm B'+Subterm C')+ Subterm D'.

CALCULATION OF SUM: SUBTERM B'+ SUBTERM C'

Subterm B'+ Subterm C'=

$$=(+15/8)c\,\mu\, r^{3/2}\sin(\theta/2)\left[200(A+5B)\right]+$$

$$+(+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+(-74+102\cos\theta)A\mu+\\+(-374+2\cos\theta)B\mu\end{bmatrix}\Rightarrow$$

$$\Rightarrow \boxed{\text{Subterm B'+ Subterm C'}=(+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+(126+102\cos\theta)A\mu+\\+(626+2\cos\theta)B\mu\end{bmatrix}}$$

CALCULATION OF SUM: -(SUBTERM B'+ SUBTERM C')+ SUBTERM D'

Subterm D' is: $\frac{R_y}{r}$, where R_y is:

$$R_y=(-15/2)c\,r^{-1/2}\sin(\theta/2)\left\{50A\lambda+63(A\mu+B\mu)+\left[50A\lambda+51A\mu+B\mu\right]\cos\theta\right\}$$

Then, -(SUBTERM B'+ SUBTERM C')+ SUBTERM D', which is:

$$(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+(126+102\cos\theta)A\mu+\\+(626+2\cos\theta)B\mu\end{bmatrix}+$$

$$+(-15/2)c\,r^{-3/2}\sin(\theta/2)\left\{50A\lambda+63(A\mu+B\mu)+\left[50A\lambda+51A\mu+B\mu\right]\cos\theta\right\}=$$

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{Bmatrix}4\cdot 50A\lambda+(126+4\cdot 63)A\mu+(626+4\cdot 63)B\mu+\\+\left[(100+4\cdot 50)A\lambda+(102+4\cdot 51)A\mu+(2+4)B\mu\right]\cos\theta\end{Bmatrix}=$$

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{Bmatrix}200A\lambda+378A\mu+878B\mu+\\+(300A\lambda+306A\mu+6B\mu)\cos\theta\end{Bmatrix}\Rightarrow$$

$\Rightarrow$-(Subterm B' of P_y + Subterm C' of P_y) + Subterm D' of P_y =

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{Bmatrix}200A\lambda+378A\mu+878B\mu+\\+(300A\lambda+306A\mu+6B\mu)\cos\theta\end{Bmatrix}$$

P_y includes the component $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$, so the final expression for P_y is:

$$P_y=\left(n_x\tau_{xy}+n_y\tau_{yy}\right)+(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{Bmatrix}200A\lambda+378A\mu+878B\mu+\\+\left(300A\lambda+306A\mu+6B\mu\right)\cos\theta\end{Bmatrix}$$

The component's $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$ order of r, after the integration, is greater than zero, which means that this term does not contribute to the final result for the J_1-Integral.

Omitting $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$, the remaining P_y is:

$$P_y=(-15/8)c\ r^{-3/2}\sin(\theta/2)\left[\left(200A\lambda+378A\mu+878B\mu\right)+\left(300A\lambda+306A\mu+6B\mu\right)\cos\theta\right]$$

P_y will be multiplied by $\frac{\partial u_y}{\partial x}$,

$$\frac{\partial u_y}{\partial x}=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)\left(2\cos\theta-1\right)\right]=$$
$$=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[(26A-374B)+50(A+5B)\cos\theta\right]$$

<u>Calculation of the product:</u> $P_y\left(\frac{\partial u_y}{\partial x}\right)$

This product is:

$$P_y\left(\frac{\partial u_y}{\partial x}\right)=$$
$$=(-15/8)c\ r^{-3/2}\sin(\theta/2)\left[\left(200A\lambda+378A\mu+878B\mu\right)+\left(300A\lambda+306A\mu+6B\mu\right)\cos\theta\right]\cdot$$
$$\cdot(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[\left(26A-374B\right)+50\left(A+5B\right)\cos\theta\right]\Rightarrow$$

$$\Rightarrow P_y\left(\frac{\partial u_y}{\partial x}\right)=$$

$$=(+225/32)c\ r^{-1}\sin^2(\theta/2)\left\{\begin{array}{l}\left[\begin{array}{l}\left(5200A^2-74800AB\right)\lambda+\\+\left(9828A^2-118544AB-328372B^2\right)\mu\end{array}\right]+\\+\left[\begin{array}{l}\left(17800A^2-62200AB\right)\lambda+\\+\left(26856A^2+24112AB+217256B^2\right)\mu\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}\left(15000A^2+75000AB\right)\lambda+\\+\left(15300A^2+76800AB\mu+1500B^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}$$

1.4 Integrations in J_1

The J_1-Integral can now be formulated, by integrating its parts in the interval [-π, π].

Integration of *W* terms in J_1

Term W1

This integral is zero, due to the order of r (vanishes at crack-tip).

Term W2

This integral is zero, too, for the same reason.

Term W3

Bearing in mind that: $dy=n_1\, d\Gamma=\cos\theta\, d\Gamma=\cos\theta\, r\, d\theta=r\cos\theta\, d\theta$, the value of the definite integral is: (in the interval [-π,π])

$$\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]dy=\int_{-\pi}^{\pi}\left(140625A^2\ r^{-1}\right)r\cos\theta\, d\theta=$$

$$=140625A^2\int_{-\pi}^{\pi}\cos\theta\ d\theta=140625A^2\left[\sin\theta\right]_{-\pi}^{\pi}=140625A^2\left[\sin\pi-\sin(-\pi)\right]=$$

$$=140625A^2\left[0-0\right]=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]dy=0}$$

Term W4

Term W4(a)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left((-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[50(A+5B)(\cos\theta-\cos(2\theta))+124(A+B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2 dy=\frac{225}{128}(149A+249B)^2\pi}$$

Term W4(b)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[-50(A+5B)(\cos\theta+\cos(2\theta))+(24A+124B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2 dy=-\frac{225}{128}(49A+249B)^2\pi}$$

Term W4(c)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(+\frac{15}{8}\right)r^{-1/2}\cos(\theta/2)\left[50(A+5B)(\cos\theta-\cos(2\theta))+(-76A+124B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2 dy=\frac{2025}{128}(17A-83B)^2\pi}$$

Term W4(d)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[-50(A+5B)(\cos\theta+\cos(2\theta))+(24A-376B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2 dy=-\frac{225}{128}(49A-251B)^2\pi}$$

Term W4(e)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)r^{-1/2}\cos(\theta/2)\left[50(A+5B)(-\cos\theta+\cos(2\theta))+(76A+376B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2 dy=\frac{225}{128}(51A+251B)^2\pi}$$

Term W4(f)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[-50(A+5B)(\cos\theta+\cos(2\theta))+(-176A-376B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 dy=-\frac{225}{128}(151A+251B)^2\pi}$$

By summing the results of the integrations of the W terms, we get:

$$\int_{-\pi}^{\pi} W dy=\int_{-\pi}^{\pi} W\ r\cos\theta d\theta=$$
$$=0+0+0+$$
$$+\frac{225}{128}(149A+249B)^2\pi+2\left(-\frac{225}{128}(49A+249B)^2\pi\right)+\frac{2025}{128}(17A-83B)^2\pi+$$
$$+\left(-\frac{225}{128}(49A-251B)^2\pi\right)+2\frac{225}{128}(51A+251B)^2\pi+\left(-\frac{225}{128}(151A+251B)^2\pi\right)\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} W \mathrm{dy}=0}$$

This means that W does not contribute to J_1. Furthermore, we notice that, when integrating the squares of the strains' gradient, we obtain 6 non-zero terms, whose sum, though, is zero!

Integration of R_q terms in J_1

$$\int_{-\pi}^{\pi} \mathrm{R}_x D\left(\frac{\partial u_x}{\partial x}\right)\mathrm{d}\Gamma = \int_{-\pi}^{\pi} \mathrm{R}_x D\left(\frac{\partial u_x}{\partial x}\right) \mathrm{r}\,\mathrm{d}\theta =$$

$$= \int_{-\pi}^{\pi} (+225/8)\mathrm{c}\mathrm{r}^{-1}\cos^2(\theta/2)\left\{\begin{array}{l} \left(481\mathrm{A}^2+1512\mathrm{AB}-3969\mathrm{B}^2\right)\mu+ \\ +\left[\begin{array}{l}\left(1850\mathrm{A}^2-3150\mathrm{AB}\right)\lambda+ \\ +\left(2138\mathrm{A}^2+76\mathrm{AB}+7938\mathrm{B}^2\right)\mu\end{array}\right]\cos\theta+ \\ +25\left[\begin{array}{l}50\left(\mathrm{A}^2+5\mathrm{AB}\right)\lambda+ \\ +\left(49\mathrm{A}^2+244\mathrm{AB}-5\mathrm{B}^2\right)\mu\end{array}\right]\cos^2\theta \end{array}\right\} \mathrm{r}\,\mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{R}_x D\left(\frac{\partial u_x}{\partial x}\right)\mathrm{d}\Gamma = \frac{5625}{16}\mathrm{c}\pi\left[124\left(\mathrm{A}^2+\mathrm{B}\right)\lambda+\left(173\mathrm{A}^2+368\mathrm{A}-5\mathrm{B}^2\right)\mu\right]}$$

and additionally,

$$\int_{-\pi}^{\pi} \mathrm{R}_y D\left(\frac{\partial u_y}{\partial x}\right)\mathrm{d}\Gamma = \int_{-\pi}^{\pi} \mathrm{R}_y D\left(\frac{\partial u_y}{\partial x}\right) \mathrm{r}\,\mathrm{d}\theta =$$

$$= \int_{-\pi}^{\pi} (+225/8)\mathrm{c}\,\mathrm{r}^{-1}\sin^2(\theta/2)\left\{\begin{array}{l} (650\mathrm{A}^2-9350\mathrm{AB})\lambda+ \\ +\left(819\mathrm{A}^2-10962\mathrm{AB}-11781\mathrm{B}^2\right)\mu+ \\ +\left[\begin{array}{l}\left(1900\mathrm{A}^2-3100\mathrm{AB}\right)\lambda+ \\ +\left(2238\mathrm{A}^2-74\mathrm{AB}+7688\mathrm{B}^2\right)\mu\end{array}\right]\cos\theta+ \\ +\left[\begin{array}{l}\left(1250\mathrm{A}^2+6250\mathrm{AB}\right)\lambda+ \\ +\left(1275\mathrm{A}^2+6400\mathrm{AB}+125\mathrm{B}^2\right)\mu\end{array}\right]\cos^2\theta \end{array}\right\} \mathrm{r}\,\mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \int_{-\pi}^{\pi} \mathrm{R}_y\, D\left(\frac{\partial u_y}{\partial x}\right) d\Gamma = -\frac{5625}{16} c\pi \left[1245\mathrm{B}^2\mu - \mathrm{A}^2(26\lambda + 27\mu) + \mathrm{AB}(374\lambda + 618\mu)\right] \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{R}_y\, D\left(\frac{\partial u_y}{\partial x}\right) d\Gamma = \frac{5625}{16} c\pi \left[\left(26\mathrm{A}^2 - 374\mathrm{AB}\right)\lambda + \left(27\mathrm{A}^2 - 618\mathrm{AB} - 1245\mathrm{B}^2\right)\mu\right]}$$

Integration of P_z terms in J_1

$$\int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) d\Gamma = \int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) \mathrm{r}\, d\theta =$$

$$= \int_{-\pi}^{\pi} (+225/32)\mathrm{c}\, \mathrm{r}^{-1} \cos^2(\theta/2) \left\{ \begin{aligned} & \left[\begin{aligned} & \left(-7400\mathrm{A}^2 + 12600\mathrm{AB}\right)\lambda + \\ & + \left(5772\mathrm{A}^2 + 55144\mathrm{AB} - 110628\mathrm{B}^2\right)\mu \end{aligned} \right] + \\ & + \left[\begin{aligned} & \left(17200\mathrm{A}^2 - 62800\mathrm{AB}\right)\lambda + \\ & + \left(25656\mathrm{A}^2 + 25912\mathrm{AB} + 220256\mathrm{B}^2\right)\mu \end{aligned} \right] \cos\theta + \\ & + \left[\begin{aligned} & 15000\left(\mathrm{A}^2 + 5\mathrm{B}\right)\lambda + \\ & + \left(14700\mathrm{A}^2 + 73200\mathrm{AB\text{-}}1500\mathrm{B}^2\right)\mu \end{aligned} \right] \cos^2\theta \end{aligned} \right\} \mathrm{r}\, d\theta \Rightarrow$$

$$\Rightarrow \int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) d\Gamma = \frac{5625}{16} c\pi \left[-25\mathrm{B}^2\mu + 3\mathrm{A}^2(58\lambda + 173\mu) + \mathrm{AB}(374\lambda + 2094\mu)\right] \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) d\Gamma = \frac{5625}{16} c\pi \left[\left(174\mathrm{A}^2 + 374\mathrm{AB}\right)\lambda + \left(519\mathrm{A}^2 + 2094\mathrm{AB} - 25\mathrm{B}^2\right)\mu\right]}$$

and also

$$\int_{-\pi}^{\pi} \mathrm{P}_y \left(\frac{\partial u_y}{\partial x}\right) d\Gamma = \int_{-\pi}^{\pi} \mathrm{P}_y \left(\frac{\partial u_y}{\partial x}\right) \mathrm{r}\, d\theta =$$

$$= \int_{-\pi}^{\pi} (+225/32)\mathrm{c}\, \mathrm{r}^{-1} \sin^2(\theta/2) \left\{ \begin{aligned} & \left[\begin{aligned} & \left(5200\mathrm{A}^2 - 74800\mathrm{AB}\right)\lambda + \\ & + \left(9828\mathrm{A}^2 - 118544\mathrm{AB} - 328372\mathrm{B}^2\right)\mu \end{aligned} \right] + \\ & + \left[\begin{aligned} & \left(17800\mathrm{A}^2 - 62200\mathrm{AB}\right)\lambda + \\ & + \left(26856\mathrm{A}^2 + 24112\mathrm{AB} + 217256\mathrm{B}^2\right)\mu \end{aligned} \right] \cos\theta + \\ & + \left[\begin{aligned} & \left(15000\mathrm{A}^2 + 75000\mathrm{AB}\right)\lambda + \\ & + \left(15300\mathrm{A}^2 + 76800\mathrm{AB}\mu + 1500\mathrm{B}^2\right)\mu \end{aligned} \right] \cos^2\theta \end{aligned} \right\} \mathrm{r}\, d\theta \Rightarrow$$

$$\Rightarrow \int_{-\pi}^{\pi} \mathrm{P}_y \left(\frac{\partial u_y}{\partial x} \right) d\Gamma = \frac{5625}{16} c\pi \left[-8725\mathrm{B}^2\mu + \mathrm{A}^2 \left(76\lambda + 81\mu\right) - 4\mathrm{AB}\left(31\lambda + 461\mu\right) \right] \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{P}_y \left(\frac{\partial u_y}{\partial x} \right) d\Gamma = \frac{5625}{16} c\pi \left[\left(76\mathrm{A}^2 - 124\mathrm{AB}\right)\lambda + \left(81\mathrm{A}^2 - 1844\mathrm{AB} - 8725\mathrm{B}^2\right)\mu \right]}$$

The total sum for J_1 is:

$$J_1 = \int_{\Gamma} \left[\mathrm{Wdy} - \mathrm{P}_q \frac{\partial u_q}{\partial x} d\Gamma - \mathrm{R}_z \mathrm{D} \left(\frac{\partial u_z}{\partial x} \right) d\Gamma \right] =$$

$$= \int_{\Gamma} \mathrm{Wdy} - \int_{\Gamma} \left[\mathrm{P}_x \frac{\partial u_x}{\partial x} d\Gamma + \mathrm{P}_y \frac{\partial u_y}{\partial x} \right] - \int_{\Gamma} \left[\mathrm{R}_x \mathrm{D} \left(\frac{\partial u_x}{\partial x} \right) d\Gamma + \mathrm{R}_y \mathrm{D} \left(\frac{\partial u_y}{\partial x} \right) d\Gamma \right] =$$

$$= 0 - \left\{ \begin{aligned} & \frac{5625}{16} c\pi \left[\left(174\mathrm{A}^2 + 374\mathrm{AB}\right)\lambda + \left(519\mathrm{A}^2 + 2094\mathrm{AB} - 25\mathrm{B}^2\right)\mu \right] + \\ & + \frac{5625}{16} c\pi \left[\left(76\mathrm{A}^2 - 124\mathrm{AB}\right)\lambda + \left(81\mathrm{A}^2 - 1844\mathrm{AB} - 8725\mathrm{B}^2\right)\mu \right] \end{aligned} \right\} -$$

$$- \left\{ \begin{aligned} & \frac{5625}{16} c\pi \left[124\left(\mathrm{A}^2 + \mathrm{B}\right)\lambda + \left(173\mathrm{A}^2 + 368\mathrm{A} - 5\mathrm{B}^2\right)\mu \right] + \\ & + \frac{5625}{16} c\pi \left[\left(26\mathrm{A}^2 - 374\mathrm{AB}\right)\lambda + \left(27\mathrm{A}^2 - 618\mathrm{AB} - 1245\mathrm{B}^2\right)\mu \right] \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow J_1 = -140625 c\pi \left[-25\mathrm{B}^2\mu + \mathrm{A}^2 \left(\lambda + 2\mu\right) \right] =$$

$$= -140625 c\pi \left[\mathrm{A}^2\lambda + \left(2\mathrm{A}^2 - 25\mathrm{B}^2\right)\mu \right] \Rightarrow$$

$$\Rightarrow \boxed{J_1 = -140625 c\pi\mu \left[\mathrm{A}^2 \frac{\lambda}{\mu} + \left(2\mathrm{A}^2 - 25\mathrm{B}^2\right) \right]}$$

CHAPTER 2

THE J_2-INTEGRAL

Presentation of the J_2-Integral

Another path independent integral is J_2. J_2 is defined in this way:

$$J_2 = \int_\Gamma \left[\mathrm{Wn_2} d\Gamma - \mathrm{P_q} \frac{\partial u_q}{\partial y} d\Gamma - \mathrm{R_z D} \left(\frac{\partial u_z}{\partial y} \right) d\Gamma \right] \Rightarrow$$

$$\Rightarrow \boxed{J_2 = \int_\Gamma \left[\mathrm{W\sin}\theta d\Gamma - \mathrm{P_q} \frac{\partial u_q}{\partial y} d\Gamma - \mathrm{R_z D} \left(\frac{\partial u_z}{\partial y} \right) d\Gamma \right]}$$

Physically, this indicates the propensity of the crack-tip to shift parallel to the crack. J_2 is also a path independent integral [Georgiadis H.G., Grentzelou C.G., (2006)].

2.1 Terms W in J_2

<u>Calculation of terms</u> W

W is the same for J_1 and J_2. The only difference is that W is multiplied by cosθ in J_1 and by sinθ in J_2.

2.2 Terms R_q in J_2

<u>Calculation of terms</u> $\mathrm{R}_q D\left(\frac{\partial u_q}{\partial y}\right)$

$\mathrm{R}_q D\left(\frac{\partial u_q}{\partial y}\right)$ is literally:

$$\mathrm{R}_q D\left(\frac{\partial u_q}{\partial y}\right) = \mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right) + \mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right)$$

The terms $\mathrm{R}_x, \mathrm{R}_y$ are identical for J_1 and J_2.

Additionally,

$$D\left(\frac{\partial u_x}{\partial y}\right)=\mathrm{n}_r\,\partial_r\left(\frac{\partial u_x}{\partial y}\right)=\mathrm{n}_x\,\partial_x\left(\frac{\partial u_x}{\partial y}\right)+\mathrm{n}_y\,\partial_y\left(\frac{\partial u_x}{\partial y}\right)=$$
$$=\mathrm{n}_x\frac{\partial^2 u_x}{\partial x\partial y}+\mathrm{n}_y\frac{\partial^2 u_x}{\partial y^2}=\cos\theta\frac{\partial^2 u_x}{\partial x\partial y}+\sin\theta\frac{\partial^2 u_x}{\partial y^2}$$

and

$$D\left(\frac{\partial u_y}{\partial y}\right)=\mathrm{n}_r\,\partial_r\left(\frac{\partial u_y}{\partial y}\right)=\mathrm{n}_x\,\partial_x\left(\frac{\partial u_y}{\partial y}\right)+\mathrm{n}_y\,\partial_y\left(\frac{\partial u_y}{\partial y}\right)=$$
$$=\mathrm{n}_x\frac{\partial^2 u_y}{\partial x\partial y}+\mathrm{n}_y\frac{\partial^2 u_y}{\partial y^2}=\cos\theta\frac{\partial^2 u_y}{\partial x\partial y}+\sin\theta\frac{\partial^2 u_y}{\partial y^2}$$

These two expressions can be estimated only after calculating the partial derivatives $\frac{\partial^2 u_x}{\partial x\partial y}$, $\frac{\partial^2 u_x}{\partial y^2}$ (for $D\left(\frac{\partial u_x}{\partial y}\right)$) and $\frac{\partial^2 u_y}{\partial x\partial y}$, $\frac{\partial^2 u_y}{\partial y^2}$ (for $D\left(\frac{\partial u_y}{\partial y}\right)$)

<u>Calculation of the partial derivatives of the displacements</u>

1. $\frac{\partial^2 u_x}{\partial x\partial y}$

The derivative $\frac{\partial^2 u_x}{\partial x\partial y}$ has already been found to be:

$$\boxed{\frac{\partial^2 u_x}{\partial x\partial y}=(-\frac{15}{8})\mathrm{r}^{-1/2}\sin(\theta/2)\left\{(49\mathrm{A}\text{-}251\mathrm{B})-25(\mathrm{A}+5\mathrm{B})\left[1+2\cos\theta+2\cos(2\theta)\right]\right\}}$$

2. $\frac{\partial u_x}{\partial y}$

$$\boxed{\frac{\partial u_x}{\partial y}=(-\frac{15}{4})\mathrm{r}^{1/2}\sin(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]}$$

3. $\frac{\partial^2 u_x}{\partial y^2}$

$$\boxed{\frac{\partial^2 u_x}{\partial y^2}=(+\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\left[(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(-5+2\cos\theta-2\cos(2\theta))\right]}$$

4. $\dfrac{\partial^2 u_y}{\partial x \partial y}$

This partial derivative has already been found to be:

$$\boxed{\frac{\partial^2 u_y}{\partial x \partial y} = \left(-\frac{15}{8}\right) r^{-1/2} \cos(\theta/2) \left\{3(17A\text{-}83B)+25(A+5B)\left[1-2\cos\theta+2\cos(2\theta)\right]\right\}}$$

5. $\dfrac{\partial u_y}{\partial y}$

$$\boxed{\frac{\partial u_y}{\partial y} = (-\frac{15}{4})\ r^{1/2}\cos(\theta/2)\left[3\left(17A\text{-}83B\right)+25\left(A+5B\right)\left(3\text{-}2\cos\theta\right)\right]}$$

6. $\dfrac{\partial^2 u_y}{\partial y^2}$

$$\boxed{\frac{\partial^2 u_y}{\partial y^2} = (-\frac{15}{8})\ r^{-1/2}\sin(\theta/2)\left[3\left(17A\text{-}83B\right)+25\left(A+5B\right)\left(5+2\cos\theta+2\cos(2\theta)\right)\right]}$$

Calculation of $D\left(\dfrac{\partial u_x}{\partial y}\right) = n_x \dfrac{\partial^2 u_x}{\partial x \partial y} + n_y \dfrac{\partial^2 u_x}{\partial y^2} = \cos\theta \dfrac{\partial^2 u_x}{\partial x \partial y} + \sin\theta \dfrac{\partial^2 u_x}{\partial y^2}$

$$D\left(\frac{\partial u_x}{\partial y}\right) = n_x \frac{\partial^2 u_x}{\partial x \partial y} + n_y \frac{\partial^2 u_x}{\partial y^2} = \cos\theta \frac{\partial^2 u_x}{\partial x \partial y} + \sin\theta \frac{\partial^2 u_x}{\partial y^2} \Rightarrow$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_x}{\partial y}\right) = (-\frac{15}{4}) r^{-1/2} \sin(\theta/2)\left[\left(13A+313B\right)+25\left(A+5B\right)\cos\theta\right]}$$

Calculation of $D\left(\dfrac{\partial u_y}{\partial y}\right) = n_x \dfrac{\partial^2 u_y}{\partial x \partial y} + n_y \dfrac{\partial^2 u_y}{\partial y^2} = \cos\theta \dfrac{\partial^2 u_y}{\partial x \partial y} + \sin\theta \dfrac{\partial^2 u_y}{\partial y^2}$

$$D\left(\frac{\partial u_y}{\partial y}\right)=\mathrm{n}_x\frac{\partial^2 u_y}{\partial x\partial y}+\mathrm{n}_y\frac{\partial^2 u_y}{\partial y^2}=\cos\theta\frac{\partial^2 u_y}{\partial x\partial y}+\sin\theta\frac{\partial^2 u_y}{\partial y^2}\Rightarrow$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_y}{\partial y}\right)=(-\frac{15}{4})\mathrm{r}^{-1/2}\cos(\theta/2)\left[(63\mathrm{A}+63\mathrm{B})-25(\mathrm{A}+5\mathrm{B})\cos\theta\right]}$$

So far, all those terms which contribute to:

$$\mathrm{R}_q D\left(\frac{\partial u_q}{\partial y}\right)=\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)+\mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right)$$

have been obtained. Following these, the products:

$\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)$ and

$\mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right)$ will be acquired.

Calculation of the product: $\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)$

$$\boxed{\begin{aligned}&\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)=\\&=(+\frac{225}{8})\mathrm{c}\mathrm{r}^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{aligned}&\left(169\mathrm{A}^2+4888\mathrm{AB}+19719\mathrm{B}^2\right)\mu+\\&+\left[\begin{aligned}&\left(650\mathrm{A}^2+15650\mathrm{AB}\right)\lambda+\\&+\left(9624\mathrm{A}^2+18524\mathrm{AB}+7562\mathrm{B}^2\right)\mu\end{aligned}\right]\cos\theta+\\&+\left[\begin{aligned}&\left(1250\mathrm{A}^2+6250\mathrm{AB}\right)\lambda+\\&+\left(1225\mathrm{A}^2+6100\mathrm{AB}-125\mathrm{B}^2\right)\mu\end{aligned}\right]\cos^2\theta\end{aligned}\right\}\end{aligned}}$$

Calculation of the product: $\mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right)$

$$
\mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right) =
$$

$$
= (+\frac{225}{8}) c\, r^{-1} \sin(\theta/2)\cos(\theta/2) \left\{ \begin{array}{l} 3150\left(\mathrm{A}^2+\mathrm{AB}\right)\lambda + 3969\left(\mathrm{A}^2+2\mathrm{AB}+\mathrm{B}\right)^2 \mu + \\ + \left[\begin{array}{l} \left(1900\mathrm{A}^2 - 3100\mathrm{AB}\right)\lambda + \\ + \left(1638\mathrm{A}^2 - 6174\mathrm{AB}\text{-}7812\mathrm{B}^2\right)\mu \end{array}\right]\cos\theta - \\ -25\left[50\left(\mathrm{A}^2+5\mathrm{AB}\right)\lambda + \left(51\mathrm{A}^2 + 256\mathrm{AB}\mu + 5\mathrm{B}^2\right)\mu\right]\cos^2\theta \end{array} \right\}
$$

2.3 Terms P_z in J_2

Calculation of terms $\mathrm{P}_z\left(\frac{\partial u_z}{\partial y}\right)$

$\mathrm{P}_z\left(\frac{\partial u_z}{\partial y}\right)$ can expanded in this manner:

$$
\mathrm{P}_z\left(\frac{\partial u_z}{\partial y}\right) = \mathrm{P}_x\left(\frac{\partial u_x}{\partial y}\right) + \mathrm{P}_y\left(\frac{\partial u_y}{\partial y}\right)
$$

P_x and P_y are identical for J_1 and J_2.

The components of P_x, P_y that vanish at the crack-tip (r→0), after being integrated, will not be included in the products $\mathrm{P}_x\left(\frac{\partial u_x}{\partial y}\right)$ and $\mathrm{P}_y\left(\frac{\partial u_y}{\partial y}\right)$.

Calculation of the product: $\mathrm{P}_x\left(\frac{\partial u_x}{\partial y}\right)$

It has been calculated that:

$$
\frac{\partial u_x}{\partial y} = (-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[-\left(49\mathrm{A}\text{-}251\mathrm{B}\right) + 25\left(\mathrm{A}+5\mathrm{B}\right)\left(3+2\cos\theta\right)\right]
$$

or by performing some manipulations, one obtains this equivalent form:

$$
\boxed{\frac{\partial u_x}{\partial y} = (-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[26\mathrm{A}+626\mathrm{B} + \left(50\mathrm{A}+250\mathrm{B}\right)\cos\theta\right]}
$$

Forming the product $\mathrm{P}_x\left(\frac{\partial u_x}{\partial y}\right)$, we acquire:

$$P_x\left(\frac{\partial u_x}{\partial y}\right)=$$

$$=(+\frac{225}{32})c\,r^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{array}{l}\left(-2600A^2-62600AB\right)\lambda+\\+\left(2028A^2+71656AB+549628B^2\right)\mu+\\+\left[\begin{array}{l}\left(2800A^2+162800AB\right)\lambda+\\+\left(11544A^2+247288AB+215744B^2\right)\mu\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}15000\left(A^2+5AB\right)\lambda+\\+\left(14700A^2+73200AB-1500B^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}$$

<u>**Calculation of the product:**</u> $P_y\left(\frac{\partial u_y}{\partial y}\right)$

The partial derivative $\frac{\partial u_y}{\partial y}$ is:

$$\frac{\partial u_y}{\partial y}=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[3\left(17A\text{-}83B\right)+25\left(A+5B\right)\left(3\text{-}2\cos\theta\right)\right]$$

or, equivalently,

$$\boxed{\frac{\partial u_y}{\partial y}=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[126\left(A+B\right)-50\left(A+5B\right)\cos\theta\right]}$$

The product $P_y\left(\frac{\partial u_y}{\partial y}\right)$ finally becomes:

$$P_y\left(\frac{\partial u_y}{\partial y}\right)=$$

$$=(+\frac{225}{32})c\,r^{-1}\sin(\theta/2)\cos(\theta/2)\left\{\begin{array}{l}\left[\begin{array}{l}25200\left(A^2+AB\right)\lambda+\\+\left(47628A^2+158256AB+110628B^2\right)\mu\end{array}\right]+\\+\left[\begin{array}{l}\left(27800A^2-12200AB\right)\lambda+\\+\left(19656A^2-99088AB-218744B^2\right)\mu\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}-15000\left(A^2+5AB\right)\lambda-\\-\left(15300A^2+76800AB+1500B^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}\Rightarrow$$

2.4 Integrations in J_2

J_2 can now be calculated, by integrating its parts in the interval [-π, π].

Integration of *W* terms in J_2

The terms consisting *W* are multiplied by sinθ.

Term W1

This integral is zero, due to the order of r (vanishes at crack-tip).

Term W2

This integral is zero, too, for the same reason.

Term W3

$$\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]\sin\theta d\Gamma=\int_{-\pi}^{\pi}\left(140625A^2\ r^{-1}\right)r\sin\theta\, d\theta=$$

$$=140625A^2\int_{-\pi}^{\pi}\sin\theta\ d\theta=-140625A^2\left[\cos\theta\right]_{-\pi}^{\pi}=-140625A^2\left[\cos\pi\text{-}\cos(\text{-}\pi)\right]=$$

$$=140625A^2\left[\cos\pi\text{-}\cos\pi\right]=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]\sin\theta d\Gamma=0}$$

Term W4

Term W4(a)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2 r\sin\theta\, d\theta=$$

$$=\int_{-\pi}^{\pi}\left((\text{-}\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[50\left(A+5B\right)\left(\cos\theta-\cos(2\theta)\right)+124\left(A+B\right)\right]\right)^2 r\sin\theta d\theta=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2 r\sin\theta d\theta=0}$$

Term W4(b)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(24\mathrm{A}+124\mathrm{B})\right]\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0}$$

Term W4(c)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(+\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left[50(\mathrm{A}+5\mathrm{B})(\cos\theta-\cos(2\theta))+(-76\mathrm{A}+124\mathrm{B})\right]\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0}$$

Term W4(d)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(24\mathrm{A}-376\mathrm{B})\right]\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0}$$

Term W4(e)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left[50(\mathrm{A}+5\mathrm{B})(-\cos\theta+\cos(2\theta))+(76\mathrm{A}+376\,\mathrm{B})\right]\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0}$$

Term W4(f)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(-176\mathrm{A}-376\mathrm{B})\right]\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 \mathrm{r}\,\sin\theta \mathrm{d}\theta=0}$$

We observe that *W* consists of zero terms.

Integration of R_q terms in J_2

$$\int_{-\pi}^{\pi}\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)\mathrm{d}\Gamma=\int_{-\pi}^{\pi}\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)\mathrm{r}\,\mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}(+\frac{225}{8})\mathrm{c}\mathrm{r}^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{array}{l}\left(169\mathrm{A}^2+4888\mathrm{AB}+19719\mathrm{B}^2\right)\mu+\\ +\left[\begin{array}{l}\left(650\mathrm{A}^2+15650\mathrm{AB}\right)\lambda+\\ +\left(9624\mathrm{A}^2+18524\mathrm{AB}+7562\mathrm{B}^2\right)\mu\end{array}\right]\cos\theta+\\ +\left[\begin{array}{l}\left(1250\mathrm{A}^2+6250\mathrm{AB}\right)\lambda+\\ +\left(1225\mathrm{A}^2+6100\mathrm{AB}-125\mathrm{B}^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}\mathrm{r}\,\mathrm{d}\theta=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\mathrm{R}_x D\left(\frac{\partial u_x}{\partial y}\right)\mathrm{d}\Gamma=0}$$

and

$$\int_{-\pi}^{\pi}\mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right)\mathrm{d}\Gamma=\int_{-\pi}^{\pi}\mathrm{R}_y D\left(\frac{\partial u_y}{\partial y}\right)\mathrm{r}\,\mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}(+\frac{225}{8})\mathrm{c}\,\mathrm{r}^{-1}\sin(\theta/2)\cos(\theta/2)\left\{\begin{array}{l}3150\left(\mathrm{A}^2+\mathrm{AB}\right)\lambda+3969\left(\mathrm{A}^2+2\mathrm{AB}+\mathrm{B}\right)^2\mu+\\ +\left[\begin{array}{l}\left(1900\mathrm{A}^2-3100\mathrm{AB}\right)\lambda+\\ +\left(1638\mathrm{A}^2-6174\mathrm{AB}-7812\mathrm{B}^2\right)\mu\end{array}\right]\cos\theta-\\ -25\left[50\left(\mathrm{A}^2+5\mathrm{AB}\right)\lambda+\left(51\mathrm{A}^2+256\mathrm{AB}\mu+5\mathrm{B}^2\right)\mu\right]\cos^2\theta\end{array}\right\}\mathrm{r}\,\mathrm{d}\theta=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} R_y\, D\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = 0}$$

Integration of P_z terms in J_2

$$\int_{-\pi}^{\pi} P_x\left(\frac{\partial u_x}{\partial y}\right) d\Gamma = \int_{-\pi}^{\pi} P_x\left(\frac{\partial u_x}{\partial y}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} \left(+\frac{225}{32}\right) c\, r^{-1} \cos(\theta/2)\sin(\theta/2) \left\{ \begin{array}{l} \left(-2600A^2 - 62600AB\right)\lambda + \\ +\left(2028A^2 + 71656AB + 549628B^2\right)\mu + \\ + \left[\begin{array}{l} \left(2800A^2 + 162800AB\right)\lambda + \\ +\left(11544A^2 + 247288AB + 215744B^2\right)\mu \end{array} \right] \cos\theta + \\ + \left[\begin{array}{l} 15000\left(A^2 + 5AB\right)\lambda + \\ +\left(14700A^2 + 73200AB - 1500B^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} r\, d\theta = 0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} P_x\left(\frac{\partial u_x}{\partial y}\right) d\Gamma = 0}$$

and finally,

$$\int_{-\pi}^{\pi} P_y\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = \int_{-\pi}^{\pi} P_y\left(\frac{\partial u_y}{\partial y}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} \left(+\frac{225}{32}\right) c\, r^{-1} \sin(\theta/2)\cos(\theta/2) \left\{ \begin{array}{l} \left[\begin{array}{l} 25200\left(A^2 + AB\right)\lambda + \\ +\left(47628A^2 + 158256AB + 110628B^2\right)\mu \end{array} \right] + \\ + \left[\begin{array}{l} \left(27800A^2 - 12200AB\right)\lambda + \\ +\left(19656A^2 - 99088AB - 218744B^2\right)\mu \end{array} \right] \cos\theta + \\ + \left[\begin{array}{l} -15000\left(A^2 + 5AB\right)\lambda - \\ -\left(15300A^2 + 76800AB + 1500B^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} r\, d\theta = 0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} P_y\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = 0}$$

Noticing these results, we see that all terms in J_2 are zero! This means that there is no energy dissipation in the direction which is perpendicular to the direction of the crack advancement!

$$\boxed{J_2 = 0}$$

CHAPTER 3

THE L-INTEGRAL

In general, there are 3 components of the L-Integral.
The L-Integral is defined this way:

$$\mathrm{L_k} = \int_{\Gamma} e_{kij}\left[n_i W x_j - P_n u_{n,i} x_j + P_i u_j - R_n D\left(u_{n,i}\right) x_j + R_i D\left(u_j\right) - R_n D_i\left(u_n\right) n_j\right] d\Gamma,$$

where k=1,2,3, e_{kij} is the alternating (Levi - Civita) tensor.

The L-Integral indicates the propensity of the in-plane turn of the crack-tip as it propagates. It is also path independent.
For mode I, k=3 only.
Therefore,

$$\begin{aligned}
\mathrm{L_3} &= \int_{\Gamma} e_{3ij}\left[n_i W x_j - P_n u_{n,i} x_j + P_i u_j - R_n D\left(u_{n,i}\right) x_j + R_i D\left(u_j\right) - R_n D_i\left(u_n\right) n_j\right] d\Gamma = \\
&= \int_{\Gamma} e_{31j}\left[n_1 W x_j - P_n u_{n,1} x_j + P_1 u_j - R_n D\left(u_{n,1}\right) x_j + R_1 D\left(u_j\right) - R_n D_1\left(u_n\right) n_j\right] d\Gamma + \\
&+ \int_{\Gamma} e_{32j}\left[n_2 W x_j - P_n u_{n,2} x_j + P_2 u_j - R_n D\left(u_{n,2}\right) x_j + R_2 D\left(u_j\right) - R_n D_2\left(u_n\right) n_j\right] d\Gamma =
\end{aligned}$$

$$\begin{aligned}
&= \int_{\Gamma} \underbrace{e_{311}}_{0}\left[n_1 W x_1 - P_n u_{n,1} x_1 + P_1 u_1 - R_n D\left(u_{n,1}\right) x_1 + R_1 D\left(u_1\right) - R_n D_1\left(u_n\right) n_1\right] d\Gamma + \\
&+ \int_{\Gamma} e_{321}\left[n_2 W x_1 - P_n u_{n,2} x_1 + P_2 u_1 - R_n D\left(u_{n,2}\right) x_1 + R_2 D\left(u_1\right) - R_n D_2\left(u_n\right) n_1\right] d\Gamma + \\
&+ \int_{\Gamma} e_{312}\left[n_1 W x_2 - P_n u_{n,1} x_2 + P_1 u_2 - R_n D\left(u_{n,1}\right) x_2 + R_1 D\left(u_2\right) - R_n D_1\left(u_n\right) n_2\right] d\Gamma + \\
&+ \int_{\Gamma} \underbrace{e_{322}}_{0}\left[n_2 W x_2 - P_n u_{n,2} x_2 + P_2 u_2 - R_n D\left(u_{n,2}\right) x_2 + R_2 D\left(u_2\right) - R_n D_2\left(u_n\right) n_2\right] d\Gamma =
\end{aligned}$$

$$= \int_{\Gamma} e_{312}\left[\begin{array}{l} W\left(n_1 x_2 - n_2 x_1\right) - P_n\left(u_{n,1} x_2 - u_{n,2} x_1\right) + \left(P_1 u_2 - P_2 u_1\right) - \\ -R_n\left[D\left(u_{n,1}\right) x_2 - D\left(u_{n,2}\right) x_1\right] + R_1 D\left(u_2\right) - R_2 D\left(u_1\right) - \\ -R_n\left[D_1\left(u_n\right) n_2 - D_2\left(u_n\right) n_1\right] \end{array}\right] d\Gamma =$$

$$= \int_{\Gamma} e_{312} \begin{bmatrix} W(n_1 x_2 - n_2 x_1) - \left[P_1 (u_{1,1} x_2 - u_{1,2} x_1) + P_2 (u_{2,1} x_2 - u_{2,2} x_1) \right] + \\ +(P_1 u_2 - P_2 u_1) - \\ -\left[R_1 \left[D(u_{1,1}) x_2 - D(u_{1,2}) x_1 \right] + R_2 \left[D(u_{2,1}) x_2 - D(u_{2,2}) x_1 \right] \right] + \\ +\left[R_1 D(u_2) - R_2 D(u_1) \right] - \\ -\left[R_1 \left[D_1(u_1) n_2 - D_2(u_1) n_1 \right] + R_2 \left[D_1(u_2) n_2 - D_2(u_1) n_1 \right] \right] \end{bmatrix} d\Gamma$$

and

$$(n_1 x_2 - n_2 x_1) = \cos\theta \, r \sin\theta - \sin\theta \, r \cos\theta = 0!,$$
$$\left[P_1 (u_{1,1} x_2 - u_{1,2} x_1) + P_2 (u_{2,1} x_2 - u_{2,2} x_1) \right] \sim r^3 + r^1 \to 0!,$$
$$(P_1 u_2 - P_2 u_1) \sim r^3 + r^1 \to 0!,$$
$$\left[R_1 \left[D(u_{1,1}) x_2 - D(u_{1,2}) x_1 \right] + R_2 \left[D(u_{2,1}) x_2 - D(u_{2,2}) x_1 \right] \right] \sim r^1 \to 0!,$$
$$\left[R_1 D(u_2) - R_2 D(u_1) \right] \sim r^1 \to 0!,$$
$$\left[R_1 \left[D_1(u_1) n_2 - D_2(u_1) n_1 \right] + R_2 \left[D_1(u_2) n_2 - D_2(u_1) n_1 \right] \right] \sim r^1 \to 0!$$

Consequently, (in the limit **r→0**)

$$\boxed{L_3 = 0}$$

The physical meaning of this is that the mode I crack in strain-gradient elastic materials propagates on a straight line.

CHAPTER 4

QUALITATIVE EXPERIMENTS WITH TEXTILE MATERIALS

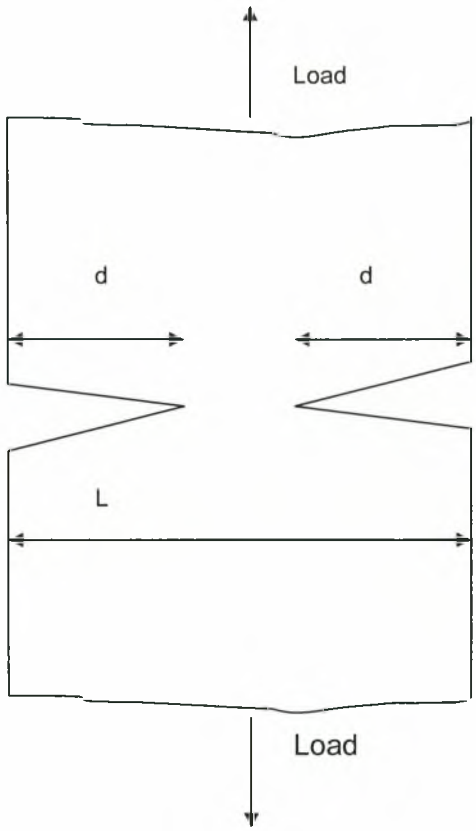
Qualitative experiments have been performed so as to make evident if the new predictions for the mechanical behaviour for materials with microstructure are valid. The textile material which has been used is a common textile, easy to find in the commerce, frequently used in military and other applications, used as cover against weather conditions etc. and is of polyesteric composition. Despite our efforts to collect more technical specifications about it, little information was obtained.

The material consisted of a hexagonal braided fiber textile net, wetted by a polymer marix.

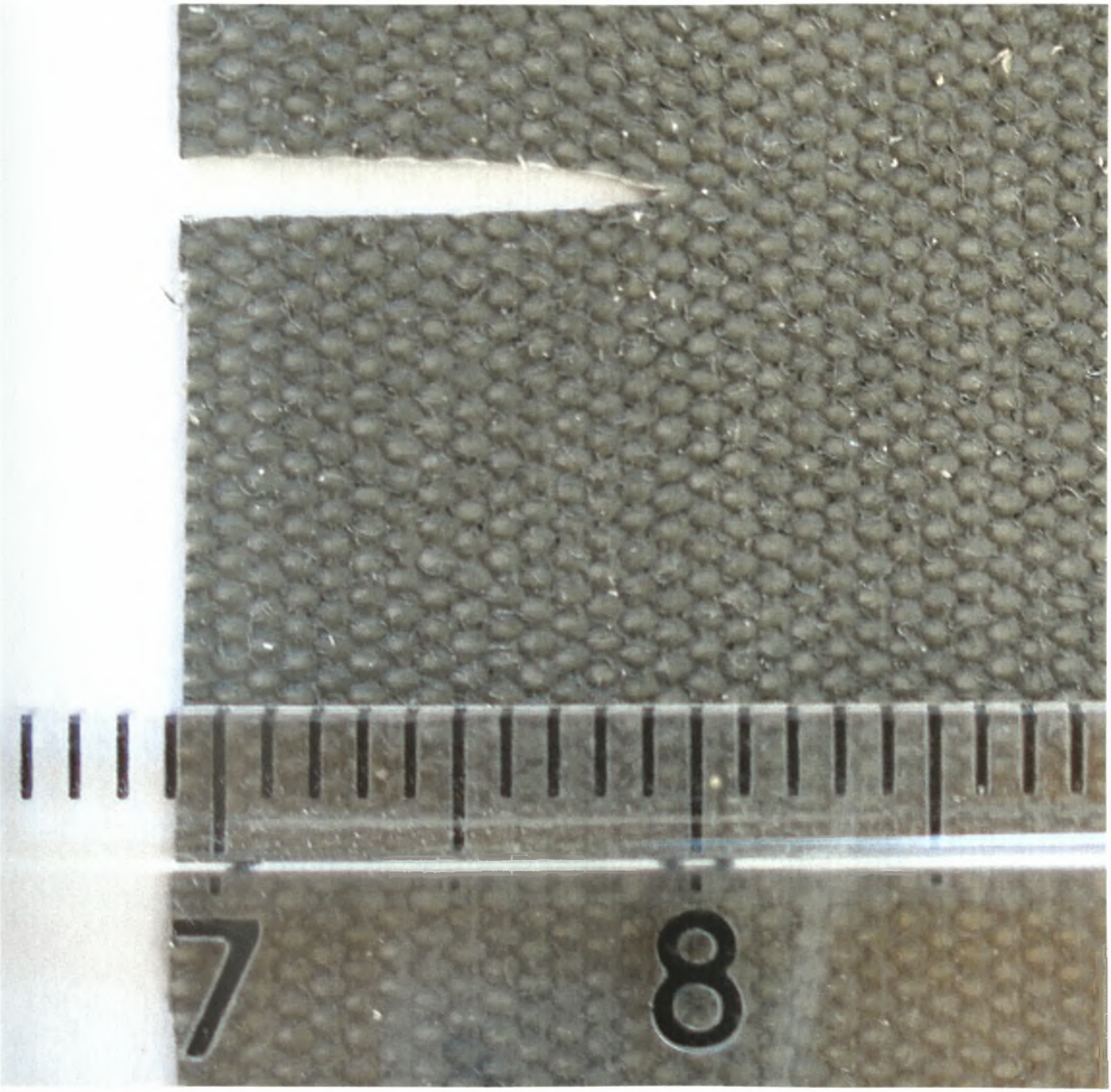
The experiments have not been performed only once, but at least 3 to 4 times each. Three types of cracked strip configurations were selected, as shown in the following figures. These are typical cracked configurations that are often encountered in the fracture analysis.

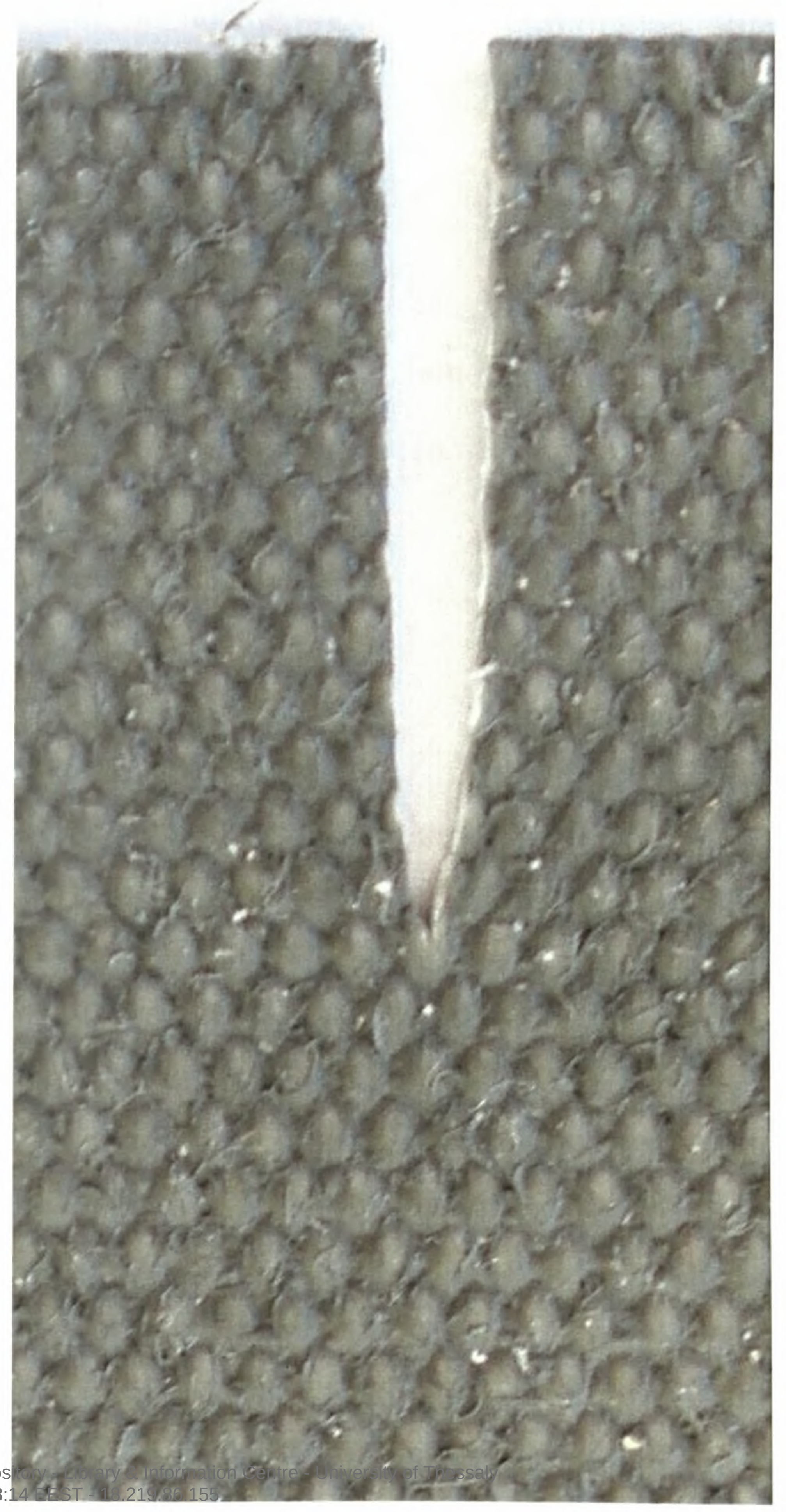
On the following pages, cracks have been created and each specimen was subjected to simple tensile load. For the specific material, $c^{1/2} \approx$ 1mm (internal microstructural length). The specimens had a uniform thickness of 0.5mm and were confined into transparent plexiglass plates to enforce plain strain conditions.

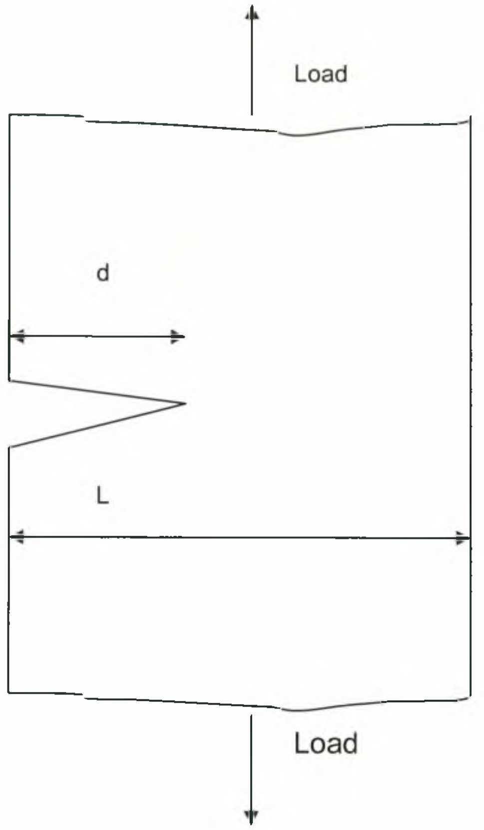
Note that the experiments indicate a <u>cusp-type</u> opening displacement near the crack-tip, as predicted by the asymptotic theory (see Introduction for the related shape of the opened crack).



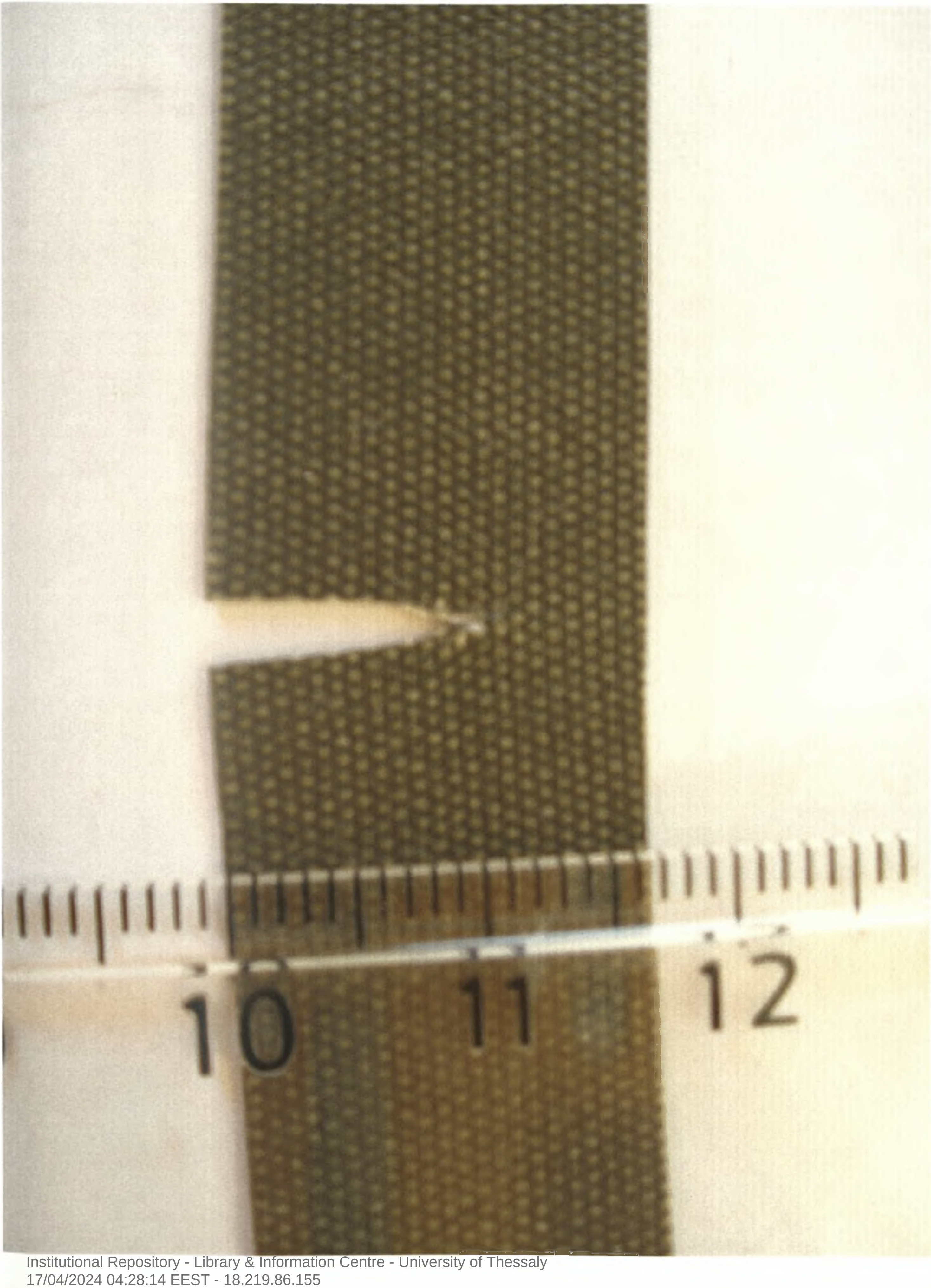
d ≈ 10mm
L ≈ 30mm

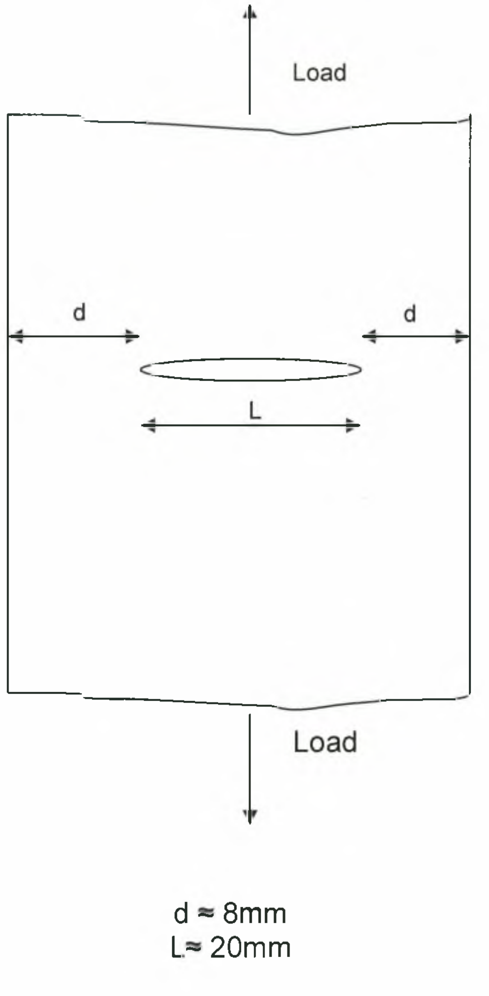

7
8

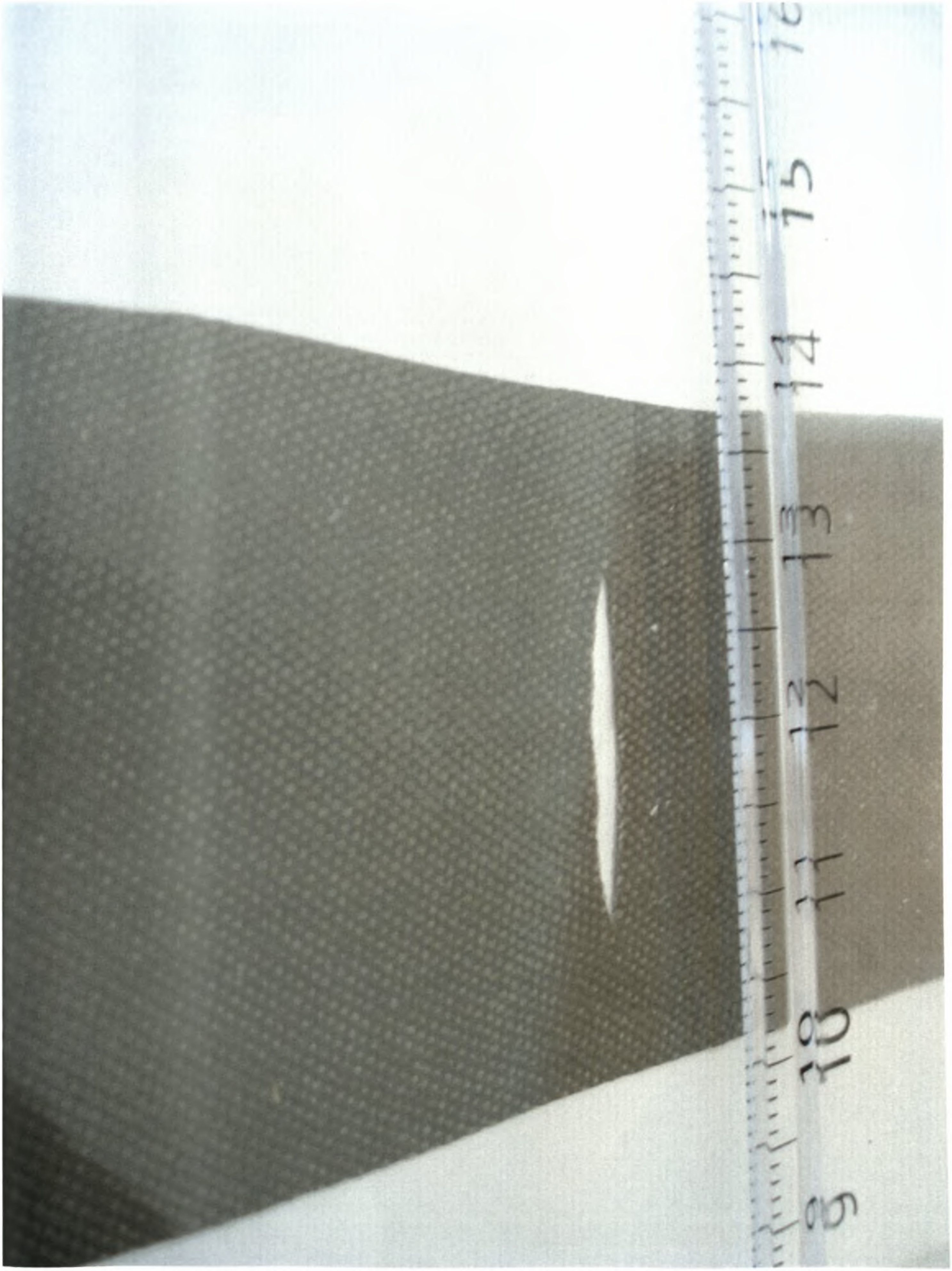




d ≈ 10mm
L ≈ 16mm


10
11
12


Load
d
d
L
Load
d ≈ 8mm
L≈ 20mm


9
10
11
12
13
14
15

CHAPTER 5

CONCLUSION AND DISCUSSION

The final result for the J_1-Integral is:

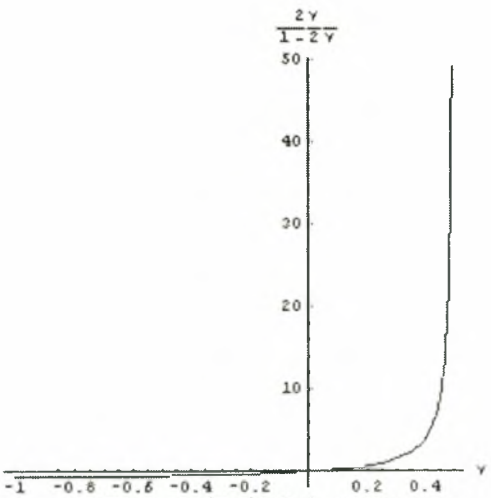
$$J_1 = -140625c\pi\left[A^2\lambda + \left(2A^2 - 25B^2\right)\mu\right] \Rightarrow$$

$$\Rightarrow \boxed{J_1 = -140625c\pi\mu\left[A^2\frac{\lambda}{\mu} + \left(2A^2 - 25B^2\right)\right]}$$

In this form, $c^{1/2}$ is the internal length, λ and μ the Lamé constants (μ *is the shear modulus*).
The dimensions of these terms are:
c [m^2],
A, B $\left[m^{-1/2}\right]$

J_1 $\left[\frac{N}{m^2}m\right]$

and the quotient $\frac{\lambda}{\mu}$ is : $\dfrac{\dfrac{\nu E}{(1+\nu)(1-2\nu)}}{\dfrac{E}{2(1+\nu)}} = \dfrac{2\nu}{(1-2\nu)}$



Plot of (2v/1-2v) vs v

Note that the present theory holds for E>0 and $-1 \le \nu < ½$. For the incompressible case ($\nu = ½$), see the analysis of [Shi, M.X., et al (2000)]
We can observe that the limit $\nu \to ½$ does not exist, which means that ν cannot obtain the value ½. Similarly, incompressibility cannot be achieved by using the equations applying for the compressible model and solely considering $\nu = ½$

The J_1-Integral is always greater than zero. Equivalently,

$$A^2\frac{\lambda}{\mu}+\left(2A^2-25B^2\right)<0 \Rightarrow A^2\left(2+\frac{\lambda}{\mu}\right)-25B^2<0 \Rightarrow$$

$$\Rightarrow A^2\left(2+\frac{\lambda}{\mu}\right)<25B^2 \Rightarrow \left(\text{since}: 2+\frac{\lambda}{\mu}>0\right)$$

$$\Rightarrow \frac{|A|}{5}\sqrt{2+\frac{\lambda}{\mu}}<|B| \Rightarrow$$

$$\Rightarrow -\frac{|A|}{5}\sqrt{2+\frac{\lambda}{\mu}}<B<\frac{|A|}{5}\sqrt{2+\frac{\lambda}{\mu}}$$

But it has already been shown that: **$19A/31 < B < -A/26$, $A<0$**.
Additionally,

$$\sqrt{2+\frac{\lambda}{\mu}}>1, \frac{1}{5}\sqrt{2+\frac{\lambda}{\mu}}>\frac{1}{5}$$

and at the same time:

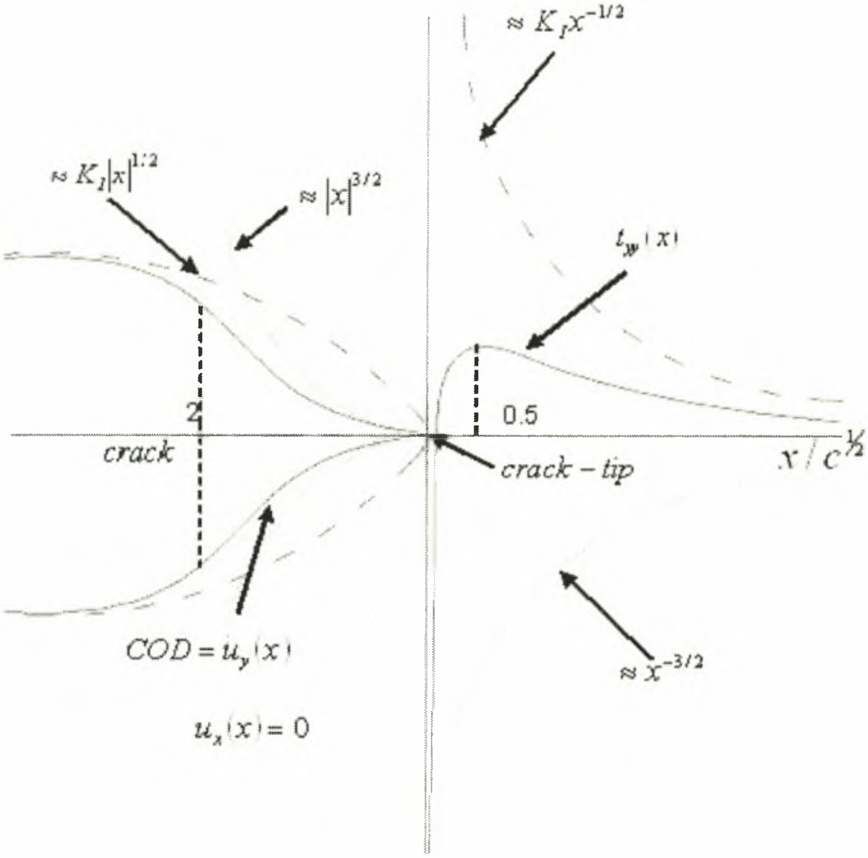
$$1/26<1/5, -19/31<-1/5$$

This means that the inequality:

$$-\frac{|A|}{5}\sqrt{2+\frac{\lambda}{\mu}}<B<\frac{|A|}{5}\sqrt{2+\frac{\lambda}{\mu}}$$

that determines the sign of J_1 is always correct, since $19A/31 < B < -A/26$.
In other words, the physical intuition that: **$J_1>0$ always** is correct!

The positive definiteness of the J_1-Integral can also be proven as a result of the minimum potential energy theorem, which gives positive definite energy release rate [Giannakopoulos, A.E. et al, (2006)] and [Georgiadis H.G., Grentzelou C.G., (2006)].



COD: crack opening displacement
t_{yy}: local normal tractions
$c^{1/2}$: characteristic internal material length

The related sizes stem from the numerical work by [Shi, M.X., et al (2000)].

Certain investigators seem to be perplexed with the result that the tractions close to the crack-tip are negative, suggesting an “unphysical” situation, and therefore are critical to the predictive capability of the model itself. We believe that the notion of “unphysical” should be elaborated and in any case it cannot be correlated with the negative traction in front of the crack-tip. Tractions are themselves a mathematical description of the mechanical behaviour.

We believe that the real quantities entering the discussion of strength and toughness are deformation and strain. These quantities are “physical” within the present formulation, because they do not contradict fundamental topological inequalities related to the crack-tip. Dilatation is physically the precursor of crack advancement. In addition, and as the present work clearly shows, the energy released from the crack-tip advancement is positive definite, as it should be from fundamental energy considerations. As crack advances, energy should be dissipated. We conclude that the present model seems to adequately predict the mechanics and physics of the crack-tip in materials that can be described in the context of simple strain-gradient elasticity (composites, textiles etc.).

REFERENCES

[1] Amanatidou, E. and Aravas, N., (2002), "Mixed finite element formulations of strain-gradient elasticity problems", Comp. Meth. Appl. Mech. Engng, vol. 191, pp. 1723-1751

[2] Georgiadis H.G., Grentzelou C.G., (2006), "Energy theorems and the J-Integral in dipolar gradient elasticity", Int. J. of Solids and Structures, vol. 43, pp. 5690-5712.

[3] Giannakopoulos, A.E., Amanatidou, E. and Aravas, N., (2006), "A reciprocity theorem in linear elasticity and the corresponding Saint-Venant principle", Int. J. of Solids and Structures, vol 43, pp. 3875-3894.

[4] Mindlin, R.D., (1964), "Micro-structure in linear elasticity", Arch. Rat. Mech. Anal., vol. 16, pp. 51-78.

[5] Shi, M.X., Huang Y., Hwang, K.C., (2000), "Fracture in a higher-order elastic continuum", J. Mech. Phys. Solids, vol. 48, pp. 2513-2538.

[6] Sifnaiou M., (2006), Problems of Notches and Cracks in the Context of Strain-gradient Elasticity Theories, Ph.D. Thesis, National Technical University of Athens.

[7] Wei, Yueguang, (2006), "A new finite element method for strain gradient theories and applications to fracture analyses", European J. Mech. A/Solids, vol. 25, pp. 897-913

[8] Williams, M.L., (1957), "On the stress distribution at the base of a stationary crack", ASME, J. Appl. Mechanics, vol. 79, p. 109-114.

APPENDIX 1

ASYMPTOTIC RESULTS

Before performing all the calculations already presented, the results proposed by Sifnaiou in her Ph.D. Thesis were checked.

First, the eigenvalue problem from which the value ω=3/2 emerged was resolved. Next, beginning from the potentials ϕ and ψ ,all relations used in this work (displacements, strains, stress) have been obtained and proved correct.

To cross-check the results, the relations containing ω, A_1, A_2, B_1, B_2, which are also leading to the same expressions, were checked for consistency. These relations were also correct, except from some minor errors concerning signs.

These results are quoted below.

Lamé potentials

$$\phi(r,\theta)=\mathrm{A}r^{5/2}\left[\cos(5\theta/2)-125\cos(\theta/2)\right]$$

$$\psi(r,\theta)=\mathrm{B}r^{5/2}\left[\sin(5\theta/2)-625\cos(\theta/2)\right]$$

Cauchy stress

$$\tau_{xx}=\lambda\left(\varepsilon_{xx}+\varepsilon_{yy}\right)+2\mu\varepsilon_{xx}$$

$$\tau_{xy}=2\mu\varepsilon_{xy}$$

$$\tau_{yy}=\lambda\left(\varepsilon_{xx}+\varepsilon_{yy}\right)+2\mu\varepsilon_{yy}$$

Dipolar stress

$$m_{xxx}=c\frac{\partial}{\partial x}[\lambda\left(\partial_x u_x+\partial_y u_y\right)+2\mu\partial_x u_x]=c\frac{\partial}{\partial x}\tau_{xx}$$

$$m_{xxy}=m_{xyx}=c\mu\frac{\partial}{\partial x}\left(\partial_y u_x+\partial_x u_y\right)=c\frac{\partial}{\partial x}\tau_{xy}$$

$$m_{xyy}=c\frac{\partial}{\partial x}[\lambda\left(\partial_x u_x+\partial_y u_y\right)+2\mu\partial_y u_y]=c\frac{\partial}{\partial x}\tau_{yy}$$

$$m_{yxx}=c\frac{\partial}{\partial y}[\lambda\left(\partial_x u_x+\partial_y u_y\right)+2\mu\partial_x u_x]=c\frac{\partial}{\partial y}\tau_{xx}$$

$$m_{yyx} = m_{yxy} = c\mu \frac{\partial}{\partial y}\left(\partial_y u_x + \partial_x u_y\right) = c\frac{\partial}{\partial y}\tau_{yx} = c\frac{\partial}{\partial y}\tau_{xy}$$

$$m_{yyy} = c\frac{\partial}{\partial y}[\lambda\left(\partial_x u_x + \partial_y u_y\right) + 2\mu\partial_y u_y] = c\frac{\partial}{\partial y}\tau_{yy}$$

<u>Displacements</u>

$$u_x = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y}, \;\; u_y = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}$$

<u>Strains</u>

$$\varepsilon_{xx} = \partial_x u_x$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\partial_y u_x + \partial_x u_y\right) = \varepsilon_{yx}$$

$$\varepsilon_{yy} = \partial_y u_y$$

The expressions containing ω, A_1, A_2, B_1, B_2 are (with ω=3/2, A_2= -125A_1, A_1=A, B_2= -625B_1, B_1=B):

$$u_x(r,\theta) = r^{\omega}\{[(\omega+1)(A_1+B_1)+A_2-B_2]\cos(\omega\theta)+\omega(A_2+B_2)\cos[(\omega-2)\theta]\}$$

$$u_y(r,\theta) = -r^{\omega}\{[(\omega+1)(A_1+B_1)-A_2+B_2]\sin(\omega\theta)+\omega(A_2+B_2)\sin[(\omega-2)\theta]\}$$

$$\varepsilon_{xx}(r,\theta) = r^{\omega-1}\omega[(\omega-1)(A_2+B_2)\cos[(\omega-3)\theta]+(A_2+B_2)\cos[(\omega-1)\theta]+ \\ +[(\omega+1)(A_1+B_1)+A_2-B_2]\cos[(\omega-1)\theta]]$$

$$\varepsilon_{xy}(r,\theta) = \varepsilon_{yx}(r,\theta) = -r^{\omega-1}\omega\Big[(\omega-1)(A_2+B_2)\sin[(\omega-3)\theta+ \\ +(\omega+1)(A_1+B_1)\sin[(\omega-1)\theta]\Big]$$

$$\varepsilon_{yy}(r,\theta) = -r^{\omega-1}\omega[(\omega-1)(A_2+B_2)\cos[(\omega-3)\theta]-(A_2+B_2)\cos[(\omega-1)\theta]+ \\ +[(\omega+1)(A_1+B_1)-A_2+B_2]\cos[(\omega-1)\theta]]$$

$$\tau_{xx}(r,\theta) = 2r^{\omega-1}\omega[(\omega-1)(A_2+B_2)\cos[(\omega-3)\theta]+ \\ +\mu(\omega+1)(A_1+B_1)\cos[(\omega-1)\theta]+2(\lambda+\mu)A_2\cos[(\omega-1)\theta]$$

$$\tau_{xy}(r,\theta)=\tau_{yx}(r,\theta)=-2r^{\omega-1}\mu\omega[(\omega-1)(A_2+B_2)\sin[(\omega-3)\theta]+$$
$$+(\omega+1)(A_1+B_1)\sin[(\omega-1)\theta]]$$

$$\tau_{yy}(r,\theta)=-2r^{\omega-1}\omega[\mu(\omega-1)(A_2+B_2)\cos[(\omega-3)\theta]+$$
$$+\mu(\omega+1)(A_1+B_1)\cos[(\omega-1)\theta]-2(\lambda+\mu)A_2\cos[(\omega-1)\theta]]$$

$$m_{xxx}(r,\theta)=2r^{\omega-2}c\omega(\omega-1)[\mu(\omega-2)(A_2+B_2)\cos[(\omega-4)\theta]+$$
$$+[(2\lambda+3\mu)A_2+\mu B_2+\mu(\omega+1)(A_1+B_1)]\cos[(\omega-2)\theta]$$

$$m_{xxy}(r,\theta)=m_{xyx}(r,\theta)=-2r^{\omega-2}c\mu\omega(\omega-1)[(\omega-2)(A_2+B_2)\sin[(\omega-4)\theta]+$$
$$+[(\omega+1)(A_1+B_1)+(A_2+B_2)]\sin[(\omega-2)\theta]$$

$$m_{xyy}(r,\theta)=-2r^{\omega-2}c\omega(\omega-1)[\mu(\omega-2)(A_2+B_2)\cos[(\omega-4)\theta]-$$
$$-[(2\lambda+\mu)A_2-\mu B_2-\mu(\omega+1)(A_1+B_1)]\cos[(\omega-2)\theta]]$$

$$m_{yxx}(r,\theta)=-2r^{\omega-2}c\omega(\omega-1)[\mu(\omega-2)(A_2+B_2)\sin[(\omega-4)\theta]+$$
$$+[(2\lambda+\mu)A_2-\mu B_2+\mu(\omega+1)(A_1+B_1)]\sin[(\omega-2)\theta]]$$

$$m_{yyx}(r,\theta)=m_{yxy}(r,\theta)=-2r^{\omega-2}c\mu\omega(\omega-1)[(\omega-2)(A_2+B_2)\cos[(\omega-4)\theta]-$$
$$-[A_2+B_2-(\omega+1)(A_1+B_1)]\cos[(\omega-2)\theta]]$$

$$m_{yyy}(r,\theta)=-2r^{\omega-2}c\omega(\omega-1)[-\mu(\omega-2)(A_2+B_2)\sin[(\omega-4)\theta]+$$
$$+[(2\lambda+3\mu)A_2+\mu B_2-\mu(\omega+1)(A_1+B_1)]\sin[(\omega-2)\theta]]$$

The asymptotic analysis is based on the expansion of the displacement field. Note that not all of the singular terms obey the boundary conditions at the crack face [Mindlin, R.D., (1964)].

For example, $m_{yxx}\left(\underset{r\to0}{\theta=0}\right)\to0$, but $m_{yyy}\left(\underset{r\to0}{\theta=0}\right)\neq0$.

Also, $t_{yy}\left(\underset{r\to0}{\theta=\pi}\right)\to0$, but $t_{xy}\left(\underset{r\to0}{\theta=\pi}\right)\neq0$.

This, however, does not invalid the J-Integral calculations. The actual boundary conditions require more terms in the expansion, a feature that is characteristic for materials that have internal length scales.

APPENDIX 2

EXPLICIT CALCULATIONS FOR THE J_1-INTEGRAL

1.1 Terms W in J_1

Calculation of terms W

W is:

$$W=(\lambda/2)\left(\varepsilon_{xx}+\varepsilon_{yy}\right)^2+\mu\left(\varepsilon_{xx}^2+2\varepsilon_{xy}+\varepsilon_{yy}^2\right)+(\lambda c/2)\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]+$$

$$\mu c\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2+2\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2+2\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2+\left(\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]$$

where the strains are:

$$\varepsilon_{xx} = (-15/4)\,\mathrm{r}^{1/2}\left[\,25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (99\mathrm{A}-\mathrm{B})\cos(\theta/2)\right]$$

$$\varepsilon_{xy} = (-15/4)\,\mathrm{r}^{1/2}\left[\,25(\mathrm{A}+5\mathrm{B})\sin(3\theta/2) + (\mathrm{A}+\mathrm{B})\sin(\theta/2)\right]$$

$$\varepsilon_{yy} = (+15/4)\,\mathrm{r}^{1/2}\left[\,25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) - (101\mathrm{A}+\mathrm{B})\cos(\theta/2)\right]$$

For practical reasons we name the terms consisting W as seen below, following their order of appearance in the above expression.

- Term $W1$: the term whose multiplying factor is (λ/2) (Note that the respective multiplying factors are not included. All results will be then multiplied by their appropriate coefficient.)
- Term $W2$: the term whose multiplying factor is μ
- Term $W3$: the term whose multiplying factor is ($\lambda c/2$)
- Term $W4$: the term whose multiplying factor is μc.

Further, in the same manner, term $W3$, consists of the terms:

Term $W3$(a): $\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2$

Term $W3$(b): $\left(\frac{\partial \varepsilon_{xx}}{\partial y}+\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2$

while term $W4$ of these:

Term $W4$(a): $\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)^2$

Term $W4$(b): $\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2$ (without the coefficient 2)

Term $W4$(c): $\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$

Term $W4$(d): $\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2$

Term $W4$(e): $\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2$ (without the coefficient 2)

Term W4(f): $\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2$

In the expressions of the strains $\varepsilon_{xx},\varepsilon_{yy}$, r is of order ½. Consequently, the expressions $(\lambda/2)\left(\varepsilon_{xx}+\varepsilon_{yy}\right)^2$ and $\mu\left(\varepsilon_{xx}^2+2\varepsilon_{xy}+\varepsilon_{yy}^2\right)$ (terms $W1$ and $W2$, respectively) are of order 1. When multiplied by: $dy=n_1\, d\Gamma=\cos\theta\, d\Gamma=\cos\theta\, r\, d\theta= r\cos\theta\, d\theta$, r should become of order 2. So integrated along a curve Γ, which is chosen to be a circle of radius r, then in the limit r→0, the integral itself will tend to 0!

Each of the rest terms will be calculated separately and then integrated (so as to form the integral $\int W dy$)

The transformation of the derivative from (x,y) Cartesian coordinates to (r,θ) polar coordinates follows the rule:

$$\frac{\partial}{\partial x}(\ ) = \cos\theta \ \frac{\partial}{\partial r}(\ ) - \frac{\sin\theta}{r}\ \frac{\partial}{\partial \theta}(\ )$$
$$\frac{\partial}{\partial y}(\ ) = \sin\theta \ \frac{\partial}{\partial r}(\ ) + \frac{\cos\theta}{r}\ \frac{\partial}{\partial \theta}(\ )$$

Term $W3$

Calculation of Term $W3$(a) $\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$

Instead of calculating $\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2$, it is easier to calculate $\left(\frac{\partial}{\partial x}(\varepsilon_{xx}+\varepsilon_{yy})\right)^2$.

As already said, the transformation of the derivative from (x,y) Cartesian coordinates to (r,θ) polar coordinates follows the rule:

$$\frac{\partial}{\partial x}(\ ) = \cos\theta \ \frac{\partial}{\partial r}(\ ) - \frac{\sin\theta}{\mathrm{r}} \ \frac{\partial}{\partial \theta}(\ )$$
$$\frac{\partial}{\partial y}(\ ) = \sin\theta \ \frac{\partial}{\partial r}(\ ) + \frac{\cos\theta}{\mathrm{r}} \ \frac{\partial}{\partial \theta}(\ )$$

$$\begin{aligned}(\varepsilon_{xx}+\varepsilon_{yy}) &= (-15/4)\ \mathrm{r}^{1/2} \left[\ 25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (99\mathrm{A}-\mathrm{B})\cos(\theta/2)\right] + \\ &+ (+15/4)\ \mathrm{r}^{1/2} \left[\ 25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) - (101\mathrm{A}+\mathrm{B})\cos(\theta/2)\right] = \\ &= (-15/4)\ \mathrm{r}^{1/2}\ [25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (99\mathrm{A}-\mathrm{B})\cos(\theta/2) - \\ &\quad - 25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (101\mathrm{A}+\mathrm{B})\cos(\theta/2)] =\end{aligned}$$

$$\begin{aligned}&= (-15/4)\ \mathrm{r}^{1/2}\ [0+ (99\mathrm{A}-\mathrm{B})\cos(\theta/2) + (101\mathrm{A}+\mathrm{B})\cos(\theta/2)\ ] = \\ &= (-15/4)\ \mathrm{r}^{1/2} \cos(\theta/2)\ [99\mathrm{A}-\mathrm{B}\ +101\mathrm{A}+\mathrm{B}] = \\ &= (-15/4)\ \mathrm{r}^{1/2} \cos(\theta/2)\ [200\mathrm{A}] \Rightarrow \\ (\varepsilon_{xx}+\varepsilon_{yy}) &= (-750\mathrm{A})\ \mathrm{r}^{1/2} \cos(\theta/2)\end{aligned}$$

$$\Rightarrow \boxed{(\varepsilon_{xx}+\varepsilon_{yy}) = (-750\mathrm{A})\ \mathrm{r}^{1/2} \cos(\theta/2)}$$

$$\begin{aligned}\frac{\partial}{\partial x}(\varepsilon_{xx}+\varepsilon_{yy}) &= \cos\theta\ \frac{\partial}{\partial r}\left[(-750\mathrm{A})\ \mathrm{r}^{1/2}\cos(\theta/2)\right] - \frac{\sin\theta}{\mathrm{r}}\ \frac{\partial}{\partial\theta}\left[(-750\mathrm{A})\ \mathrm{r}^{1/2}\cos(\theta/2)\right] = \\ &= \cos\theta\ (-750\mathrm{A})\ \frac{1}{2}\ \mathrm{r}^{-1/2} \cos(\theta/2) - \frac{\sin\theta}{\mathrm{r}}\ (-750\mathrm{A})\ \mathrm{r}^{1/2}\ [-\sin(\theta/2)]\ \frac{1}{2} = \\ &= (-375\mathrm{A})\mathrm{r}^{-1/2} \cos\theta \cos(\theta/2) - (+375\mathrm{A})\mathrm{r}^{-1/2} \sin\theta \sin(\theta/2) = \\ &= (-375\mathrm{A})\ \mathrm{r}^{-1/2} \left[\ \cos\theta \cos(\theta/2) + \sin\theta \sin(\theta/2)\ \right]\end{aligned}$$

and by making use of the identity

$$\cos a \cos b + \sin a \sin b = \cos(a-b)$$

we get for a=θ, b= θ/2 :

$$\cos\theta \cos(\theta/2) + \sin\theta \sin(\theta/2) = \cos[\theta-(\theta/2)] = \cos(\theta/2),$$

So,

$$\frac{\partial}{\partial x}(\varepsilon_{xx}+\varepsilon_{yy}) = \text{(-375A)}\ r^{-\frac{1}{2}}\left[\cos(\theta/2)\right]$$

By taking the square of it:

$$\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2 = 375^2\,A^2\ r^{-1}\ \cos^2(\theta/2) \Rightarrow$$

$$\Rightarrow \left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2 = 140625A^2\ r^{-1}\ \cos^2(\theta/2)$$

Calculation of Term W3(b) $\left(\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy})\right)^2$

This term will be calculated as the previous one.

$$\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy}) = \sin\theta\,\frac{\partial}{\partial r}\left[\text{(-750A)}\ r^{1/2}\cos(\theta/2)\right]+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\left[\text{(-750A)}\ r^{1/2}\cos(\theta/2)\right] =$$

$$= \sin\theta\text{(-750A)}\frac{1}{2}\ r^{-1/2}\cos(\theta/2)+\frac{\cos\theta}{r}\ \text{(-750A)}\ r^{1/2}\ [-\sin(\theta/2)]\ \frac{1}{2} =$$

$$= \text{(-375A)}r^{-1/2}\ \sin\theta\ \cos(\theta/2)+\text{(375A)}r^{-1/2}\ \cos\theta\ \sin(\theta/2) =$$

$$= \text{(-375A)}\ r^{-1/2}\ \left[\sin\theta\ \cos(\theta/2) - \cos\theta\ \sin(\theta/2)\right]$$

In this case, we use the identity: $\sin a \cos b - \cos a \sin b = \sin(a-b)$ [for $a=\theta$, $b=\theta/2$]:

$$\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2) = \sin[\theta - \theta/2] = \sin(\theta/2)$$

Substitution of this into the expression under examination, gives

$$\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy}) = \text{(-375A)}\ r^{-\frac{1}{2}}\ \sin(\theta/2)\ ,$$

the square of which is:

$$\left(\frac{\partial}{\partial y}(\varepsilon_{xx}+\varepsilon_{yy})\right)^2 = 140625\ A^2\ r^{-1}\ \sin^2(\theta/2)$$

It is remarkable to notice that the term *W*3 converts into:

$$\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_{xx}}{\partial y}+\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 = 140625\mathrm{A}^2\ \mathrm{r}^{-1}\ \cos^2(\theta/2) + 140625\ \mathrm{A}^2\ \mathrm{r}^{-1}\sin^2(\theta/2) =$$

$$= 140625\mathrm{A}^2\ \mathrm{r}^{-1}\ \left(\cos^2(\theta/2)+\sin^2(\theta/2)\right) = 140625\mathrm{A}^2\ \mathrm{r}^{-1}\cdot 1 \Rightarrow$$

$$\Rightarrow \boxed{\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_{xx}}{\partial y}+\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 = 140625\mathrm{A}^2\ \mathrm{r}^{-1}}$$

Term *W4*

Calculation of Term *W4*(b): $\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2$

$$\frac{\partial \varepsilon_{xx}}{\partial x} = \frac{\partial}{\partial x}\left[(-\frac{15}{4})\ \mathrm{r}^{1/2}\ \left[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right]\right] =$$

$$=\cos\theta\ \frac{\partial}{\partial r}\left[(-\frac{15}{4})\ \mathrm{r}^{1/2}\left[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right]\right] -$$

$$-\frac{\sin\theta}{\mathrm{r}}\ \frac{\partial}{\partial\theta}\left[(-\frac{15}{4})\ \mathrm{r}^{1/2}\left[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right]\right]=$$

$$=\cos\theta(-\frac{15}{4})\ \frac{1}{2}\ \mathrm{r}^{-1/2}\left[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right] -$$

$$-\ \frac{\sin\theta}{\mathrm{r}}(-\frac{15}{4})\ \mathrm{r}^{1/2}\left[25(\mathrm{A}+5\mathrm{B})[-\sin(3\theta/2)]\frac{3}{2}+(99\mathrm{A}\text{-}\mathrm{B})[-\sin(\theta/2)]\frac{1}{2}\right] =$$

$$=\cos\theta\left\{(-\frac{15}{8})\mathrm{r}^{-1/2}\left[\ 25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right]\right\}+$$

$$+\sin\theta\left\{(-\frac{15}{8})\mathrm{r}^{-1/2}\left[75(\mathrm{A}+5\mathrm{B})\sin(3\theta/2) + (99\mathrm{A}\text{-}\mathrm{B})\sin(\theta/2)\right]\right\}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\left\{\begin{matrix}\cos\theta\left[\ 25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right]+ \\ +\sin\theta\left[75(\mathrm{A}+5\mathrm{B})\sin(3\theta/2) + (99\mathrm{A}\text{-}\mathrm{B})\sin(\theta/2)\right]\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow \frac{\partial \varepsilon_{xx}}{\partial x}=(-\frac{15}{8})\mathrm{r}^{-1/2}\left\{\begin{matrix}\cos\theta\left[\ 25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2) + (99\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\right]+ \\ +\sin\theta\left[75(\mathrm{A}+5\mathrm{B})\sin(3\theta/2) + (99\mathrm{A}\text{-}\mathrm{B})\sin(\theta/2)\right]\end{matrix}\right\}$$

$$\Rightarrow \frac{\partial \varepsilon_{xx}}{\partial x} = \left(-\frac{15}{8} \right) r^{-1/2} \left(\cos\theta C_1 + \sin\theta C_2 \right)$$

where the coefficients C_1, C_2 are defined as described below:

$$C_1 = 25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)$$
$$C_2 = 75(A+5B)\sin(3\theta/2) + (99A\text{-}B)\sin(\theta/2)$$

This expression for $\frac{\partial \varepsilon_{xx}}{\partial x}$ can be further simplified if we perform some algebraic manipulations on the term in brackets.

$$\cos\theta\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2) \right] +$$
$$+\sin\theta\left[75(A+5B)\sin(3\theta/2) + (99A\text{-}B)\sin(\theta/2) \right] =$$

$$= 25(A+5B)\left[\cos\theta\cos(3\theta/2) + 3\sin\theta\sin(3\theta/2)\right] +$$
$$+(99A\text{-}B)\left[\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] =$$

$$= 25(A+5B)\left[\cos\theta\cos(3\theta/2) + \sin\theta\sin(3\theta/2) + 2\sin\theta\sin(3\theta/2)\right] +$$
$$+(99A\text{-}B)\cos(\theta/2) =$$

$$= 25(A+5B)\left[\cos(\theta/2) + 2\sin\theta\sin(3\theta/2)\right] + (99A\text{-}B)\cos(\theta/2) =$$
$$= \cos(\theta/2)\left[25(A+5B)\left(1 + 2\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)} \right) + (99A\text{-}B) \right]$$

The quotient $\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)}$ is:

$$\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)} = \frac{2\sin(\theta/2)\cancel{\cos(\theta/2)}\sin(3\theta/2)}{\cancel{\cos(\theta/2)}} = 2\sin(\theta/2)\sin(3\theta/2) =$$
$$= 2\sin(\theta/2)\left(\sin\theta\cos(\theta/2) + \cos\theta\sin(\theta/2)\right) =$$
$$= 2\left(\sin\theta\cos(\theta/2)\sin(\theta/2) + \cos\theta\sin^2(\theta/2)\right) =$$
$$= 2\left(\frac{1}{2}\sin\theta\sin\theta + \cos\theta\sin^2(\theta/2)\right) = 2\left(\frac{1}{2}\frac{1\text{-}\cos(2\theta)}{2} + \cos\theta\frac{1\text{-}\cos\theta}{2}\right) =$$
$$= \frac{1\text{-}\cos(2\theta)}{2} + \cos\theta(1\text{-}\cos\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta) + \cos\theta - \cos^2\theta =$$

$$=\frac{1}{2}-\frac{1}{2}\cos(2\theta)+\cos\theta-\frac{1+\cos(2\theta)}{2}=$$

$$=\frac{1}{2}-\frac{1}{2}\cos(2\theta)+\cos\theta-\frac{1}{2}-\frac{1}{2}\cos(2\theta)=$$

$$=\cos\theta-\cos(2\theta)\Rightarrow$$

$$\Rightarrow\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)}=\cos\theta-\cos(2\theta)$$

By virtue of this, the previous expression turns into:

$$\cos(\theta/2)\left[25(A+5B)(1+2(\cos\theta-\cos(2\theta)))+(99A\text{-}B)\right]$$

so as to make $\frac{\partial\varepsilon_{xx}}{\partial x}$:

$$\frac{\partial\varepsilon_{xx}}{\partial x}=(-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[25(A+5B)(1+2(\cos\theta-\cos(2\theta)))+(99A\text{-}B)\right]$$

or

$$\boxed{\frac{\partial\varepsilon_{xx}}{\partial x}=(-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[50(A+5B)(\cos\theta-\cos(2\theta))+124(A+B)\right]}$$

<u>Calculation of Term *W4*(b):</u> $\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2$

$$\frac{\partial\varepsilon_{xy}}{\partial x}=\frac{\partial}{\partial x}\left[(-\frac{15}{4})\,r^{1/2}\left[\,25(A+5B)\sin(3\theta/2)\;+\;(A+B)\sin(\theta/2)\right]\right]\;=$$

$$=\cos\theta\,\frac{\partial}{\partial r}\left[(-\frac{15}{4})\,r^{1/2}\left[\,25(A+5B)\sin(3\theta/2)\;+\;(A+B)\sin(\theta/2)\right]\right]\;-$$

$$-\frac{\sin\theta}{r}\;\frac{\partial}{\partial\theta}\left[(-\frac{15}{4})\,r^{1/2}\left[\,25(A+5B)\sin(3\theta/2)\;+\;(A+B)\sin(\theta/2)\right]\right]=$$

$$=\cos\theta(-\frac{15}{4})\;\frac{1}{2}\;r^{-1/2}\left[\,25(A+5B)\sin(3\theta/2)\;+\;(A+B)\sin(\theta/2)\right]\;-$$

$$-\;\frac{\sin\theta}{r}(-\frac{15}{4})\,r^{1/2}\left[25(A+5B)[\cos(3\theta/2)]\frac{3}{2}\;+(A+B)[\cos(\theta/2)]\frac{1}{2}\;\right]\;=$$

$$= \cos\theta \left\{ (-\frac{15}{8}) r^{-1/2} \left[25(A+5B)\sin(3\theta/2) + (A+B)\sin(\theta/2) \right] \right\} -$$

$$-\sin\theta \left\{ (-\frac{15}{8}) r^{-1/2} \left[75(A+5B)\cos(3\theta/2) + (A+B)\cos(\theta/2) \right] \right\} =$$

$$= (-\frac{15}{8}) r^{-1/2} \left\{ \begin{array}{l} \cos\theta \left[25(A+5B)\sin(3\theta/2) + (A+B)\sin(\theta/2) \right] - \\ -\sin\theta \left[75(A+5B)\cos(3\theta/2) + (A+B)\cos(\theta/2) \right] \end{array} \right\} \Rightarrow$$

$$\Rightarrow \frac{\partial \varepsilon_{xy}}{\partial x} = \left(-\frac{15}{8} \right) r^{-1/2} \left\{ \begin{array}{l} \cos\theta \cdot \left[25(A+5B)\sin(3\theta/2) + (A+B)\sin(\theta/2) \right] - \\ -\sin\theta \cdot \left[75(A+5B)\cos(3\theta/2) + (A+B)\cos(\theta/2) \right] \end{array} \right\}$$

or

$$\frac{\partial \varepsilon_{xy}}{\partial x} = \left(-\frac{15}{8} \right) r^{-1/2} \left(\cos\theta C_3 - \sin\theta C_4 \right)$$

where

$$C_3 = 25(A+5B)\sin(3\theta/2) + (A+B)\sin(\theta/2)$$
$$C_4 = 75(A+5B)\cos(3\theta/2) + (A+B)\cos(\theta/2)$$

The form $\left\{ \begin{array}{l} \cos\theta \left[25(A+5B)\sin(3\theta/2) + (A+B)\sin(\theta/2) \right] - \\ -\sin\theta \left[75(A+5B)\cos(3\theta/2) + (A+B)\cos(\theta/2) \right] \end{array} \right\}$ can be simplified.

$$\cos\theta \left[25(A+5B)\sin(3\theta/2) + (A+B)\sin(\theta/2) \right] -$$
$$-\sin\theta \left[75(A+5B)\cos(3\theta/2) + (A+B)\cos(\theta/2) \right] =$$

$$= 25(A+5B)\left[\cos\theta \sin(3\theta/2) - 3\sin\theta\cos(3\theta/2) \right] +$$
$$+ (A+B)\left[\cos\theta \sin(\theta/2) - \sin\theta\cos(\theta/2) \right] =$$

$$= 25(A+5B)\left[\cos\theta \sin(3\theta/2) - \sin\theta\cos(3\theta/2) - 2\sin\theta\cos(3\theta/2) \right] +$$
$$+ (A+B)\left[\cos\theta \sin(\theta/2) - \sin\theta\cos(\theta/2) \right] =$$

$$= 25(A+5B)\left[\sin(\theta/2) - 2\sin\theta\cos(3\theta/2) \right] +$$
$$+ (A+B)\left[-\sin(\theta/2) \right] =$$
$$= \sin(\theta/2) \left[25(A+5B)\left(1 - 2\frac{\sin\theta\cos(3\theta/2)}{\sin(\theta/2)} \right) - (A+B) \right]$$

The factor $\frac{\sin\theta\cos(3\theta/2)}{\sin(\theta/2)}$ becomes:

$$\frac{\sin\theta\cos(3\theta/2)}{\sin(\theta/2)}=\frac{2\sin(\theta/2)\cos(\theta/2)\cos(3\theta/2)}{\sin(\theta/2)}=$$

$$=2\cos(\theta/2)\big(\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)\big)=$$

$$=2[\cos\theta\cos^2(\theta/2)-\underbrace{\cos(\theta/2)\sin(\theta/2)}_{\frac{1}{2}\sin\theta}\sin\theta]=$$

$$=2\left(\cos\theta\frac{1+\cos\theta}{2}-\frac{1}{2}\sin\theta\sin\theta\right)=\cos\theta+\cos^2\theta-\sin^2\theta=$$

$$=\cos\theta+\cos(2\theta)$$

Thus, the aforementioned set of brackets is:

$$\sin(\theta/2)\Big[25(\mathrm{A}+5\mathrm{B})\big(1-2(\cos\theta+\cos(2\theta))\big)-(\mathrm{A}+\mathrm{B})\Big]$$

The simplified expression for $\frac{\partial\varepsilon_{xy}}{\partial x}$ is:

$$\frac{\partial\varepsilon_{xy}}{\partial x}=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\Big[25(\mathrm{A}+5\mathrm{B})\big(1-2(\cos\theta+\cos(2\theta))\big)-(\mathrm{A}+\mathrm{B})\Big]$$

or

$$\boxed{\frac{\partial\varepsilon_{xy}}{\partial x}=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\Big[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(24\mathrm{A}+124\mathrm{B})\Big]}$$

Calculation of Term *W*4(c): $\left(\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2$

$$\frac{\partial\varepsilon_{yy}}{\partial x}=\frac{\partial}{\partial x}\left[(+\frac{15}{4})\,\mathrm{r}^{1/2}\Big[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)\text{ - }(101\mathrm{A}+\mathrm{B})\cos(\theta/2)\Big]\right]=$$

$$=\cos\theta\,\frac{\partial}{\partial r}\left[(+\frac{15}{4})\,\mathrm{r}^{1/2}\Big[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(\text{-}101\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\Big]\right]-$$

$$-\frac{\sin\theta}{\mathrm{r}}\,\frac{\partial}{\partial\theta}\left[(+\frac{15}{4})\,\mathrm{r}^{1/2}\Big[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(\text{-}101\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\Big]\right]=$$

$$=\cos\theta(+\frac{15}{4})\,\frac{1}{2}\,\mathrm{r}^{-1/2}\Big[25(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+(\text{-}101\mathrm{A}\text{-}\mathrm{B})\cos(\theta/2)\Big]-$$

$$-\frac{\sin\theta}{\mathrm{r}}(+\frac{15}{4})\,\mathrm{r}^{1/2}\left[25(\mathrm{A}+5\mathrm{B})[-\sin(3\theta/2)]\frac{3}{2}+(\text{-}101\mathrm{A}\text{-}\mathrm{B})[-\sin(\theta/2)]\frac{1}{2}\right]=$$

$$=\cos\theta\left\{(+\frac{15}{8})r^{-1/2}\left[25\left(A+5B\right)\cos(3\theta/2)+(-101A\text{-}B)\cos(\theta/2)\right]\right\}+$$

$$+\sin\theta\left\{(+\frac{15}{8})r^{-1/2}\left[75\left(A+5B\right)\sin(3\theta/2)+(-101A\text{-}B)\sin(\theta/2)\right]\right\}=$$

$$=(+\frac{15}{8})r^{-1/2}\begin{Bmatrix}\cos\theta\left[25\left(A+5B\right)\cos(3\theta/2)+(-101A\text{-}B)\cos(\theta/2)\right]+\\+\sin\theta\left[75\left(A+5B\right)\sin(3\theta/2)+(-101A\text{-}B)\sin(\theta/2)\right]\end{Bmatrix}\Rightarrow$$

$$\Rightarrow\frac{\partial\varepsilon_{yy}}{\partial x}=\left(+\frac{15}{8}\right)r^{-1/2}\begin{Bmatrix}\cos\theta\left[25\left(A+5B\right)\cos(3\theta/2)+(-101A\text{-}B)\cos(\theta/2)\right]+\\+\sin\theta\left[75\left(A+5B\right)\sin(3\theta/2)+(-101A\text{-}B)\sin(\theta/2)\right]\end{Bmatrix}$$

or

$$\frac{\partial\varepsilon_{yy}}{\partial x}=\left(+\frac{15}{8}\right)r^{-1/2}\left(\cos\theta C_5+\sin\theta C_6\right)$$

with

$$C_5=25\left(A+5B\right)\cos(3\theta/2)+(-101A\text{-}B)\cos(\theta/2)$$
$$C_6=75\left(A+5B\right)\sin(3\theta/2)+(-101A\text{-}B)\sin(\theta/2)$$

By manipulating

$$\cos\theta\left[25\left(A+5B\right)\cos(3\theta/2)+(-101A\text{-}B)\cos(\theta/2)\right]+$$
$$+\sin\theta\left[75\left(A+5B\right)\sin(3\theta/2)+(-101A\text{-}B)\sin(\theta/2)\right]$$

one can get:

$$\cos\theta\left[25\left(A+5B\right)\cos(3\theta/2)+\left(-101A\text{-}B\right)\cos(\theta/2)\right]+$$
$$+\sin\theta\left[75\left(A+5B\right)\sin(3\theta/2)+\left(-101A\text{-}B\right)\sin(\theta/2)\right]=$$

$$=25\left(A+5B\right)\left[\cos\theta\cos(3\theta/2)+3\sin\theta\sin(3\theta/2)\right]+$$
$$+\left(-101A\text{-}B\right)\left[\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right]=$$

$$=25\left(A+5B\right)\left[\cos\theta\cos(3\theta/2)+\sin\theta\sin(3\theta/2)+2\sin\theta\sin(3\theta/2)\right]+$$
$$+\left(-101A\text{-}B\right)\cos(\theta/2)=$$

$$= 25(A+5B)\left[\cos(\theta/2) + 2\sin\theta\sin(3\theta/2)\right] +$$
$$+(\text{-}101A\text{-}B)\cos(\theta/2) =$$
$$= \cos(\theta/2)\left[25(A+5B)\left(1+2\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)}\right) + (\text{-}101A\text{-}B)\right] =$$
$$= \cos(\theta/2)\left[25(A+5B)(1+2(\cos\theta-\cos(2\theta))) + (\text{-}101A\text{-}B)\right]$$

since it has been made evident (while simplifying $\frac{\partial\varepsilon_{xx}}{\partial x}$) that:

$$\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)} = \cos\theta - \cos(2\theta).$$

Consequently, $\frac{\partial\varepsilon_{yy}}{\partial x}$:

$$\frac{\partial\varepsilon_{yy}}{\partial x} = \left(+\frac{15}{8}\right) r^{-1/2}\cos(\theta/2)\left[25(A+5B)(1+2\cos\theta-2\cos(2\theta)) + (\text{-}101A\text{-}B)\right]$$

or

$$\boxed{\frac{\partial\varepsilon_{yy}}{\partial x} = \left(+\frac{15}{8}\right) r^{-1/2}\cos(\theta/2)\left[50(A+5B)(\cos\theta-\cos(2\theta)) + (\text{-}76A+124B)\right]}$$

<u>Calculation of Term *W4*(d):</u> $\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2$

$$\frac{\partial\varepsilon_{xx}}{\partial y} = \frac{\partial}{\partial y}\left[(\text{-}\frac{15}{4})\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right]\right] =$$
$$= \sin\theta\,\frac{\partial}{\partial r}\left[(\text{-}\frac{15}{4})\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right]\right] +$$
$$+\frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\left[(\text{-}\frac{15}{4})\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right]\right] =$$
$$= \sin\theta(\text{-}\frac{15}{4})\,\frac{1}{2}\, r^{-1/2}\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right] +$$
$$+\frac{\cos\theta}{r}(\text{-}\frac{15}{4})\, r^{1/2}\left[25(A+5B)[\text{-}\sin(3\theta/2)]\frac{3}{2} + (99A\text{-}B)[\text{-}\sin(\theta/2)]\frac{1}{2}\right] =$$

$$= \sin\theta \left\{ (-\frac{15}{8}) r^{-1/2} \left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2) \right] \right\} -$$

$$-\cos\theta \left\{ (-\frac{15}{8}) r^{-1/2} \left[75(A+5B)\sin(3\theta/2) + (99A\text{-}B)\sin(\theta/2) \right] \right\} =$$

$$= (-\frac{15}{8}) r^{-1/2} \left\{ \begin{array}{l} \sin\theta\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right] - \\ -\cos\theta\left[75(A+5B)\sin(3\theta/2) + (99A\text{-}B)\sin(\theta/2)\right] \end{array} \right\} \Rightarrow$$

$$\Rightarrow \frac{\partial \varepsilon_{xx}}{\partial y} = \left(-\frac{15}{8}\right) r^{-1/2} \left\{ \begin{array}{l} \sin\theta\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right] - \\ -\cos\theta\left[75(A+5B)\sin(3\theta/2) + (99A\text{-}B)\sin(\theta/2)\right] \end{array} \right\}$$

or

$$\frac{\partial \varepsilon_{xx}}{\partial y} = \left(-\frac{15}{8}\right) r^{-1/2} \left(\sin\theta C_1 - \cos\theta C_2\right)$$

Further simplification of $\frac{\partial \varepsilon_{xx}}{\partial y}$ leads to:

$$\sin\theta\left[25(A+5B)\cos(3\theta/2) + (99A\text{-}B)\cos(\theta/2)\right] -$$

$$-\cos\theta\left[75(A+5B)\sin(3\theta/2) + (99A\text{-}B)\sin(\theta/2)\right] =$$

$$= 25(A+5B)\left[\sin\theta\cos(3\theta/2) - 3\cos\theta\sin(3\theta/2)\right] +$$

$$+ (99A\text{-}B)\left[\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2)\right] =$$

$$= 25(A+5B)\left[\sin\theta\cos(3\theta/2) - \cos\theta\sin(3\theta/2) - 2\cos\theta\sin(3\theta/2)\right] +$$

$$+ (99A\text{-}B)\sin(\theta/2) =$$

$$= 25(A+5B)\left[-\sin(\theta/2) - 2\cos\theta\sin(3\theta/2)\right] +$$

$$+ (99A\text{-}B)\sin(\theta/2) =$$

$$= \sin(\theta/2)\left[25(A+5B)\left(-1 - 2\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}\right) + (99A\text{-}B)\right]$$

A more compact formula is obtained this way:

$$\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}=\frac{\left(1\text{-}2\sin^2(\theta/2)\right)\sin(3\theta/2)}{\sin(\theta/2)}=$$
$$=\frac{\sin(3\theta/2)}{\sin(\theta/2)}-2\sin(\theta/2)\sin(3\theta/2)=$$

Also:

$$\sin(\theta/2)\sin(3\theta/2)=$$
$$=\left[\sin\theta\cos(\theta/2)+\cos\theta\sin(\theta/2)\right]\left[\sin\theta\cos(\theta/2)-\cos\theta\sin(\theta/2)\right]=$$
$$=\sin^2\theta\cos^2(\theta/2)-\cancel{\sin\theta\cos(\theta/2)\cos\theta\sin(\theta/2)}+$$
$$+\cancel{\cos\theta\sin(\theta/2)\sin\theta\cos(\theta/2)}-\cos^2\theta\sin^2(\theta/2)=$$
$$=\sin^2\theta\cos^2(\theta/2)-\cos^2\theta\sin^2(\theta/2)=$$

$$=\frac{1-\cos(2\theta)}{2}\frac{1+\cos\theta}{2}-\frac{1+\cos(2\theta)}{2}\frac{1-\cos\theta}{2}=$$
$$=\frac{1}{4}\left(\cancel{1}+\cos\theta-\cos(2\theta)-\cancel{\cos(2\theta)\cos\theta}-\cancel{1}+\cos\theta-\cos(2\theta)+\cancel{\cos(2\theta)\cos\theta}\right)=$$
$$=\frac{1}{4}\left(2\cos\theta-2\cos(2\theta)\right)=\frac{1}{2}\left(\cos\theta-\cos(2\theta)\right)\Rightarrow$$
$$\Rightarrow\sin(\theta/2)\sin(3\theta/2)=\frac{1}{2}\left(\cos\theta-\cos(2\theta)\right)$$

This identity also gives:

$$\sin(\theta/2)\sin(3\theta/2)=\frac{1}{2}\left(\cos\theta-\cos(2\theta)\right)\Rightarrow\frac{\sin(3\theta/2)}{\sin(\theta/2)}=\frac{1}{2}\frac{\left(\cos\theta-\cos(2\theta)\right)}{\sin^2(\theta/2)}\Rightarrow$$

$$\Rightarrow\frac{\sin(3\theta/2)}{\sin(\theta/2)}=\frac{1}{2}\left[\frac{\cos\theta}{\sin^2(\theta/2)}-\frac{\cos(2\theta)}{\sin^2(\theta/2)}\right]=\frac{1}{2}\left[\frac{1-2\sin^2(\theta/2)}{\sin^2(\theta/2)}-\frac{\cos^2\theta-\sin^2\theta}{\sin^2(\theta/2)}\right]=$$
$$=\frac{1}{2}\left[\frac{1}{\sin^2(\theta/2)}-2-\frac{\left(1-2\sin^2(\theta/2)\right)^2-4\sin^2(\theta/2)\cos^2(\theta/2)}{\sin^2(\theta/2)}\right]=$$
$$=\frac{1}{2}\left[\frac{1}{\sin^2(\theta/2)}-2-\frac{\left(1-2\sin^2(\theta/2)\right)^2}{\sin^2(\theta/2)}+4\cos^2(\theta/2)\right]=$$
$$=\frac{1}{2}\left[\frac{1}{\sin^2(\theta/2)}-2-\frac{1-4\sin^2(\theta/2)+4\sin^4(\theta/2)}{\sin^2(\theta/2)}+4\cos^2(\theta/2)\right]=$$

$$=\frac{1}{2}\left[\frac{1}{\sin^2(\theta/2)}-2-\frac{1}{\sin^2(\theta/2)}+4-4\sin^2(\theta/2)+4\cos^2(\theta/2)\right]=$$

$$=\frac{1}{2}\left[2-4\sin^2(\theta/2)+4\cos^2(\theta/2)\right]=\frac{1}{2}\left[2+4\cos\theta\right]=1+2\cos\theta\Rightarrow$$

$$\Rightarrow\frac{\sin(3\theta/2)}{\sin(\theta/2)}=1+2\cos\theta$$

Substituting:

$$\sin(\theta/2)\sin(3\theta/2)=\frac{1}{2}\left(\cos\theta-\cos(2\theta)\right)\quad\text{and}\quad\frac{\sin(3\theta/2)}{\sin(\theta/2)}=1+2\cos\theta$$

into

$$\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}=\frac{\left(1\text{-}2\sin^2(\theta/2)\right)\sin(3\theta/2)}{\sin(\theta/2)}=\frac{\sin(3\theta/2)}{\sin(\theta/2)}-2\sin(\theta/2)\sin(3\theta/2)=$$

$$=1+2\cos\theta-2\frac{1}{2}\left(\cos\theta-\cos(2\theta)\right)=1+\cos\theta+\cos(2\theta)\Rightarrow$$

$$\Rightarrow\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}=1+\cos\theta+\cos(2\theta)$$

After these manipulations:

$$\sin(\theta/2)\left[25\left(A+5B\right)\left(-1-2\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}\right)+\left(99A\text{-}B\right)\right]=$$

$$=\sin(\theta/2)\left[25\left(A+5B\right)\left(-1-2\left(1+\cos\theta+\cos(2\theta)\right)\right)+\left(99A\text{-}B\right)\right]=$$

$$=\sin(\theta/2)\left[25\left(A+5B\right)\left(-1-2-2\cos\theta-2\cos(2\theta)\right)+\left(99A\text{-}B\right)\right]=$$

$$=\sin(\theta/2)\left[\text{-}25\left(A+5B\right)\left(3+2\cos\theta+2\cos(2\theta)\right)+\left(99A\text{-}B\right)\right]$$

and the resulting formula for $\frac{\partial\varepsilon_{xx}}{\partial y}$ is:

$$\frac{\partial\varepsilon_{xx}}{\partial y}=\left(-\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[\text{-}25\left(A+5B\right)\left(3+2\cos\theta+2\cos(2\theta)\right)+\left(99A\text{-}B\right)\right]$$

or

$$\boxed{\frac{\partial\varepsilon_{xx}}{\partial y}=\left(-\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[\text{-}50\left(A+5B\right)\left(\cos\theta+\cos(2\theta)\right)+\left(24A\text{-}376B\right)\right]}$$

Calculation of Term *W*4(e): $\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2$

$$\frac{\partial \varepsilon_{xy}}{\partial y}=\frac{\partial}{\partial y}\left[(-\frac{15}{4})\ r^{1/2}\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]\right]=$$

$$=\sin\theta\ \frac{\partial}{\partial r}\left[(-\frac{15}{4})\ r^{1/2}\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]\right]+$$

$$+\frac{\cos\theta}{r}\ \frac{\partial}{\partial\theta}\left[(-\frac{15}{4})\ r^{1/2}\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]\right]=$$

$$=\sin\theta(-\frac{15}{4})\ \frac{1}{2}\ r^{-1/2}\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]+$$

$$+\frac{\cos\theta}{r}(-\frac{15}{4})\ r^{1/2}\left[25(A+5B)\cos(3\theta/2)\frac{3}{2}+(A+B)\cos(\theta/2)\frac{1}{2}\right]=$$

$$=\sin\theta\left\{(-\frac{15}{8})r^{-1/2}\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]\right\}+$$

$$+\cos\theta\left\{(-\frac{15}{8})r^{-1/2}\left[75(A+5B)\cos(3\theta/2)+(A+B)\cos(\theta/2)\right]\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}\sin\theta\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]+\\+\cos\theta\left[75(A+5B)\cos(3\theta/2)+(A+B)\cos(\theta/2)\right]\end{array}\right\}\Rightarrow$$

$$\Rightarrow\frac{\partial \varepsilon_{xy}}{\partial y}=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}\sin\theta\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]+\\+\cos\theta\left[75(A+5B)\cos(3\theta/2)+(A+B)\cos(\theta/2)\right]\end{array}\right\}.$$

Analyzing the bracket:

$$\sin\theta\left[25(A+5B)\sin(3\theta/2)+(A+B)\sin(\theta/2)\right]+$$
$$+\cos\theta\left[75(A+5B)\cos(3\theta/2)+(A+B)\cos(\theta/2)\right]=$$

$$=25(A+5B)\left[\sin\theta\sin(3\theta/2)+3\cos\theta\cos(3\theta/2)\right]+$$
$$+(A+B)\left[\sin\theta\sin(\theta/2)+\cos\theta\cos(\theta/2)\right]=$$

$$=25(\mathrm{A}+5\mathrm{B})\left[\sin\theta\sin(3\theta/2)+\cos\theta\cos(3\theta/2)+2\cos\theta\cos(3\theta/2)\right]+$$
$$+(\mathrm{A}+\mathrm{B})\cos(\theta/2)=$$
$$=25(\mathrm{A}+5\mathrm{B})\left[\cos(\theta/2)+2\cos\theta\cos(3\theta/2)\right]+(\mathrm{A}+\mathrm{B})\cos(\theta/2)=$$
$$=25(\mathrm{A}+5\mathrm{B})\left[\cos(\theta/2)+2\cos\theta\cos(3\theta/2)\right]+(\mathrm{A}+\mathrm{B})\cos(\theta/2)=$$
$$=\cos(\theta/2)\left[25(\mathrm{A}+5\mathrm{B})\left[1+2\frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}\right]+(\mathrm{A}+\mathrm{B})\right]$$

The decomposition of $\frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}$ is:

$$\frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}=\left[\cos^2(\theta/2)-\sin^2(\theta/2)\right]\frac{\cos(3\theta/2)}{\cos(\theta/2)}=$$
$$=\cos(\theta/2)\cos(3\theta/2)-\frac{\sin^2(\theta/2)\cos(3\theta/2)}{\cos(\theta/2)}$$

The forms $\frac{\cos(3\theta/2)}{\cos(\theta/2)}$ and $\cos(\theta/2)\cos(3\theta/2)$ will be simplified.

- $\frac{\cos(3\theta/2)}{\cos(\theta/2)}=$

$$=\frac{\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)}{\cos(\theta/2)}=\cos\theta-\frac{\sin\theta\sin(\theta/2)}{\cos(\theta/2)}=$$
$$=\cos\theta-\frac{2\sin^2(\theta/2)\cancel{\cos(\theta/2)}}{\cancel{\cos(\theta/2)}}=\cos\theta-2\sin^2(\theta/2)=$$
$$=\cos\theta-\cancel{2}\frac{1-\cos\theta}{\cancel{2}}=-1+2\cos\theta\Rightarrow$$
$$\Rightarrow\frac{\cos(3\theta/2)}{\cos(\theta/2)}=-1+2\cos\theta$$

and also

$$\frac{\cos(3\theta/2)}{\cos(\theta/2)}=-1+2\cos\theta\Rightarrow\cos(\theta/2)\cos(3\theta/2)=(-1+2\cos\theta)\cos^2(\theta/2)$$

This leads to:

- $\cos(\theta/2)\cos(3\theta/2)=$

$$= -\cos^2(\theta/2) + 2\cos\theta\cos^2(\theta/2) =$$

$$-\frac{1+\cos\theta}{2} + \not{2}\cos\theta\frac{1+\cos\theta}{\not{2}} = -\frac{1}{2} - \frac{1}{2}\cos\theta + \cos\theta + \cos^2\theta =$$

$$= -\frac{1}{2} + \frac{1}{2}\cos\theta + \frac{1+\cos(2\theta)}{2} = \frac{1}{2}\left(\cos\theta + \cos(2\theta)\right) \Rightarrow$$

$$\Rightarrow \cos(\theta/2)\cos(3\theta/2) = \frac{1}{2}\left(\cos\theta + \cos(2\theta)\right)$$

Thus, $\dfrac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)} =$

$$= \left[\cos^2(\theta/2) - \sin^2(\theta/2)\right]\frac{\cos(3\theta/2)}{\cos(\theta/2)} =$$

$$= \cos(\theta/2)\cos(3\theta/2) - \frac{\sin^2(\theta/2)\cos(3\theta/2)}{\cos(\theta/2)} =$$

$$= \frac{1}{2}\left(\cos\theta + \cos(2\theta)\right) - \sin^2(\theta/2)\left(-1 + 2\cos\theta\right) =$$

$$= \frac{1}{2}\left(\cos\theta + \cos(2\theta)\right) + \sin^2(\theta/2) - 2\sin^2(\theta/2)\cos\theta =$$

$$= \frac{1}{2}\cos\theta + \frac{1}{2}\cos(2\theta) + \frac{1-\cos\theta}{2} - \not{2}\frac{1-\cos\theta}{\not{2}}\cos\theta =$$

$$= \frac{1}{\not{2}}\not{\cos\theta} + \frac{1}{2}\cos(2\theta) + \frac{1}{2} - \frac{1}{\not{2}}\not{\cos\theta} - \cos\theta + \cos^2\theta =$$

$$= \frac{1}{2} - \cos\theta + \frac{1}{2}\cos(2\theta) + \frac{1+\cos(2\theta)}{2} = 1 - \cos\theta + \cos(2\theta) \Rightarrow$$

$$\Rightarrow \frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)} = 1 - \cos\theta + \cos(2\theta)$$

Eventually,

$$\cos(\theta/2)\left[25\left(A+5B\right)\left[1 + 2\frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}\right] + \left(A+B\right)\right]$$

becomes:

$$= \cos(\theta/2)\left[25\left(A+5B\right)\left[1 + 2\left(1 - \cos\theta + \cos(2\theta)\right)\right] + \left(A+B\right)\right]$$

and so:

$$\frac{\partial \varepsilon_{xy}}{\partial y} = (-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[25\left(A+5B\right)\left[1 + 2\left(1 - \cos\theta + \cos(2\theta)\right)\right] + \left(A+B\right)\right]$$

or

$$\frac{\partial \varepsilon_{xy}}{\partial y} = (-\frac{15}{8}) r^{-1/2} \cos(\theta/2)\left[50(A+5B)(-\cos\theta + \cos(2\theta)) + (76A + 376B)\right]$$

Calculation of Term *W4*(f): $\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2$

$$\frac{\partial \varepsilon_{yy}}{\partial y} = \frac{\partial}{\partial y}\left[(+\frac{15}{4})\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) - (101A+B)\cos(\theta/2)\right]\right] =$$

$$= \sin\theta\, \frac{\partial}{\partial r}\left[(\frac{15}{4})\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) - (101A+B)\cos(\theta/2)\right]\right] +$$

$$+ \frac{\cos\theta}{r}\, \frac{\partial}{\partial \theta}\left[(\frac{15}{4})\, r^{1/2}\left[25(A+5B)\cos(3\theta/2) - (101A+B)\cos(\theta/2)\right]\right] =$$

$$= \sin\theta (15/4)\, \frac{1}{2}\, r^{-1/2}\left[25(A+5B)\cos(3\theta/2) - (101A+B)\cos(\theta/2)\right] +$$

$$+ \frac{\cos\theta}{r}(\frac{15}{4})\, r^{1/2}\left[25(A+5B)[-\sin(3\theta/2)]\frac{3}{2} - (101A+B)[-\sin(\theta/2)]\frac{1}{2}\right] =$$

$$= \sin\theta\left\{(\frac{15}{8}) r^{-1/2}\left[25(A+5B)\cos(3\theta/2) + (-101A-B)\cos(\theta/2)\right]\right\} -$$

$$-\cos\theta\left\{(\frac{15}{8}) r^{-1/2}\left[75(A+5B)\sin(3\theta/2) + (-101A-B)\sin(\theta/2)\right]\right\} =$$

$$= (\frac{15}{8}) r^{-1/2}\left\{\begin{array}{l}\sin\theta\left[25(A+5B)\cos(3\theta/2) + (-101A-B)\cos(\theta/2)\right] - \\ -\cos\theta\left[75(A+5B)\sin(3\theta/2) + (-101A-B)\sin(\theta/2)\right]\end{array}\right\} \Rightarrow$$

$$\Rightarrow \frac{\partial \varepsilon_{yy}}{\partial y} = \left(\frac{15}{8}\right) r^{-1/2}\left\{\begin{array}{l}\sin\theta\left[25(A+5B)\cos(3\theta/2) + (-101A-B)\cos(\theta/2)\right] - \\ -\cos\theta\left[75(A+5B)\sin(3\theta/2) + (-101A-B)\sin(\theta/2)\right]\end{array}\right\}$$

or

$$\frac{\partial \varepsilon_{yy}}{\partial y} = \left(+\frac{15}{8}\right) r^{-1/2}\left(\sin\theta\, C_5 - \cos\theta\, C_6\right)$$

The simplification of

$$\left\{\begin{array}{l}\sin\theta\left[25(A+5B)\cos(3\theta/2) + (-101A-B)\cos(\theta/2)\right] - \\ -\cos\theta\left[75(A+5B)\sin(3\theta/2) + (-101A-B)\sin(\theta/2)\right]\end{array}\right\}$$

leads to:

$$\sin\theta\left[25(A+5B)\cos(3\theta/2)+(-101A\text{-}B)\cos(\theta/2)\right]-$$
$$-\cos\theta\left[75(A+5B)\sin(3\theta/2)+(-101A\text{-}B)\sin(\theta/2)\right]=$$

$$=25(A+5B)\left[\sin\theta\cos(3\theta/2)-3\cos\theta\sin(3\theta/2)\right]+$$
$$+(-101A\text{-}B)\left[\sin\theta\cos(\theta/2)-\cos\theta\sin(\theta/2)\right]=$$

$$=25(A+5B)\left[\sin\theta\cos(3\theta/2)-\cos\theta\sin(3\theta/2)-2\cos\theta\sin(3\theta/2)\right]+$$
$$+(-101A\text{-}B)\sin(\theta/2)=$$

$$=25(A+5B)\left[-\sin(\theta/2)-2\cos\theta\sin(3\theta/2)\right]+$$
$$+(-101A\text{-}B)\sin(\theta/2)=$$

$$=\sin(\theta/2)\left[25(A+5B)\left(-1-2\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}\right)+(-101A\text{-}B)\right]$$

While calculating $\frac{\partial\varepsilon_{xx}}{\partial y}$, it was proved that:

$$\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}=1+\cos\theta+\cos(2\theta),$$

which allows the expression

$$\sin(\theta/2)\left[25(A+5B)\left(-1-2\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}\right)+(-101A\text{-}B)\right]$$

to be rewritten as:

$$\sin(\theta/2)\left[25(A+5B)\left(-1-2(1+\cos\theta+\cos(2\theta))\right)+(-101A\text{-}B)\right]=$$
$$=\sin(\theta/2)\left[25(A+5B)(-3-2\cos\theta-2\cos(2\theta))+(-101A\text{-}B)\right]$$

The resulting brief form for $\frac{\partial\varepsilon_{yy}}{\partial y}$ is:

$$\frac{\partial\varepsilon_{yy}}{\partial y}=\left(+\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[25(A+5B)(-3-2\cos\theta-2\cos(2\theta))+(-101A\text{-}B)\right]$$

or, alternatively

$$\boxed{\frac{\partial\varepsilon_{yy}}{\partial y}=\left(+\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[-50(A+5B)(\cos\theta+\cos(2\theta))+(-176A\text{-}376B)\right]}$$

1.2 Terms R_q in J_1

Calculation of terms $R_q\, D\left(\frac{\partial u_q}{\partial x}\right)$

The term $R_q\, D\left(\frac{\partial u_q}{\partial x}\right)$ can be expanded in the following way:

$$R_q\, D\left(\frac{\partial u_q}{\partial x}\right) = R_x\, D\left(\frac{\partial u_x}{\partial x}\right) + R_y\, D\left(\frac{\partial u_y}{\partial x}\right)\ .$$

Consequently, these are the terms that should be calculated:

$$R_x, D\left(\frac{\partial u_x}{\partial x}\right), R_y, D\left(\frac{\partial u_y}{\partial x}\right)\ .$$

These are their definitions:

$$\begin{aligned} R_x &= n_r n_p m_{rpx} = \\ &= n_x n_p m_{xpx} + n_y n_p m_{ypx} = \\ &= n_x n_x m_{xxx} + n_x n_y m_{xyx} + n_y n_x m_{yxx} + n_y n_y m_{yyx} \end{aligned}$$

$$\begin{aligned} D\left(\frac{\partial u_x}{\partial x}\right) &= n_r\, \partial_r \left(\frac{\partial u_x}{\partial x}\right) = n_x\, \partial_x \left(\frac{\partial u_x}{\partial x}\right) + n_y\, \partial_y \left(\frac{\partial u_x}{\partial x}\right) = \\ &= n_x \frac{\partial^2 u_x}{\partial x^2} + n_y \frac{\partial^2 u_x}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_x}{\partial x^2} + \sin\theta \frac{\partial^2 u_x}{\partial x \partial y} \end{aligned}$$

(given that $n_x = \cos\theta$ and $n_y = \sin\theta$)
and

$$\begin{aligned} R_y &= n_r n_p m_{rpy} = \\ &= n_x n_p m_{xpy} + n_y n_p m_{ypy} = \\ &= n_x n_x m_{xxy} + n_x n_y m_{xyy} + n_y n_x m_{yxy} + n_y n_y m_{yyy} \end{aligned}$$

$$\begin{aligned} D\left(\frac{\partial u_y}{\partial x}\right) &= n_r\, \partial_r \left(\frac{\partial u_y}{\partial x}\right) = n_x\, \partial_x \left(\frac{\partial u_y}{\partial x}\right) + n_y\, \partial_y \left(\frac{\partial u_y}{\partial x}\right) = \\ &= n_x \frac{\partial^2 u_y}{\partial x^2} + n_y \frac{\partial^2 u_y}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_y}{\partial x^2} + \sin\theta \frac{\partial^2 u_y}{\partial x \partial y} \end{aligned}$$

The dipolar stresses, needed to form the terms R_x, R_y are defined as [Sifnaiou M., (2006)]:

$$m_{xxx} = (+15/4)\, c\, r^{-1/2} \left[\, 25\mu\left(A+5B\right)\cos(5\theta/2) \;-\; (100A\lambda+149A\mu+249B\mu)\cos(\theta/2)\right]$$

$$m_{xxy} = m_{xyx} = (+15/4)\, c\,\mu\, r^{-1/2} \left[\, 25\left(A+5B\right)\sin(5\theta/2) \;-\; (49A+249B)\sin(\theta/2)\right]$$

$$m_{xyy} = (-15/4)\, c\, r^{-1/2} \left[\, 25\mu\left(A+5B\right)\cos(5\theta/2) \;+\; (100A\lambda+51A\mu-249B\mu)\cos(\theta/2)\right]$$

$$m_{yxx} = (+15/4)\, c\, r^{-1/2} \left[\, 25\mu\left(A+5B\right)\sin(5\theta/2) \;-\; (100A\lambda+49A\mu-251B\mu)\sin(\theta/2)\right]$$

$$m_{yyx} = m_{yxy} = (-15/4)\, c\,\mu\, r^{-1/2} \left[\, 25\left(A+5B\right)\cos(5\theta/2) \;+\; (51A+251B)\cos(\theta/2)\right]$$

$$m_{yyy} = (-15/4)\, c\, r^{-1/2} \left[\, 25\mu\left(A+5B\right)\sin(5\theta/2) \;+\; (100A\lambda+151A\mu+251B\mu)\sin(\theta/2)\right]$$

<u>Calculation of the subterms that compose the term</u> $R_q\, D\left(\dfrac{\partial u_q}{\partial x}\right)$

<u>CALCULATION OF</u> R_x

$$R_x = n_x n_x m_{xxx} + n_x n_y m_{xyx} + n_y n_x m_{yxx} + n_y n_y m_{yyx} =$$
$$= \cos^2\theta\, m_{xxx} + \cos\theta\sin\theta\, m_{xyx} + \sin\theta\cos\theta\, m_{yxx} + \sin^2\theta\, m_{yyx} =$$

$$= \cos^2\theta (15/4)\, c\, r^{-1/2}\left[25\mu\left(A+5B\right)\cos(5\theta/2) - (100A\lambda+149A\mu+249B\mu)\cos(\theta/2)\right] +$$
$$+ \cos\theta\sin\theta\, (15/4)\, c\,\mu\, r^{-1/2}\left[25\left(A+5B\right)\sin(5\theta/2) - (49A+249B)\sin(\theta/2)\right] +$$
$$+ \sin\theta\cos\theta (15/4)\, c\, r^{-1/2}\left[25\mu\left(A+5B\right)\sin(5\theta/2) - (100A\lambda+49A\mu-251B\mu)\sin(\theta/2)\right] +$$
$$+ \sin^2\theta (-15/4)\, c\,\mu\, r^{-1/2}\left[25\left(A+5B\right)\cos(5\theta/2) + (51A+251B)\cos(\theta/2)\right] =$$

$$=(15/4)\, c\, r^{-1/2}\, 25\mu\left(A+5B\right)\begin{bmatrix} \cos^2\theta\cos(5\theta/2) \quad + \\ +\cos\theta\sin\theta\sin(5\theta/2)+ \\ +\sin\theta\cos\theta\sin(5\theta/2)- \\ -\sin^2\theta\cos(5\theta/2) \end{bmatrix}+$$

$$+(15/4)\, c\, r^{-1/2}\begin{bmatrix} -(100A\lambda+149A\mu+249B\mu)\cos^2\theta\cos(\theta/2) \; - \\ -\;(49A+249B)\mu\cos\theta\sin\theta\sin(\theta/2)- \\ -(100A\lambda+49A\mu-251B\mu)\sin\theta\cos\theta\sin(\theta/2)- \\ -(51A+251B)\mu\sin^2\theta\cos(\theta/2) \end{bmatrix}=$$

The terms in the first set of brackets yield to:

$$= \cos^2\theta\cos(5\theta/2) \quad +\cos\theta\sin\theta\sin(5\theta/2)+\sin\theta\cos\theta\sin(5\theta/2) \; -\sin^2\theta\cos(5\theta/2)=$$
$$=\cos^2\theta\cos(5\theta/2)+2\cos\theta\sin\theta\sin(5\theta/2)-\sin^2\theta\cos(5\theta/2)=$$
$$=\cos(5\theta/2)\left(\cos^2\theta-\sin^2\theta\right)+\sin(2\theta)\sin(5\theta/2)=$$
$$=\cos(5\theta/2)\cos(2\theta)+\sin(2\theta)\sin(5\theta/2)=$$
$$=\cos(\frac{5\theta}{2}-2\theta)=\cos(\theta/2)$$

while those in the second one:

$$-(100A\lambda+149A\mu+249B\mu)\cos^2\theta\cos(\theta/2) \; -$$
$$-\;(49A+249B)\mu\cos\theta\sin\theta\sin(\theta/2)-$$
$$-(100A\lambda+49A\mu-251B\mu)\sin\theta\cos\theta\sin(\theta/2)-$$
$$-(51A+251B)\mu\sin^2\theta\cos(\theta/2)=$$

$$=-(100A\lambda)\left(\cos^2\theta\cos(\theta/2)+\sin\theta\cos\theta\sin(\theta/2)\right)+$$
$$+A\mu\left(-149\cos^2\theta\cos(\theta/2)-49\cos\theta\sin\theta\sin(\theta/2)-49\sin\theta\cos\theta\sin(\theta/2)-51\sin^2\theta\cos(\theta/2)\right)+$$
$$+B\mu\left(-249\cos^2\theta\cos(\theta/2)-249\cos\theta\sin\theta\sin(\theta/2)+251\sin\theta\cos\theta\sin(\theta/2)-251\sin^2\theta\cos(\theta/2)\right)=$$

$$=-(100A\lambda)\cos\theta\left(\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right)+$$
$$+A\mu\left(-149\cos^2\theta\cos(\theta/2)-98\cos\theta\sin\theta\sin(\theta/2)-51\sin^2\theta\cos(\theta/2)\right)+$$
$$+B\mu\left(-249\cos\theta\left(\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right)+251\sin\theta\left(\cos\theta\sin(\theta/2)-\sin\theta\cos(\theta/2)\right)\right)=$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) +$$
$$+A\mu\left[-98\cos^2\theta\cos(\theta/2) - 51\cos^2\theta\cos(\theta/2) - 98\cos\theta\sin\theta\sin(\theta/2) - 51\sin^2\theta\cos(\theta/2)\right] +$$
$$+B\mu\left[-249\cos\theta\cos(\theta/2) + 251\sin\theta\left(-\sin(\theta/2)\right)\right] =$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) +$$
$$+A\mu\left[-98\cos\theta\left(\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right) - 51\cos(\theta/2)\left(\cos^2\theta + \sin^2\theta\right)\right] +$$
$$+B\mu\left[-249\cos\theta\cos(\theta/2) + 251\sin\theta\left(-\sin(\theta/2)\right)\right] =$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) +$$
$$+A\mu\left[-98\cos\theta\cos(\theta/2) - 51\cos(\theta/2)\right] +$$
$$+B\mu\left[-249\cos\theta\cos(\theta/2) - 251\sin\theta\sin(\theta/2)\right] =$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) +$$
$$+A\mu\left[-98\cos\theta\cos(\theta/2) - 51\cos(\theta/2)\right] +$$
$$+B\mu\left[-249\cos\theta\cos(\theta/2) - 249\sin\theta\sin(\theta/2) - 2\sin\theta\sin(\theta/2)\right] =$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) +$$
$$+A\mu\left[-98\cos\theta\cos(\theta/2) - 51\cos(\theta/2)\right] +$$
$$+B\mu\left[-249\left(\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right) - 2\sin\theta\sin(\theta/2)\right] =$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) +$$
$$+A\mu\left[-98\cos\theta\cos(\theta/2) - 51\cos(\theta/2)\right] +$$
$$+B\mu\left[-249\cos(\theta/2) - 2\sin\theta\sin(\theta/2)\right] =$$

$$= -(100A\lambda)\cos\theta\cos(\theta/2) -$$
$$- A\mu\cos(\theta/2)\left[98\cos\theta + 51\right] -$$
$$- B\mu\left[249\cos(\theta/2) + 2\sin\theta\sin(\theta/2)\right]$$

By combining the first and second set of terms, one gets:

$$R_x =$$
$$= (15/4)\, c\, r^{-1/2}\, 25\mu\left(A + 5B\right)\left[\cos(\theta/2)\right] +$$

$$+ (15/4)\, c\, r^{-1/2} \begin{bmatrix} -(100A\lambda)\cos\theta\cos(\theta/2) - \\ - A\mu\cos(\theta/2)\left[98\cos\theta + 51\right] - \\ - B\mu\left[249\cos(\theta/2) + 2\sin\theta\sin(\theta/2)\right] \end{bmatrix} =$$

$$=+(15/4)\ c\ r^{-1/2}\begin{bmatrix} 25\mu(A+5B)[\cos(\theta/2)]- \\ -(100A\lambda)\cos\theta\cos(\theta/2)- \\ -A\mu\cos(\theta/2)[98\cos\theta+51]- \\ -B\mu[249\cos(\theta/2)+2\sin\theta\sin(\theta/2)] \end{bmatrix}=$$

$$+(15/4)\ c\ r^{-1/2}\cos(\theta/2)\begin{bmatrix} 25\mu(A+5B)- \\ -(100A\lambda)\cos\theta- \\ -A\mu[98\cos\theta+51]- \\ -B\mu[249+2\sin\theta\tan(\theta/2)] \end{bmatrix}$$

But since:

$$\sin\theta\tan(\theta/2)=\sin\theta\frac{\sin(\theta/2)}{\cos(\theta/2)}=2\sin(\theta/2)\cos(\theta/2)\frac{\sin(\theta/2)}{\cos(\theta/2)}=2\sin^2(\theta/2)=1-\cos\theta,$$

$$R_x=+(15/4)\ c\ r^{-1/2}\cos(\theta/2)\begin{bmatrix} 25\mu(A+5B)- \\ -(100A\lambda)\cos\theta- \\ -A\mu[98\cos\theta+51]- \\ -B\mu[249+2(1-\cos\theta)] \end{bmatrix}$$

The term $\begin{bmatrix} 25\mu(A+5B)- \\ -(100A\lambda)\cos\theta- \\ -A\mu[98\cos\theta+51]- \\ -B\mu[249+2(1-\cos\theta)] \end{bmatrix}$ becomes:

$$\begin{aligned} &25A\mu+125B\mu-(100A\lambda)\cos\theta-98A\mu\cos\theta-51A\mu-B\mu[249+2-2\cos\theta]= \\ &=A\mu(25-51)+B\mu(125-251)-\cos\theta(100A\lambda+98A\mu-2B\mu)= \\ &=-26A\mu-126B\mu-\cos\theta(100A\lambda+98A\mu-2B\mu)= \\ &=-2(13A+63B)\mu-2\cos\theta(50A\lambda+49A\mu-B\mu)= \\ &=-2\left[(13A+63B)\mu+(50A\lambda+49A\mu-B\mu)\cos\theta\right] \end{aligned}$$

Finally, R_x is:

$$R_x=+(15/4)\ c\ r^{-1/2}\cos(\theta/2)(-2)\left[(13A+63B)\mu+(50A\lambda+49A\mu-B\mu)\cos\theta\right]\Rightarrow$$

$$\Rightarrow \boxed{R_x=(-15/2)\ c\ r^{-1/2}\cos(\theta/2)\left[(13A+63B)\mu+(50A\lambda+49A\mu-B\mu)\cos\theta\right]}$$

<u>CALCULATION OF</u> R_y

$$\mathrm{R}_y = n_x n_x m_{xxy} + n_x n_y m_{xyy} + n_y n_x m_{yxy} + n_y n_y m_{yyy} =$$
$$= \cos^2\theta\, m_{xxy} + \cos\theta\sin\theta\, m_{xyy} + \sin\theta\cos\theta\, m_{yxy} + \sin^2\theta\, m_{yyy} =$$
$$= \cos^2\theta\, m_{xxy} + \cos\theta\sin\theta\,\left(m_{xyy} + m_{yxy}\right) + \sin^2\theta\, m_{yyy}$$

The sum $m_{xyy} + m_{yxy}$ will be estimated first, before proceeding with the rest of the calculations.

$$m_{xyy} + m_{yxy} =$$
$$=(-15/4)\, c\, r^{-1/2}\left[25\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2) + (100\mathrm{A}\lambda + 51\mathrm{A}\mu - 249\mathrm{B}\mu)\cos(\theta/2)\right] +$$
$$+ (-15/4)\, c\,\mu\, r^{-1/2}\left[\,25(\mathrm{A}+5\mathrm{B})\cos(5\theta/2) + (51\mathrm{A}+251\mathrm{B})\cos(\theta/2)\right] =$$
$$=(-15/4)\, c\, r^{-1/2}\left\{\begin{array}{l}\cos(5\theta/2)\left[25\mu(\mathrm{A}+5\mathrm{B}) + 25\mu(\mathrm{A}+5\mathrm{B})\right] + \\ +\cos(\theta/2)\left[\begin{array}{r}100\mathrm{A}\lambda + 51\mathrm{A}\mu - 249\mathrm{B}\mu + \\ +51\mathrm{A}\mu + 251\mathrm{B}\mu\end{array}\right]\end{array}\right\} =$$

$$= (-15/4)\, c\, r^{-1/2}\left[50\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2) + (100\mathrm{A}\lambda + 102\mathrm{A}\mu + 2\mathrm{B}\mu)\cos(\theta/2)\right]$$

By substituting the result of this sum into the expression for R_y, these are obtained:

$$\mathrm{R}_y = \cos^2\theta\, m_{xxy} + \cos\theta\sin\theta\,\left(m_{xyy} + m_{yxy}\right) + \sin^2\theta\, m_{yyy} =$$
$$= \cos^2\theta\,(+15/4)\, c\,\mu\, r^{-1/2}\left[\,25(\mathrm{A}+5\mathrm{B})\sin(5\theta/2) - (49\mathrm{A}+249\mathrm{B})\sin(\theta/2)\right] +$$
$$+\cos\theta\sin\theta\,(-15/4)\, c\, r^{-1/2}\left[50\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2) + (100\mathrm{A}\lambda + 102\mathrm{A}\mu + 2\mathrm{B}\mu)\cos(\theta/2)\right] +$$
$$+\sin^2\theta\,(-15/4)\, c\, r^{-1/2}\left[25\mu(\mathrm{A}+5\mathrm{B})\sin(5\theta/2) + (100\mathrm{A}\lambda + 151\mathrm{A}\mu + 251\mathrm{B}\mu)\sin(\theta/2)\right] =$$

$$= (+15/4)\, c\, r^{-1/2}\; 25\mu(\mathrm{A}+5\mathrm{B})\left[\begin{array}{l}\cos^2\theta\sin(5\theta/2) - \\ -2\cos\theta\sin\theta\cos(5\theta/2) - \\ -\sin^2\theta\sin(5\theta/2)\end{array}\right] +$$

$$+(+15/4)\, c\, r^{-1/2}\left[\begin{array}{l}-(49\mathrm{A}\mu + 249\mathrm{B}\mu)\cos^2\theta\,\sin(\theta/2) - \\ -(100\mathrm{A}\lambda + 102\mathrm{A}\mu + 2\mathrm{B}\mu)\cos\theta\sin\theta\cos(\theta/2) - \\ -(100\mathrm{A}\lambda + 151\mathrm{A}\mu + 251\mathrm{B}\mu)\sin^2\theta\,\sin(\theta/2)\end{array}\right] =$$

The terms enclosed in the first set of brackets give:

$$
\begin{aligned}
&= \cos^2\theta \sin(5\theta/2) - 2\cos\theta\sin\theta\cos(5\theta/2) - \sin^2\theta\ \sin(5\theta/2) = \\
&= \sin(5\theta/2)\left[\cos^2\theta - \sin^2\theta\right] - 2\cos\theta\sin\theta\cos(5\theta/2) = \\
&= \sin(5\theta/2)\cos(2\theta) - \sin(2\theta)\cos(5\theta/2) = \\
&= \sin\left(\frac{5\theta}{2} - 2\theta\right) = \sin(\theta/2)
\end{aligned}
$$

whereas the terms enclosed in the second ones are:

$$
\begin{aligned}
&-49\mathrm{A}\mu\cos^2\theta\ \sin(\theta/2) - 249\mathrm{B}\mu\cos^2\theta\ \sin(\theta/2) - \\
&-100\mathrm{A}\lambda\cos\theta\sin\theta\cos(\theta/2) - 102\mathrm{A}\mu\cos\theta\sin\theta\cos(\theta/2) - 2\mathrm{B}\mu\cos\theta\sin\theta\cos(\theta/2) - \\
&-100\mathrm{A}\lambda\sin^2\theta\ \sin(\theta/2) - 151\mathrm{A}\mu\sin^2\theta\ \sin(\theta/2) - 251\mathrm{B}\mu\sin^2\theta\ \sin(\theta/2) =
\end{aligned}
$$

$$
\begin{aligned}
&= (-100\mathrm{A}\lambda)\sin\theta\left[\cos\theta\ \cos(\theta/2) + \sin\theta\ \sin(\theta/2)\right] - \\
&-\mathrm{A}\mu\left[\ 49\cos^2\theta\ \sin(\theta/2) + 102\mathrm{A}\mu\cos\theta\sin\theta\cos(\theta/2) + 151\sin^2\theta\ \sin(\theta/2)\right] + \\
&+\mathrm{B}\mu\left[-249\cos^2\theta\ \sin(\theta/2) - 2\cos\theta\sin\theta\cos(\theta/2) - 251\sin^2\theta\ \sin(\theta/2)\right] =
\end{aligned}
$$

Noticing the last representation, its first line becomes:

$$
\begin{aligned}
&(-100\mathrm{A}\lambda)\sin\theta\left[\cos\theta\ \cos(\theta/2) + \sin\theta\ \sin(\theta/2)\right] = \\
&= (-100\mathrm{A}\lambda)\sin\theta\cos(\theta/2) = \\
&= (-100\mathrm{A}\lambda)2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2) = \\
&= (-100\mathrm{A}\lambda)\sin(\theta/2)\times 2\cos^2(\theta/2) = \\
&= (-100\mathrm{A}\lambda)\sin(\theta/2)\left(1+\cos(\theta)\right)
\end{aligned}
$$

its second one:

$$
-\mathrm{A}\mu\left[\ 49\cos^2\theta\ \sin(\theta/2) + 102\cos\theta\sin\theta\cos(\theta/2) + 151\sin^2\theta\ \sin(\theta/2)\right] =
$$

$$
= -\mathrm{A}\mu\begin{bmatrix} 49\cos^2\theta\ \sin(\theta/2) + 102\cos\theta\sin\theta\cos(\theta/2) + \\ +49\sin^2\theta\ \sin(\theta/2) + 102\sin^2\theta\ \sin(\theta/2) \end{bmatrix} =
$$

$$=-\mathrm{A}\mu\begin{bmatrix} 49\sin(\theta/2)\left(\cos^2\theta+\sin^2\theta\right)+ \\ +102\sin\theta\left(\cos\theta\cos(\theta/2)+\sin\theta\,\sin(\theta/2)\right) \end{bmatrix}=$$

$$=-\mathrm{A}\mu\left[\,49\sin(\theta/2)+102\sin\theta\cos(\theta/2)\right]=$$

$$=-\mathrm{A}\mu\left[\,49\sin(\theta/2)+102\times2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2)\right]=$$

$$=-\mathrm{A}\mu\left[\,49\sin(\theta/2)+102\times2\sin(\theta/2)\cos^2(\theta/2)\right]=$$

$$=-\mathrm{A}\mu\sin(\theta/2)\left[\,49+102\times2\cos^2(\theta/2)\right]=$$

$$=-\mathrm{A}\mu\sin(\theta/2)\left[\,49+102\left(1+\cos\theta\right)\right]$$

and finally its third one:

$$+\mathrm{B}\mu\left[-249\cos^2\theta\,\sin(\theta/2)-2\cos\theta\sin\theta\cos(\theta/2)-251\sin^2\theta\,\sin(\theta/2)\right]=$$

$$=\mathrm{B}\mu\begin{bmatrix} -249\cos^2\theta\,\sin(\theta/2)-2\cos\theta\sin\theta\cos(\theta/2)- \\ -249\sin^2\theta\,\sin(\theta/2)-2\sin^2\theta\,\sin(\theta/2) \end{bmatrix}=$$

$$=\mathrm{B}\mu\left\{-249\sin(\theta/2)\left[\cos^2\theta+\sin^2\theta\,\right]-2\sin\theta\left[\cos\theta\cos(\theta/2)+\sin\theta\,\sin(\theta/2)\right]\right\}=$$

$$=\mathrm{B}\mu\left[-249\sin(\theta/2)-2\sin\theta\cos(\theta/2)\right]=$$

$$=\mathrm{B}\mu\left[-249\sin(\theta/2)-2\times2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2)\right]=$$

$$=\mathrm{B}\mu\left[-249-2\sin(\theta/2)\times2\cos^2(\theta/2)\right]=$$

$$=\mathrm{B}\mu\,\sin(\theta/2)\left[-249-2\left(1+\cos\theta\right)\right]$$

After having performed these simplifications, the expression

$$(-100\mathrm{A}\lambda)\sin\theta\left[\cos\theta\,\cos(\theta/2)+\sin\theta\,\sin(\theta/2)\right]-$$
$$-\mathrm{A}\mu\left[\,49\cos^2\theta\,\sin(\theta/2)+102\mathrm{A}\mu\cos\theta\sin\theta\cos(\theta/2)+151\sin^2\theta\,\sin(\theta/2)\right]+$$
$$+\mathrm{B}\mu\left[-249\cos^2\theta\,\sin(\theta/2)-2\cos\theta\sin\theta\cos(\theta/2)-251\sin^2\theta\,\sin(\theta/2)\right]$$

becomes:

$$(-100\mathrm{A}\lambda)\sin(\theta/2)(1+\cos(\theta)-$$
$$-\mathrm{A}\mu\sin(\theta/2)\left[\,49+102(1+\cos\theta)\right]$$
$$+\mathrm{B}\mu\,\sin(\theta/2)\left[-249-2(1+\cos\theta)\right]$$

Finally, the full expression for R_y is:

$$\mathrm{R}_y=(+15/4)\,\mathrm{c\,r}^{-1/2}\;25\mu(\mathrm{A}+5\mathrm{B})\sin(\theta/2)+$$

$$+(+15/4)\,\mathrm{c\,r}^{-1/2}\begin{bmatrix}(-100\mathrm{A}\lambda)\sin(\theta/2)(1+\cos(\theta)-\\ -\mathrm{A}\mu\sin(\theta/2)\left[\,49+102(1+\cos\theta)\right]+\\ +\mathrm{B}\mu\,\sin(\theta/2)\left[-249-2(1+\cos\theta)\right]\end{bmatrix}=$$

$$=(+15/4)\,\mathrm{c\,r}^{-1/2}\;\sin(\theta/2)\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})+\\ +(-100\mathrm{A}\lambda)(1+\cos(\theta)-\\ -\mathrm{A}\mu\left[\,49+102(1+\cos\theta)\right]-\\ -\mathrm{B}\mu\left[249+2(1+\cos\theta)\right]\end{Bmatrix}$$

The term in brackets is:

$$25\mu(\mathrm{A}+5\mathrm{B})-100\mathrm{A}\lambda(1+\cos(\theta)-\mathrm{A}\mu\left[\,49+102(1+\cos\theta)\right]-\mathrm{B}\mu\left[249+2(1+\cos\theta)\right]=$$
$$=-100\mathrm{A}\lambda+\mathrm{A}\mu\left[25-49-102\right]+\mathrm{B}\mu\left[125-249-2\right]-$$
$$-\cos\theta\left[100\mathrm{A}\lambda+102\mathrm{A}\mu+2\mathrm{B}\mu\right]=$$
$$=-100\mathrm{A}\lambda-126\mathrm{A}\mu-126\mathrm{B}\mu-\cos\theta\left[100\mathrm{A}\lambda+102\mathrm{A}\mu+2\mathrm{B}\mu\right]=$$
$$=-2\left\{50\mathrm{A}\lambda+63(\mathrm{A}\mu+\mathrm{B}\mu)+\cos\theta\left[50\mathrm{A}\lambda+51\mathrm{A}\mu+\mathrm{B}\mu\right]\right\}$$

The final form for R_y is:

$$\mathrm{R}_y=(+15/4)\,\mathrm{c\,r}^{-1/2}\;\sin(\theta/2)\cdot$$
$$\cdot(-2)\left\{50\mathrm{A}\lambda+63(\mathrm{A}\mu+\mathrm{B}\mu)+\cos\theta\left[50\mathrm{A}\lambda+51\mathrm{A}\mu+\mathrm{B}\mu\right]\right\}\Rightarrow$$

$$\Rightarrow \boxed{R_y = (-15/2)c\,r^{-1/2}\sin(\theta/2)\left\{50A\lambda + 63(A\mu + B\mu) + (50A\lambda + 51A\mu + B\mu)\cos\theta\right\}}$$

Calculation of the partial derivatives of the displacements

Before continuing, it is presupposed to determine the following partial derivatives: $\frac{\partial u_x}{\partial x}, \frac{\partial^2 u_x}{\partial x^2}, \frac{\partial^2 u_x}{\partial x \partial y}$ and $\frac{\partial u_y}{\partial x}, \frac{\partial^2 u_y}{\partial x^2}, \frac{\partial^2 u_y}{\partial x \partial y}$

CALCULATION OF THE PARTIAL DERIVATIVES

$\frac{\partial u_x}{\partial x}, \frac{\partial^2 u_x}{\partial x^2}, \frac{\partial^2 u_x}{\partial x \partial y}$ and $\frac{\partial u_y}{\partial x}, \frac{\partial^2 u_y}{\partial x^2}, \frac{\partial^2 u_y}{\partial x \partial y}$

DEFINITIONS OF u_x and u_y

$$u_x = (-5/2)r^{3/2}\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right]$$

$$u_y = (-15/2)r^{3/2}\left[(17A\text{-}83B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right]$$

1. $\frac{\partial u_x}{\partial x}$

$$\frac{\partial u_x}{\partial x} = \cos\theta\,\frac{\partial}{\partial r}(u_x) - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}(u_x) =$$

$$= \cos\theta\,\frac{\partial}{\partial r}\left[(-\frac{5}{2})\,r^{3/2}\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right]\right] -$$

$$-\frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\left[(-\frac{5}{2})\,r^{3/2}\,\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right]\right] =$$

$$= \cos\theta(-\frac{5}{2})\frac{3}{2}r^{1/2}\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right] -$$

$$-\frac{\sin\theta}{r}(-\frac{5}{2})r^{3/2}\left[(49A\text{-}251B)(\text{-}1)\sin(3\theta/2)\frac{3}{2} + 75(A+5B)(\text{-}1)\sin(\theta/2)\frac{1}{2}\right] \Rightarrow$$

$$\Rightarrow \frac{\partial u_x}{\partial x} = (-\frac{15}{4})\cos\theta\, r^{1/2}\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right] +$$

$$+ (-\frac{15}{4})\sin\theta\, r^{1/2}\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] =$$

$$= (-\frac{15}{4}) r^{1/2} \begin{Bmatrix} \cos\theta\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right] + \\ +\sin\theta\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] \end{Bmatrix}$$

By further expansion of the term in brackets, we obtain:

$$\cos\theta\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right] +$$
$$+\sin\theta\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] =$$

$$= (49A\text{-}251B)\left[\cos\theta\cos(3\theta/2) + \sin\theta\sin(3\theta/2)\right] +$$
$$+ 25(A+5B)\left[3\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] =$$

$$= (49A\text{-}251B)\left[\cos(\theta/2)\right] +$$
$$+25(A+5B)\left[2\cos\theta\cos(\theta/2) + \cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] =$$
$$[\text{since: } \cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2) = \cos(\theta - \frac{\theta}{2}) = \cos(\theta/2)]$$
$$= (49A\text{-}251B)\left[\cos(\theta/2)\right] +$$
$$+25(A+5B)\left[2\cos\theta\cos(\theta/2) + \cos(\theta/2)\right] =$$

$$= \cos(\theta/2)\left[(49A\text{-}251B) + 25(A+5B)(2\cos\theta + 1)\right]$$

After this simplification, the term $\frac{\partial u_x}{\partial x}$ is defined as:

$$\boxed{\frac{\partial u_x}{\partial x} = (-\frac{15}{4}) r^{1/2}\cos(\theta/2)\left[(49A\text{-}251B) + 25(A+5B)(2\cos\theta + 1)\right]}$$

2. $\frac{\partial^2 u_x}{\partial x^2}$

$$\frac{\partial^2 u_x}{\partial x^2} = \cos\theta\ \frac{\partial}{\partial r}\left(\frac{\partial u_x}{\partial x}\right) - \frac{\sin\theta}{\text{r}}\ \frac{\partial}{\partial\theta}\left(\frac{\partial u_x}{\partial x}\right) =$$

$$= \cos\theta\ \frac{\partial}{\partial r}\left[(-\frac{15}{4})\text{r}^{1/2}\cos(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right]\right] -$$

$$-\frac{\sin\theta}{\text{r}}\ \frac{\partial}{\partial\theta}\left[(-\frac{15}{4})\text{r}^{1/2}\cos(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right]\right] =$$

$$= \cos\theta(-\frac{15}{8})\text{r}^{-1/2}\cos(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] -$$

$$-\frac{\sin\theta}{\text{r}}(-\frac{15}{4})\text{r}^{1/2}\begin{bmatrix} -\sin(\theta/2)\frac{1}{2}\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] + \\ +\cos(\theta/2)\left[0+25(\text{A}+5\text{B})(\text{-}2\sin\theta+0)\right] \end{bmatrix} =$$

$$= (-\frac{15}{8})\text{r}^{-1/2}\cos\theta\cos(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] +$$

$$+(+\frac{15}{4})\text{r}^{-1/2}\sin\theta\begin{bmatrix} -\frac{1}{2}\sin(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] - \\ -50(\text{A}+5\text{B})\cos(\theta/2)\sin\theta \end{bmatrix} =$$

$$= (-\frac{15}{8})\text{r}^{-1/2}\cos\theta\cos(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] +$$

$$+(-\frac{15}{8})\text{r}^{-1/2}\sin\theta\begin{bmatrix} \sin(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] + \\ +100(\text{A}+5\text{B})\cos(\theta/2)\sin\theta \end{bmatrix} =$$

$$= (-\frac{15}{8})\text{r}^{-1/2}\begin{Bmatrix} \cos\theta\cos(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] + \\ +\sin\theta\sin(\theta/2)\left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right] + \\ +100(\text{A}+5\text{B})\sin^2\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (-\frac{15}{8})\text{r}^{-1/2}\begin{Bmatrix} \left[(49\text{A-}251\text{B})+25(\text{A}+5\text{B})(2\cos\theta+1)\right]\left[\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right] + \\ +100(\text{A}+5\text{B})\sin^2\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (-\frac{15}{8}) r^{-1/2} \begin{Bmatrix} \left[(49A\text{-}251B) + 25(A+5B)(2\cos\theta + 1)\right]\cos(\theta/2) + \\ +100(A+5B)\sin^2\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (-\frac{15}{8}) r^{-1/2} \cos(\theta/2) \begin{Bmatrix} \left[(49A\text{-}251B) + 25(A+5B)(2\cos\theta + 1)\right] + \\ +100(A+5B)\sin^2\theta \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \frac{\partial^2 u_x}{\partial x^2} = (-\frac{15}{8}) r^{-1/2} \cos(\theta/2) \left\{(49A\text{-}251B) + 25(A+5B)(2\cos\theta + 1 + 4\sin^2\theta)\right\}$$

Also:

$$2\cos\theta + 1 + 4\sin^2\theta = 2\cos\theta + 1 + 4\frac{1-\cos(2\theta)}{2} =$$

$$= 2\cos\theta + 1 + 2 - 2\cos(2\theta) = 3 + 2\cos\theta - 2\cos(2\theta)$$

$$\Rightarrow \boxed{\frac{\partial^2 u_x}{\partial x^2} = (-\frac{15}{8}) r^{-1/2} \cos(\theta/2) \left\{(49A\text{-}251B) + 25(A+5B)\left[3 + 2\cos\theta - 2\cos(2\theta)\right]\right\}}$$

3. $\frac{\partial^2 u_x}{\partial x \partial y}$

$$\frac{\partial^2 u_x}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}\right) = \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial u_x}{\partial x}\right) + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}\left(\frac{\partial u_x}{\partial x}\right) =$$

$$= \sin\theta(-\frac{15}{8}) r^{-1/2} \begin{Bmatrix} \cos\theta\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right] + \\ +\sin\theta\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] \end{Bmatrix} +$$

$$+ \frac{\cos\theta}{r}(-\frac{15}{8}) r^{1/2} \begin{Bmatrix} -\cos\theta\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] + \\ +\sin\theta\left[(49A\text{-}251B)\cos(3\theta/2) - 125(A+5B)\cos(\theta/2)\right] \end{Bmatrix} =$$

$$= (-\frac{15}{8}) r^{-1/2} \begin{Bmatrix} \sin\theta\cos\theta\left[(49A\text{-}251B)\cos(3\theta/2) + 75(A+5B)\cos(\theta/2)\right] + \\ +\sin^2\theta\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] \end{Bmatrix} +$$

$$+ (-\frac{15}{8}) r^{-1/2} \begin{Bmatrix} -\cos^2\theta\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] + \\ +\sin\theta\cos\theta\left[(49A\text{-}251B)\cos(3\theta/2) - 125(A+5B)\cos(\theta/2)\right] \end{Bmatrix} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \begin{Bmatrix} \left(\sin^2\theta - \cos^2\theta\right)\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] + \\ +\sin\theta\cos\theta\left[2(49A\text{-}251B)\cos(3\theta/2) - 50(A+5B)\cos(\theta/2)\right] \end{Bmatrix} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \begin{Bmatrix} \left(-\cos(2\theta)\right)\left[(49A\text{-}251B)\sin(3\theta/2) + 25(A+5B)\sin(\theta/2)\right] + \\ +\sin(2\theta)\left[(49A\text{-}251B)\cos(3\theta/2) - 25(A+5B)\cos(\theta/2)\right] \end{Bmatrix} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \begin{Bmatrix} -(49A\text{-}251B)\cos(2\theta)\sin(3\theta/2) - 25(A+5B)\cos(2\theta)\sin(\theta/2) + \\ +(49A\text{-}251B)\sin(2\theta)\cos(3\theta/2) - 25(A+5B)\sin(2\theta)\cos(\theta/2) \end{Bmatrix} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \begin{Bmatrix} (49A\text{-}251B)\left[-\cos(2\theta)\sin(3\theta/2) + \sin(2\theta)\cos(3\theta/2)\right] - \\ -25(A+5B)\left[\cos(2\theta)\sin(\theta/2) + \sin(2\theta)\cos(\theta/2)\right] \end{Bmatrix}$$

but since:

$$\left[-\cos(2\theta)\sin(3\theta/2) + \sin(2\theta)\cos(3\theta/2)\right] = \sin(2\theta - \frac{3\theta}{2}) = \sin(\theta/2)$$

$$\left[\cos(2\theta)\sin(\theta/2) + \sin(2\theta)\cos(\theta/2)\right] = \sin(2\theta + \frac{\theta}{2}) = \sin(5\theta/2)$$

$$\Rightarrow \frac{\partial^2 u_x}{\partial x \partial y} = (-\frac{15}{8})\, r^{-1/2} \left\{(49A\text{-}251B)\sin(\theta/2) - 25(A+5B)\sin(5\theta/2)\right\}$$

If we choose to make the term sin(θ/2) common multiplier, the division $\frac{\sin(5\theta/2)}{\sin(\theta/2)}$ yields to:

$$\frac{\sin(5\theta/2)}{\sin(\theta/2)} = \frac{\sin(2\theta)\cos(\theta/2) + \cos(2\theta)\sin(\theta/2)}{\sin(\theta/2)} =$$
$$= \frac{\sin(2\theta)\cos(\theta/2)}{\sin(\theta/2)} + \cos(2\theta) = \frac{2\sin\theta\cos\theta\cos(\theta/2)}{\sin(\theta/2)} + \cos(2\theta) =$$
$$= \frac{2\cdot 2\sin(\theta/2)\cos(\theta/2)\cos\theta\cos(\theta/2)}{\sin(\theta/2)} + \cos(2\theta) =$$

$$= 4\cos^2(\theta/2)\cos\theta + \cos(2\theta) = 4\frac{1+\cos\theta}{2}\cos\theta + \cos(2\theta) =$$
$$= 2\cos\theta + 2\cos^2\theta + \cos(2\theta) = 2\cos\theta + [1+\cos(2\theta)] + \cos(2\theta) =$$
$$= 1 + 2\cos\theta + 2\cos(2\theta)$$

Finally, we get:

$$\boxed{\frac{\partial^2 u_x}{\partial x \partial y} = (-\frac{15}{8})r^{-1/2}\sin(\theta/2)\left\{(49A\text{-}251B) - 25(A+5B)\left[1+2\cos\theta+2\cos(2\theta)\right]\right\}}$$

4. $\frac{\partial u_y}{\partial x}$

$$\frac{\partial u_y}{\partial x} = \cos\theta\,\frac{\partial}{\partial r}\left(u_y\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(u_y\right) =$$

$$= \cos\theta\,\frac{\partial}{\partial r}\left[(-\frac{15}{2})r^{3/2}\left[(17A\text{-}83B)\sin(3\theta/2)+25(A+5B)\sin(\theta/2)\right]\right] -$$

$$-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left[(-\frac{15}{2})r^{3/2}\left[(17A\text{-}83B)\sin(3\theta/2)+25(A+5B)\sin(\theta/2)\right]\right] =$$

$$= \cos\theta(-\frac{15}{2})\frac{3}{2}r^{1/2}\left[(17A\text{-}83B)\sin(3\theta/2)+25(A+5B)\sin(\theta/2)\right] -$$

$$-\frac{\sin\theta}{r}(-\frac{15}{2})r^{3/2}\left[(17A\text{-}83B)\cos(3\theta/2)\frac{3}{2}+25(A+5B)\cos(\theta/2)\frac{1}{2}\right] \Rightarrow$$

$$\Rightarrow \frac{\partial u_y}{\partial x} = \cos\theta(-\frac{15}{4})r^{1/2}3\left[(17A\text{-}83B)\sin(3\theta/2)+25(A+5B)\sin(\theta/2)\right] +$$

$$+\frac{\sin\theta}{r}(+\frac{15}{4})r^{3/2}\left[3(17A\text{-}83B)\cos(3\theta/2)+25(A+5B)\cos(\theta/2)\right] =$$

$$= (-\frac{15}{4})r^{1/2}\left\{\begin{array}{l}\cos\theta\left[3(17A\text{-}83B)\sin(3\theta/2)+75(A+5B)\sin(\theta/2)\right]- \\ -\sin\theta\left[3(17A\text{-}83B)\cos(3\theta/2)+25(A+5B)\cos(\theta/2)\right]\end{array}\right\}$$

The term in brackets can be simplified by performing these manipulations:

$$\cos\theta\left[3(17A\text{-}83B)\sin(3\theta/2)+75(A+5B)\sin(\theta/2)\right]-$$
$$-\sin\theta\left[3(17A\text{-}83B)\cos(3\theta/2)+25(A+5B)\cos(\theta/2)\right]=$$

$$= 3(17A\text{-}83B)\left[\cos\theta\sin(3\theta/2)\text{-}\sin\theta\cos(3\theta/2)\right]+$$
$$+25(A+5B)\left[3\cos\theta\sin(\theta/2)\text{-}\sin\theta\cos(\theta/2)\right]=$$

[because: $\cos\theta\sin(3\theta/2)\text{-}\sin\theta\cos(3\theta/2) = \sin(\frac{3\theta}{2} - \theta)$]

$= 3(17A\text{-}83B)[\sin(\theta/2)] + 25(A+5B)[2\cos\theta\sin(\theta/2) + \cos\theta\sin(\theta/2)\text{-}\sin\theta\cos(\theta/2)] =$

[but: $\cos\theta\sin(\theta/2)\text{-}\sin\theta\cos(\theta/2) = \sin(\frac{\theta}{2} - \theta) = -\sin(\frac{\theta}{2})$]

$= 3(17A\text{-}83B)\sin(\theta/2) + 25(A+5B)[2\cos\theta\sin(\theta/2)\text{-}\sin(\theta/2)] =$

$$= 3(17A\text{-}83B)\sin(\theta/2) + 25(A+5B)\sin(\theta/2)\left[2\cos\theta\text{-}1\right]$$

$$= \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]$$

Having performed the above, $\frac{\partial u_y}{\partial x}$ becomes:

$$\boxed{\frac{\partial u_y}{\partial x} = (-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]}$$

5. $\frac{\partial^2 u_y}{\partial x^2}$

$$\frac{\partial^2 u_y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u_y}{\partial x}\right) =$$

$$= \cos\theta\,\frac{\partial}{\partial r}\left(\frac{\partial u_y}{\partial x}\right) - \frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\left(\frac{\partial u_y}{\partial x}\right) =$$

$$= \cos\theta\,\frac{\partial}{\partial r}\left\{\left(-\frac{15}{4}\right)r^{1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\} -$$

$$-\frac{\sin\theta}{r}\,\frac{\partial}{\partial \theta}\left\{\left(-\frac{15}{4}\right)r^{1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\}$$

At this point, the following partial derivatives will be estimated and then substituted in the expressions written above.

- $\frac{\partial}{\partial r}\left\{\left(-\frac{15}{4}\right)r^{1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\}$

- $\frac{\partial}{\partial \theta}\left\{\left(-\frac{15}{4}\right)r^{1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\}$

- Estimation of: $\frac{\partial}{\partial r}\left\{\left(-\frac{15}{4}\right) r^{1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]\right\}$.

$$\frac{\partial}{\partial r}\left\{\left(-\frac{15}{4}\right) r^{1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]\right\}=$$

$$=\left(-\frac{15}{4}\right)\frac{1}{2} r^{-1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]=$$

$$=\left(-\frac{15}{8}\right) r^{-1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]$$

- Estimation of: $\frac{\partial}{\partial \theta}\left\{\left(-\frac{15}{4}\right) r^{1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]\right\}$.

$$\frac{\partial}{\partial \theta}\left\{\left(-\frac{15}{4}\right) r^{1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]\right\}=$$

$$=\left(-\frac{15}{4}\right) r^{1/2}\left\{\begin{array}{l}\cos(\theta/2)\frac{1}{2}\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]+\\ +\sin(\theta/2)\bullet\frac{\partial}{\partial\theta}\left(3(17A-83B)+25(A+5B)(2\cos\theta-1)\right)\end{array}\right\}=$$

$$=\left(-\frac{15}{4}\right) r^{1/2}\left\{\begin{array}{l}\cos(\theta/2)\frac{1}{2}\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]+\\ +\sin(\theta/2)\left[25(A+5B)(-2\sin\theta)\right]\end{array}\right\}=$$

$$=\left(-\frac{15}{4}\right) r^{1/2}\left\{\begin{array}{l}\cos(\theta/2)\frac{1}{2}\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]+\\ +\sin(\theta/2)\left[-50(A+5B)\sin\theta\right]\end{array}\right\}$$

When these partial derivatives are substituted in the expression of $\frac{\partial^2 u_y}{\partial x^2}$, we obtain:

$$\frac{\partial^2 u_y}{\partial x^2}=\cos\theta\left(-\frac{15}{8}\right) r^{-1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]-$$
$$-\frac{\sin\theta}{r}\left(-\frac{15}{4}\right) r^{1/2}\left\{\begin{array}{l}\cos(\theta/2)\frac{1}{2}\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]+\\ +\sin(\theta/2)\left[-50(A+5B)\sin\theta\right]\end{array}\right\}=$$

$$=\cos\theta\left(-\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]+$$

$$+\sin\theta\left(+\frac{15}{8}\right)r^{-1/2}\left\{\begin{array}{l}\cos(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]- \\ -100(A+5B)\sin\theta\sin(\theta/2)\end{array}\right\}=$$

$$=\left(-\frac{15}{8}\right)r^{-1/2}\left\{\begin{array}{l}\cos\theta\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]- \\ -\sin\theta\cos(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]+ \\ +100(A+5B)\sin^2\theta\sin(\theta/2)\end{array}\right\}=$$

$$=\left(-\frac{15}{8}\right)r^{-1/2}\left\{\begin{array}{l}\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]\left[\cos\theta\sin(\theta/2)-\sin\theta\cos(\theta/2)\right]+ \\ +100(A+5B)\sin^2\theta\sin(\theta/2)\end{array}\right\}=$$

$$\left[\text{since: }\cos\theta\sin(\theta/2)-\sin\theta\cos(\theta/2)=\sin\left(\frac{\theta}{2}-\theta\right)=-\sin(\theta/2)\right]$$

$$=\left(-\frac{15}{8}\right)r^{-1/2}\left\{\begin{array}{l}\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]\left[-\sin(\theta/2)\right]+ \\ +100(A+5B)\sin^2\theta\sin(\theta/2)\end{array}\right\}\Rightarrow$$

$$\Rightarrow\frac{\partial^2 u_y}{\partial x^2}=\left(-\frac{15}{8}\right)r^{-1/2}\left[-\sin(\theta/2)\right]\left\{\begin{array}{l}\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]- \\ -100(A+5B)\sin^2\theta\end{array}\right\}=$$

$$=\left(+\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left\{\begin{array}{l}3(17A\text{-}83B)+50(A+5B)\cos\theta-25(A+5B)- \\ -100(A+5B)\sin^2\theta\end{array}\right\}$$

By further analyzing the term $\left\{\begin{array}{l}3(17A\text{-}83B)+50(A+5B)\cos\theta-25(A+5B)- \\ -100(A+5B)\sin^2\theta\end{array}\right\}$,

the following representation derives:

$$\begin{array}{l}3(17A\text{-}83B)+50(A+5B)\cos\theta-25(A+5B)- \\ -100(A+5B)\sin^2\theta= \\ =3(17A\text{-}83B)+50(A+5B)\cos\theta-25(A+5B)- \\ -100(A+5B)\dfrac{1\text{-}\cos(2\theta)}{2}= \\ =3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta-1-2+2\cos(2\theta)\right)= \\ =3(17A\text{-}83B)+25(A+5B)\left(-3+2\cos\theta+2\cos(2\theta)\right)\end{array}$$

Eventually, $\frac{\partial^2 u_y}{\partial x^2}$ is calculated to be:

$$\boxed{\frac{\partial^2 u_y}{\partial x^2} = \left(+\frac{15}{8}\right) r^{-1/2} \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(-3+2\cos\theta+2\cos(2\theta)\right)\right]}$$

6. $\frac{\partial^2 u_y}{\partial x \partial y}$

The partial derivative $\frac{\partial u_y}{\partial x}$ has already been calculated to be:

$$\frac{\partial u_y}{\partial x} = (-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]$$

So,

$$\frac{\partial^2 u_y}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial x}\right) = \sin\theta \frac{\partial}{\partial r}\left(\frac{\partial u_y}{\partial x}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_y}{\partial x}\right) =$$

$$= \sin\theta \frac{\partial}{\partial r}\left\{(-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\} +$$

$$+ \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left\{(-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\}$$

The aforementioned expression includes the term

$$\frac{\partial}{\partial \theta}\left\{(-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(2\cos\theta\text{-}1)\right]\right\}$$

that has been estimated previously to be:

$$\frac{\partial}{\partial \theta}\left\{(-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right]\right\} =$$

$$= \left(-\frac{15}{4}\right) r^{1/2} \left\{\begin{array}{l} \cos(\theta/2)\frac{1}{2}\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right] + \\ +\sin(\theta/2)\left[-50(A+5B)\sin\theta\right] \end{array}\right\} =$$

$$= \left(-\frac{15}{8}\right) r^{1/2} \left\{\begin{array}{l} \cos(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(2\cos\theta\text{-}1\right)\right] - \\ -100(A+5B)\sin\theta\sin(\theta/2) \end{array}\right\}$$

This way

$$\frac{\partial^2 u_y}{\partial x \partial y} = \sin\theta\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left(2\cos\theta\text{-}1\right)\right]+$$

$$+\frac{\cos\theta}{\mathrm{r}}\left(-\frac{15}{8}\right)\mathrm{r}^{1/2}\left\{\begin{array}{l}\cos(\theta/2)\left[3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left(2\cos\theta\text{-}1\right)\right]- \\ -100(\mathrm{A}+5\mathrm{B})\sin\theta\sin(\theta/2)\end{array}\right\}=$$

$$=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\left\{\begin{array}{l}\sin\theta\sin(\theta/2)\left[3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left(2\cos\theta\text{-}1\right)\right]+ \\ +\cos\theta\cos(\theta/2)\left[3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left(2\cos\theta\text{-}1\right)\right]- \\ -100(\mathrm{A}+5\mathrm{B})\cos\theta\sin\theta\sin(\theta/2)\end{array}\right\}=$$

$$=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\left\{\begin{array}{l}\left[3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left(2\cos\theta\text{-}1\right)\right]\left[\sin\theta\sin(\theta/2)+\cos\theta\cos(\theta/2)\right]- \\ -100(\mathrm{A}+5\mathrm{B})\cos\theta\sin\theta\sin(\theta/2)\end{array}\right\}$$

$$\left(\begin{array}{l}\sin\theta\sin(\theta/2)+\cos\theta\cos(\theta/2)=\cos(\theta-\frac{\theta}{2})=\cos(\theta/2) \\ \cos\theta\sin\theta=\frac{1}{2}\sin(2\theta) \\ \cos\theta\sin\theta\sin(\theta/2)=\cos\theta\ 2\sin(\theta/2)\ \cos(\theta/2)\ \sin(\theta/2)=2\cos\theta\cos(\theta/2)\sin^2(\theta/2)\end{array}\right)$$

This derivative can be rewritten in this form:

$$\frac{\partial^2 u_y}{\partial x \partial y}=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\left\{\begin{array}{l}\left[3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(2\cos\theta\text{-}1)\right]\cos(\theta/2)- \\ -200(\mathrm{A}+5\mathrm{B})\cos\theta\cos(\theta/2)\sin^2(\theta/2)\end{array}\right\}\Rightarrow$$

$$\Rightarrow\frac{\partial^2 u_y}{\partial x \partial y}=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left\{\begin{array}{l}3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(2\cos\theta\text{-}1)- \\ -200(\mathrm{A}+5\mathrm{B})\cos\theta\sin^2(\theta/2)\end{array}\right\}$$

$$\Rightarrow\frac{\partial^2 u_y}{\partial x \partial y}=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left\{3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left[2\cos\theta\text{-}1\text{-}8\cos\theta\sin^2(\theta/2)\right]\right\}$$

but:

$$2\cos\theta\text{-}1\text{-}8\cos\theta\sin^2(\theta/2)=2\cos\theta\left[1-4\sin^2(\theta/2)\right]-1=$$

$$=2\cos\theta\left[1-4\frac{1-\cos\theta}{2}\right]-1=2\cos\theta\left[1-2+2\cos\theta\right]-1=$$

$$=-2\cos\theta+4\cos^2\theta-1=-2\cos\theta+4\frac{1+\cos(2\theta)}{2}-1=$$

$$=-2\cos\theta+2+2\cos(2\theta)-1=1-2\cos\theta+2\cos(2\theta)$$

and consequently:

$$\boxed{\frac{\partial^2 u_y}{\partial x \partial y} = \left(-\frac{15}{8}\right) r^{-1/2} \cos(\theta/2)\left\{3(17A\text{-}83B)+25(A+5B)\left[1-2\cos\theta+2\cos(2\theta)\right]\right\}}$$

<u>Calculation of</u> $D\left(\frac{\partial u_x}{\partial x}\right) = n_x \frac{\partial^2 u_x}{\partial x^2} + n_y \frac{\partial^2 u_x}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_x}{\partial x^2} + \sin\theta \frac{\partial^2 u_x}{\partial x \partial y}$

$$D\left(\frac{\partial u_x}{\partial x}\right) = n_x \frac{\partial^2 u_x}{\partial x^2} + n_y \frac{\partial^2 u_x}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_x}{\partial x^2} + \sin\theta \frac{\partial^2 u_x}{\partial x \partial y} =$$

$$= \cos\theta(-\frac{15}{8}) r^{-1/2} \cos(\theta/2)\left\{(49A\text{-}251B)+25(A+5B)\left[3+2\cos\theta-2\cos(2\theta)\right]\right\} +$$

$$+ \sin\theta(-\frac{15}{8}) r^{-1/2} \sin(\theta/2)\left\{(49A\text{-}251B)-25(A+5B)\left[1+2\cos\theta+2\cos(2\theta)\right]\right\} =$$

$$= (-\frac{15}{8}) r^{-1/2} \left\{\begin{array}{l} (49A\text{-}251B)\left[\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right] + \\ +25(A+5B)\left[\begin{array}{l}\cos\theta\cos(\theta/2)\left[3+2\cos\theta-2\cos(2\theta)\right] - \\ -\sin\theta\sin(\theta/2)\left[1+2\cos\theta+2\cos(2\theta)\right]\end{array}\right]\end{array}\right\} =$$

$$= (-\frac{15}{8}) r^{-1/2} \left\{\begin{array}{l} (49A\text{-}251B)\cos(\theta/2) + \\ +25(A+5B)\cdot \\ \cdot\left[\begin{array}{l}3\cos\theta\cos(\theta/2)+2\cos^2\theta\cos(\theta/2)-2\cos\theta\cos(\theta/2)\cos(2\theta) - \\ -\sin\theta\sin(\theta/2)-2\sin\theta\sin(\theta/2)\cos\theta-2\sin\theta\sin(\theta/2)\cos(2\theta)\end{array}\right]\end{array}\right\} =$$

$$= (-\frac{15}{8}) r^{-1/2} \left\{\begin{array}{l} (49A\text{-}251B)\cos(\theta/2) + \\ +25(A+5B)\times \\ \times\left[\begin{array}{l}2\cos\theta\cos(\theta/2)+\underline{\cos\theta\cos(\theta/2)} + \\ +2\cos\theta\left[\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)\right] - \\ \underline{-\sin\theta\sin(\theta/2)}-2\cos(2\theta)\left[\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)\right]\end{array}\right]\end{array}\right\} =$$

$$= (-\frac{15}{8}) r^{-1/2} \left\{\begin{array}{l} (49A\text{-}251B)\cos(\theta/2) + \\ +25(A+5B)\left|\begin{array}{l}2\cos\theta\cos(\theta/2)+\underline{\cos(3\theta/2)}+2\cos\theta\cos(3\theta/2) - \\ -2\cos(2\theta)\cos(\theta/2)\end{array}\right|\end{array}\right\}$$

The parenthesis multiplying the term $25(A+5B)$ becomes:

$$
\begin{aligned}
&2\cos\theta\cos(\theta/2)+\cos(3\theta/2+2\cos\theta\cos(3\theta/2)-2\cos(2\theta)\cos(\theta/2)=\\
&=2\cos\theta\left[\cos(\theta/2)+\cos(3\theta/2)\right]+\\
&+\cos\theta\cos(\theta/2)\text{-}\sin\theta\sin(\theta/2)\text{-}\cos(2\theta)\cos(\theta/2)\text{-}\cos(2\theta)\cos(\theta/2)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\cos(\theta/2)+\cos(3\theta/2)=\\
&=\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)+\\
&+\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)=\\
&=2\cos\theta\cos(\theta/2)
\end{aligned}
$$

so

$$
\begin{aligned}
&=2\cos\theta\left[2\cos\theta\cos(\theta/2)\right]+\\
&+\cos(\theta/2)\left[\cos\theta\text{ - }\cos(2\theta)\right]-\\
&-\sin\theta\sin(\theta/2)-\cos(2\theta)\cos(\theta/2)=
\end{aligned}
$$

$$
\begin{aligned}
&=2\cos\theta\left[2\cos\theta\cos(\theta/2)\right]+\\
&+\cos(\theta/2)\left[\cos\theta\text{ - }2\cos^2\theta+1\right]-\\
&-2\sin^2(\theta/2)\cos(\theta/2)-\cos(2\theta)\cos(\theta/2)=
\end{aligned}
$$

$$
\begin{aligned}
&=\cos(\theta/2)\left|4\cos^2\theta+\cos\theta\text{-}2\cos^2\theta+1-2\sin^2(\theta/2)-\cos(2\theta)\right|=\\
&=\cos(\theta/2)\left|2\cos^2\theta+\cos\theta+1-\cancel{2}\,\frac{1-\cos\theta}{\cancel{2}}-\cos(2\theta)\right|=\\
&=\cos(\theta/2)\left|\cancel{2}\,\frac{1+\cos(2\theta)}{\cancel{2}}+\cos\theta+1-1+\cos\theta-\cos(2\theta)\right|=\\
&=\cos(\theta/2)\left[1+\cancel{\cos(2\theta)}+2\cos\theta-\cancel{\cos(2\theta)}\right]=\\
&=\cos(\theta/2)(1+2\cos\theta)
\end{aligned}
$$

Thus, the whole term

$$
D\left(\frac{\partial u_x}{\partial x}\right)=\mathrm{n}_x\frac{\partial^2 u_x}{\partial x^2}+\mathrm{n}_y\frac{\partial^2 u_x}{\partial x\partial y}=\cos\theta\frac{\partial^2 u_x}{\partial x^2}+\sin\theta\frac{\partial^2 u_x}{\partial x\partial y}
$$

is

$$= (-\frac{15}{8})\, r^{-1/2} \left| \begin{matrix} (49A\text{-}251B)\cos(\theta/2) + \\ +25(A+5B)\cos(\theta/2)(1+2\cos\theta) \end{matrix} \right| =$$

$$= (-\frac{15}{8})\, r^{-1/2} \cos(\theta/2) \left\{ (49A\text{-}251B) + 25(A+5B)(1+2\cos\theta) \right\} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \cos(\theta/2) \left\{ (74A\text{-}126B) + 50(A+5B)\cos\theta \right\} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \cos(\theta/2) \left\{ 2(37A\text{-}63B) + 50(A+5B)\cos\theta \right\} \Rightarrow$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_x}{\partial x}\right) = (-\frac{15}{4})\, r^{-1/2} \cos(\theta/2) \left\{ (37A\text{-}63B) + 25(A+5B)\cos\theta \right\}}$$

<u>Calculation of:</u> $D\left(\frac{\partial u_y}{\partial x}\right) = n_x \frac{\partial^2 u_y}{\partial x^2} + n_y \frac{\partial^2 u_y}{\partial x \partial y} = \cos\theta \frac{\partial^2 u_y}{\partial x^2} + \sin\theta \frac{\partial^2 u_y}{\partial x \partial y}$

$$D\left(\frac{\partial u_y}{\partial x}\right) = \cos\theta \frac{\partial^2 u_y}{\partial x^2} + \sin\theta \frac{\partial^2 u_y}{\partial x \partial y} =$$

$$= \cos\theta \left(+\frac{15}{8}\right) r^{-1/2} \sin(\theta/2) \left[3(17A\text{-}83B) + 25(A+5B)\left(-3 + 2\cos\theta + 2\cos(2\theta)\right) \right] +$$

$$+ \sin\theta \left(-\frac{15}{8}\right) r^{-1/2} \cos(\theta/2) \left\{ 3(17A\text{-}83B) + 25(A+5B)\left[1 - 2\cos\theta + 2\cos(2\theta)\right] \right\} =$$

$$= (-\frac{15}{8})\, r^{-1/2} \left| \begin{matrix} 3(17A\text{-}83B)\left[-\cos\theta\sin(\theta/2) + \sin\theta\cos(\theta/2)\right] + \\ +25(A+5B) \left| \begin{matrix} \left(3 - 2\cos\theta - 2\cos(2\theta)\right)\cos\theta\sin(\theta/2) + \\ + \left[1 - 2\cos\theta + 2\cos(2\theta)\right]\sin\theta\cos(\theta/2) \end{matrix} \right| \end{matrix} \right| =$$

$$= (-\frac{15}{8})\, r^{-1/2} \left| \begin{matrix} 3(17A\text{-}83B)\sin(\theta/2) + \\ +25(A+5B) \left| \begin{matrix} 3\cos\theta\sin(\theta/2) - 2\cos^2\theta\sin(\theta/2) - \\ -2\cos\theta\cos(2\theta)\sin(\theta/2) + \\ +\sin\theta\cos(\theta/2) - \\ -2\cos\theta\sin\theta\cos(\theta/2) + \\ +2\cos(\theta/2)\cos(2\theta)\sin\theta \end{matrix} \right| \end{matrix} \right| =$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\sin(\theta/2)+\\ +25(A+5B)\left\{\begin{array}{l}2\cos\theta\sin(\theta/2)+\underline{\cos\theta\sin(\theta/2)}-\\ -2\cos\theta\left[\cos\theta\sin(\theta/2)+\sin\theta\cos(\theta/2)\right]+\\ +\underline{\sin\theta\cos(\theta/2)}-\\ -2\cos(2\theta)\left[\cos\theta\sin(\theta/2)-\sin\theta\cos(\theta/2)\right]\end{array}\right\}\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\sin(\theta/2)+\\ +25(A+5B)\left\{\begin{array}{l}2\cos\theta\sin(\theta/2)+\underline{\sin(3\theta/2)}-\\ -2\cos\theta\sin(3\theta/2)-2\cos(2\theta)\left[-\sin(\theta/2)\right]\end{array}\right\}\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\sin(\theta/2)+\\ +25(A+5B)\left\{\begin{array}{l}2\cos\theta\sin(\theta/2)+\underline{\sin(3\theta/2)}-\\ -2\cos\theta\sin(3\theta/2)+2\cos(2\theta)\sin(\theta/2)\end{array}\right\}\end{array}\right\}$$

The factor
$2\cos\theta\sin(\theta/2)+\sin(3\theta/2)-2\cos\theta\sin(3\theta/2)+2\cos(2\theta)\sin(\theta/2)$

can be rewritten in this manner:

$$\begin{aligned}&2\cos\theta\sin(\theta/2)+\sin(3\theta/2)-2\cos\theta\sin(3\theta/2)+2\cos(2\theta)\sin(\theta/2)=\\&=2\cos\theta\left[\sin(\theta/2)-\sin(3\theta/2)\right]+\sin\theta\cos(\theta/2)+\cos\theta\sin(\theta/2)+\\&+\cos(2\theta)\sin(\theta/2)+\cos(2\theta)\sin(\theta/2)\end{aligned}$$

The subtraction $\sin(\theta/2)-\sin(3\theta/2)$ gives:

$$\begin{aligned}&\sin(\theta/2)-\sin(3\theta/2)=\\&=\sin\theta\cos(\theta/2)-\cos\theta\sin(\theta/2)\text{-}\left[\sin\theta\cos(\theta/2)+\cos\theta\sin(\theta/2)\right]=\\&=\sin\theta\cos(\theta/2)-\cos\theta\sin(\theta/2)-\sin\theta\cos(\theta/2)-\cos\theta\sin(\theta/2)=\\&=-2\cos\theta\sin(\theta/2)\end{aligned}$$

By replacing it in the aforementioned expression, this is reached:

$$= 2\cos\theta\left[-2\cos\theta\sin(\theta/2)\right] + \sin\theta\cos(\theta/2) +$$

$$+\sin(\theta/2)\left[\cos\theta + \cos(2\theta)\right] + +\cos(2\theta)\sin(\theta/2) =$$

$$= 2\cos\theta\left[-2\cos\theta\sin(\theta/2)\right] + 2\sin(\theta/2)\cos^2(\theta/2) +$$

$$+\sin(\theta/2)\left[\cos\theta + 2\cos^2\theta - 1\right] + \cos(2\theta)\sin(\theta/2) =$$

$$= \sin(\theta/2)\left\{-4\cos^2\theta + 2\cos^2(\theta/2) + \cos\theta + 2\cos^2\theta - 1 + \cos(2\theta)\right\} =$$

$$= \sin(\theta/2)\left\{-2\cos^2\theta + \not{2}\,\frac{1+\cos\theta}{\not{2}} + \cos\theta - 1 + \cos(2\theta)\right\} =$$

$$= \sin(\theta/2)\left\{-\not{2}\,\frac{1+\cos(2\theta)}{\not{2}} + \not{1} + \cos\theta + \cos\theta - \not{1} + \cos(2\theta)\right\} =$$

$$= \sin(\theta/2)\left\{-1 - \cos(2\theta) + 2\cos\theta + \cos(2\theta)\right\} =$$

$$= \sin(\theta/2)(-1 + 2\cos\theta)$$

The term $D\left(\frac{\partial u_y}{\partial x}\right) = \cos\theta\frac{\partial^2 u_y}{\partial x^2} + \sin\theta\frac{\partial^2 u_y}{\partial x \partial y}$ takes the more simple form:

$$= (-\frac{15}{8})\mathrm{r}^{-1/2}\left\{\begin{matrix}3(17\mathrm{A}\text{-}83\mathrm{B})\sin(\theta/2) + \\ +25(\mathrm{A}+5\mathrm{B})\sin(\theta/2)(-1+2\cos\theta)\end{matrix}\right\} =$$

$$= (-\frac{15}{8})\mathrm{r}^{-1/2}\sin(\theta/2)\left\{3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(-1+2\cos\theta)\right\} =$$

$$= (-\frac{15}{8})\mathrm{r}^{-1/2}\sin(\theta/2)\left\{(51\mathrm{A}\text{-}25\mathrm{A}\text{-}249\mathrm{B}\text{-}125\mathrm{B}) + 50(\mathrm{A}+5\mathrm{B})\cos\theta\right\} =$$

$$= (-\frac{15}{8})\mathrm{r}^{-1/2}\sin(\theta/2)\left\{(26\mathrm{A}\text{-}374\mathrm{B})+50(\mathrm{A}+5\mathrm{B})\cos\theta\right\} \Rightarrow$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_y}{\partial x}\right) = (-\frac{15}{4})\mathrm{r}^{-1/2}\sin(\theta/2)\left\{(13\mathrm{A}\text{-}187\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\cos\theta\right\}}$$

At this point, each one of the tems $\mathrm{R}_x, D\left(\frac{\partial u_x}{\partial x}\right), \mathrm{R}_y, D\left(\frac{\partial u_y}{\partial x}\right)$ has been calculated. The next step is to estimate the products:

$$\mathrm{R}_x\, D\left(\frac{\partial u_x}{\partial x}\right), \mathrm{R}_y\, D\left(\frac{\partial u_y}{\partial x}\right)$$

<u>Calculation of the product:</u> $R_x D\left(\frac{\partial u_x}{\partial x}\right)$

$$R_x D\left(\frac{\partial u_x}{\partial x}\right)=$$

$$=(-15/2)\, c\, r^{-1/2}\cos(\theta/2)\left[(13A+63B)\mu+(50A\lambda+49A\mu-B\mu)\cos\theta\right]\cdot$$

$$\cdot(-\frac{15}{4})r^{-1/2}\cos(\theta/2)\{(37A\text{-}63B)+25(A+5B)\cos\theta\}=$$

$$=(\frac{225}{8})c\, r^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}(13A+63B)\mu(37A\text{-}63B)+\\+(13A+63B)\mu 25(A+5B)\cos\theta+\\+(50A\lambda+49A\mu-B\mu)\cos\theta(37A\text{-}63B)+\\+(50A\lambda+49A\mu-B\mu)25(A+5B)\cos^2\theta\end{array}\right\}=$$

$$=(\frac{225}{8})c\, r^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}(13A+63B)(37A\text{-}63B)\mu+\\+\left[\begin{array}{l}(13A+63B)25(A+5B)\mu+\\+(50A\lambda+49A\mu-B\mu)(37A\text{-}63B)\end{array}\right]\cos\theta+\\+\left[(50A\lambda+49A\mu-B\mu)25(A+5B)\right]\cos^2\theta\end{array}\right\}$$

In order to make the manipulations easier, each one of the 3 sub-terms (the first is the one not containing cosθ, the second one is the one multiplied by cosθ and the last is the one multiplied by cos$^2\theta$), that compose this product, will be calculated separately.

<u>First sub-term</u>

$$(13A+63B)(37A\text{-}63B)\mu=$$

$$=\left(13\cdot 37A^2 - 13\cdot 63AB + 63\cdot 37AB - 63^2B^2\right)\mu=$$

$$=\left(481A^2+1512AB - 3969B^2\right)\mu$$

<u>Second sub-term (multiplied by cosθ)</u>

$$(13A+63B)25(A+5B)\mu+(50A\lambda+49A\mu-B\mu)(37A\text{-}63B)=$$

$$=\left(13\cdot 25A^2+13\cdot 125AB+63\cdot 25AB+63\cdot 125B^2\right)\mu+$$

$$+50\cdot 37A^2\lambda-50\cdot 63AB\lambda+49\cdot 37A^2\mu-49\cdot 63AB\mu-37AB\mu+63B^2\mu=$$

$$=\left(325A^2+3200AB+7875B^2\right)\mu+$$

$$+\left(1850A^2-3200AB\right)\lambda+\left(1813A^2-3087AB-37B^2\right)\mu=$$

$$=\left(2138A^2+76AB+7938B^2\right)\mu+\left(1850A^2-3150AB\right)\lambda$$

Third sub-term (multiplied by $\cos^2\theta$)

$$(50A\lambda + 49A\mu - B\mu)25(A + 5B) =$$
$$=25\left(50A^2\lambda + 250AB\lambda + 49A^2\mu + 49\cdot 5AB\mu - AB\mu - 5B^2\mu\right) =$$
$$=25\left[\left(50A^2 + 250AB\right)\lambda + \left(49A^2 + 244AB - 5B^2\right)\mu\right] =$$
$$=25\left[50\left(A^2 + 5AB\right)\lambda + \left(49A^2 + 244AB - 5B^2\right)\mu\right]$$

By combining these 3 sub-terms, the following representation emerges for

$R_x D\left(\frac{\partial u_x}{\partial x}\right)$:

$$R_x D\left(\frac{\partial u_x}{\partial x}\right) =$$
$$=\left(\frac{225}{8}\right)cr^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}\left(481A^2 + 1512AB - 3969B^2\right)\mu + \\ +\left[\left(1850A^2 - 3150AB\right)\lambda + \left(2138A^2 + 76AB + 7938B^2\right)\mu\right]\cos\theta + \\ +25\left[50\left(A^2 + 5AB\right)\lambda + \left(49A^2 + 244AB - 5B^2\right)\mu\right]\cos^2\theta\end{array}\right\}$$

Calculation of the product: $R_y D\left(\frac{\partial u_y}{\partial x}\right)$

$$R_y D\left(\frac{\partial u_y}{\partial x}\right) =$$
$$=\left(-\frac{15}{2}\right)c\,r^{-1/2}\sin(\theta/2)\left\{50A\lambda + 63\left(A\mu + B\mu\right) + \left[50A\lambda + 51A\mu + B\mu\right]\cos\theta\right\}\cdot$$
$$\cdot\left(-\frac{15}{4}\right)r^{-1/2}\sin(\theta/2)\left\{(13A - 187B) + 25(A + 5B)\cos\theta\right\} =$$

$$=\left(\frac{225}{8}\right)c\,r^{-1}\sin^2(\theta/2)\left\{\begin{array}{l}63\left(A\mu + B\mu\right)(13A - 187B) + 63\left(A\mu + B\mu\right)25(A + 5B)\cos\theta + \\ +\left(50A\lambda + 51A\mu + B\mu\right)(13A - 187B)\cos\theta + \\ +25\left(50A\lambda + 51A\mu + B\mu\right)\cos^2\theta + 50A\lambda(13A - 187B) + \\ +50A\lambda\times 25(A + 5B)\cos\theta\end{array}\right\} =$$

$$= (\frac{225}{8}) c\, r^{-1} \sin^2(\theta/2) \left| \begin{array}{l} 63(A\mu + B\mu)(13A - 187B) + 50A\lambda(13A - 187B) + \\ + \left| \begin{array}{l} 63(A\mu + B\mu)25(A + 5B) + \\ + (50A\lambda + 51A\mu + B\mu)(13A - 187B) + \\ + 50A\lambda \times 25(A + 5B) \end{array} \right| \cos\theta + \\ + \left[25(50A\lambda + 51A\mu + B\mu)(A + 5B)\right] \cos^2\theta \end{array} \right|$$

Utilizing the same method as before, the 3 sub-terms are:

First sub-term

$$63(A\mu + B\mu)(13A - 187B) + 50A\lambda(13A - 187B) =$$
$$= \left(63 \cdot 13A^2 - 63 \cdot 187AB + 63 \cdot 13AB - 63 \cdot 187AB - 63 \cdot 187B^2\right)\mu +$$
$$+ (50 \cdot 13A^2 - 50 \cdot 187AB)\lambda =$$

$$= \left(819A^2 - 10962AB - 11781B^2\right)\mu + (650A^2 - 9350AB)\lambda$$

Second sub-term (multiplied by cosθ)

$$63(A\mu + B\mu)25(A + 5B) + (50A\lambda + 51A\mu + B\mu)(13A - 187B) +$$
$$+ 50A\lambda \cdot 25(A + 5B) =$$

$$= \left(63 \cdot 25A^2 + 63 \cdot 125AB + 63 \cdot 25AB + 63 \cdot 125B^2\right)\mu +$$
$$+ 50 \cdot 13A^2\lambda - 50 \cdot 187AB\lambda + 51 \cdot 13A^2\mu - 51 \cdot 187AB\mu +$$
$$+ 13AB\mu - 187B^2\mu +$$
$$+ 1250A^2\lambda + 1250 \cdot 5AB\lambda =$$

$$= \left(1575A^2 + 9450AB + 7875B^2\right)\mu +$$
$$+ 650A^2\lambda - 9350AB\lambda + 663A^2\mu - 9537AB\mu + 13AB\mu - 187B^2\mu$$
$$+ 1250A^2\lambda + 6250AB\lambda =$$

$$= \left(1575A^2 + 9450AB + 7875B^2\right)\mu +$$
$$+ \left(650A^2 - 9350AB\right)\lambda + \left(663A^2 - 9524AB - 187B^2\right)\mu +$$
$$+ \left(1250A^2 + 6250AB\right)\lambda =$$

$$= \left[(1575 + 663)A^2 + (9450 - 9524)AB + (7875 - 187)B^2\right]\mu +$$
$$+ \left[(650 + 1250)A^2 + (-9350 + 6250)AB\right]\lambda =$$

$$= \left(1900A^2 - 3100AB\right)\lambda + \left(2238A^2 - 74AB + 7688B^2\right)\mu$$

<u>Third sub-term (multiplied by $\cos^2\theta$)</u>

$$25(50A\lambda + 51A\mu + B\mu)(A + 5B) =$$
$$= 50 \cdot 25A^2\lambda + 50 \cdot 125AB\lambda + 50 \cdot 25A^2\mu + 51 \cdot 125AB\mu +$$
$$+ 25AB\mu + 125B^2\mu =$$
$$= \left(1250A^2 + 6250AB\right)\lambda + \left(1275A^2 + 6400AB + 125B^2\right)\mu$$

After all these manipulations,

$$R_y\, D\left(\frac{\partial u_y}{\partial x}\right) =$$
$$= \left(\frac{225}{8}\right) c\, r^{-1} \sin^2(\theta/2) \left\{ \begin{array}{l} (650A^2 - 9350AB)\lambda + \left(819A^2 - 10962AB - 11781B^2\right)\mu + \\ + \left[\left(1900A^2 - 3100AB\right)\lambda + \left(2238A^2 - 74AB + 7688B^2\right)\mu\right]\cos\theta + \\ + \left[\left(1250A^2 + 6250AB\right)\lambda + \left(1275A^2 + 6400AB + 125B^2\right)\mu\right]\cos^2\theta \end{array} \right\}$$

1.3 Terms P_z in J_1

<u>Calculation of terms</u> $P_z\left(\frac{\partial u_z}{\partial x}\right)$

The term $P_z\left(\frac{\partial u_z}{\partial x}\right)$ is expanded this way, following the summation convention:

$$P_z\left(\frac{\partial u_x}{\partial x}\right)=P_x\left(\frac{\partial u_x}{\partial x}\right)+P_y\left(\frac{\partial u_y}{\partial x}\right)$$

Its definition is:

$$\begin{aligned} P_z &= n_p\left(\tau_{pz} - \partial_r m_{rpz}\right) - D_p\left(n_r m_{rpz}\right) + \left(D_j n_j\right) n_r n_p m_{rpz} = \\ &= n_p \tau_{pz} - n_p \partial_r m_{rpz} - D_p\left(n_r m_{rpz}\right) + \left(D_j n_j\right) n_r n_p m_{rpz} \end{aligned}$$

<u>CALCULATION OF</u> P_x

$$\begin{aligned} P_x &= n_p \tau_{px} - n_p \partial_r m_{rpx} - D_p\left(n_r m_{rpx}\right) + \left(D_j n_j\right) n_r n_p m_{rpx} = \\ &= n_p \tau_{px} - n_p \partial_r m_{rpx} - D_p\left(n_r m_{rpx}\right) + \left(D_j n_j\right) R_x = \\ &= n_x \tau_{xx} - n_x \partial_r m_{rxx} - D_x\left(n_r m_{rxx}\right) + \left(D_j n_j\right) R_x + \\ &\quad n_y \tau_{yx} - n_y \partial_r m_{ryx} - D_y\left(n_r m_{ryx}\right) = \\ &= \left(n_x\tau_{xx}+n_y\tau_{yx}\right)-\left(n_x\partial_r m_{rxx}+n_y\partial_r m_{ryx}\right)-\left[D_x\left(n_r m_{rxx}\right)+ D_y\left(n_r m_{ryx}\right)\right]+\left(D_j n_j\right) R_x = \end{aligned}$$

$$=\left(n_x\tau_{xx}+n_y\tau_{yx}\right)-\left(n_x\partial_x m_{xxx}+n_y\partial_x m_{xyx}\right)-\left[D_x\left(n_x m_{xxx}\right)+ D_y\left(n_x m_{xyx}\right)\right]+\left(D_j n_j\right) R_x -$$

$$-\left(n_x\partial_y m_{yxx}+n_y\partial_y m_{yyx}\right)-\left[D_x\left(n_y m_{yxx}\right)+ D_y\left(n_y m_{yyx}\right)\right]\Rightarrow$$

$$\Rightarrow P_x=\left(n_x\tau_{xx}+n_y\tau_{yx}\right)-\begin{bmatrix} n_x\partial_x m_{xxx}+n_y\partial_x m_{xyx}+ \\ +n_x\partial_y m_{yxx}+n_y\partial_y m_{yyx}\end{bmatrix}-\begin{bmatrix} D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)+ \\ + D_y\left(n_x m_{xyx}+n_y m_{yyx}\right)\end{bmatrix}+ \\ +\left(D_j n_j\right)R_x$$

For practical reasons, the first set of terms wil be referred to as Subterm A of P_x, the second as Subterm B etc, following their order of appearance in the above expression.

The full term P_x, consisting of the Subterms A, B, C, D, is going to be multiplied by the derivative of the appropriate displacement (either by $\frac{\partial u_x}{\partial x}$ for the J_1-Integral or by $\frac{\partial u_x}{\partial y}$ for the J_2-Integral).

Its component $\left(n_x \tau_{xx} + n_y \tau_{yx}\right)$ (Subterm A) is ~ $r^{1/2}$.

The product $\left(n_x \tau_{xx} + n_y \tau_{yx}\right)\frac{\partial u_x}{\partial x} d\Gamma \sim r^{1/2} r^{1/2} r\, d\theta \sim r^2 \to 0$ for $r \to 0$.

Consequently, this Subterm's integral will be 0!

CALCULATION OF SUBTERM B: $\begin{bmatrix} n_x \partial_x m_{xxx} + n_y \partial_x m_{xyx} + \\ + n_x \partial_y m_{yxx} + n_y \partial_y m_{yyx} \end{bmatrix}$

$$n_x \partial_x m_{xxx} + n_y \partial_x m_{xyx} + n_x \partial_y m_{yxx} + n_y \partial_y m_{yyx} =$$

$$= \cos\theta \left[\cos\theta \frac{\partial}{\partial r} m_{xxx} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} m_{xxx} \right] + \sin\theta \left[\cos\theta \frac{\partial}{\partial r} m_{xyx} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} m_{xyx} \right] +$$

$$+ \cos\theta \left[\sin\theta \frac{\partial}{\partial r} m_{yxx} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} m_{yxx} \right] + \sin\theta \left[\sin\theta \frac{\partial}{\partial r} m_{yyx} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} m_{yyx} \right] =$$

$$= \cos^2\theta \frac{\partial}{\partial r} m_{xxx} - \frac{\cos\theta \sin\theta}{r} \frac{\partial}{\partial \theta} m_{xxx} + \sin\theta \cos\theta \frac{\partial}{\partial r} m_{xyx} - \frac{\sin^2\theta}{r} \frac{\partial}{\partial \theta} m_{xyx} +$$

$$+ \cos\theta \sin\theta \frac{\partial}{\partial r} m_{yxx} + \frac{\cos^2\theta}{r} \frac{\partial}{\partial \theta} m_{yxx} + \sin^2\theta \frac{\partial}{\partial r} m_{yyx} + \frac{\cos\theta \sin\theta}{r} \frac{\partial}{\partial \theta} m_{yyx} =$$

$$= \cos^2\theta \frac{\partial}{\partial r} m_{xxx} + \sin\theta \cos\theta \left[\frac{\partial}{\partial r} m_{xyx} + \frac{\partial}{\partial r} m_{yxx} \right] - \frac{\cos\theta \sin\theta}{r} \left[\frac{\partial}{\partial \theta} m_{xxx} - \frac{\partial}{\partial \theta} m_{yyx} \right] -$$

$$- \frac{\sin^2\theta}{r} \frac{\partial}{\partial \theta} m_{xyx} + \frac{\cos^2\theta}{r} \frac{\partial}{\partial \theta} m_{yxx} + \sin^2\theta \frac{\partial}{\partial r} m_{yyx}$$

The partial derivatives contained in this expression are calculated first.

$$\to \frac{\partial}{\partial r} m_{xxx} = (-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2} \begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2)\ - \\ -\ (100\mathrm{A}\lambda+149\mathrm{A}\mu+249\mathrm{B}\mu)\cos(\theta/2) \end{bmatrix}$$

$$\to \frac{\partial}{\partial r} m_{xyx} = (-15/8)\ \mathrm{c}\ \mu\ \mathrm{r}^{-3/2}\left[25\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)-(49\mathrm{A}+249\mathrm{B})\sin(\theta/2)\right]$$

$$\to \frac{\partial}{\partial r} m_{yxx} = (-15/8)\ \mathrm{c}\,\mathrm{r}^{-3/2} \begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)\ - \\ -\ (100\mathrm{A}\lambda+49\mathrm{A}\mu-251\mathrm{B}\mu)\sin(\theta/2) \end{bmatrix}$$

$$\to \frac{\partial}{\partial \theta} m_{xxx} = (+15/4)\mathrm{c}\mathrm{r}^{-1/2} \begin{bmatrix} -25\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)\frac{5}{2}+ \\ +(100\mathrm{A}\lambda+149\mathrm{A}\mu+249\mathrm{B}\mu)\sin(\theta/2)\frac{1}{2} \end{bmatrix} =$$

$$= (+15/8)\mathrm{c}\mathrm{r}^{-1/2} \begin{bmatrix} -125\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)+ \\ +(100\mathrm{A}\lambda+149\mathrm{A}\mu+249\mathrm{B}\mu)\sin(\theta/2) \end{bmatrix}$$

$$\to \frac{\partial}{\partial \theta} m_{yyx} = (-15/4)\ \mathrm{c}\ \mu\ \mathrm{r}^{-1/2}\left[-25\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)\frac{5}{2}-(51\mathrm{A}+251\mathrm{B})\sin(\theta/2)\frac{1}{2}\right] =$$

$$= (-15/8)\ \mathrm{c}\ \mu\ \mathrm{r}^{-1/2}\left[-125\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)-(51\mathrm{A}+251\mathrm{B})\sin(\theta/2)\right]$$

$$\to \frac{\partial}{\partial \theta} m_{xyx} = (+15/4)\mathrm{c}\ \mu\ \mathrm{r}^{-1/2}\left[25\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2)\frac{5}{2}-(49\mathrm{A}+249\mathrm{B})\cos(\theta/2)\frac{1}{2}\right] =$$

$$= (+15/8)\ \mathrm{c}\ \mu\ \mathrm{r}^{-1/2}\left[125\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2)-(49\mathrm{A}+249\mathrm{B})\cos(\theta/2)\right]$$

$$\to \frac{\partial}{\partial \theta} m_{yxx} = (+15/4)\ \mathrm{c}\,\mathrm{r}^{-1/2} \begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2)\frac{5}{2}- \\ -(100\mathrm{A}\lambda+49\mathrm{A}\mu-251\mathrm{B}\mu)\cos(\theta/2)\frac{1}{2} \end{bmatrix} =$$

$$= (+15/8)\ \mathrm{c}\,\mathrm{r}^{-1/2} \begin{bmatrix} 125\mu\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2)- \\ -(100\mathrm{A}\lambda+49\mathrm{A}\mu-251\mathrm{B}\mu)\cos(\theta/2) \end{bmatrix}$$

$$\to \frac{\partial}{\partial r} m_{yyx} = (15/8)\ \mathrm{c}\ \mu\ \mathrm{r}^{-3/2}\left[25\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2)-(51\mathrm{A}+251\mathrm{B})\cos(\theta/2)\right]$$

Thus, the Subterm B of P_x is:

$$n_x\partial_x m_{xxx} + n_y\partial_x m_{xyx} + n_x\partial_y m_{yxx} + n_y\partial_y m_{yyx} =$$

$$= \cos^2\theta(-15/8)\,c\,r^{-3/2}\begin{bmatrix} 25\mu(A+5B)\cos(5\theta/2) - \\ -(100A\lambda+149A\mu+249B\mu)\cos(\theta/2)\end{bmatrix}+$$

$$+\sin\theta\cos\theta\begin{Bmatrix}(-15/8)c\,\mu\,r^{-3/2}\left[25(A+5B)\sin(5\theta/2)-(49A+249B)\sin(\theta/2)\right]+ \\ +(-15/8)\,c\,r^{-3/2}\begin{bmatrix}25\mu(A+5B)\sin(5\theta/2) - \\ -(100A\lambda+49A\mu-251B\mu)\sin(\theta/2)\end{bmatrix}\end{Bmatrix}-$$

$$-\frac{\cos\theta\sin\theta}{r}\begin{Bmatrix}(+15/8)cr^{-1/2}\begin{bmatrix}-125\mu(A+5B)\sin(5\theta/2)+ \\ +(100A\lambda+149A\mu+249B\mu)\sin(\theta/2)\end{bmatrix}- \\ -(-15/8)\,c\,\mu\,r^{-1/2}\left[-125(A+5B)\sin(5\theta/2)-(51A+251B)\sin(\theta/2)\right]\end{Bmatrix}-$$

$$-\frac{\sin^2\theta}{r}(+15/8)\,c\,\mu\,r^{-1/2}\left[125(A+5B)\cos(5\theta/2)-(49A+249B)\cos(\theta/2)\right]+$$

$$+\frac{\cos^2\theta}{r}(+15/8)\,c\,r^{-1/2}\begin{bmatrix}125\mu(A+5B)\cos(5\theta/2)- \\ -(100A\lambda+49A\mu-251B\mu)\cos(\theta/2)\end{bmatrix}+$$

$$+\sin^2\theta(+15/8)\,c\,\mu\,r^{-3/2}\left[25(A+5B)\cos(5\theta/2)-(51A+251B)\cos(\theta/2)\right]=$$

$$=(-15/8)\cos^2\theta\,c\,r^{-3/2}\begin{bmatrix}25\mu(A+5B)\cos(5\theta/2) - \\ -(100A\lambda+149A\mu+249B\mu)\cos(\theta/2)\end{bmatrix}+$$

$$+(-15/8)\sin\theta\cos\theta\,c\,r^{-3/2}\left[50\mu(A+5B)\sin(5\theta/2)-\sin(\theta/2)(100A\lambda+98A\mu-2B\mu)\right]-$$

$$-(+15/8)cr^{-3/2}\cos\theta\sin\theta\left[-250\mu(A+5B)\sin(5\theta/2)+\sin(\theta/2)(100A\lambda+98A\mu-2B\mu)\right]-$$

$$-(+15/8)\,c\,\mu\,r^{-3/2}\sin^2\theta\left[125(A+5B)\cos(5\theta/2)-(49A+249B)\cos(\theta/2)\right]+$$

$$+(15/8)\, c\, r^{-3/2}\cos^2\theta \begin{bmatrix} 125\mu(A+5B)\cos(5\theta/2)- \\ -(100A\lambda+49A\mu-251B\mu)\cos(\theta/2) \end{bmatrix}+$$

$$+(15/8)\, c\,\mu\, r^{-3/2}\sin^2\theta\left[25(A+5B)\cos(5\theta/2)-(51A+251B)\cos(\theta/2)\right]=$$

This form is composed of 6 terms, each occupying a line. If we combine the terms in the first with the fifth term, the second with the third and the fourth with the sixth, we obtain:

$$(+15/8)c\, r^{-3/2}\cos^2\theta \begin{bmatrix} \cos(5\theta/2)\left[-25\mu(A+5B)+125\mu(A+5B)\right] - \\ -\cos(\theta/2)\begin{pmatrix} -100A\lambda-149A\mu-249B\mu+ \\ +100A\lambda+49A\mu-251B\mu \end{pmatrix} \end{bmatrix}+$$

$$+(-15/8)c\, r^{-3/2}\sin\theta\cos\theta \begin{bmatrix} \sin(5\theta/2)\left[-250\mu(A+5B)\right]+ \\ +\sin(\theta/2)\begin{pmatrix} -100A\lambda-98A\mu+2B\mu+ \\ +100A\lambda+98A\mu-2B\mu \end{pmatrix} \end{bmatrix}+$$

$$+(+15/8)\, c\,\mu\, r^{-3/2}\sin^2\theta \cdot \begin{vmatrix} \cos(5\theta/2)\left[-125(A+5B)+25(A+5B)\right]+ \\ +\cos(\theta/2)(49A+249B+51A+251B) \end{vmatrix}=$$

$$=(+15/8)c\, r^{-3/2}\cos^2\theta\left[100\mu(A+5B)\cos(5\theta/2)-(-100)\mu(A+5B)\cos(\theta/2)\right]+$$

$$+(-15/8)c\, r^{-3/2}\sin\theta\cos\theta\left[200\mu(A+5B)\sin(5\theta/2)+0\right]+$$

$$+(+15/8)\, c\,\mu\, r^{-3/2}\sin^2\theta\left[-100(A+5B)\cos(5\theta/2)+100(A+5B)\cos(\theta/2)\right]=$$

$$=(+15/8)c\, r^{-3/2}\begin{bmatrix} 100\mu(A+5B)\cos(5\theta/2)(\cos^2\theta-\sin^2\theta)+ \\ +100\mu(A+5B)\cos(\theta/2)(\cos^2\theta+\sin^2\theta)+ \\ +200\mu(A+5B)\sin\theta\cos\theta\sin(5\theta/2) \end{bmatrix}=$$

$$=(+15/8)c\, r^{-3/2}\begin{bmatrix} 100\mu(A+5B)\cos(5\theta/2)\cos(2\theta)+ \\ +100\mu(A+5B)\cos(\theta/2)\times 1+ \\ +100\mu(A+5B)\sin(2\theta)\sin(5\theta/2) \end{bmatrix}=$$

$$=(+15/8)c\,r^{-3/2}\begin{bmatrix}100\mu(A+5B)\cos(\frac{5\theta}{2}-2\theta)+\\+100\mu(A+5B)\cos(\theta/2)\end{bmatrix}=$$

$$=(+15/8)c\,r^{-3/2}\begin{bmatrix}100\mu(A+5B)\cos(\theta/2)+\\+100\mu(A+5B)\cos(\theta/2)\end{bmatrix}\Rightarrow$$

$$\Rightarrow \boxed{\text{Subterm B of } P_x =(+15/8)c\,\mu\, r^{-3/2}\cos(\theta/2)\left[200(A+5B)\right]}$$

CALCULATION OF SUBTERM C: $\begin{bmatrix}D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)+\\+D_y\left(n_x m_{xyx}+n_y m_{yyx}\right)\end{bmatrix}$

Subterm C can be decomposed in two parts : $D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)$ and $D_y\left(n_x m_{xyx}+n_y m_{yyx}\right)$.

First part: $D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)$

$$\begin{aligned}
&D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)=\\
&=\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-n_x D\left(n_x m_{xxx}+n_y m_{yxx}\right)=\\
&=\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-n_x n_r\,\partial_r\left(n_x m_{xxx}+n_y m_{yxx}\right)=\\
&=\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-n_x\left[n_x\,\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)+n_y\,\partial_y\left(n_x m_{xxx}+n_y m_{yxx}\right)\right]=\\
&=\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-n_x^2\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-n_x n_y\partial_y\left(n_x m_{xxx}+n_y m_{yxx}\right)=
\end{aligned}$$

$$\begin{aligned}
&=\left(1-n_x^2\right)\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-n_x n_y\partial_y\left(n_x m_{xxx}+n_y m_{yxx}\right)=\\
&=\left(1-\cos^2\theta\right)\partial_x\left(n_x m_{xxx}+n_y m_{yxx}\right)-\cos\theta\,\sin\theta\,\partial_y\left(n_x m_{xxx}+n_y m_{yxx}\right)\Rightarrow
\end{aligned}$$

$$\boxed{\begin{aligned}&\Rightarrow D_x\left(n_x m_{xxx}+n_y m_{yxx}\right)=\\&=\sin^2\theta\,\partial_x\left(\cos\theta\, m_{xxx}+\sin\theta\, m_{yxx}\right)-\cos\theta\,\sin\theta\,\partial_y\left(\cos\theta\, m_{xxx}+\sin\theta\, m_{yxx}\right)\end{aligned}}$$

Second part: $D_y\left(n_x m_{xyx}+n_y m_{yyx}\right)$

$$D_y\left(n_x m_{xyx} + n_y m_{yyx} \right) =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) - n_y D\left(n_x m_{xyx} + n_y m_{yyx} \right) =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) - n_y n_r\, \partial_r\left(n_x m_{xyx} + n_y m_{yyx} \right) =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) - n_y\left[n_x\, \partial_x\left(n_x m_{xyx} + n_y m_{yyx} \right) + n_y\, \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) \right] =$$
$$= \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) - n_y\, n_x\, \partial_x\left(n_x m_{xyx} + n_y m_{yyx} \right) - n_y^{\,2}\left(n_x m_{xyx} + n_y m_{yyx} \right) =$$

$$= \left(1 - n_y^{\,2}\right) \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) - n_y\, n_x\, \partial_x\left(n_x m_{xyx} + n_y m_{yyx} \right) =$$
$$= \left(1 - \sin^2\theta\right) \partial_y\left(n_x m_{xyx} + n_y m_{yyx} \right) - \sin\theta\cos\theta\, \partial_x\left(n_x m_{xyx} + n_y m_{yyx} \right) \Rightarrow$$

$$\boxed{\begin{aligned} &\Rightarrow D_y\left(n_x m_{xyx} + n_y m_{yyx} \right) = \\ &= \cos^2\theta\, \partial_y\left(\cos\theta\, m_{xyx} + \sin\theta m_{yyx}\right) - \sin\theta\cos\theta\, \partial_x\left(\cos\theta\, m_{xyx} + \sin\theta m_{yyx}\right) \end{aligned}}$$

The explicit calculation of each part can now be performed.

First part:

$$D_x\left(n_x m_{xxx} + n_y m_{yxx} \right) =$$
$$= \sin^2\theta\, \partial_x\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx} \right) - \cos\theta\, \sin^2\theta\, \partial_y\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx} \right)$$

First, these sums will be calculated:

sum 1: $\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right)$

sum 2: $\left(\cos\theta\, m_{xyx} + \sin\theta\, m_{yyx}\right)$

Calculation of sum 1: $\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right)$

$$\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx} =$$
$$\cos\theta \cdot (+15/4) c r^{-1/2}\left[25\mu\left(\mathrm{A} + 5\mathrm{B}\right)\cos(5\theta/2) - (100\mathrm{A}\lambda + 149\mathrm{A}\mu + 249\mathrm{B}\mu)\cos(\theta/2)\right] +$$
$$+ \sin\theta \cdot (15/4) c\, r^{-1/2}\left[25\mu\left(\mathrm{A} + 5\mathrm{B}\right)\sin(5\theta/2) - (100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)\sin(\theta/2)\right] =$$

$$= (+15/4)\, c\, r^{-1/2} \left\{ \begin{array}{l} 25\mu\left(\mathrm{A} + 5\mathrm{B}\right)\left[\cos\theta\cos(5\theta/2) + \sin\theta\sin(5\theta/2)\right] - \\ -(100\mathrm{A}\lambda + 149\mathrm{A}\mu + 249\mathrm{B}\mu)\cos\theta\cos(\theta/2) - \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)\sin\theta\sin(\theta/2) \end{array} \right\} =$$

$$= (+15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\cos(3\theta/2) - \\ -\begin{pmatrix} 100A\lambda + 49A\mu - 251B\mu \\ +100A\mu + 500B\mu \end{pmatrix} \cos\theta\cos(\theta/2) - \\ -(100A\lambda + 49A\mu - 251B\mu)\sin\theta\sin(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\cos(3\theta/2) - \\ -(100A\lambda + 49A\mu - 251B\mu)\left[\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right] - \\ -(100A\mu + 500B\mu)\cos\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\cos(3\theta/2) - \\ -(100A\lambda + 49A\mu - 251B\mu)\cos(\theta/2) - \\ -100\mu(A+5B)\cos\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\left(\cos(3\theta/2) - 4\cos\theta\cos(\theta/2)\right) - \\ -(100A\lambda + 49A\mu - 251B\mu)\cos(\theta/2) \end{Bmatrix} =$$

(but since: $\dfrac{\cos(3\theta/2)}{\cos(\theta/2)} = -1 + 2\cos\theta$, as already shown while simplifying $\dfrac{\partial\varepsilon_{xy}}{\partial y}$)

$$= (+15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\cos(\theta/2)(-1+2\cos\theta - 4\cos\theta) - \\ -(100A\lambda + 49A\mu - 251B\mu)\cos(\theta/2) \end{Bmatrix} \Rightarrow$$

the result for sum 1 is:

$$\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx} = (15/4)\, c\, r^{-1/2} \cos(\theta/2) \begin{Bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda + 49A\mu - 251B\mu) \end{Bmatrix}$$

Calculation of sum 2: $\left(\cos\theta\, m_{xyx} + \sin\theta\, m_{yyx}\right)$

Now, sum 2, $\left(\cos\theta\, m_{xyx} + \sin\theta\, m_{yyx}\right)$:

$$\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)=$$

$$=\cos\theta\cdot(+15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\Big[25\big(\mathrm{A}+5\mathrm{B}\big)\sin(5\theta/2)-(49\mathrm{A}+249\mathrm{B})\sin(\theta/2)\Big]+$$

$$+\sin\theta\cdot(-15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\Big[25\big(\mathrm{A}+5\mathrm{B}\big)\cos(5\theta/2)+(51\mathrm{A}+251\mathrm{B})\cos(\theta/2)\Big]=$$

$$=(+15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left\{\begin{aligned}&25\big(\mathrm{A}+5\mathrm{B}\big)\big[\sin(5\theta/2)\cos\theta-\cos(5\theta/2)\sin\theta\big]-\\&-(49\mathrm{A}+249\mathrm{B})\cos\theta\sin(\theta/2)-\\&-\begin{pmatrix}49\mathrm{A}+249\mathrm{B}+\\+2\mathrm{A}+2\mathrm{B}\end{pmatrix}\sin\theta\cos(\theta/2)\end{aligned}\right\}=$$

$$=(+15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left\{\begin{aligned}&25\big(\mathrm{A}+5\mathrm{B}\big)\sin(3\theta/2)-\\&-(49\mathrm{A}+249\mathrm{B})\big[\cos\theta\sin(\theta/2)+\sin\theta\cos(\theta/2)\big]-\\&-\big(2\mathrm{A}+2\mathrm{B}\big)\sin\theta\cos(\theta/2)\end{aligned}\right\}=$$

$$=(+15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left\{\begin{aligned}&25\big(\mathrm{A}+5\mathrm{B}\big)\sin(3\theta/2)-\\&-(49\mathrm{A}+249\mathrm{B})\sin(3\theta/2)-\\&-2\big(\mathrm{A}+\mathrm{B}\big)\sin\theta\cos(\theta/2)\end{aligned}\right\}=$$

$$=(+15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left\{\begin{aligned}&-\big(24\mathrm{A}+124\mathrm{B}\big)\sin(3\theta/2)-\\&-2\big(\mathrm{A}+\mathrm{B}\big)\sin\theta\cos(\theta/2)\end{aligned}\right\}=$$

$$=(-15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left\{\begin{aligned}&\big(24\mathrm{A}+124\mathrm{B}\big)\sin(3\theta/2)+\\&+2\big(\mathrm{A}+\mathrm{B}\big)\times2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2)\end{aligned}\right\}\Rightarrow$$

(given that: $\dfrac{\sin(3\theta/2)}{\sin(\theta/2)}=1+2\cos\theta$)

the result for sum 2: $\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)$ is:

$$\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)=(-15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\sin(\theta/2)\left\{\begin{aligned}&\big(24\mathrm{A}+124\mathrm{B}\big)\big(1+2\cos\theta\big)+\\&+4\big(\mathrm{A}+\mathrm{B}\big)\cos^2(\theta/2)\end{aligned}\right\}$$

In order to assist estimate Subterm C of P_x, the partial derivatives of sum 1 and sum 2 with respect to x and y will first be calculated.

Partial derivative of sum 1 with respect to x

$$\frac{\partial}{\partial x}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) =$$

$$= \cos\theta \frac{\partial}{\partial r}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) =$$

$$= \cos\theta \frac{\partial}{\partial r}\left\{(+15/4)\ \mathrm{c}\ \mathrm{r}^{-1/2}\cos(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix}\right\} -$$

$$-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left\{(+15/4)\ \mathrm{c}\ \mathrm{r}^{-1/2}\cos(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix}\right\} =$$

$$= (-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\cos\theta\cos(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix} -$$

$$-\frac{\sin\theta}{r}\cdot(+15/4)\ \mathrm{c}\ \mathrm{r}^{-1/2}\left\{\begin{matrix} \left(-\frac{1}{2}\right)\sin(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix} + \\ +\cos(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(0+2\sin\theta) - \\ -0 \end{bmatrix} \end{matrix}\right\} =$$

$$= (-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\cos\theta\cos(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix} -$$

$$-(15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\sin\theta\cdot\left\{\begin{matrix} -\sin(\theta/2)\begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix} + \\ +100\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin\theta\cos(\theta/2) \end{matrix}\right\} =$$

$$= (-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\left\{\begin{matrix} \begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix}\left[\cos\theta\cos(\theta/2) - \sin\theta\sin(\theta/2)\right] + \\ +100\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin^2\theta\cos(\theta/2) \end{matrix}\right\} =$$

$$= (-15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2}\left\{\begin{matrix} \begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)(-1-2\cos\theta) - \\ -\left(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu\right) \end{bmatrix}\cos(3\theta/2) + \\ +100\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin^2\theta\cos(\theta/2) \end{matrix}\right\} =$$

(taking into account that $\frac{\cos(3\theta/2)}{\cos(\theta/2)} = -1 + 2\cos\theta$)

$$= (\text{-}15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2} \cos(\theta/2) \begin{Bmatrix} \begin{bmatrix} 25\mu(\mathrm{A}+5\mathrm{B})(-1-2\cos\theta)- \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu) \end{bmatrix} (-1+2\cos\theta) + \\ +100\mu(\mathrm{A}+5\mathrm{B})\sin^2\theta \end{Bmatrix} =$$

$$= (\text{-}15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2} \cos(\theta/2) \begin{Bmatrix} 25\mu(\mathrm{A}+5\mathrm{B})\left[(-1-2\cos\theta)(-1+2\cos\theta) + 4\sin^2\theta\right] - \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)(-1+2\cos\theta) \end{Bmatrix} =$$

$$= (\text{-}15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2} \cos(\theta/2) \begin{Bmatrix} 25\mu(\mathrm{A}+5\mathrm{B})(1-4\cos^2\theta + 4\sin^2\theta) - \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)(-1+2\cos\theta) \end{Bmatrix} =$$

$$= (\text{-}15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2} \cos(\theta/2) \begin{Bmatrix} 25\mu(\mathrm{A}+5\mathrm{B})(1-4\cos(2\theta)) - \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)(-1+2\cos\theta) \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) =$$

$$= (\text{-}15/8)\ \mathrm{c}\ \mathrm{r}^{-3/2} \cos(\theta/2) \begin{Bmatrix} (100\mathrm{A}\lambda + 74\mathrm{A}\mu - 126\mathrm{B}\mu) - \\ -2(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu)\cos\theta - \\ -100\mu(\mathrm{A}+5\mathrm{B})\cos(2\theta) \end{Bmatrix}$$

<u>Partial derivative of sum 1 with respect to y</u>

$$\frac{\partial}{\partial y}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) =$$

$$= \sin\theta \frac{\partial}{\partial r}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) =$$

$$= \sin\theta \frac{\partial}{\partial r}\left\{ (+15/4)\ \mathrm{c}\ \mathrm{r}^{-1/2}\cos(\theta/2) \begin{bmatrix} 25\mu(\mathrm{A}+5\mathrm{B})(-1-2\cos\theta) - \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu) \end{bmatrix} \right\} +$$

$$+ \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left\{ (+15/4)\ \mathrm{c}\ \mathrm{r}^{-1/2}\cos(\theta/2) \begin{bmatrix} 25\mu(\mathrm{A}+5\mathrm{B})(-1-2\cos\theta) - \\ -(100\mathrm{A}\lambda + 49\mathrm{A}\mu - 251\mathrm{B}\mu) \end{bmatrix} \right\} =$$

$$= (-15/8)\ c\ r^{-3/2} \sin\theta \cos(\theta/2) \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda + 49A\mu - 251B\mu) \end{bmatrix} +$$

$$+ \frac{\cos\theta}{r} \cdot (+15/4)\ c\ r^{-1/2} \left\{ \begin{array}{l} \left(-\frac{1}{2}\right)\sin(\theta/2) \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda+49A\mu-251B\mu) \end{bmatrix} + \\ +\cos(\theta/2) \begin{bmatrix} 25\mu(A+5B)(0+2\sin\theta) - \\ -0 \end{bmatrix} \end{array} \right\} =$$

$$= (-15/8)\ c\ r^{-3/2} \sin\theta \cos(\theta/2) \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda + 49A\mu - 251B\mu) \end{bmatrix} +$$

$$+(+15/8)\ c\ r^{-3/2} \cos\theta \left\{ \begin{array}{l} -\sin(\theta/2) \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda+49A\mu-251B\mu) \end{bmatrix} + \\ +100\mu(A+5B)\sin\theta\cos(\theta/2) \end{array} \right\} =$$

$$=(-15/8)\ c\ r^{-3/2} \left\{ \begin{array}{l} \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda+49A\mu-251B\mu) \end{bmatrix} \left[\sin\theta\cos(\theta/2)+\cos\theta\sin(\theta/2)\right] - \\ -100\mu(A+5B)\cos\theta\sin\theta\cos(\theta/2) \end{array} \right\} =$$

$$=(-15/8)\ c\ r^{-3/2} \left\{ \begin{array}{l} \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda+49A\mu-251B\mu) \end{bmatrix} \sin(3\theta/2) - \\ -100\mu(A+5B)\cos\theta\sin\theta\cos(\theta/2) \end{array} \right\} =$$

(since $\frac{\sin(3\theta/2)}{\sin(\theta/2)} = 1 + 2\cos\theta$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$)

$$=(-15/8)\ c\ r^{-3/2} \sin(\theta/2) \left\{ \begin{array}{l} \begin{bmatrix} 25\mu(A+5B)(-1-2\cos\theta) - \\ -(100A\lambda+49A\mu-251B\mu) \end{bmatrix} (1+2\cos\theta) - \\ -100\mu(A+5B)\times 2\cos\theta\cos^2(\theta/2) \end{array} \right\} =$$

$$=(-15/8)\ c\ r^{-3/2} \sin(\theta/2) \left\{ \begin{bmatrix} 25\mu(A+5B)\left[(-1-2\cos\theta)(1+2\cos\theta) - 8\cos\theta\cos^2(\theta/2)\right] - \\ -(100A\lambda+49A\mu-251B\mu)(1+2\cos\theta) \end{bmatrix} \right\} =$$

$$=(-15/8)cr^{-3/2}\sin(\theta/2)\left\{\begin{bmatrix}25\mu(A+5B)\left[\left(-1-4\cos\theta-4\cos^2\theta-8\cos\theta\cos^2(\theta/2)\right)\right]- \\ -(100A\lambda+49A\mu-251B\mu)(1+2\cos\theta)\end{bmatrix}\right\}$$

The representation $\left(-1-4\cos\theta-4\cos^2\theta-8\cos\theta\cos^2(\theta/2)\right)$ can be simplified into:

$$\begin{aligned}
&-1-4\cos\theta-4\cos^2\theta-8\cos\theta\cos^2(\theta/2)= \\
&=-1-4\cos\theta-4\frac{1+\cos(2\theta)}{2}-8\cos\theta\frac{1+\cos\theta}{2}= \\
&=-1-4\cos\theta-2-2\cos(2\theta)-4\cos\theta-4\cos^2\theta= \\
&=-3-8\cos\theta-2\cos(2\theta)-4\frac{1+\cos(2\theta)}{2}= \\
&=-5-8\cos\theta-4\cos(2\theta)
\end{aligned}$$

so

$$(-15/8)cr^{-3/2}\sin(\theta/2)\begin{bmatrix}25\mu(A+5B)(-5-8\cos\theta-4\cos(2\theta))- \\ -(100A\lambda+49A\mu-251B\mu)(1+2\cos\theta)\end{bmatrix}=$$

$$=(-15/8)cr^{-3/2}\sin(\theta/2)\begin{bmatrix}-125\mu(A+5B)-(100A\lambda+49A\mu-251B\mu)+ \\ +\left[-200\mu(A+5B)-2(100A\lambda+49A\mu-251B\mu)\right]\cos\theta+ \\ +\left[-100\mu(A+5B)\right]\cos(2\theta)\end{bmatrix}\Rightarrow$$

$$\Rightarrow\frac{\partial}{\partial y}\left(\cos\theta\, m_{xxx}+\sin\theta\, m_{yxx}\right)=$$

$$=(+15/8)cr^{-3/2}\sin(\theta/2)\begin{bmatrix}(100A\lambda+174A\mu+374B\mu)+ \\ +(200A\lambda+298A\mu+498B\mu)\cos\theta+ \\ +100\mu(A+5B)\cos(2\theta)\end{bmatrix}$$

<u>Partial derivative of sum 2 with respect to x</u>

$$\frac{\partial}{\partial x}\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)=$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)=$$

$$=\cos\theta\frac{\partial}{\partial r}\left\{(-15/4)c\mu r^{-1/2}\sin(\theta/2)\begin{Bmatrix}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{Bmatrix}\right\}-$$

$$-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left\{(-15/4)c\mu r^{-1/2}\sin(\theta/2)\begin{Bmatrix}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{Bmatrix}\right\}=$$

$$=\cos\theta\cdot(+15/8)c\mu r^{-3/2}\sin(\theta/2)\begin{Bmatrix}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{Bmatrix}-$$

$$-\frac{\sin\theta}{r}\cdot(-15/4)c\mu r^{-1/2}\left\{\begin{matrix}\frac{1}{2}\cos(\theta/2)\begin{Bmatrix}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{Bmatrix}+\\ \\+\sin(\theta/2)\begin{Bmatrix}(24A+124B)(-2\sin\theta)+\\+4(A+B)2\cos(\theta/2)(-\sin(\theta/2))\frac{1}{2}\end{Bmatrix}\end{matrix}\right\}=$$

$$=(+15/8)c\mu r^{-3/2}\cos\theta\sin(\theta/2)\begin{Bmatrix}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{Bmatrix}+$$

$$+(+15/8)c\mu r^{-3/2}\sin\theta\left\{\begin{matrix}\cos(\theta/2)\begin{Bmatrix}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{Bmatrix}+\\ \\+\sin(\theta/2)\begin{Bmatrix}-4\sin\theta(24A+124B)-\\-8(A+B)\cos(\theta/2)\sin(\theta/2)\end{Bmatrix}\end{matrix}\right\}=$$

$$=(+15/8)c\mu r^{-3/2}\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}[\cos\theta\sin(\theta/2)+\sin\theta\cos(\theta/2)]+\\ \\+\sin\theta\sin(\theta/2)\left\{\begin{array}{l}-4\sin\theta(24A+124B)-\\-8(A+B)\cos(\theta/2)\sin(\theta/2)\end{array}\right\}\end{array}\right\}=$$

$(8\cos(\theta/2)\sin(\theta/2)=4\sin\theta)$

$$=(+15/8)c\mu r^{-3/2}\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}\sin(3\theta/2)-\\ \\-4\sin^2\theta\sin(\theta/2)\{(24A+124B)+(A+B)\}\end{array}\right\}=$$

$$=(+15/8)c\mu r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}(1+2\cos\theta)-\\ \\-4\sin^2\theta\{(24A+124B)+(A+B)\}\end{array}\right\}=$$

$$=(+15/8)c\mu r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[(1+2\cos\theta)^2-4\sin^2\theta\right]+\\ \\+4(A+B)\left[\cos^2(\theta/2)(1+2\cos\theta)-\sin^2\theta\right]\end{array}\right\}$$

Some simplifications can be performed:

$$\begin{aligned}&(1+2\cos\theta)^2-4\sin^2\theta=\\&=1+4\cos\theta+4\cos^2\theta-4(1-\cos^2\theta)=\\&=-3+4\cos\theta+8\cos^2\theta=\\&=-3+4\cos\theta+8\frac{1+\cos(2\theta)}{2}=\\&=1+4\cos\theta+4\cos(2\theta)\end{aligned}$$

and

$$\cos^2(\theta/2)\left(1+2\cos\theta\right)-\sin^2\theta=$$
$$=\frac{1+\cos\theta}{2}\left(1+2\cos\theta\right)-\frac{1-\cos(2\theta)}{2}=$$
$$=\frac{1}{2}\left(1+2\cos\theta+\cos\theta+2\cos^2\theta-1+\cos(2\theta)\right)=$$
$$=\frac{1}{2}\left(3\cos\theta+2\cos^2\theta+\cos(2\theta)\right)=$$
$$=\frac{1}{2}\left(3\cos\theta+\cancel{2}\,\frac{1+\cos(2\theta)}{\cancel{2}}+\cos(2\theta)\right)=$$
$$=\frac{1}{2}\left(1+3\cos\theta+2\cos(2\theta)\right)$$

After these simplifications:

$$(+15/8)\mathrm{c}\mu\,\mathrm{r}^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left(24\mathrm{A}+124\mathrm{B}\right)\left(1+4\cos\theta+4\cos(2\theta)\right)+\\+4\left(\mathrm{A}+\mathrm{B}\right)\left[\frac{1}{2}\left(1+3\cos\theta+2\cos(2\theta)\right)\right]\end{array}\right\}=$$

$$=(+15/8)\mathrm{c}\mu\,\mathrm{r}^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left(24\mathrm{A}+124\mathrm{B}\right)\left(1+4\cos\theta+4\cos(2\theta)\right)+\\+2\left(\mathrm{A}+\mathrm{B}\right)\left(1+3\cos\theta+2\cos(2\theta)\right)\end{array}\right\}=$$

$$=(+15/8)\mathrm{c}\mu\,\mathrm{r}^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left(24\mathrm{A}+124\mathrm{B}\right)+2\left(\mathrm{A}+\mathrm{B}\right)+\\+\left[4\left(24\mathrm{A}+124\mathrm{B}\right)+6\left(\mathrm{A}+\mathrm{B}\right)\right]\cos\theta+\\+\left[4\left(24\mathrm{A}+124\mathrm{B}\right)+4\left(\mathrm{A}+\mathrm{B}\right)\right]\cos(2\theta)\end{array}\right\}\Rightarrow$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\cos\theta\, m_{xyx}+\sin\theta\, m_{yyx}\right)=$$
$$=(+15/8)\,\mathrm{c}\,\mu\,\mathrm{r}^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left(26\mathrm{A}+126\mathrm{B}\right)+\\+\left(102\mathrm{A}+502\mathrm{B}\right)\cos\theta+\\+\left(100\mathrm{A}+500\mathrm{B}\right)\cos(2\theta)\end{array}\right\}$$

Partial derivative of sum 2 with respect to y

$$\frac{\partial}{\partial y}\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xyx}+\sin\theta m_{yyx}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left\{(\text{-}15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\sin(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(1+2\cos\theta)+\\+4(\mathrm{A{+}B})\cos^2(\theta/2)\end{Bmatrix}\right\}+$$

$$+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left\{(\text{-}15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\sin(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(1+2\cos\theta)+\\+4(\mathrm{A{+}B})\cos^2(\theta/2)\end{Bmatrix}\right\}=$$

$$=\sin\theta\cdot(+15/8)\mathrm{c}\mu\,\mathrm{r}^{-3/2}\sin(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(1+2\cos\theta)+\\+4(\mathrm{A{+}B})\cos^2(\theta/2)\end{Bmatrix}+$$

$$+\frac{\cos\theta}{r}\cdot(\text{-}15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left\{\begin{matrix}\frac{1}{2}\cos(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(1+2\cos\theta)+\\+4(\mathrm{A{+}B})\cos^2(\theta/2)\end{Bmatrix}+\\ \\+\sin(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(-2\sin\theta)+\\+4(\mathrm{A{+}B})2\cos(\theta/2)(-\sin(\theta/2))\frac{1}{2}\end{Bmatrix}\end{matrix}\right\}=$$

$$=(+15/8)\mathrm{c}\mu\,\mathrm{r}^{-3/2}\sin\theta\sin(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(1+2\cos\theta)+\\+4(\mathrm{A{+}B})\cos^2(\theta/2)\end{Bmatrix}+$$

$$+(\text{-}15/8)\mathrm{c}\mu\,\mathrm{r}^{-3/2}\cos\theta\left\{\begin{matrix}\cos(\theta/2)\begin{Bmatrix}(24\mathrm{A}+124\mathrm{B})(1+2\cos\theta)+\\+4(\mathrm{A{+}B})\cos^2(\theta/2)\end{Bmatrix}+\\ \\+\sin(\theta/2)\begin{Bmatrix}-4(24\mathrm{A}+124\mathrm{B})\sin\theta-\\-8(\mathrm{A{+}B})\cos(\theta/2)\sin(\theta/2)\end{Bmatrix}\end{matrix}\right\}=$$

$$= (+15/8)c\mu r^{-3/2}\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}\left[\sin\theta\sin(\theta/2)-\cos\theta\cos(\theta/2)\right]-\\ \\ -\left\{\begin{array}{l}-4(24A+124B)\sin\theta-\\-8(A+B)\cos(\theta/2)\sin(\theta/2)\end{array}\right\}\cos\theta\sin(\theta/2)\end{array}\right\}=$$

$$= (+15/8)c\mu r^{-3/2}\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}\left[-\cos(3\theta/2)\right]+\\ \\ +4\left\{\begin{array}{l}(24A+124B)\sin\theta+\\+2(A+B)\cos(\theta/2)\sin(\theta/2)\end{array}\right\}\cos\theta\sin(\theta/2)\end{array}\right\}=$$

$$= (+15/8)c\mu r^{-3/2}\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}\left[-\cos(3\theta/2)\right]+\\ \\ +4\cos\theta\sin\theta\sin(\theta/2)\left\{(24A+124B)+(A+B)\right\}\end{array}\right\}=$$

(because $\frac{\cos(3\theta/2)}{\cos(\theta/2)}=-1+2\cos\theta$)

$$= (+15/8)c\mu r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}\left\{\begin{array}{l}(24A+124B)(1+2\cos\theta)+\\+4(A+B)\cos^2(\theta/2)\end{array}\right\}(1-2\cos\theta)+\\ \\ +8\cos\theta\sin^2(\theta/2)\left\{(24A+124B)+(A+B)\right\}\end{array}\right\}=$$

$$=(+15/8)c\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[(1+2\cos\theta)(1-2\cos\theta)+8\cos\theta\sin^2(\theta/2)\right]+\\ +4(A+B)\left[\cos^2(\theta/2)(1-2\cos\theta)+2\cos\theta\sin^2(\theta/2)\right]\end{array}\right\}=$$

$$=(+15/8)c\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[1-4\cos^2\theta+8\cos\theta\sin^2(\theta/2)\right]+\\ +4(A+B)\left[\cos^2(\theta/2)-2\cos\theta\cos^2(\theta/2)+2\cos\theta\sin^2(\theta/2)\right]\end{array}\right\}=$$

$$=(+15/8)c\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[1-4\frac{1+\cos(2\theta)}{2}+8\cos\theta\frac{1-\cos\theta}{2}\right]+\\ +4(A+B)\left[\frac{1+\cos\theta}{2}-2\cos\theta\frac{1+\cos\theta}{2}+2\cos\theta\frac{1-\cos\theta}{2}\right]\end{array}\right\}=$$

$$=(+15/8)c\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[1-2-2\cos(2\theta)+4\cos\theta-4\cos^2\theta\right]+\\ +4(A+B)\frac{1}{2}\left[1+\cos\theta-2\cos\theta-2\cos^2\theta+2\cos\theta-2\cos^2\theta\right]\end{array}\right\}=$$

$$=(+15/8)c\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[-1+4\cos\theta-2\cos(2\theta)-4\frac{1+\cos(2\theta)}{2}\right]+\\ +4(A+B)\frac{1}{2}\left[1+\cos\theta-4\frac{1+\cos(2\theta)}{2}\right]\end{array}\right\}=$$

$$=(+15/8)c\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(24A+124B)\left[-3+4\cos\theta-4\cos(2\theta)\right]+\\ +4(A+B)\frac{1}{2}\left[-1+\cos\theta-2\cos(2\theta)\right]\end{array}\right\}=$$

$$= (+15/8)c\mu r^{-3/2}\cos(\theta/2)\begin{Bmatrix} -3(24A+124B)-2(A+B)+ \\ +\left[4(24A+124B)+2(A+B)\right]\cos\theta+ \\ +\left[4(24A+124B)+4(A+B)\right]\cos(2\theta) \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\cos\theta\, m_{xyx} + \sin\theta m_{yyx}\right) =$$

$$= (+15/8)c\mu r^{-3/2}\cos(\theta/2)\begin{Bmatrix} -(74A+374B)+ \\ +(98A+498B)\cos\theta - \\ -100(A+5B)\cos(2\theta) \end{Bmatrix}$$

With the calculation of these 4 expressions, i.e. the partrial derivatives of sum 1 and sum 2 with respect to x and y, the estimation of Subterm C of P_x is now feasible.

Subterm C of P_x:

$$D_x\left(n_x m_{xxx} + n_y m_{yxx}\right) + D_y\left(n_x m_{xyx} + n_y m_{yyx}\right) =$$
$$= \sin^2\theta\,\partial_x\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) - \cos\theta\,\sin^2\theta\,\partial_y\left(\cos\theta\, m_{xxx} + \sin\theta\, m_{yxx}\right) +$$
$$+ \cos^2\theta\,\partial_y\left(\cos\theta\, m_{xyx} + \sin\theta m_{yyx}\right) - \sin\theta\cos\theta\,\partial_x\left(\cos\theta\, m_{xyx} + \sin\theta m_{yyx}\right) =$$

$$= \sin^2\theta\cdot(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix} (100A\lambda+74A\mu-126B\mu)- \\ -2(100A\lambda+49A\mu-251B\mu)\cos\theta - \\ -100\mu(A+5B)\cos(2\theta) \end{Bmatrix} -$$

$$-\cos\theta\,\sin\theta\cdot(+15/8)cr^{-3/2}\sin(\theta/2)\begin{bmatrix} (100A\lambda+174A\mu+374B\mu)+ \\ +(200A\lambda+298A\mu+498B\mu)\cos\theta + \\ +100\mu(A+5B)\cos(2\theta) \end{bmatrix} +$$

$$+\cos^2\theta\cdot(+15/8)c\mu r^{-3/2}\cos(\theta/2)\begin{vmatrix}-(74A+374B)+\\+(98A+498B)\cos\theta-\\-100(A+5B)\cos(2\theta)\end{vmatrix}-$$

$$-\sin\theta\cos\theta\cdot(+15/8)c\,\mu\, r^{-3/2}\sin(\theta/2)\begin{vmatrix}(26A+126B)+\\+(102A+502B)\cos\theta+\\+(100A+500B)\cos(2\theta)\end{vmatrix}=$$

$$=(+15/8)\,c\,r^{-3/2}\cos(\theta/2)\begin{vmatrix}\left[-(100A\lambda+74A\mu-126B\mu)\sin^2\theta-(74A+374B)\mu\cos^2\theta\right]+\\+\left[2(100A\lambda+49A\mu-251B\mu)\sin^2\theta+(98A+498B)\mu\cos^2\theta\right]\cos\theta+\\+\left[100\mu(A+5B)\sin^2\theta-100\mu(A+5B)\cos^2\theta\right]\cos(2\theta)\end{vmatrix}-$$

$$-(+15/8)cr^{-3/2}\sin\theta\cos\theta\sin(\theta/2)\begin{vmatrix}\left[(100A\lambda+174A\mu+374B\mu)+(26A+126B)\mu\right]+\\+\left[(200A\lambda+298A\mu+498B\mu)+(102A+502B)\mu\right]\cos\theta+\\+\left[100\mu(A+5B)+100\mu(A+5B)\right]\cos(2\theta)\end{vmatrix}=$$

For practical reasons, the penultimate set of brackets is called term (i) and the last one term (ii).
Term (i) is:

- The term in the first line:

$$\begin{aligned}&\text{-}100A\lambda\sin^2\theta+A\mu\left[-74\cos^2\theta-74\sin^2\theta\right]+B\mu\left[-374\cos^2\theta+126\sin^2\theta\right]=\\&=\text{-}100A\lambda\sin^2\theta-74A\mu+B\mu\left[-248\cos^2\theta-126\cos^2\theta+126\sin^2\theta\right]=\\&=\text{-}100A\lambda\sin^2\theta-74A\mu+B\mu\left[-248\frac{1+\cos(2\theta)}{2}-126\cos(2\theta)\right]=\\&=\text{-}100A\lambda\sin^2\theta-74A\mu+B\mu\left[-124-124\cos(2\theta)-126\cos(2\theta)\right]=\\&=\text{-}100A\lambda\sin^2\theta-74A\mu+B\mu\left[-124-250\cos(2\theta)\right]\end{aligned}$$

- The term in the second line (multiplied by cosθ):

$$2(100A\lambda+49A\mu-251B\mu)\sin^2\theta+(98A+498B)\mu\cos^2\theta=$$
$$=200A\lambda\sin^2\theta+A\mu\left(98\cos^2\theta+98\sin^2\theta\right)+B\mu\left[498\cos^2\theta-502\sin^2\theta\right]=$$
$$=200A\lambda\sin^2\theta+98A\mu+B\mu\left[498\cos^2\theta-498\sin^2\theta-4\sin^2\theta\right]=$$
$$=200A\lambda\sin^2\theta+98A\mu+B\mu\left[498\cos(2\theta)-4\frac{1-\cos(2\theta)}{2}\right]=$$
$$=200A\lambda\sin^2\theta+98A\mu+B\mu\left[-2+500\cos(2\theta)\right]$$

- The third line (multiplied by $\cos(2\theta)$):

$$100\mu(A+5B)\sin^2\theta-100\mu(A+5B)\cos^2\theta=$$
$$=-100\mu(A+5B)\cos(2\theta)$$

The full form of term (i) is:

$$(+15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-\left[100A\lambda\sin^2\theta+74A\mu+B\mu(124+250\cos(2\theta))\right]+\\+\left[200A\lambda\sin^2\theta+98A\mu+(-2+500\cos(2\theta))B\mu\right]\cos\theta+\\+\left[-100\mu(A+5B)\cos(2\theta)\right]\cos(2\theta)\end{Bmatrix}$$

The same procedure is followed for term (ii).

Term (ii) is:

$$-(+15/8)cr^{-3/2}\sin\theta\cos\theta\sin(\theta/2)\begin{Bmatrix}\left[(100A\lambda+174A\mu+374B\mu)+(26A+126B)\mu\right]+\\+\left[(200A\lambda+298A\mu+498B\mu)+(102A+502B)\mu\right]\cos\theta+\\+\left[100\mu(A+5B)+100\mu(A+5B)\right]\cos(2\theta)\end{Bmatrix}=$$

and can be written as:

- The term in the first line:

$$(100A\lambda+174A\mu+374B\mu)+(26A+126B)\mu=$$
$$=100A\lambda+200A\mu+500B\mu$$

- The term in the second line (multiplied by $\cos\theta$):

$$(200A\lambda+298A\mu+498B\mu)+(102A+502B)\mu=$$
$$=200A\lambda+400A\mu+1000B\mu$$

- The third line (multiplied by cos(2θ)):

$$100\mu(A+5B)+100\mu(A+5B)=$$
$$=200\mu(A+5B)$$

The full form of term (ii) is:

$$-(+15/8)cr^{-3/2}\sin\theta\cos\theta\sin(\theta/2)\left\{\begin{array}{l}(100A\lambda+200A\mu+500B\mu)+\\+(200A\lambda+400A\mu+1000B\mu)\cos\theta+\\+\left[200\mu(A+5B)\right]\cos(2\theta)\end{array}\right\}=$$

$$=-(+15/8)cr^{-3/2}\cdot2\sin(\theta/2)\cos(\theta/2)\times\cos\theta\sin(\theta/2)\left\{\begin{array}{l}(100A\lambda+200A\mu+500B\mu)+\\+(200A\lambda+400A\mu+1000B\mu)\cos\theta+\\+\left[200\mu(A+5B)\right]\cos(2\theta)\end{array}\right\}=$$

$$=-(+15/8)cr^{-3/2}\cdot2\sin^2(\theta/2)\cos(\theta/2)\cos\theta\left\{\begin{array}{l}(100A\lambda+200A\mu+500B\mu)+\\+(200A\lambda+400A\mu+1000B\mu)\cos\theta+\\+\left[200\mu(A+5B)\right]\cos(2\theta)\end{array}\right\}$$

The full Subterm C of P_x is the sum of term (i) and term (ii) :

$$(+15/8)\,c\,r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}-\left[100A\lambda\sin^2\theta+74A\mu+B\mu(124+250\cos(2\theta))\right]+\\+\left[200A\lambda\sin^2\theta+98A\mu+(-2+500\cos(2\theta))B\mu\right]\cos\theta+\\+\left[-100\mu(A+5B)\cos(2\theta)\right]\cos(2\theta)\end{array}\right\}-$$

$$-(+15/8)cr^{-3/2}\cdot2\sin^2(\theta/2)\cos(\theta/2)\cos\theta\left\{\begin{array}{l}(100A\lambda+200A\mu+500B\mu)+\\+(200A\lambda+400A\mu+1000B\mu)\cos\theta+\\+\left[200\mu(A+5B)\right]\cos(2\theta)\end{array}\right\}=$$

$$=(+15/8)\, c\, r^{-3/2} \cos(\theta/2)\left\{\begin{array}{l} -\left[100A\lambda \sin^2\theta + 74A\mu + B\mu\left(124+250\cos(2\theta)\right)\right] - \\ -2\sin^2(\theta/2)\cos\theta\left(100A\lambda + 200A\mu + 500B\mu\right) + \\ \\ +\left\{\begin{array}{l} \left[200A\lambda\sin^2\theta + 98A\mu + \left(-2+500\cos(2\theta)\right)B\mu\right] - \\ -2\sin^2(\theta/2)\cos\theta\left[200A\lambda + 400A\mu + 1000B\mu\right] \end{array}\right\}\cos\theta + \\ \\ +\left\{\begin{array}{l} \left[-100\mu\left(A+5B\right)\cos(2\theta)\right] - \\ -2\sin^2(\theta/2)\cos\theta\left[200\mu\left(A+5B\right)\right] \end{array}\right\}\cos(2\theta) \end{array}\right\}$$

The term enclosed in the first brackets,

$$\begin{array}{l} -\left[100A\lambda \sin^2\theta + 74A\mu + B\mu\left(124+250\cos(2\theta)\right)\right] - \\ -2\sin^2(\theta/2)\cos\theta\left(100A\lambda + 200A\mu + 500B\mu\right) \end{array},$$

is further simplified this way:

$$\begin{aligned} & 100A\lambda\left[-\sin^2\theta - 2\sin^2(\theta/2)\cos\theta\right] = \\ & = 100A\lambda\left[-\frac{1-\cos(2\theta)}{2} - 2\frac{1-\cos\theta}{2}\cos\theta\right] = \\ & = 100A\lambda\left[-\frac{1}{2} + \frac{1}{2}\cos(2\theta) - \cos\theta + \cos^2\theta\right] = \\ & = 100A\lambda\left[-\frac{1}{2} + \frac{1}{2}\cos(2\theta) - \cos\theta + \frac{1+\cos(2\theta)}{2}\right] = \\ & = 100A\lambda\left[-\frac{1}{2} + \frac{1}{2}\cos(2\theta) - \cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta)\right] = \\ & = 100A\lambda\left[+\frac{1}{2}\cos(2\theta) - \cos\theta + \frac{1}{2}\cos(2\theta)\right] = \\ & = 100A\lambda\left[-\cos\theta + \cos(2\theta)\right] \end{aligned}$$

and

$$
\begin{aligned}
&A\mu\left[-74-400\sin^2(\theta/2)\cos\theta\right]= \\
&= A\mu\left[-74-400\frac{1-\cos\theta}{2}\cos\theta\right]= \\
&= A\mu\left[-74-200\cos\theta+200\cos^2\theta\right]= \\
&= A\mu\left[-74-200\cos\theta+200\frac{1+\cos(2\theta)}{2}\right]= \\
&= A\mu\left[26-200\cos\theta+100\cos(2\theta)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&B\mu\left[-\left(124+250\cos(2\theta)\right)-1000\sin^2(\theta/2)\cos\theta\right]= \\
&= B\mu\left[-\left(124+250\cos(2\theta)\right)-1000\frac{1-\cos\theta}{2}\cos\theta\right]= \\
&= B\mu\left[-\left(124+250\cos(2\theta)\right)-500\cos\theta+500\cos^2\theta\right]= \\
&= B\mu\left[-124-250\cos(2\theta)-500\cos\theta+500\frac{1+\cos(2\theta)}{2}\right]= \\
&= B\mu\left[-124-\cancel{250\cos(2\theta)}-500\cos\theta+250+\cancel{250\cos(2\theta)}\right]= \\
&= B\mu\left[126-500\cos\theta\right]
\end{aligned}
$$

Similarly, the term enclosed in the second brackets, multiplied by cosθ,

$$
\begin{aligned}
&\left[200A\lambda\sin^2\theta+98A\mu+\left(-2+500\cos(2\theta)\right)B\mu\right]- \\
&-2\sin^2(\theta/2)\cos\theta\left[200A\lambda+400A\mu+1000B\mu\right]
\end{aligned}
$$

becomes, bearing in mind that:

$$
\begin{aligned}
&2\sin^2(\theta/2)\cos\theta=2\frac{1-\cos\theta}{2}\cos\theta=\cos\theta-\cos^2\theta=\cos\theta-\frac{1+\cos(2\theta)}{2}= \\
&=\cos\theta-\frac{1}{2}-\frac{1}{2}\cos(2\theta)=\frac{1}{2}\left(-1+2\cos\theta-\cos(2\theta)\right),
\end{aligned}
$$

then,

$$\mathrm{A}\lambda\left[200\sin^2\theta-200\frac{1}{2}(-1+2\cos\theta-\cos(2\theta))\right]=$$
$$=\mathrm{A}\lambda\left[200\frac{1-\cos(2\theta)}{2}+100-200\cos\theta+100\cos(2\theta)\right]=$$
$$=\mathrm{A}\lambda\left[100-100\cos(2\theta)+100-200\cos\theta+100\cos(2\theta)\right]=$$
$$=\mathrm{A}\lambda\left[200-200\cos\theta\right]=$$
$$=200\mathrm{A}\lambda(1-\cos\theta)$$

and

$$\mathrm{A}\mu\left[98-400\frac{1}{2}(-1+2\cos\theta-\cos(2\theta))\right]=$$
$$=\mathrm{A}\mu\left[98+200-400\cos\theta+200\cos(2\theta)\right]=$$
$$=\mathrm{A}\mu\left[298-400\cos\theta+200\cos(2\theta)\right]$$

and

$$\mathrm{B}\mu\left[-2+500\cos(2\theta)-1000\frac{1}{2}(-1+2\cos\theta-\cos(2\theta))\right]=$$
$$=\mathrm{B}\mu\left[-2+500\cos(2\theta)+500-1000\cos\theta+500\cos(2\theta)\right]=$$
$$=\mathrm{B}\mu\left[498-1000\cos\theta+1000\cos(2\theta)\right]$$

The term enclosed in the third brackets, multiplied by $\cos^2\theta$,

$$\left[-100\mu(\mathrm{A}+5\mathrm{B})\cos(2\theta)\right]-$$
$$-2\sin^2(\theta/2)\cos\theta\left[200\mu(\mathrm{A}+5\mathrm{B})\right]$$

is:

$$-100\mu(\mathrm{A}+5\mathrm{B})\cos(2\theta)-200\mu(\mathrm{A}+5\mathrm{B})\frac{1}{2}(-1+2\cos\theta-\cos(2\theta))=$$
$$=-100\mu(\mathrm{A}+5\mathrm{B})\cos(2\theta)-100\mu(\mathrm{A}+5\mathrm{B})(-1+2\cos\theta-\cos(2\theta))=$$
$$=-100\mu(\mathrm{A}+5\mathrm{B})\left[\cos(2\theta)+(-1+2\cos\theta-\cos(2\theta))\right]=$$
$$=-100\mu(\mathrm{A}+5\mathrm{B})(-1+2\cos\theta)$$

Finally, the overall term, consisting of the three aforementioned brackets is:

$$
(+15/8)\ c\ r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}
\left[\begin{array}{l}100A\lambda\left[-\cos\theta+\cos(2\theta)\right]+\\ +A\mu\left[26-200\cos\theta+100\cos(2\theta)\right]+\\ +B\mu\left[126-500\cos\theta\right]\end{array}\right]+\\
+\left[\begin{array}{l}200A\lambda\left(1-\cos\theta\right)+\\ +A\mu\left[298-400\cos\theta+200\cos(2\theta)\right]+\\ +B\mu\left[498-1000\cos\theta+1000\cos(2\theta)\right]\end{array}\right]\cos\theta+\\
+\left[-100\mu\left(A+5B\right)\left(-1+2\cos\theta\right)\right]\cos(2\theta)
\end{array}\right\}
$$

If the terms are grouped with respect to $A\lambda, A\mu, B\mu$, the following expression is acquired:

$$
(+15/8)\ c\ r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}
100A\lambda\left[-\cos\theta+\cos(2\theta)+2\cos\theta\left(1-\cos\theta\right)\right]+\\
+A\mu\left\{\begin{array}{l}\left[26-200\cos\theta+100\cos(2\theta)\right]+\\ +\left[298-400\cos\theta+200\cos(2\theta)\right]\cos\theta+\\ +100\left(1-2\cos\theta\right)\cos(2\theta)\end{array}\right\}+\\
+B\mu\left\{\begin{array}{l}\left[126-500\cos\theta\right]+\\ +\left[498-1000\cos\theta+1000\cos(2\theta)\right]\cos\theta+\\ +500\left(1-2\cos\theta\right)\cos(2\theta)\end{array}\right\}
\end{array}\right\}
$$

The brackets multiplying $100A\lambda$ are:

$$
\begin{aligned}
&-\cos\theta+\cos(2\theta)+2\cos\theta\left(1-\cos\theta\right)=\\
&=-\cos\theta+\cos(2\theta)+2\cos\theta-2\cos^2\theta=\\
&=\cos\theta+\cos(2\theta)-2\frac{1+\cos(2\theta)}{2}=\\
&=-1+\cos\theta
\end{aligned}
$$

the brackets multiplying $A\mu$ are:

$$
\begin{aligned}
&\left[26-200\cos\theta+100\cos(2\theta)\right]+\\
&+\left[298-400\cos\theta+200\cos(2\theta)\right]\cos\theta+\\
&+100\left(1-2\cos\theta\right)\cos(2\theta)=
\end{aligned}
$$

$$
\begin{aligned}
&=26-200\cos\theta+100\cos(2\theta)+\\
&+298\cos\theta-400\cos^2\theta+\cancel{200\cos(2\theta)\cos\theta}+\\
&+100\cos(2\theta)-\cancel{200\cos(2\theta)\cos\theta}=
\end{aligned}
$$

$$
\begin{aligned}
&=26+98\cos\theta-400\cos^2\theta+200\cos(2\theta)=\\
&=26+98\cos\theta-400\frac{1+\cos(2\theta)}{2}+200\cos(2\theta)=\\
&=26+98\cos\theta-200-\cancel{200\cos(2\theta)}+\cancel{200\cos(2\theta)}=\\
&=-174+98\cos\theta
\end{aligned}
$$

and the brackets multiplying $B\mu$ are:

$$
\begin{aligned}
&\left[126-500\cos\theta\right]+\left[498-1000\cos\theta+1000\cos(2\theta)\right]\cos\theta+\\
&+500\left(1-2\cos\theta\right)\cos(2\theta)=\\
&=126-500\cos\theta+498\cos\theta-1000\cos^2\theta+\cancel{1000\cos(2\theta)\cos\theta}+\\
&+500\cos(2\theta)-\cancel{1000\cos\theta\cos(2\theta)}=\\
&=126-2\cos\theta-1000\frac{1+\cos(2\theta)}{2}+500\cos(2\theta)=\\
&=126-2\cos\theta-500-\cancel{500\cos(2\theta)}+\cancel{500\cos(2\theta)}\\
&=-374-2\cos\theta
\end{aligned}
$$

Finally, Subterm C of P_x in its most compact form becomes:

$$
\text{Subterm C of } P_x = (+15/8)\, c\, r^{-3/2}\cos(\theta/2)\cdot\begin{Bmatrix}100A\lambda\left(-1+\cos\theta\right)+\\ +A\mu\left(-174+98\cos\theta\right)+\\ +B\mu\left(-374-2\cos\theta\right)\end{Bmatrix}
$$

CALCULATION OF SUBTERM D: $\left(D_j n_j\right)R_x$

$$\left(D_j n_j\right)\mathrm{R}_x = \left(D_x n_x + D_y n_y\right)\mathrm{R}_x$$

Each one of $D_x n_x, D_y n_y$ will be estimated separately.

$$\begin{aligned}
D_x n_x &= \partial_x\left(n_x\right) - n_x D\left(n_x\right) = \partial_x\left(n_x\right) - n_x n_r \partial_r\left(n_x\right) = \\
&= \partial_x\left(n_x\right) - n_x\left[n_x\partial_x\left(n_x\right) + n_y\partial_y\left(n_x\right)\right] = \\
&= \partial_x\left(n_x\right) - n_x^2\partial_x\left(n_x\right) - n_x n_y \partial_y\left(n_x\right) = \\
&= \left(1-n_x^2\right)\partial_x\left(n_x\right) - n_x n_y \partial_y\left(n_x\right) = \\
&= \left(1-\cos^2\theta\right)\partial_x\left(\cos\theta\right) - \cos\theta\sin\theta\,\partial_y\left(\cos\theta\right) \Rightarrow
\end{aligned}$$

$$\Rightarrow D_x n_x = \sin^2\theta\ \partial_x\left(\cos\theta\right) - \cos\theta\sin\theta\,\partial_y\left(\cos\theta\right)$$

The derivatives $\partial_x\left(\cos\theta\right)$, $\partial_y\left(\cos\theta\right)$ should be estimated in order to continue.

$$\begin{aligned}
\partial_x\left(\cos\theta\right) &= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\right) = \\
&= 0 - \frac{\sin\theta}{r}\left(-\sin\theta\right) = \frac{\sin^2\theta}{r} \Rightarrow
\end{aligned}$$

$$\Rightarrow\ \partial_x\left(\cos\theta\right) = \frac{\sin^2\theta}{r}$$

and

$$\begin{aligned}
\partial_y\left(\cos\theta\right) &= \sin\theta\frac{\partial}{\partial r}\left(\cos\theta\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\right) = \\
&= 0 + \frac{\cos\theta}{r}\left(-\sin\theta\right) = -\frac{\cos\theta\sin\theta}{r} \Rightarrow
\end{aligned}$$

$$\Rightarrow\ \partial_y\left(\cos\theta\right) = -\frac{\cos\theta\sin\theta}{r}$$

Consequently,

$$\begin{aligned}
D_x n_x &= \sin^2\theta\ \frac{\sin^2\theta}{r} - \cos\theta\sin\theta\left(-\frac{\cos\theta\sin\theta}{r}\right) = \\
&= \frac{\sin^4\theta}{r} + \frac{\cos^2\theta\sin^2\theta}{r} = \\
&= \frac{\sin^2\theta}{r}\left(\sin^2\theta + \cos^2\theta\right) \Rightarrow
\end{aligned}$$

$$\Rightarrow D_x n_x = \frac{\sin^2\theta}{r}$$

In the same manner,

$$D_y n_y = \partial_y(n_y) - n_y D(n_y) = \partial_y(n_y) - n_y n_r \partial_r(n_y) =$$
$$= \partial_y(n_y) - n_y\left[n_x \partial_x(n_y) + n_y \partial_y(n_y)\right] =$$
$$= \partial_y(n_y) - n_y n_x \partial_x(n_y) - n_y^2 \partial_y(n_y) =$$
$$= \left(1 - n_y^2\right)\partial_y(n_y) - n_y n_x \partial_x(n_y) =$$
$$= \left(1 - \sin^2\theta\right)\partial_y(\sin\theta) - \sin\theta\cos\theta\,\partial_x(\sin\theta) \Rightarrow$$

$$\Rightarrow D_y n_y = \cos^2\theta\ \partial_y(\sin\theta) - \sin\theta\cos\theta\,\partial_x(\sin\theta)$$

The partial derivatives $\partial_y(\sin\theta)$, $\partial_x(\sin\theta)$ will be obtained first.

$$\partial_y(\sin\theta) = \sin\theta \frac{\partial}{\partial r}(\sin\theta) + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}(\sin\theta) =$$
$$= 0 + \frac{\cos\theta}{r}(\cos\theta) = \frac{\cos^2\theta}{r} \Rightarrow$$

$$\Rightarrow \partial_y(\sin\theta) = \frac{\cos^2\theta}{r}$$

and

$$\partial_x(\sin\theta) = \cos\theta \frac{\partial}{\partial r}(\sin\theta) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(\sin\theta) =$$
$$= 0 - \frac{\sin\theta}{r}(\cos\theta) = -\frac{\sin\theta\cos\theta}{r} \Rightarrow$$

$$\Rightarrow \partial_x(\sin\theta) = -\frac{\sin\theta\cos\theta}{r}$$

Therefore, $D_y n_y$ is:

$$D_y n_y = \cos^2\theta \frac{\cos^2\theta}{r} - \sin\theta\cos\theta \frac{(-\sin\theta\cos\theta)}{r} =$$
$$= \frac{\cos^4\theta}{r} + \frac{\sin^2\theta\cos^2\theta}{r} =$$
$$= \frac{\cos^2\theta}{r}\left(\cos^2\theta + \sin^2\theta\right) \Rightarrow$$

$$\Rightarrow D_y n_y = \frac{\cos^2\theta}{r}$$

The sum $\left(D_j n_j\right) = D_x n_x + D_y n_y$ leads to:

$$\left(D_j n_j\right) = D_x n_x + D_y n_y =$$

$$= \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} =$$

$$= \frac{1}{r} \Rightarrow$$

$$\Rightarrow \left(D_j n_j\right) = \frac{1}{r}$$

Eventually, Subterm D of P_x takes the form:

$$\left(D_j n_j\right)\mathrm{R}_x = \left(D_x n_x + D_y n_y\right)\mathrm{R}_x \Rightarrow$$

$$\Rightarrow \boxed{\left(D_j n_j\right)\mathrm{R}_x = \frac{\mathrm{R}_x}{r}}$$

Having estimated all Subterms of P_x, its full form can be acquired. First, the sum: Subterm B+ Subterm C will be calculated.
Then, P_x = -(Subterm B+ Subterm C) + Subterm D.

<u>CALCULATION OF SUM: SUBTERM B+ SUBTERM C</u>

Subterm B+ Subterm C=

$$= (+15/8)\mathrm{c}\,\mu\,\mathrm{r}^{-3/2}\cos(\theta/2)\times 200\left(\mathrm{A}+5\mathrm{B}\right)+$$

$$+(+15/8)\;\mathrm{c}\;\mathrm{r}^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}100\mathrm{A}\lambda\left(-1+\cos\theta\right)+\\ +\mathrm{A}\mu\left(-174+98\cos\theta\right)+\\ +\mathrm{B}\mu\left(-374-2\cos\theta\right)\end{array}\right\} =$$

$$= (+15/8)\;\mathrm{c}\;\mathrm{r}^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}100\mathrm{A}\lambda\left(-1+\cos\theta\right)+\\ +\mathrm{A}\mu\left(200-174+98\cos\theta\right)+\\ +\mathrm{B}\mu\left(1000-374-2\cos\theta\right)\end{array}\right\} =$$

$$= (+15/8)\ c\ r^{-3/2} \cos(\theta/2) \begin{Bmatrix} 100A\lambda(-1+\cos\theta)+ \\ +A\mu(26+98\cos\theta)+ \\ +B\mu(626-2\cos\theta) \end{Bmatrix} \Rightarrow$$

$\Rightarrow$ Subterm B+ Subterm C=

$$= (+15/8)\ c\ r^{-3/2} \cos(\theta/2) \begin{Bmatrix} 100A\lambda(-1+\cos\theta)+ \\ +A\mu(26+98\cos\theta)+ \\ +B\mu(626-2\cos\theta) \end{Bmatrix}$$

<u>CALCULATION OF SUM: -(SUBTERM B+ SUBTERM C)+ SUBTERM D</u>

Now,

-(Subterm B of P_x + Subterm C of P_x) + Subterm D of P_x =

$$(-15/8)\ c\ r^{-3/2} \cos(\theta/2) \begin{Bmatrix} 100A\lambda(-1+\cos\theta)+ \\ +A\mu(26+98\cos\theta)+ \\ +B\mu(626-2\cos\theta) \end{Bmatrix} +$$

$$+(-15/2)\ c\ r^{-1/2} \cos(\theta/2)\left[(13A+63B)\mu+(50A\lambda+49A\mu-B\mu)\cos\theta\right]=$$

$$=(-15/8)\ c\ r^{-3/2} \cos(\theta/2) \begin{Bmatrix} 100A\lambda(-1+\cos\theta)+ \\ +A\mu(26+98\cos\theta)+ \\ +B\mu(626-2\cos\theta) \end{Bmatrix} +$$

$$+(-15/8)\ c\ r^{-1/2} \cos(\theta/2)\left[4(13A+63B)\mu+4(50A\lambda+49A\mu-B\mu)\cos\theta\right]=$$

$$=(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}(-100A\lambda+26A\mu+626B\mu)+4(13A+63B)\mu+ \\ +\left[(100A\lambda+98A\mu-2B\mu)+4(50A\lambda+49A\mu-B\mu)\right]\cos\theta\end{Bmatrix}=$$

$$=(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+A\mu(26+52)+B\mu(626+252)+ \\ +\left[A\lambda(100+200)+A\mu(98+196)+B\mu(-2-4)\right]\cos\theta\end{Bmatrix}=$$

$$=(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+78A\mu+878B\mu+ \\ +(300A\lambda+249A\mu-6B\mu)\cos\theta\end{Bmatrix}\Rightarrow$$

$\Rightarrow$-(Subterm B of P_x + Subterm C of P_x) + Subterm D of P_x =

$$=(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+78A\mu+878B\mu+ \\ +(300A\lambda+294A\mu-6B\mu)\cos\theta\end{Bmatrix}$$

It should be reminded that P_x also contains the term $(n_x\tau_{xx}+n_y\tau_{yx})$. The full form for P_x is:

$$P_x=(n_x\tau_{xx}+n_y\tau_{yx})+$$
$$+(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+78A\mu+678B\mu+ \\ +(300A\lambda+294A\mu-6B\mu)\cos\theta\end{Bmatrix}$$

The product $P_x\left(\frac{\partial u_x}{\partial x}\right)$ should now be estimated. The component $(n_x\tau_{xx}+n_y\tau_{yx})$ of P_x will not be included in this estimation of the product (since this term, after being integrated, vanishes for r→0).

<u>Calculation of the product:</u> $P_x\left(\frac{\partial u_x}{\partial x}\right)$

$$P_x\left(\frac{\partial u_x}{\partial x}\right)=$$

$$(-15/8)\ c\ r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+78A\mu+878B\mu+ \\ +(300A\lambda+294A\mu-6B\mu)\cos\theta\end{Bmatrix}\cdot$$

$$\cdot(-\frac{15}{4})r^{1/2}\cos(\theta/2)\left[(49A\text{-}251B)+25(A+5B)(2\cos\theta+1)\right]=$$

$$=\left(\frac{225}{32}\right)c\,r^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}\left(-100A\lambda+78A\mu+878B\mu\right)\left(49A-251B\right)+\\+\left(-100A\lambda+78A\mu+878B\mu\right)\left(25A+125B\right)+\\+\left(-100A\lambda+78A\mu+878B\mu\right)\left(50A+250B\right)\cos\theta+\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(49A-251B\right)\cos\theta\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(25A+125B\right)\cos\theta\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(50A+250B\right)\cos^2\theta\end{array}\right\}=$$

$$=\left(\frac{225}{32}\right)c\,r^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}\left(-100A\lambda+78A\mu+878B\mu\right)\left(49A-251B+25A+125B\right)+\\+\left(-100A\lambda+78A\mu+878B\mu\right)\left(50A+250B\right)\cos\theta+\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(49A-251B+25A+125B\right)\cos\theta\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(50A+250B\right)\cos^2\theta\end{array}\right\}=$$

$$=\left(\frac{225}{32}\right)c\,r^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}\left(-100A\lambda+78A\mu+878B\mu\right)\left(74A-126B\right)+\\+\left(-100A\lambda+78A\mu+878B\mu\right)\left(50A+250B\right)\cos\theta+\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(74A-126B\right)\cos\theta\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(50A+250B\right)\cos^2\theta\end{array}\right\}=$$

$$=\left(\frac{225}{32}\right)c\,r^{-1}\cos^2(\theta/2)\left\{\begin{array}{l}\left(-100A\lambda+78A\mu+878B\mu\right)\left(74A-126B\right)+\\+\left[\begin{array}{l}\left(-100A\lambda+78A\mu+878B\mu\right)\left(50A+250B\right)+\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(74A-126B\right)\end{array}\right]\cos\theta+\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(50A+250B\right)\cos^2\theta\end{array}\right\}$$

The product $\left(-100A\lambda+78A\mu+878B\mu\right)\left(74A-126B\right)$ becomes:

$$\begin{aligned}&\left(-100A\lambda+78A\mu+678B\mu\right)\left(74A-126B\right)=\\&=-7400A^2\lambda+12600AB\lambda+78\cdot74A^2\mu-78\cdot126AB\mu+\\&\qquad\qquad+878\cdot74AB\mu-878\cdot126B^2\mu=\\&=\left(-7400A^2+12600AB\right)\lambda+\left(5772A^2+(-9828+64972)AB-110626B^2\right)\mu=\\&=\left(-7400A^2+12600AB\right)\lambda+\left(5772A^2+55144AB-110626B^2\right)\mu\end{aligned}$$

The sum $\left[\begin{array}{l}\left(-100A\lambda+78A\mu+878B\mu\right)\left(50A+250B\right)+\\+\left(300A\lambda+294A\mu-6B\mu\right)\left(74A-126B\right)\end{array}\right]$ becomes:

$$(-100A\lambda+78A\mu+878B\mu)(50A+250B)+$$
$$+(300A\lambda+294A\mu-6B\mu)(74A\text{-}126B)=$$

$$=-5000A^2\lambda-25000AB\lambda+78\cdot50A^2\mu+78\cdot250AB\mu+$$
$$+878\cdot50AB\mu+878\cdot250B^2\mu+$$
$$+300\cdot74A^2\lambda-300\cdot126AB\lambda+294\cdot74A^2\mu-294\cdot126AB\mu-$$
$$-6\cdot74AB\mu+6\cdot126B^2\mu=$$

$$=(22200-5000)A^2\lambda+(-25000-37800)AB\lambda+$$
$$+(3900+21756)A^2\mu+(19500+43900-37044-444)AB\mu+(219500+756)B^2\mu=$$

$$=(17200A^2-62800AB)\lambda+(25656A^2+25912AB+220256B^2)\mu$$

and finally $(300A\lambda+294A\mu-6B\mu)(50A+250B)$ transforms into:

$$(300A\lambda+294A\mu-6B\mu)(50A+250B)=$$
$$=300\cdot50A^2\lambda+300\cdot250AB\lambda+294\cdot50A^2\mu+294\cdot250AB\mu-$$
$$-6\cdot50AB\mu-6\cdot250B^2\mu=$$
$$=(15000A^2+75000AB)\lambda+(14700A^2+73200AB\text{-}1500B^2)\mu=$$
$$=15000(A^2+5AB)\lambda+(14700A^2+73200AB\text{-}1500B^2)\mu$$

After these manipulations, the product $P_x\left(\frac{\partial u_x}{\partial x}\right)$ is:

$$P_x\left(\frac{\partial u_x}{\partial x}\right)=$$
$$=(+225/32)\,c\,r^{-1}\cos^2(\theta/2)\cdot$$
$$\cdot\left\{\begin{aligned}&(-7400A^2+12600AB)\lambda+(5772A^2+55144AB-110628B^2)\mu+\\&+\left[(17200A^2-62800AB)\lambda+(25656A^2+25912AB+220256B^2)\mu\right]\cos\theta+\\&+\left[15000(A^2+5B)\lambda+(14700A^2+73200AB\text{-}1500B^2)\mu\right]\cos^2\theta\end{aligned}\right\}$$

CALCULATION OF P_y

$$P_y = n_p\tau_{py} - n_p\partial_r m_{rpy} - D_p\left(n_r m_{rpy}\right) + \left(D_j n_j\right) n_r n_p m_{rpy} =$$

$$= n_p\tau_{py} - n_p\partial_r m_{rpy} - D_p\left(n_r m_{rpy}\right) + \left(D_j n_j\right) R_y =$$

$$= n_x\tau_{xy} - n_x\partial_r m_{rxy} - D_x\left(n_r m_{rxy}\right) + \left(D_j n_j\right) R_y +$$

$$n_y\tau_{yy} - n_y\partial_r m_{ryy} - D_y\left(n_r m_{ryy}\right) =$$

$$= \left(n_x\tau_{xy} + n_y\tau_{yy}\right) - \left(n_x\partial_r m_{rxy} + n_y\partial_r m_{ryy}\right) - \left[D_x\left(n_r m_{rxy}\right) + D_y\left(n_r m_{ryy}\right)\right] + \left(D_j n_j\right) R_y =$$

$$= \left(n_x\tau_{xy} + n_y\tau_{yy}\right) - \left(n_x\partial_x m_{xxy} + n_y\partial_x m_{xyy}\right) - \left[D_x\left(n_x m_{xxy}\right) + D_y\left(n_x m_{xyy}\right)\right] + \left(D_j n_j\right) R_y -$$

$$-\left(n_x\partial_y m_{yxy} + n_y\partial_y m_{yyy}\right) - \left[D_x\left(n_y m_{yxy}\right) + D_y\left(n_y m_{yyy}\right)\right] \Rightarrow$$

$$\Rightarrow P_y = \left(n_x\tau_{xy} + n_y\tau_{yy}\right) - \begin{bmatrix} n_x\partial_x m_{xxy} + n_y\partial_x m_{xyy} + \\ + n_x\partial_y m_{yxy} + n_y\partial_y m_{yyy} \end{bmatrix} - \begin{bmatrix} D_x\left(n_x m_{xxy} + n_y m_{yxy}\right) + \\ + D_y\left(n_x m_{xyy} + n_y m_{yyy}\right) \end{bmatrix} + \left(D_j n_j\right) R_y$$

The process followed for the calculation of P_x will be performed for the calculation of P_y, too.

The term $\left(n_x\tau_{xy} + n_y\tau_{yy}\right)$ (Subterm A') is ~ $r^{1/2}$.

The product $\left(n_x\tau_{xy} + n_y\tau_{yy}\right)\dfrac{\partial u_y}{\partial x} d\Gamma \sim r^{1/2} r^{1/2} r\, d\theta \sim r^2 \to 0$ for $r \to 0$.

Consequently, this Subterm does not contribute to the final result.

<u>CALCULATION OF SUBTERM B'</u>: $\begin{bmatrix} n_x\partial_x m_{xxy} + n_y\partial_x m_{xyy} + \\ + n_x\partial_y m_{yxy} + n_y\partial_y m_{yyy} \end{bmatrix}$

$$n_x\partial_x m_{xxy} + n_y\partial_x m_{xyy} + n_x\partial_y m_{yxy} + n_y\partial_y m_{yyy} =$$

$$= \cos\theta\left[\cos\theta\frac{\partial}{\partial r} m_{xxy} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} m_{xxy}\right] + \sin\theta\left[\cos\theta\frac{\partial}{\partial r} m_{xyy} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} m_{xyy}\right] +$$

$$+ \cos\theta\left[\sin\theta\frac{\partial}{\partial r} m_{yxy} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} m_{yxy}\right] + \sin\theta\left[\sin\theta\frac{\partial}{\partial r} m_{yyy} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} m_{yyy}\right] =$$

$$=\cos^2\theta\frac{\partial}{\partial r}m_{xxy}-\frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta}m_{xxy}+\sin\theta\cos\theta\frac{\partial}{\partial r}m_{xyy}-\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}m_{xyy}+$$

$$+\cos\theta\sin\theta\frac{\partial}{\partial r}m_{yxy}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta}m_{yxy}+\sin^2\theta\frac{\partial}{\partial r}m_{yyy}+\frac{\cos\theta\sin\theta}{r}\frac{\partial}{\partial\theta}m_{yyy}\Rightarrow$$

$$\Rightarrow n_x\partial_x m_{xxy}+n_y\partial_x m_{xyy}+n_x\partial_y m_{yxy}+n_y\partial_y m_{yyy}=$$

$$=\cos^2\theta\frac{\partial}{\partial r}m_{xxy}+\sin\theta\cos\theta\left[\frac{\partial}{\partial r}m_{xyy}+\frac{\partial}{\partial r}m_{yxy}\right]-\frac{\cos\theta\sin\theta}{r}\left[\frac{\partial}{\partial\theta}m_{xxy}-\frac{\partial}{\partial\theta}m_{yyy}\right]-$$

$$-\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}m_{xyy}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta}m_{yxy}+\sin^2\theta\frac{\partial}{\partial r}m_{yyy}$$

The partial derivatives included in the above expression will be calculated before performing the rest of the calculations.

$$\rightarrow\frac{\partial}{\partial r}m_{xxy}=(-15/8)\,\mathrm{c}\mu\,\mathrm{r}^{-3/2}\left[25(\mathrm{A}+5\mathrm{B})\sin(5\theta/2)-(49\mathrm{A}+249\mathrm{B})\sin(\theta/2)\right]$$

$$\rightarrow\frac{\partial}{\partial r}m_{xyy}=(+15/8)\,\mathrm{c}\,\mathrm{r}^{-3/2}\left[25\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)\cos(\theta/2)\right]$$

$$\rightarrow\frac{\partial}{\partial r}m_{yxy}=(+15/8)\,\mathrm{c}\mu\,\mathrm{r}^{-3/2}\left[25(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)+(51\mathrm{A}+251\mathrm{B})\cos(\theta/2)\right]$$

$$\rightarrow\frac{\partial}{\partial\theta}m_{xxy}=(+15/4)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left[\begin{array}{l}25(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)\frac{5}{2}-\\-(49\mathrm{A}+249\mathrm{B})\cos(\theta/2)\frac{1}{2}\end{array}\right]=$$

$$=(+15/8)\mathrm{c}\mu\,\mathrm{r}^{-1/2}\left[125(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)-(49\mathrm{A}+249\mathrm{B})\cos(\theta/2)\right]$$

$$\rightarrow\frac{\partial}{\partial\theta}m_{yyy}=(-15/4)\mathrm{c}\,\mathrm{r}^{-1/2}\left[\begin{array}{l}25\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)\frac{5}{2}+\\+(100\mathrm{A}\lambda+151\mathrm{A}\mu+251\mathrm{B}\mu)\cos(\theta/2)\frac{1}{2}\end{array}\right]=$$

$$=(-15/8)\mathrm{c}\,\mathrm{r}^{-1/2}\left[\begin{array}{l}125\mu(\mathrm{A}+5\mathrm{B})\cos(5\theta/2)+\\+(100\mathrm{A}\lambda+151\mathrm{A}\mu+251\mathrm{B}\mu)\cos(\theta/2)\end{array}\right]$$

$$\rightarrow \frac{\partial}{\partial\theta} m_{xyy} = (-15/4)\mathrm{c}\,\mathrm{r}^{-1/2} \begin{bmatrix} -25\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)\frac{5}{2} - \\ -(100\mathrm{A}\lambda + 51\mathrm{A}\mu - 249\mathrm{B}\mu)\sin(\theta/2)\frac{1}{2} \end{bmatrix} =$$

$$= (+15/8)\mathrm{c}\,\mathrm{r}^{-1/2} \begin{bmatrix} 125\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2) + \\ +(100\mathrm{A}\lambda + 51\mathrm{A}\mu - 249\mathrm{B}\mu)\sin(\theta/2) \end{bmatrix}$$

$$\rightarrow \frac{\partial}{\partial\theta} m_{yxy} = (-15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2} \begin{bmatrix} -25\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2)\frac{5}{2} - \\ -(51\mathrm{A} + 251\mathrm{B})\sin(\theta/2)\frac{1}{2} \end{bmatrix} =$$

$$= (+15/8)\mathrm{c}\mu\ \mathrm{r}^{-1/2}\left[125\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2) + (51\mathrm{A}+251\mathrm{B})\sin(\theta/2)\right]$$

$$\rightarrow \frac{\partial}{\partial r} m_{yyy} = (+15/8)\mathrm{c}\,\mathrm{r}^{-3/2} \begin{bmatrix} 25\mu\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2) + \\ +(100\mathrm{A}\lambda + 151\mathrm{A}\mu + 251\mathrm{B}\mu)\sin(\theta/2) \end{bmatrix}$$

These partial derivatives are substituted in the expression for Subterm B' deducted already.

$$n_x\partial_x m_{xxy} + n_y\partial_x m_{xyy} + n_x\partial_y m_{yxy} + n_y\partial_y m_{yyy} =$$

$$= \cos^2\theta\frac{\partial}{\partial r}m_{xxy} + \sin\theta\cos\theta\left[\frac{\partial}{\partial r}m_{xyy} + \frac{\partial}{\partial r}m_{yxy}\right] - \frac{\cos\theta\sin\theta}{r}\left[\frac{\partial}{\partial\theta}m_{xxy} - \frac{\partial}{\partial\theta}m_{yyy}\right] -$$

$$-\frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}m_{xyy} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial\theta}m_{yxy} + \sin^2\theta\frac{\partial}{\partial r}m_{yyy} =$$

$$= \cos^2\theta \cdot (-15/8)\, c\mu\, r^{-3/2}\left[25(A+5B)\sin(5\theta/2) - (49A+249B)\sin(\theta/2)\right] +$$

$$+\sin\theta\cos\theta\left\{\begin{array}{l}(+15/8)\, c\, r^{-3/2}\left[25\mu(A+5B)\cos(5\theta/2) + (100A\lambda+51A\mu-249B\mu)\cos(\theta/2)\right] + \\ +(+15/8)\, c\mu\, r^{-3/2}\left[25(A+5B)\cos(5\theta/2) + (51A+251B)\cos(\theta/2)\right]\end{array}\right\} -$$

$$-\frac{\cos\theta\sin\theta}{r}\left\{\begin{array}{l}(+15/8)c\mu\, r^{-1/2}\left[125(A+5B)\cos(5\theta/2) - (49A+249B)\cos(\theta/2)\right] - \\ -(-15/8)c\, r^{-1/2}\left[\begin{array}{l}125\mu(A+5B)\cos(5\theta/2) + \\ +(100A\lambda+151A\mu+251B\mu)\cos(\theta/2)\end{array}\right]\end{array}\right\} -$$

$$-\frac{\sin^2\theta}{r}\cdot(+15/8)c\, r^{-1/2}\left[\begin{array}{l}125\mu(A+5B)\sin(5\theta/2) + \\ +(100A\lambda+51A\mu-249B\mu)\sin(\theta/2)\end{array}\right] +$$

$$+\frac{\cos^2\theta}{r}\cdot(+15/8)c\mu\, r^{-1/2}\left[125(A+5B)\sin(5\theta/2) + (51A+251B)\sin(\theta/2)\right] +$$

$$+\sin^2\theta\cdot(+15/8)c\, r^{-3/2}\left[\begin{array}{l}25\mu(A+5B)\sin(5\theta/2) + \\ +(100A\lambda+151A\mu+251B\mu)\sin(\theta/2)\end{array}\right] =$$

$$= (-15/8)\, c\mu\, r^{-3/2}\cos^2\theta\left[25(A+5B)\sin(5\theta/2) - (49A+249B)\sin(\theta/2)\right] +$$

$$+(+15/8)\, c\, r^{-3/2}\sin\theta\cos\theta\left[50\mu(A+5B)\cos(5\theta/2) + \underbrace{\left(\begin{array}{r}100A\lambda+51A\mu-249B\mu + \\ +51A\mu+251B\mu\end{array}\right)}_{100A\lambda+102A\mu+2B\mu}\cos(\theta/2)\right] -$$

$$-(+15/8)c\, r^{-3/2}\cos\theta\sin\theta\left[250\mu(A+5B)\cos(5\theta/2) + \underbrace{\left(\begin{array}{r}-49A\mu-249B\mu + \\ +100A\lambda+151A\mu+251B\mu\end{array}\right)}_{100A\lambda+102A\mu+2B\mu}\cos(\theta/2)\right] -$$

$$-(+15/8)c\,r^{-3/2}\sin^2\theta\begin{bmatrix}125\mu\left(A+5B\right)\sin(5\theta/2)+\\+(100A\lambda+51A\mu-249B\mu)\sin(\theta/2)\end{bmatrix}+$$

$$+(+15/8)c\mu\ r^{3/2}\cos^2\theta\left[125\left(A+5B\right)\sin(5\theta/2)+(51A+251B)\sin(\theta/2)\right]+$$

$$+(+15/8)c\,r^{-3/2}\sin^2\theta\begin{bmatrix}25\mu\left(A+5B\right)\sin(5\theta/2)+\\+(100A\lambda+151A\mu+251B\mu)\sin(\theta/2)\end{bmatrix}=$$

$$=(+15/8)c\mu\ r^{3/2}\cos^2\theta\begin{bmatrix}\underbrace{\left[125\left(A+5B\right)-25\left(A+5B\right)\right]}_{100\left(A+5B\right)}\sin(5\theta/2)+\\+\underbrace{\left[51A+251B+49A+249B\right]}_{100A+500B=100\left(A+5B\right)}\sin(\theta/2)\end{bmatrix}+$$

$$+(15/8)c\,r^{-3/2}\cos\theta\sin\theta\begin{bmatrix}\left[-200\mu\left(A+5B\right)\right]\cos(5\theta/2)+\\+\underbrace{\begin{pmatrix}100A\lambda+102A\mu+2B\mu-\\-100A\lambda-102A\mu-2B\mu\end{pmatrix}}_{0}\cos(\theta/2)\end{bmatrix}+$$

$$+(15/8)c\,r^{-3/2}\sin^2\theta\begin{bmatrix}\left[-100\mu\left(A+5B\right)\right]\sin(5\theta/2)+\\+\underbrace{\begin{pmatrix}100A\lambda+151A\mu+251B\mu-\\-100A\lambda-51A\mu+249B\mu\end{pmatrix}}_{100A\mu+500B\mu=100\mu\left(A+5B\right)}\sin(\theta/2)\end{bmatrix}=$$

$$=(+15/8)c\mu\ r^{3/2}\cos^2\theta\left[100\left(A+5B\right)\sin(5\theta/2)+100\left(A+5B\right)\sin(\theta/2)\right]+$$

$$+(+15/8)c\,r^{-3/2}\cos\theta\sin\theta\left[-200\mu\left(A+5B\right)\cos(5\theta/2)\right]+$$

$$+(+15/8)c\,r^{-3/2}\sin^2\theta\left[-100\mu\left(A+5B\right)\sin(5\theta/2)+100\mu\left(A+5B\right)\sin(\theta/2)\right]=$$

$$=(+15/8)c\ r^{3/2}\begin{Bmatrix}100\mu\left(A+5B\right)\sin(5\theta/2)\left[\cos^2\theta-\sin^2\theta\right]+\\+\mu\sin(\theta/2)\left[100\left(A+5B\right)\cos^2\theta+100\left(A+5B\right)\sin^2\theta\right]+\\+(-200)\mu\left(A+5B\right)\cos(5\theta/2)\sin\theta\cos\theta\end{Bmatrix}=$$

$$=(+15/8)c\ r^{3/2}\begin{Bmatrix}100\mu\left(A+5B\right)\sin(5\theta/2)\cos(2\theta)+\\+100\mu\left(A+5B\right)\sin(\theta/2)+\\+(-200)\mu\left(A+5B\right)\cos(5\theta/2)\sin\theta\cos\theta\end{Bmatrix}=$$

$$=(+15/8)c\ r^{3/2}\begin{Bmatrix}100\mu\left(A+5B\right)\sin(5\theta/2)\cos(2\theta)+\\+100\mu\left(A+5B\right)\sin(\theta/2)+\\+(-100)\mu\left(A+5B\right)\cos(5\theta/2)\sin(2\theta)\end{Bmatrix}=$$

$$=(+15/8)c\ r^{3/2}\begin{Bmatrix}100\mu\left(A+5B\right)\left[\sin(5\theta/2)\cos(2\theta)-\cos(5\theta/2)\sin(2\theta)\right]+\\+100\mu\left(A+5B\right)\sin(\theta/2)\end{Bmatrix}=$$

$$=(+15/8)c\ r^{3/2}\begin{Bmatrix}100\mu\left(A+5B\right)\sin(\theta/2)+\\+100\mu\left(A+5B\right)\sin(\theta/2)\end{Bmatrix}=$$

$$=(+15/8)c\,\mu r^{3/2}\left[200\left(A+5B\right)\sin(\theta/2)\right]\Rightarrow$$

$\Rightarrow$ Subterm B' of $P_y=(+15/8)c\,\mu r^{3/2}\sin(\theta/2)\left[200\left(A+5B\right)\right]$

CALCULATION OF SUBTERM C': $\begin{bmatrix}D_x\left(n_x m_{xxy}+n_y m_{yxy}\right)+\\+D_y\left(n_x m_{xyy}+n_y m_{yyy}\right)\end{bmatrix}$

The decomposition of this Subterm in 2 parts gives: $D_x\left(n_x m_{xxy}+n_y m_{yxy}\right)$ and $D_y\left(n_x m_{xyy}+n_y m_{yyy}\right)$.

First part: $D_x\left(n_x m_{xxy}+n_y m_{yxy}\right)$

$$D_x\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$
$$= \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - n_x D\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$
$$= \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - n_x n_r \partial_r\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$
$$= \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - n_x\left[n_x \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) + n_y \partial_y\left(n_x m_{xxy} + n_y m_{yxy} \right) \right] =$$
$$= \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - n_x^2 \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - n_x n_y \partial_y\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$

$$= \left(1 - n_x^2\right) \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - n_x n_y \partial_y\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$
$$= \left(1 - \cos^2\theta\right) \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - \cos\theta \sin\theta\, \partial_y\left(n_x m_{xxy} + n_y m_{yxy} \right) \Rightarrow$$

$$\Rightarrow D_x\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$
$$= \sin^2\theta\, \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - \cos\theta \sin\theta\, \partial_y\left(n_x m_{xxy} + n_y m_{yxy} \right)$$

Second part: $D_y\left(n_x m_{xyy} + n_y m_{yyy} \right)$

$$D_y\left(n_x m_{xyy} + n_y m_{yyy} \right) =$$
$$= \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - n_y D\left(n_x m_{xyy} + n_y m_{yyy} \right) =$$
$$= \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - n_y n_r \partial_r\left(n_x m_{xyy} + n_y m_{yyy} \right) =$$
$$= \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - n_y\left[n_x \partial_x\left(n_x m_{xyy} + n_y m_{yyy} \right) + n_y \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) \right] =$$
$$= \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - n_y n_x \partial_x\left(n_x m_{xyy} + n_y m_{yyy} \right) - n_y^2\left(n_x m_{xyy} + n_y m_{yyy} \right) =$$

$$= \left(1 - n_y^2\right) \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - n_y n_x \partial_x\left(n_x m_{xyy} + n_y m_{yyy} \right) =$$
$$= \left(1 - \sin^2\theta\right) \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - \sin\theta \cos\theta\, \partial_x\left(n_x m_{xyy} + n_y m_{yyy} \right) \Rightarrow$$

$$\Rightarrow D_y\left(n_x m_{xyy} + n_y m_{yyy} \right) =$$
$$= \cos^2\theta\, \partial_y\left(n_x m_{xyy} + n_y m_{yyy} \right) - \sin\theta \cos\theta\, \partial_x\left(n_x m_{xyy} + n_y m_{yyy} \right)$$

Each part's explicit calculation follows.

First part:

$$D_x\left(n_x m_{xxy} + n_y m_{yxy} \right) =$$
$$= \sin^2\theta\, \partial_x\left(n_x m_{xxy} + n_y m_{yxy} \right) - \cos\theta \sin\theta\, \partial_y\left(n_x m_{xxy} + n_y m_{yxy} \right)$$

For practical reasons, the following auxiliary sums will be estimated first.

sum 1' : $\left(\cos\theta\, m_{xxy} + \sin\theta\, m_{yxy}\right)$

sum 2' : $\left(\cos\theta\, m_{xyy} + \sin\theta\, m_{yyy}\right)$

Calculation of sum 1': $\left(\cos\theta\, m_{xxy} + \sin\theta\, m_{yxy}\right)$

$$\cos\theta\, m_{xxy} + \sin\theta\, m_{yxy} =$$
$$= \cos\theta \cdot (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\left[25\left(\mathrm{A}+5\mathrm{B}\right)\sin(5\theta/2) - (49\mathrm{A}+249\mathrm{B})\sin(\theta/2)\right] +$$
$$+ \sin\theta \cdot (-15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\left[25\left(\mathrm{A}+5\mathrm{B}\right)\cos(5\theta/2) + (51\mathrm{A}+251\mathrm{B})\cos(\theta/2)\right] =$$

$$= (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\begin{Bmatrix} 25\left(\mathrm{A}+5\mathrm{B}\right)\left[\cos\theta\sin(5\theta/2) - \sin\theta\cos(5\theta/2)\right] - \\ -\left(49\mathrm{A}+249\mathrm{B}\right)\cos\theta\sin(\theta/2) - \\ -\left(51\mathrm{A}+251\mathrm{B}\right)\sin\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\begin{Bmatrix} 25\left(\mathrm{A}+5\mathrm{B}\right)\sin(3\theta/2) - \\ -\left(49\mathrm{A}+249\mathrm{B}\right)\cos\theta\sin(\theta/2) - \\ -\left(49\mathrm{A}+249\mathrm{B}\right)\sin\theta\cos(\theta/2) - \\ -\left(2\mathrm{A}+2\mathrm{B}\right)\sin\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\begin{Bmatrix} 25\left(\mathrm{A}+5\mathrm{B}\right)\sin(3\theta/2) - \\ -\left(49\mathrm{A}+249\mathrm{B}\right)\left[\cos\theta\sin(\theta/2 + \sin\theta\cos(\theta/2)\right] - \\ -\left(2\mathrm{A}+2\mathrm{B}\right)\sin\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\begin{Bmatrix} 25\left(\mathrm{A}+5\mathrm{B}\right)\sin(3\theta/2) - \\ -\left(49\mathrm{A}+249\mathrm{B}\right)\sin(3\theta/2) - \\ -\left(2\mathrm{A}+2\mathrm{B}\right)\sin\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\begin{Bmatrix} \sin(3\theta/2)\left[25\mathrm{A}+125\mathrm{B}-49\mathrm{A}-249\mathrm{B}\right] \\ -\left(2\mathrm{A}+2\mathrm{B}\right)\sin\theta\cos(\theta/2) \end{Bmatrix} =$$
$$= (+15/4)\ \mathrm{c}\,\mu\ \mathrm{r}^{-1/2}\begin{Bmatrix} \sin(3\theta/2)\left[-24\mathrm{A}-124\mathrm{B}\right] \\ -\left(2\mathrm{A}+2\mathrm{B}\right)\cdot 2\sin(\theta/2)\cos(\theta/2)\cos(\theta/2) \end{Bmatrix} =$$

(by using the identity: $\sin(3\theta/2) = \sin(\theta/2)\left(1+2\cos\theta\right)$)

$$=(+15/4)\, c\,\mu\, r^{-1/2} \sin(\theta/2) \begin{Bmatrix} (1+2\cos\theta)[-24A-124B] \\ -(2A+2B)\cdot 2\cos^2(\theta/2) \end{Bmatrix} =$$

$$=(+15/4)\, c\,\mu\, r^{-1/2} \sin(\theta/2) \begin{Bmatrix} (-24A-124B)+2(-24A-124B)\cos\theta - \\ -(2A+2B)\cdot \not{2}\,\dfrac{1+\cos\theta}{\not{2}} \end{Bmatrix} =$$

$$=(+15/4)\, c\,\mu\, r^{-1/2} \sin(\theta/2) \begin{Bmatrix} (-24A-124B)-(2A+2B)+ \\ +\left[2(-24A-124B)-(2A+2B)\right]\cos\theta \end{Bmatrix} =$$

$$=(+15/4)\, c\,\mu\, r^{-1/2} \sin(\theta/2) \begin{Bmatrix} (-26A-126B)+ \\ +\left[2(-25A-125B)\right]\cos\theta \end{Bmatrix} =$$

$$=(+15/4)\, c\,\mu\, r^{-1/2} \sin(\theta/2)\left[(-26A-126B)-50(A+5B)\cos\theta\right] \Rightarrow$$

the result for sum 1' is:

$$\cos\theta\, m_{xxy} + \sin\theta\, m_{yxy} =$$
$$=(+15/4)\, c\,\mu\, r^{-1/2} \sin(\theta/2)\left[(-26A-126B)-50(A+5B)\cos\theta\right]$$

Calculation of sum 2': $\left(\cos\theta\, m_{xyy} + \sin\theta\, m_{yyy}\right)$

$$\left(\cos\theta\, m_{xyy} + \sin\theta\, m_{yyy}\right) =$$
$$\cos\theta\cdot(-15/4)\, c\, r^{-1/2}\left[25\mu(A+5B)\cos(5\theta/2) + (100A\lambda + 51A\mu - 249B\mu)\cos(\theta/2)\right] +$$
$$+\sin\theta\cdot(-15/4)\, c\, r^{-1/2}\left[25\mu(A+5B)\sin(5\theta/2) + (100A\lambda + 151A\mu + 251B\mu)\sin(\theta/2)\right] =$$

$$=(-15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\left[\cos\theta\cos(5\theta/2)+\sin\theta\sin(5\theta/2)\right] + \\ +(100A\lambda+51A\mu-249B\mu)\cos\theta\cos(\theta/2) + \\ +(100A\lambda+151A\mu+251B\mu)\sin\theta\sin(\theta/2) \end{Bmatrix} =$$

$$=(-15/4)\, c\, r^{-1/2} \begin{Bmatrix} 25\mu(A+5B)\cos(3\theta/2) + \\ +(100A\lambda+51A\mu-249B\mu)\cos\theta\cos(\theta/2) + \\ +\begin{pmatrix} 100A\lambda+51A\mu-249B\mu+ \\ +100A\mu+500B\mu \end{pmatrix}\sin\theta\sin(\theta/2) \end{Bmatrix} =$$

$$=(\text{-}15/4)\,\mathrm{c}\,\mathrm{r}^{-1/2}\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+\\+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)\cos(\theta/2)+\\+(100\mathrm{A}\mu+500\mathrm{B}\mu)\sin\theta\sin(\theta/2)\end{Bmatrix}=$$

$$=(\text{-}15/4)\,\mathrm{c}\,\mathrm{r}^{-1/2}\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})\cos(3\theta/2)+\\+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)\cos(\theta/2)+\\+(100\mathrm{A}\mu+500\mathrm{B}\mu)\cdot 2\sin^2(\theta/2)\cos(\theta/2)\end{Bmatrix}=$$

(since: $\cos(3\theta/2)=\cos(\theta/2)(-1+2\cos\theta)$)

$$=(\text{-}15/4)\,\mathrm{c}\,\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})(-1+2\cos\theta)+\\+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)+\\+100\mu(\mathrm{A}+5\mathrm{B})\cdot\cancel{2}\,\dfrac{1-\cos\theta}{\cancel{2}}\end{Bmatrix}=$$

$$=(\text{-}15/4)\,\mathrm{c}\,\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})(-1+2\cos\theta+4-4\cos\theta)+\\+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)\end{Bmatrix}=$$

$$=(\text{-}15/4)\,\mathrm{c}\,\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})(3-2\cos\theta)+\\+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)\end{Bmatrix}\Rightarrow$$

the result for sum 2' is:

$$\cos\theta\,m_{xyy}+\sin\theta\,m_{yyy}=(\text{-}15/4)\,\mathrm{c}\,\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}25\mu(\mathrm{A}+5\mathrm{B})(3-2\cos\theta)+\\+(100\mathrm{A}\lambda+51\mathrm{A}\mu-249\mathrm{B}\mu)\end{Bmatrix}$$

The next step is to obtain the partial derivatives of sum 1' and sum 2' with respect to x and y.

<u>Partial derivative of sum 1' with respect to x</u>

$$\frac{\partial}{\partial x}\left(\cos\theta\,m_{xxy}+\sin\theta\,m_{yxy}\right)=$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\,m_{xxy}+\sin\theta\,m_{yxy}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\,m_{xxy}+\sin\theta\,m_{yxy}\right)=$$

$$= \cos\theta \frac{\partial}{\partial r}\left\{(+15/4)\ c\,\mu\ r^{-1/2} \sin(\theta/2)\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right]\right\} -$$

$$-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left\{(+15/4)\ c\,\mu\ r^{-1/2} \sin(\theta/2)\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right]\right\} =$$

$$= (-15/8)\ c\,\mu\ r^{-3/2} \cos\theta \sin(\theta/2)\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right] -$$

$$-\frac{\sin\theta}{r}(+15/4)\ c\,\mu\ r^{-1/2}\left\{\begin{matrix}\cos(\theta/2)\frac{1}{2}\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right] + \\ +\sin(\theta/2)\left[0 + 50\left(A+5B\right)\sin\theta\right]\end{matrix}\right\} =$$

$$= (-15/8)\ c\,\mu\ r^{-3/2} \cos\theta \sin(\theta/2)\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right] -$$

$$-(+15/8)\ c\,\mu\ r^{-3/2} \sin\theta\left\{\begin{matrix}\cos(\theta/2)\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right] + \\ +\sin(\theta/2)\left[100\left(A+5B\right)\sin\theta\right]\end{matrix}\right\} =$$

$$= (-15/8)\ c\,\mu\ r^{-3/2}\left\{\begin{matrix}\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right]\begin{bmatrix}\cos\theta\sin(\theta/2) + \\ +\sin\theta\cos(\theta/2)\end{bmatrix} + \\ +100\left(A+5B\right)\sin^2\theta\sin(\theta/2)\end{matrix}\right\} =$$

$$= (-15/8)\ c\,\mu\ r^{-3/2}\left\{\begin{matrix}\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right]\sin(3\theta/2) + \\ +100\left(A+5B\right)\sin^2\theta\sin(\theta/2)\end{matrix}\right\} =$$

(since: $\sin(3\theta/2) = \sin(\theta/2)\left(1 + 2\cos\theta\right)$)

$$= (-15/8)\ c\,\mu\ r^{-3/2}\sin(\theta/2)\left\{\begin{matrix}\left[\left(-26A - 126B\right) - 50\left(A+5B\right)\cos\theta\right]\left(1 + 2\cos\theta\right) + \\ +100\left(A+5B\right)\sin^2\theta\end{matrix}\right\} =$$

$$= (-15/8)\ c\,\mu\ r^{-3/2}\sin(\theta/2)\begin{bmatrix}\left(-26A - 126B\right)\left(1 + 2\cos\theta\right) + \\ +50\left(A+5B\right)\left[-\cos\theta\left(1 + 2\cos\theta\right) + 2\sin^2\theta\right]\end{bmatrix}$$

but

$$-\cos\theta\left(1 + 2\cos\theta\right) + 2\sin^2\theta =$$
$$= -\cos\theta - 2\cos^2\theta + 2\sin^2\theta =$$
$$= -\cos\theta - 2\cos(2\theta)$$

thus

$$(-15/8)\ c\,\mu\ r^{-3/2}\sin(\theta/2)\begin{bmatrix}(-26A-126B)(1+2\cos\theta)+\\ +50(A+5B)(-\cos\theta-2\cos(2\theta))\end{bmatrix}=$$

$$=(-15/8)\ c\,\mu\ r^{-3/2}\sin(\theta/2)\begin{bmatrix}(-26A-126B)+\\ +\left[2(-26A-126B)-50(A+5B)\right]\cos\theta-\\ -100(A+5B)\cos(2\theta)\end{bmatrix}=$$

$$=(-15/8)\ c\,\mu\ r^{-3/2}\sin(\theta/2)\begin{bmatrix}(-26A-126B)+\\ +(-102A-502B)\cos\theta-\\ -100(A+5B)\cos(2\theta)\end{bmatrix}$$

or equivalently the partial derivative of sum 1' with respect to x is:

$$\frac{\partial}{\partial x}\left(\cos\theta\, m_{xxy}+\sin\theta\, m_{yxy}\right)=(-15/8)\ c\,\mu\ r^{-3/2}\sin(\theta/2)\begin{bmatrix}(-26A-126B)+\\ +(-102A-502B)\cos\theta-\\ -100(A+5B)\cos(2\theta)\end{bmatrix}$$

<u>Partial derivative of sum 1' with respect to y</u>

$$\frac{\partial}{\partial y}\left(\cos\theta\, m_{xxy}+\sin\theta\, m_{yxy}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\cos\theta\, m_{xxy}+\sin\theta\, m_{yxy}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xxy}+\sin\theta\, m_{yxy}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left\{(15/4)\ c\,\mu\ r^{-1/2}\sin(\theta/2)\left[(-26A-126B)-50(A+5B)\cos\theta\right]\right\}+$$

$$+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left\{(15/4)\ c\,\mu\ r^{-1/2}\sin(\theta/2)\left[(-26A-126B)-50(A+5B)\cos\theta\right]\right\}=$$

$$=(-15/8)\ c\,\mu\ r^{-1/2}\sin\theta\sin(\theta/2)\left[(-26A-126B)-50(A+5B)\cos\theta\right]+$$

$$+\frac{\cos\theta}{r}(+15/4)\ c\,\mu\ r^{-1/2}\begin{Bmatrix}\cos(\theta/2)\frac{1}{2}\left[(-26A-126B)-50(A+5B)\cos\theta\right]+\\ +\sin(\theta/2)\left[0+50(A+5B)\sin\theta\right]\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\begin{Bmatrix}\left[(-26A-126B)-50(A+5B)\cos\theta\right]\begin{bmatrix}-\sin\theta\sin(\theta/2)+\\+\cos\theta\cos(\theta/2)\end{bmatrix}+\\+100(A+5B)\cos\theta\sin\theta\sin(\theta/2)\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\begin{Bmatrix}\left[(-26A-126B)-50(A+5B)\cos\theta\right]\cos(3\theta/2)+\\+100(A+5B)\cos\theta\cdot 2\sin^2\theta\cos(\theta/2)\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}\left[(-26A-126B)-50(A+5B)\cos\theta\right](-1+2\cos\theta)+\\+100(A+5B)\cdot 2\cos\theta\sin^2\theta\end{Bmatrix}$$

The factor $2\cos\theta\sin^2\theta$ is:

$$2\cos\theta\sin^2\theta=2\cos\theta\frac{1-\cos\theta}{2}=\cos\theta-\cos^2\theta=$$
$$=\cos\theta-\frac{1+\cos(2\theta)}{2}=-\frac{1}{2}+\cos\theta-\frac{1}{2}\cos(2\theta)=$$
$$=\frac{1}{2}(-1+2\cos\theta-\cos(2\theta))$$

and

$$(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}\left[(-26A-126B)-50(A+5B)\cos\theta\right](-1+2\cos\theta)+\\+100(A+5B)\cdot 2\cos\theta\sin^2\theta\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}\left[(-26A-126B)-50(A+5B)\cos\theta\right](-1+2\cos\theta)+\\+100(A+5B)\frac{1}{2}(-1+2\cos\theta-\cos(2\theta))\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-(-26A-126B)+2(-26A-126B)\cos\theta+\\+50(A+5B)\cos\theta-50(A+5B)\cdot\underbrace{2\cos^2\theta}_{1+\cos(2\theta)}+\\+(-50)(A+5B)+100(A+5B)\cos\theta-50(A+5B)\cos(2\theta)\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}26A+126B-50A-250B+\\+\left[-52A-252B+50(A+5B)+100(A+5B)\right]\cos\theta+\\+(-50)(A+5B)\cos(2\theta)-50(A+5B)(1+\cos(2\theta))\end{Bmatrix}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\left\{\begin{aligned}&\left[-24A-124B-50(A+5B)\right]+\\&+\left[50A+100A-52A+250B+500B-252B\right]\cos\theta+\\&+\left[-50(A+5B)-50(A+5B)\right]\cos(2\theta)\end{aligned}\right\}=$$

$$=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\left\{\begin{aligned}&(-74A-374B)+\\&+(98A+498B)\cos\theta-\\&-100(A+5B)\cos(2\theta)\end{aligned}\right\}\Rightarrow$$

sum's 1' partial derivative with respect to y is:

$$\frac{\partial}{\partial y}\left(\cos\theta\, m_{xxy}+\sin\theta\, m_{yxy}\right)=(+15/8)\,c\,\mu\,r^{-3/2}\cos(\theta/2)\left\{\begin{aligned}&(-74A-374B)+\\&+(98A+498B)\cos\theta-\\&-100(A+5B)\cos(2\theta)\end{aligned}\right\}$$

<u>Partial derivative of sum 2' with respect to x</u>

$$\frac{\partial}{\partial x}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)=$$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)=$$

$$=\cos\theta\frac{\partial}{\partial r}\left\{(-15/4)\,c\,r^{-1/2}\cos(\theta/2)\left\{\begin{aligned}&25\mu(A+5B)(3-2\cos\theta)+\\&+(100A\lambda+51A\mu-249B\mu)\end{aligned}\right\}\right\}-$$

$$-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left\{(-15/4)\,c\,r^{-1/2}\cos(\theta/2)\left\{\begin{aligned}&25\mu(A+5B)(3-2\cos\theta)+\\&+(100A\lambda+51A\mu-249B\mu)\end{aligned}\right\}\right\}=$$

$$=(+15/8)\,c\,r^{-3/2}\cos(\theta/2)\cos\theta\left\{\begin{aligned}&25\mu(A+5B)(3-2\cos\theta)+\\&+(100A\lambda+51A\mu-249B\mu)\end{aligned}\right\}-$$

$$-\frac{\sin\theta}{r}(-15/4)\,c\,r^{-1/2}\left\{\begin{aligned}&-\sin(\theta/2)\frac{1}{2}\left\{\begin{aligned}&25\mu(A+5B)(3-2\cos\theta)+\\&+(100A\lambda+51A\mu-249B\mu)\end{aligned}\right\}+\\&+\cos(\theta/2)\left\{25\mu(A+5B)(0+2\sin\theta)+0\right\}\end{aligned}\right\}=$$

$$= (+15/8)\, c\, r^{-3/2} \cos(\theta/2)\cos\theta \begin{Bmatrix} 25\mu(A+5B)(3-2\cos\theta) + \\ +(100A\lambda + 51A\mu - 249B\mu) \end{Bmatrix} +$$

$$+(-15/8)\, c\, r^{-3/2} \sin\theta \begin{Bmatrix} -\sin(\theta/2)\begin{Bmatrix} 25\mu(A+5B)(3-2\cos\theta) + \\ +(100A\lambda + 51A\mu - 249B\mu) \end{Bmatrix} + \\ +100\mu(A+5B)\cos(\theta/2)\sin\theta \end{Bmatrix} =$$

$$= (+15/8)\, c\, r^{-3/2} \begin{Bmatrix} \begin{Bmatrix} 25\mu(A+5B)(3-2\cos\theta) + \\ +(100A\lambda + 51A\mu - 249B\mu) \end{Bmatrix} \left[\cos\theta\cos(\theta/2) - \sin\theta\sin(\theta/2)\right] + \\ +100\mu(A+5B)\sin^2\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/8)\, c\, r^{-3/2} \begin{Bmatrix} \begin{Bmatrix} 25\mu(A+5B)(3-2\cos\theta) + \\ +(100A\lambda + 51A\mu - 249B\mu) \end{Bmatrix} \cos(3\theta/2) + \\ +100\mu(A+5B)\sin^2\theta\cos(\theta/2) \end{Bmatrix} =$$

$$= (+15/8)\, c\, r^{-3/2} \cos(\theta/2) \begin{Bmatrix} \begin{Bmatrix} 25\mu(A+5B)(3-2\cos\theta) + \\ +(100A\lambda + 51A\mu - 249B\mu) \end{Bmatrix} (-1+2\cos\theta) + \\ +100\mu(A+5B)\sin^2\theta \end{Bmatrix} =$$

$$= (+15/8)\, c\, r^{-3/2} \cos(\theta/2) \begin{Bmatrix} -25\mu(A+5B)(3-2\cos\theta) + 50\mu(A+5B)(3-2\cos\theta)\cos\theta - \\ -(100A\lambda + 51A\mu - 249B\mu) + 2(100A\lambda + 51A\mu - 249B\mu)\cos\theta + \\ +100\mu(A+5B)\sin^2\theta \end{Bmatrix} =$$

$$= (+15/8)\, c\, r^{-3/2} \cos(\theta/2) \begin{Bmatrix} -75\mu(A+5B) - (100A\lambda + 51A\mu - 249B\mu) + \\ +\left[50\mu(A+5B) + 150\mu(A+5B) + 2(100A\lambda + 51A\mu - 249B\mu)\right]\cos\theta - \\ -100\mu(A+5B)\cos^2\theta + 100\mu(A+5B)\sin^2\theta \end{Bmatrix} =$$

$$= (+15/8)\, c\, r^{-3/2} \cos(\theta/2) \begin{Bmatrix} \left[100A\lambda + (-75-51)A\mu + (-375+249)B\mu\right] + \\ +\left[200A\lambda + (50+150+102)A\mu + (250+750-498)B\mu\right]\cos\theta - \\ -100\mu(A+5B)\cos(2\theta) \end{Bmatrix} =$$

$$=(+15/8)\,c\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-\left(100A\lambda+126A\mu+126B\mu\right)+\\+\left(200A\lambda+302A\mu+502B\mu\right)\cos\theta-\\-100\mu\left(A+5B\right)\cos(2\theta)\end{Bmatrix}\Rightarrow$$

the partial derivative of sum 2' with respect to x is:

$$\frac{\partial}{\partial x}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)=$$

$$=(+15/8)\,c\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-\left(100A\lambda+126A\mu+126B\mu\right)+\\+\left(200A\lambda+302A\mu+502B\mu\right)\cos\theta-\\-100\mu\left(A+5B\right)\cos(2\theta)\end{Bmatrix}$$

<u>Partial derivative of sum 2' with respect to y</u>

$$\frac{\partial}{\partial y}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left\{(-15/4)\,c\,r^{-1/2}\cos(\theta/2)\begin{Bmatrix}25\mu\left(A+5B\right)\left(3-2\cos\theta\right)+\\+\left(100A\lambda+51A\mu-249B\mu\right)\end{Bmatrix}\right\}+$$

$$+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\left\{(-15/4)\,c\,r^{-1/2}\cos(\theta/2)\begin{Bmatrix}25\mu\left(A+5B\right)\left(3-2\cos\theta\right)+\\+\left(100A\lambda+51A\mu-249B\mu\right)\end{Bmatrix}\right\}=$$

$$=(+15/8)\,c\,r^{-3/2}\sin\theta\cos(\theta/2)\begin{Bmatrix}25\mu\left(A+5B\right)\left(3-2\cos\theta\right)+\\+\left(100A\lambda+51A\mu-249B\mu\right)\end{Bmatrix}+$$

$$+\frac{\cos\theta}{r}(-15/4)\,c\,r^{-1/2}\left\{\begin{matrix}-\sin(\theta/2)\frac{1}{2}\begin{Bmatrix}25\mu\left(A+5B\right)\left(3-2\cos\theta\right)+\\+\left(100A\lambda+51A\mu-249B\mu\right)\end{Bmatrix}+\\+\cos(\theta/2)\underbrace{\left[25\mu\left(A+5B\right)\left(0+2\sin\theta\right)+0\right]}_{50\mu\left(A+5B\right)\sin\theta}\end{matrix}\right\}=$$

$$= (+15/8)\, c\, r^{-3/2} \left\{ \begin{array}{l} \left\{ \begin{array}{l} 25\mu(A+5B)(3-2\cos\theta)+ \\ +(100A\lambda+51A\mu-249B\mu) \end{array} \right\} \left[\begin{array}{l} \sin\theta\cos(\theta/2)+ \\ +\cos\theta\sin(\theta/2) \end{array} \right] - \\ -100\mu(A+5B)\sin\theta\cos\theta\cos(\theta/2) \end{array} \right\} =$$

$$= (+15/8)\, c\, r^{-3/2} \left\{ \begin{array}{l} \left\{ \begin{array}{l} 25\mu(A+5B)(3-2\cos\theta)+ \\ +(100A\lambda+51A\mu-249B\mu) \end{array} \right\} \sin(3\theta/2) - \\ -100\mu(A+5B)\cdot 2\sin(\theta/2)\cos^2(\theta/2)\cos\theta \end{array} \right\}$$

By using the identities:

$$\sin(3\theta/2) = \sin(\theta/2)(1+2\cos\theta)$$

and

$$2\cos^2(\theta/2)\cos\theta = \not{2}\,\frac{1+\cos\theta}{\not{2}}\cos\theta =$$
$$= \cos\theta + \cos^2\theta = \cos\theta + \frac{1+\cos(2\theta)}{2} =$$
$$= \frac{1}{2} + \cos\theta + \frac{1}{2}\cos(2\theta) = \frac{1}{2}(1+2\cos\theta+\cos(2\theta))$$

one can obtain:

$$(+15/8)\, c\, r^{-3/2} \sin(\theta/2) \left\{ \begin{array}{l} \left\{ \begin{array}{l} 25\mu(A+5B)(3-2\cos\theta)+ \\ +(100A\lambda+51A\mu-249B\mu) \end{array} \right\} (1+2\cos\theta) - \\ -100\mu(A+5B)\cdot\frac{1}{2}(1+2\cos\theta+\cos(2\theta)) \end{array} \right\} =$$

$$= (+15/8) c\, r^{-3/2} \sin(\theta/2) \left\{ \begin{array}{l} 25\mu(A+5B)(3-2\cos\theta) + 50\mu(A+5B)(3-2\cos\theta)\cos\theta + \\ +(100A\lambda+51A\mu-249B\mu) + 2(100A\lambda+51A\mu-249B\mu)\cos\theta - \\ -50\mu(A+5B)(1+2\cos\theta+\cos(2\theta)) \end{array} \right\} =$$

$$= (+15/8) c\, r^{-3/2} \sin(\theta/2) \left\{ \begin{array}{l} 75\mu(A+5B) - 50\mu(A+5B)\cos\theta + \\ +150\mu(A+5B)\cos\theta - 100\mu(A+5B)\cos^2\theta + \\ +(100A\lambda+51A\mu-249B\mu) + 2(100A\lambda+51A\mu-249B\mu)\cos\theta - \\ -50\mu(A+5B)(1+2\cos\theta+\cos(2\theta)) \end{array} \right\} =$$

$$= (+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l} 100A\lambda+(75+51-50)A\mu+(75\times5-249-250)B\mu+ \\ +\left[\begin{array}{l}200A\lambda+(-50+150+102-100)A\mu+ \\ +(-250+750-498-500)B\mu\end{array}\right]\cos\theta- \\ -50\mu(A+5B)\cos(2\theta)-100\mu(A+5B)\dfrac{1+\cos(2\theta)}{2}\end{array}\right\} =$$

$$= (+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l} (100A\lambda+76A\mu-124B\mu)+ \\ +(200A\lambda+102A\mu-498B\mu)\cos\theta- \\ -50\mu(A+5B)\cos(2\theta)-50\mu(A+5B)-50\mu(A+5B)\cos(2\theta)\end{array}\right\} =$$

$$= (+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l} [100A\lambda+(76-50)A\mu+(-124-250)B\mu]+ \\ +(200A\lambda+102A\mu-498B\mu)\cos\theta- \\ -100\mu(A+5B)\cos(2\theta)\end{array}\right\} =$$

$$= (+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l} (100A\lambda+26A\mu-374B\mu)+ \\ +(200A\lambda+102A\mu-498B\mu)\cos\theta- \\ -100\mu(A+5B)\cos(2\theta)\end{array}\right\} \Rightarrow$$

the partial derivative of sum 2' with respect to y is:

$$\frac{\partial}{\partial y}\left(\cos\theta\, m_{xyy}+\sin\theta\, m_{yyy}\right) = (+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l} (100A\lambda+26A\mu-374B\mu)+ \\ +(200A\lambda+102A\mu-498B\mu)\cos\theta- \\ -100\mu(A+5B)\cos(2\theta)\end{array}\right\}$$

Having calculated the partial derivatives of sum1' and sum 2' with respect to x and y, the full form of Subterm C' of P_y can be reached.

$$\begin{array}{l} D_x\left(n_x m_{xxy}+n_y m_{yxy}\right)+D_y\left(n_x m_{xyy}+n_y m_{yyy}\right)= \\ \sin^2\theta\,\partial_x\left(n_x m_{xxy}+n_y m_{yxy}\right)-\cos\theta\sin\theta\,\partial_y\left(n_x m_{xxy}+n_y m_{yxy}\right)+ \\ +\cos^2\theta\,\partial_y\left(n_x m_{xyy}+n_y m_{yyy}\right)-\sin\theta\cos\theta\,\partial_x\left(n_x m_{xyy}+n_y m_{yyy}\right)= \end{array}$$

$$=\sin^2\theta(-15/8)\,c\,\mu\, r^{-3/2}\sin(\theta/2)\left[\begin{array}{l}(-26A-126B)+\\+(-102A-502B)\cos\theta-\\-100(A+5B)\cos(2\theta)\end{array}\right]-$$

$$-\cos\theta\,\sin\theta(+15/8)\,c\,\mu\, r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}(-74A-374B)+\\+(98A+498B)\cos\theta-\\-100(A+5B)\cos(2\theta)\end{array}\right\}+$$

$$+\cos^2\theta(+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}(100A\lambda+26A\mu-374B\mu)+\\+(200A\lambda+102A\mu-498B\mu)\cos\theta-\\-100\mu(A+5B)\cos(2\theta)\end{array}\right\}-$$

$$-\sin\theta\cos\theta\,(+15/8)\,c\,r^{-3/2}\cos(\theta/2)\left\{\begin{array}{l}-(100A\lambda+126A\mu+126B\mu)+\\+(200A\lambda+302A\mu+502B\mu)\cos\theta-\\-100\mu(A+5B)\cos(2\theta)\end{array}\right\}=$$

$$=(+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}-(-26A-126B)\mu\sin^2\theta+(100A\lambda+26A\mu-374B\mu)\cos^2\theta+\\+\left[\begin{array}{l}-(-102A-502B)\mu\sin^2\theta+\\+(200A\lambda+102A\mu-498B\mu)\cos^2\theta\end{array}\right]\cos\theta+\\+\left[100\mu(A+5B)\sin^2\theta-100\mu(A+5B)\cos^2\theta\right]\cos(2\theta)\end{array}\right\}-$$

$$-(15/8)\,c\,r^{-3/2}\sin\theta\cos\theta\cos(\theta/2)\left\{\begin{array}{l}(-74A-374B)\mu-(100A\lambda+126A\mu+126B\mu)+\\+\left[\begin{array}{l}(98A+498B)\mu+\\+(200A\lambda+302A\mu+502B\mu)\end{array}\right]\cos\theta+\\+\left[-100\mu(A+5B)-100\mu(A+5B)\right]\cos(2\theta)\end{array}\right\}$$

To help perform the calculations, the penultimate set of brackets is called term (i)' and the last one term (ii)'.

Term (i)'
Each part of term (i)' will be simplified separately.

- The representation:
  $-(-26A-126B)\mu\sin^2\theta+(100A\lambda+26A\mu-374B\mu)\cos^2\theta$ becomes:

$$-(-26\mathrm{A}-126\mathrm{B})\mu\sin^2\theta+(100\mathrm{A}\lambda+26\mathrm{A}\mu-374\mathrm{B}\mu)\cos^2\theta=$$
$$=100\mathrm{A}\lambda\cos^2\theta+\left(26\sin^2\theta+26\cos^2\theta\right)\mathrm{A}\mu+\left(126\sin^2\theta-347\cos^2\theta\right)\mathrm{B}\mu=$$
$$=100\mathrm{A}\lambda\cos^2\theta+26\mathrm{A}\mu+\left(126\sin^2\theta-126\cos^2\theta-248\cos^2\theta\right)\mathrm{B}\mu=$$
$$=100\mathrm{A}\lambda\cos^2\theta+26\mathrm{A}\mu+\left(-126\cos(2\theta)-248\frac{1+\cos(2\theta)}{2}\right)\mathrm{B}\mu=$$
$$=100\mathrm{A}\lambda\cos^2\theta+26\mathrm{A}\mu+(-124-250\cos(2\theta))\mathrm{B}\mu$$

- the representation:
  $\left[-(-102\mathrm{A}-502\mathrm{B})\mu\sin^2\theta+(200\mathrm{A}\lambda+102\mathrm{A}\mu-498\mathrm{B}\mu)\cos^2\theta\right]$ becomes:

$$\left[-(-102\mathrm{A}-502\mathrm{B})\mu\sin^2\theta+(200\mathrm{A}\lambda+102\mathrm{A}\mu-498\mathrm{B}\mu)\cos^2\theta\right]=$$
$$=200\mathrm{A}\lambda\cos^2\theta+\left(102\sin^2\theta+102\cos^2\theta\right)\mathrm{A}\mu+\left(502\sin^2\theta-498\cos^2\theta\right)\mathrm{B}\mu=$$
$$=200\mathrm{A}\lambda\cos^2\theta+102\mathrm{A}\mu+\left(4\sin^2\theta+498\sin^2\theta-498\cos^2\theta\right)\mathrm{B}\mu=$$
$$=200\mathrm{A}\lambda\cos^2\theta+102\mathrm{A}\mu+\left(4\frac{1-\cos(2\theta)}{2}-498\cos(2\theta)\right)\mathrm{B}\mu=$$
$$=200\mathrm{A}\lambda\cos^2\theta+102\mathrm{A}\mu+(2-500\cos(2\theta))\mathrm{B}\mu$$

- the representation:
  $\left[100\mu(\mathrm{A}+5\mathrm{B})\sin^2\theta-100\mu(\mathrm{A}+5\mathrm{B})\cos^2\theta\right]$ becomes:

$$100\mu(\mathrm{A}+5\mathrm{B})\sin^2\theta-100\mu(\mathrm{A}+5\mathrm{B})\cos^2\theta=$$
$$=-100\mu(\mathrm{A}+5\mathrm{B})\cos(2\theta)$$

Term (i)' transforms into:

$$(+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left[100\mathrm{A}\lambda\cos^2\theta+26\mathrm{A}\mu+(-124-250\cos(2\theta))\mathrm{B}\mu\right]+\\+\left[\begin{array}{l}200\mathrm{A}\lambda\cos^2\theta+102\mathrm{A}\mu+\\+(2-500\cos(2\theta))\mathrm{B}\mu\end{array}\right]\cos\theta+\\+\left[-100\mu(\mathrm{A}+5\mathrm{B})\cos(2\theta)\right]\cos(2\theta)\end{array}\right\}$$

In the same fashion, term (ii)'

$$-(15/8)\, c\, r^{-3/2} \sin\theta\cos\theta\cos(\theta/2)\cdot\left\{\begin{array}{l}(-74\mathrm{A}-374\mathrm{B})\mu-(100\mathrm{A}\lambda+126\mathrm{A}\mu+126\mathrm{B}\mu)+\\ +\left[\begin{array}{l}(98\mathrm{A}+498\mathrm{B})\mu+\\ +(200\mathrm{A}\lambda+302\mathrm{A}\mu+502\mathrm{B}\mu)\end{array}\right]\cos\theta+\\ +\left[-100\mu(\mathrm{A}+5\mathrm{B})-100\mu(\mathrm{A}+5\mathrm{B})\right]\cos(2\theta)\end{array}\right\}$$

is:

- Its first factor: $(-74\mathrm{A}-374\mathrm{B})\mu-(100\mathrm{A}\lambda+126\mathrm{A}\mu+126\mathrm{B}\mu)$

becomes:

$$(-74\mathrm{A}-374\mathrm{B})\mu-(100\mathrm{A}\lambda+126\mathrm{A}\mu+126\mathrm{B}\mu)=$$
$$=-(100\mathrm{A}\lambda+200\mathrm{A}\mu+500\mathrm{B}\mu)$$

- Its second factor: $(98\mathrm{A}+498\mathrm{B})\mu+(200\mathrm{A}\lambda+302\mathrm{A}\mu+502\mathrm{B}\mu)$

is:

$$(98\mathrm{A}+498\mathrm{B})\mu+(200\mathrm{A}\lambda+302\mathrm{A}\mu+502\mathrm{B}\mu)=$$
$$=(200\mathrm{A}\lambda+400\mathrm{A}\mu+1000\mathrm{B}\mu)$$

and finally

- Its third factor: $\left[-100\mu(\mathrm{A}+5\mathrm{B})-100\mu(\mathrm{A}+5\mathrm{B})\right]$ takes the form:

$$-100\mu(\mathrm{A}+5\mathrm{B})-100\mu(\mathrm{A}+5\mathrm{B})=$$
$$=-200\mu(\mathrm{A}+5\mathrm{B})$$

The simplified form of <u>term (ii)'</u> is:

$$(-15/8)\, c\, r^{-3/2} \sin\theta\cos\theta\cos(\theta/2)\cdot\left\{\begin{array}{l}-(100\mathrm{A}\lambda+200\mathrm{A}\mu+500\mathrm{B}\mu)+\\ +(200\mathrm{A}\lambda+400\mathrm{A}\mu+1000\mathrm{B}\mu)\cos\theta+\\ +\left[-200\mu(\mathrm{A}+5\mathrm{B})\right]\cos(2\theta)\end{array}\right\}$$

where:

$$\sin\theta\cos\theta\cos(\theta/2)=2\sin(\theta/2)\cos^2(\theta/2)\cos\theta=$$
$$=\sin(\theta/2)\cdot 2\frac{1+\cos\theta}{2}\cos\theta=\sin(\theta/2)\cdot \not{2}\frac{1+\cos\theta}{\not{2}}\cos\theta=$$
$$=\sin(\theta/2)\left(\cos\theta+\cos^2\theta\right)=\sin(\theta/2)\left(\cos\theta+\frac{1+\cos(2\theta)}{2}\right)=$$
$$=\sin(\theta/2)\frac{1}{2}(1+2\cos\theta+\cos(2\theta))$$

- The sum of <u>term (i)'</u> and <u>term (ii)'</u> is:

$$(+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left[100A\lambda\cos^2\theta+26A\mu+(-124-250\cos(2\theta))B\mu\right]+\\+\left[\begin{array}{l}200A\lambda\cos^2\theta+102A\mu+\\+(2-500\cos(2\theta))B\mu\end{array}\right]\cos\theta+\\+\left[-100\mu(A+5B)\cos(2\theta)\right]\cos(2\theta)\end{array}\right\}-$$

$$-(15/8)\,c\,r^{-3/2}\cdot\sin(\theta/2)\frac{1}{2}(1+2\cos\theta+\cos(2\theta))\cdot$$

$$\cdot\left\{\begin{array}{l}-(100A\lambda+200A\mu+500B\mu)+\\+(200A\lambda+400A\mu+1000B\mu)\cos\theta+\\+\left[-200\mu(A+5B)\right]\cos(2\theta)\end{array}\right\}=$$

$$=(+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left[\begin{array}{l}100A\lambda\cos^2\theta+26A\mu+(-124-250\cos(2\theta))B\mu+\\+\left(-\frac{1}{2}\right)(1+2\cos\theta+\cos(2\theta))(-1)(100A\lambda+200A\mu+500B\mu)\end{array}\right]+\\+\left[\begin{array}{l}200A\lambda\cos^2\theta+102A\mu+(2-500\cos(2\theta))B\mu+\\+\left(-\frac{1}{2}\right)(1+2\cos\theta+\cos(2\theta))(200A\lambda+400A\mu+1000B\mu)\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}-100\mu(A+5B)\cos(2\theta)+\\+\left(-\frac{1}{2}\right)(1+2\cos\theta+\cos(2\theta))(-200)\mu(A+5B)\end{array}\right]\cos(2\theta)\end{array}\right\}$$

- The first component when simplified becomes:

$$100A\lambda\cos^2\theta+26A\mu+(-124-250\cos(2\theta))B\mu+$$

$$+\left(-\frac{1}{2}\right)(1+2\cos\theta+\cos(2\theta))(-1)(100A\lambda+200A\mu+500B\mu)=$$

$$=100A\lambda\left[\cos^2\theta+\frac{1}{2}(1+2\cos\theta+\cos(2\theta))\right]+$$

$$+A\mu\left[26+200\frac{1}{2}(1+2\cos\theta+\cos(2\theta))\right]+$$

$$+B\mu\left[-124-250\cos(2\theta)+500\frac{1}{2}(1+2\cos\theta+\cos(2\theta))\right]=$$

$$=100A\lambda\left[\frac{1}{2}+\frac{1}{2}\cos(2\theta)+\frac{1}{2}+\cos\theta+\frac{1}{2}\cos(2\theta)\right]+$$
$$+A\mu\left[26+100+200\cos\theta+100\cos(2\theta)\right]+$$
$$+B\mu\left[-124-250\cos(2\theta)+250+500\cos\theta+250\cos(2\theta)\right]=$$

$$=100A\lambda\left[1+\cos\theta+\cos(2\theta)\right]+$$
$$+A\mu\left[126+200\cos\theta+100\cos(2\theta)\right]+$$
$$+B\mu\left(126+500\cos\theta\right)$$

- The second component (the one multiplied by $\cos\theta$) when simplified becomes:

$$200A\lambda\cos^2\theta+102A\mu+\left(2-500\cos(2\theta)\right)B\mu+$$
$$+\left(-\frac{1}{2}\right)\left(1+2\cos\theta+\cos(2\theta)\right)\left(200A\lambda+400A\mu+1000B\mu\right)=$$
$$=200A\lambda\left[\cos^2\theta-\frac{1}{2}\left(1+2\cos\theta+\cos(2\theta)\right)\right]+$$
$$+A\mu\left[102-200\left(1+2\cos\theta+\cos(2\theta)\right)\right]+$$
$$+B\mu\left[2-500\cos(2\theta)-500\left(1+2\cos\theta+\cos(2\theta)\right)\right]=$$

$$=200A\lambda\left[\frac{1+\cos(2\theta)}{2}-\frac{1}{2}-\cos\theta-\frac{1}{2}\cos(2\theta)\right]+$$
$$+A\mu\left[102-200-400\cos\theta-200\cos(2\theta)\right]+$$
$$+B\mu\left[2-500\cos(2\theta)-500-1000\cos\theta-500\cos(2\theta)\right]=$$
$$=-200A\lambda\cos\theta+$$
$$+A\mu\left[-98-400\cos\theta-200\cos(2\theta)\right]+$$
$$+B\mu\left[-498-1000\cos\theta-1000\cos(2\theta)\right]$$

- The third component (the one multiplied by $\cos^2\theta$) is simplified into the form:

$$-100\mu\left(A+5B\right)\cos(2\theta)+$$
$$+\left(-\frac{1}{2}\right)\left(1+2\cos\theta+\cos(2\theta)\right)\left(-200\right)\mu\left(A+5B\right)=$$
$$=-100\mu\left(A+5B\right)\cos(2\theta)+$$
$$+100\mu\left(A+5B\right)\left(1+2\cos\theta+\cos(2\theta)\right)=$$
$$=-100\mu\left(A+5B\right)\left[-\cos(2\theta)+1+2\cos\theta+\cos(2\theta)\right]=$$
$$=-100\mu\left(A+5B\right)\left(1+2\cos\theta\right)$$

By unifying the three components, the obtained overall form is:

$$(+15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{array}{l}\left[\begin{array}{l}100A\lambda\left[1+\cos\theta+\cos(2\theta)\right]+\\+A\mu\left[126+200\cos\theta+100\cos(2\theta)\right]+\\+B\mu\left(126+500\cos\theta\right)\end{array}\right]+\\+\left[\begin{array}{l}-200A\lambda\cos\theta+\\+A\mu\left[-98-400\cos\theta-200\cos(2\theta)\right]+\\+B\mu\left[-498-1000\cos\theta-1000\cos(2\theta)\right]\end{array}\right]\cos\theta+\\+\left[-100\mu\left(A+5B\right)\left(1+2\cos\theta\right)\right]\cos(2\theta)\end{array}\right\}$$

Alternatively, by grouping it in terms of $A\lambda, A\mu, B\mu$:

- ❖ $A\lambda$:

$$100A\lambda\left(1+\cos\theta+\cos(2\theta)-2\cos^2\theta\right)=$$

$$=100A\lambda\left(1+\cos\theta+\cos(2\theta)-\cancel{2}\frac{1+\cos(2\theta)}{\cancel{2}}\right)=$$

$$=100A\lambda\cos\theta$$

- ❖ $A\mu$:

$$A\mu\left[\begin{array}{l}126+200\cos\theta+100\cos(2\theta)-98\cos\theta-400\cos^2\theta-200\cos\theta\cos(2\theta)+\\+100\left(1+2\cos\theta\right)\cos(2\theta)\end{array}\right]=$$

$$=A\mu\left[\begin{array}{l}126+102\cos\theta+100\cos(2\theta)-400\frac{1+\cos(2\theta)}{2}-\cancel{200\cos\theta\cos(2\theta)}+\\+100\cos(2\theta)+\cancel{200\cos\theta\cos(2\theta)}\end{array}\right]=$$

$$=A\mu\left[126+102\cos\theta+\cancel{100\cos(2\theta)}-200-\cancel{200\cos(2\theta)}+\cancel{100\cos(2\theta)}\right]=$$

$$=A\mu\left(-74+102\cos\theta\right)$$

- ❖ $B\mu$:

$$B\mu\left(\begin{array}{l}126+500\cos\theta-498\cos\theta-1000\cos^2\theta-\cancel{1000\cos\theta\cos(2\theta)}+\\+500\cos(2\theta)+\cancel{1000\cos\theta\cos(2\theta)}\end{array}\right)=$$

$$=B\mu\left(126+2\cos\theta-1000\frac{1+\cos(2\theta)}{2}+500\cos(2\theta)\right)=$$

$$=B\mu\left(126+2\cos\theta-500-\cancel{500\cos(2\theta)}+\cancel{500\cos(2\theta)}\right)=$$

$$=B\mu\left(-374+2\cos\theta\right)$$

Finally, Subterm C' of P_y in its most compact form is:

$$\text{Subterm C' of } P_y = (+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix} 100A\lambda\cos\theta + \\ +(-74+102\cos\theta)A\mu + \\ +(-374+2\cos\theta)B\mu \end{bmatrix}$$

It should also be reminded that P_y has a component $(n_x\tau_{xy} + n_y\tau_{yy})$. This component's order of r, after the integration, is greater than zero, which means that this term does not contribute to the final result for the J_1-Integral.

CALCULATION OF SUBTERM D: $(D_j n_j)R_y$

It has already been shown, while calculating the Subterm D of P_x that the operator $(D_j n_j)$ is:

$$(D_j n_j) = D_x n_x + D_y n_y =$$
$$= \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} = \frac{1}{r} \Rightarrow$$
$$\Rightarrow (D_j n_j) = \frac{1}{r}$$

In virtue of this, Subterm D' of P_y is:

$$\Rightarrow \text{Subterm D' } (D_j n_j)R_y = \frac{R_y}{r}$$

With all 4 Subterms of P_y estimated, the full form of P_y can be acquired. First, the Subterms B' and C' will be summed. Then, P_y=-(Subterm B'+Subterm C')+ Subterm D'.

CALCULATION OF SUM: SUBTERM B'+ SUBTERM C'

Subterm B'+ Subterm C'=

$$=(+15/8)c\,\mu\, r^{3/2}\sin(\theta/2)\left[200\left(A+5B\right)\right]+$$

$$+(15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+\left(-74+102\cos\theta\right)A\mu+\\+\left(-374+2\cos\theta\right)B\mu\end{bmatrix}=$$

$$=(+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+\left(-74+102\cos\theta+200\right)A\mu+\\+\left(-374+2\cos\theta+1000\right)B\mu\end{bmatrix}=$$

$$=(+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+\left(126+102\cos\theta\right)A\mu+\\+\left(626+2\cos\theta\right)B\mu\end{bmatrix}\Rightarrow$$

$\Rightarrow$ Subterm B'+ Subterm C'= $(+15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+\left(126+102\cos\theta\right)A\mu+\\+\left(626+2\cos\theta\right)B\mu\end{bmatrix}$

<u>CALCULATION OF SUM: -(SUBTERM B'+ SUBTERM C')+ SUBTERM D'</u>

Subterm D' is: $\frac{R_y}{r}$, where R_y is:

$$R_y=(-15/2)c\,r^{-1/2}\sin(\theta/2)\left\{50A\lambda+63\left(A\mu+B\mu\right)+\left[50A\lambda+51A\mu+B\mu\right]\cos\theta\right\}$$

Then, -(SUBTERM B'+ SUBTERM C')+ SUBTERM D', which is:

$$(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{bmatrix}100A\lambda\cos\theta+\\+\left(126+102\cos\theta\right)A\mu+\\+\left(626+2\cos\theta\right)B\mu\end{bmatrix}+$$

$$+(-15/2)c\,r^{-3/2}\sin(\theta/2)\left\{50A\lambda+63\left(A\mu+B\mu\right)+\left[50A\lambda+51A\mu+B\mu\right]\cos\theta\right\}=$$

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{matrix}4\cdot50A\lambda+\left(126+4\cdot63\right)A\mu+\left(626+4\cdot63\right)B\mu+\\+\left[\left(100+4\cdot50\right)A\lambda+\left(102+4\cdot51\right)A\mu+\left(2+4\right)B\mu\right]\cos\theta\end{matrix}\right\}=$$

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\left\{\begin{matrix}200A\lambda+378A\mu+878B\mu+\\+\left(300A\lambda+306A\mu+6B\mu\right)\cos\theta\end{matrix}\right\}\Rightarrow$$

$\Rightarrow$-(Subterm B' of P_y + Subterm C' of P_y) + Subterm D' of P_y =

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{Bmatrix}200A\lambda+378A\mu+878B\mu+\\+(300A\lambda+306A\mu+6B\mu)\cos\theta\end{Bmatrix}$$

P_y includes the component $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$, so the final expression for P_y is:

$$P_y=\left(n_x\tau_{xy}+n_y\tau_{yy}\right)+(-15/8)c\,r^{-3/2}\sin(\theta/2)\begin{Bmatrix}200A\lambda+378A\mu+878B\mu+\\+(300A\lambda+306A\mu+6B\mu)\cos\theta\end{Bmatrix}$$

The component's $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$ order of r, after the integration, is greater than zero, which means that this term does not contribute to the final result for the J_1-Integral.

Omitting $\left(n_x\tau_{xy}+n_y\tau_{yy}\right)$, the remaining P_y is:

$$P_y=(-15/8)c\,r^{-3/2}\sin(\theta/2)\left[(200A\lambda+378A\mu+878B\mu)+(300A\lambda+306A\mu+6B\mu)\cos\theta\right]$$

P_y will be multiplied by $\dfrac{\partial u_y}{\partial x}$,

$$\frac{\partial u_y}{\partial x}=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[3(17A-83B)+25(A+5B)(2\cos\theta-1)\right]=$$

$$=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[(26A-374B)+50(A+5B)\cos\theta\right]$$

<u>Calculation of the product:</u> $P_y\left(\dfrac{\partial u_y}{\partial x}\right)$

This product is:

$$P_y\left(\frac{\partial u_y}{\partial x}\right)=$$

$$=(-15/8)c\,r^{-3/2}\sin(\theta/2)\left[(200A\lambda+378A\mu+878B\mu)+(300A\lambda+306A\mu+6B\mu)\cos\theta\right]\cdot$$

$$\cdot(-15/4)r^{1/2}\sin(\theta/2)\left[(26A-374B)+50(A+5B)\cos\theta\right]=$$

$$= (+225/32)c\ r^{-1}\sin^2(\theta/2)\left\{\begin{array}{l}(200A\lambda+378A\mu+878B\mu)(26A\text{-}374B)+\\ +\left[\begin{array}{l}(200A\lambda+378A\mu+878B\mu)(50A+250B)+\\ +(300A\lambda+306A\mu+6B\mu)(26A\text{-}374B)\end{array}\right]\cos\theta+\\ +\left[(300A\lambda+306A\mu+6B\mu)(50A+250B)\right]\cos^2\theta\end{array}\right\}$$

The decomposition of the last set of brackets into 3 parts gives:

First part

$$\begin{array}{l}(200A\lambda+378A\mu+878B\mu)(26A\text{-}374B)=\\ =200\cdot 26A^2\lambda-200\cdot 374AB\lambda+378\cdot 26A^2\mu-378\cdot 374AB\mu+\\ \qquad +878\cdot 26AB\mu-878\cdot 374B^2\mu=\\ =\left(5200A^2-74800AB\right)\lambda+\left(9828A^2-118544AB-328372B^2\right)\mu\end{array}$$

Second part

$$\begin{array}{l}(200A\lambda+378A\mu+878B\mu)(50A+250B)+\\ +(300A\lambda+306A\mu+6B\mu)(26A\text{-}374B)=\\ =200\cdot 50A^2\lambda+200\cdot 250AB\lambda+378\cdot 50A^2\mu+378\cdot 250AB\mu+\\ \qquad +878\cdot 50AB\mu+878\cdot 250B^2\mu+\\ +300\cdot 26A^2\lambda-300\cdot 374AB\lambda+306\cdot 26A^2\mu-306\cdot 374AB\mu+\\ \qquad +6\cdot 26AB\mu-6\cdot 374B^2\mu=\\ =\left(17800A^2-62200AB\right)\lambda+\left(26856A^2+24112AB+217256B^2\right)\mu\end{array}$$

Third part

$$\begin{array}{l}(300A\lambda+306A\mu+6B\mu)(50A+250B)=\\ =300\cdot 50A^2\lambda+300\cdot 250AB\lambda+306\cdot 50A^2\mu+306\cdot 250AB\mu+\\ \qquad +6\cdot 50AB\mu+6\cdot 250B^2\mu=\\ =\left(15000A^2+75000AB\right)\lambda+\left(15300A^2+76800AB\mu+1500B^2\right)\mu\end{array}$$

These 3 parts combined give:

$$P_y\left(\frac{\partial u_y}{\partial x}\right)=$$

$$=(+225/32)c\, r^{-1}\sin^2(\theta/2)\left\{\begin{array}{l}\left[\begin{array}{l}\left(5200A^2-74800AB\right)\lambda+\\+\left(9828A^2-118544AB-328372B^2\right)\mu\end{array}\right]+\\+\left[\begin{array}{l}\left(17800A^2-62200AB\right)\lambda+\\+\left(26856A^2+24112AB+217256B^2\right)\mu\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}\left(15000A^2+75000AB\right)\lambda+\\+\left(15300A^2+76800AB\mu+1500B^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}$$

1.4 Integrations in J_1

The J_1-Integral can now be formulated, by integrating its parts in the interval [-π, π].

Integration of *W* terms in J_1

Term W1

This integral is zero, due to the order of r (vanishes at crack-tip).

Term W2

This integral is zero, too, for the same reason.

Term W3

Bearing in mind that: $dy=n_1\, d\Gamma=\cos\theta\, d\Gamma=\cos\theta\, r\, d\theta=r\cos\theta\, d\theta$, the value of the definite integral is: (in the interval [-π,π])

$$\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]dy=\int_{-\pi}^{\pi}\left(140625A^2\, r^{-1}\right)r\cos\theta\, d\theta=$$

$$=140625A^2\int_{-\pi}^{\pi}\cos\theta\, d\theta=140625A^2\left[\sin\theta\right]_{-\pi}^{\pi}=140625A^2\left[\sin\pi-\sin(-\pi)\right]=$$

$$=140625A^2\left[0-0\right]=0\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left[\left(\frac{\partial \varepsilon_{xx}}{\partial x}+\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_{xx}}{\partial y}+\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2\right]dy=0}$$

Term W4

Term W4(a)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left((-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[50(A+5B)(\cos\theta-\cos(2\theta))+124(A+B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xx}}{\partial x}\right)^2 dy=\frac{225}{128}(149A+249B)^2\pi}$$

Term W4(b)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[-50(A+5B)(\cos\theta+\cos(2\theta))+(24A+124B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xy}}{\partial x}\right)^2 dy=-\frac{225}{128}(49A+249B)^2\pi}$$

Term W4(c)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 dy=$$

$$=\int_{-\pi}^{\pi}\left(\left(+\frac{15}{8}\right)r^{-1/2}\cos(\theta/2)\left[50(A+5B)(\cos\theta-\cos(2\theta))+(-76A+124B)\right]\right)^2 r\cos\theta d\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 dy=\frac{2025}{128}(17A-83B)^2\pi}$$

Term W4(d)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2 \mathrm{dy}=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50\left(\mathrm{A}+5\mathrm{B}\right)\left(\cos\theta+\cos(2\theta)\right)+\left(24\mathrm{A}\text{-}376\mathrm{B}\right)\right]\right)^2 \mathrm{r}\cos\theta\mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial y}\right)^2 \mathrm{dy}=-\frac{225}{128}\left(49\mathrm{A}-251\mathrm{B}\right)^2\pi}$$

Term W4(e)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2 \mathrm{dy}=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left[50\left(\mathrm{A}+5\mathrm{B}\right)\left(-\cos\theta+\cos(2\theta)\right)+\left(76\mathrm{A}+376\mathrm{B}\right)\right]\right)^2 \mathrm{r}\cos\theta\mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial y}\right)^2 \mathrm{dy}=\frac{225}{128}\left(51\mathrm{A}+251\mathrm{B}\right)^2\pi}$$

Term W4(f)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2 \mathrm{dy}=$$

$$=\int_{-\pi}^{\pi}\left(\left(\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50\left(\mathrm{A}+5\mathrm{B}\right)\left(\cos\theta+\cos(2\theta)\right)+\left(-176\mathrm{A}\text{-}376\mathrm{B}\right)\right]\right)^2 \mathrm{r}\cos\theta\mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2 \mathrm{dy}=-\frac{225}{128}\left(151\mathrm{A}+251\mathrm{B}\right)^2\pi}$$

By summing the results of the integrations of the W terms, we get:

$$\int_{-\pi}^{\pi} W\mathrm{dy}=\int_{-\pi}^{\pi} W\ \mathrm{r}\cos\theta\mathrm{d}\theta=$$
$$=0+0+0+$$
$$+\frac{225}{128}\left(149\mathrm{A}+249\mathrm{B}\right)^2\pi+2\left(-\frac{225}{128}\left(49\mathrm{A}+249\mathrm{B}\right)^2\pi\right)+\frac{2025}{128}\left(17\mathrm{A}-83\mathrm{B}\right)^2\pi+$$
$$+\left(-\frac{225}{128}\left(49\mathrm{A}-251\mathrm{B}\right)^2\pi\right)+2\frac{225}{128}\left(51\mathrm{A}+251\mathrm{B}\right)^2\pi+\left(-\frac{225}{128}\left(151\mathrm{A}+251\mathrm{B}\right)^2\pi\right)\Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} W \mathrm{d}y = 0}$$

This means that W does not contribute to J_1. Furthermore, we notice that, when integrating the squares of the strains' gradient, we obtain 6 non-zero terms, whose sum, though, is zero!

Integration of R_q terms in J_1

$$\int_{-\pi}^{\pi} \mathrm{R}_x D\left(\frac{\partial u_x}{\partial x}\right) \mathrm{d}\Gamma = \int_{-\pi}^{\pi} \mathrm{R}_x D\left(\frac{\partial u_x}{\partial x}\right) \mathrm{r}\, \mathrm{d}\theta =$$

$$= \int_{-\pi}^{\pi} (+225/8)\mathrm{c}\mathrm{r}^{-1}\cos^2(\theta/2) \left\{ \begin{array}{l} \left(481\mathrm{A}^2 + 1512\mathrm{AB} - 3969\mathrm{B}^2\right)\mu + \\ + \left[\begin{array}{l} \left(1850\mathrm{A}^2 - 3150\mathrm{AB}\right)\lambda + \\ + \left(2138\mathrm{A}^2 + 76\mathrm{AB} + 7938\mathrm{B}^2\right)\mu \end{array} \right] \cos\theta + \\ +25 \left[\begin{array}{l} 50\left(\mathrm{A}^2 + 5\mathrm{AB}\right)\lambda + \\ + \left(49\mathrm{A}^2 + 244\mathrm{AB} - 5\mathrm{B}^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} \mathrm{r}\, \mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{R}_x D\left(\frac{\partial u_x}{\partial x}\right) \mathrm{d}\Gamma = \frac{5625}{16} \mathrm{c}\pi \left[124\left(\mathrm{A}^2 + \mathrm{B}\right)\lambda + \left(173\mathrm{A}^2 + 368\mathrm{A} - 5\mathrm{B}^2\right)\mu\right]}$$

and additionally,

$$\int_{-\pi}^{\pi} \mathrm{R}_y D\left(\frac{\partial u_y}{\partial x}\right) \mathrm{d}\Gamma = \int_{-\pi}^{\pi} \mathrm{R}_y D\left(\frac{\partial u_y}{\partial x}\right) \mathrm{r}\, \mathrm{d}\theta =$$

$$= \int_{-\pi}^{\pi} (+225/8)\mathrm{c}\,\mathrm{r}^{-1}\sin^2(\theta/2) \left\{ \begin{array}{l} (650\mathrm{A}^2 - 9350\mathrm{AB})\lambda + \\ + \left(819\mathrm{A}^2 - 10962\mathrm{AB} - 11781\mathrm{B}^2\right)\mu + \\ + \left[\begin{array}{l} \left(1900\mathrm{A}^2 - 3100\mathrm{AB}\right)\lambda + \\ + \left(2238\mathrm{A}^2 - 74\mathrm{AB} + 7688\mathrm{B}^2\right)\mu \end{array} \right] \cos\theta + \\ + \left[\begin{array}{l} \left(1250\mathrm{A}^2 + 6250\mathrm{AB}\right)\lambda + \\ + \left(1275\mathrm{A}^2 + 6400\mathrm{AB} + 125\mathrm{B}^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} \mathrm{r}\, \mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \int_{-\pi}^{\pi} \mathrm{R}_y\, D\left(\frac{\partial u_y}{\partial x}\right) \mathrm{d}\Gamma = -\frac{5625}{16} \mathrm{c}\pi \left[1245\mathrm{B}^2\mu - \mathrm{A}^2\left(26\lambda + 27\mu\right) + \mathrm{AB}\left(374\lambda + 618\mu\right)\right] \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{R}_y\, D\left(\frac{\partial u_y}{\partial x}\right) \mathrm{d}\Gamma = \frac{5625}{16} \mathrm{c}\pi \left[\left(26\mathrm{A}^2 - 374\mathrm{AB}\right)\lambda + \left(27\mathrm{A}^2 - 618\mathrm{AB} - 1245\mathrm{B}^2\right)\mu\right]}$$

Integration of P_x terms in J_1

$$\int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) \mathrm{d}\Gamma = \int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) \mathrm{r}\, \mathrm{d}\theta =$$

$$= \int_{-\pi}^{\pi} (+225/32)\,\mathrm{c}\, \mathrm{r}^{-1} \cos^2(\theta/2) \left\{ \begin{array}{l} \left[\begin{array}{l} \left(-7400\mathrm{A}^2 + 12600\mathrm{AB}\right)\lambda + \\ + \left(5772\mathrm{A}^2 + 55144\mathrm{AB} - 110628\mathrm{B}^2\right)\mu \end{array}\right] + \\ + \left[\begin{array}{l} \left(17200\mathrm{A}^2 - 62800\mathrm{AB}\right)\lambda + \\ + \left(25656\mathrm{A}^2 + 25912\mathrm{AB} + 220256\mathrm{B}^2\right)\mu \end{array}\right] \cos\theta + \\ + \left[\begin{array}{l} 15000\left(\mathrm{A}^2 + 5\mathrm{B}\right)\lambda + \\ + \left(14700\mathrm{A}^2 + 73200\mathrm{AB} - 1500\mathrm{B}^2\right)\mu \end{array}\right] \cos^2\theta \end{array} \right\} \mathrm{r}\, \mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) \mathrm{d}\Gamma = \frac{5625}{16} \mathrm{c}\pi \left[-25\mathrm{B}^2\mu + 3\mathrm{A}^2\left(58\lambda + 173\mu\right) + \mathrm{AB}\left(374\lambda + 2094\mu\right)\right] \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} \mathrm{P}_x \left(\frac{\partial u_x}{\partial x}\right) \mathrm{d}\Gamma = \frac{5625}{16} \mathrm{c}\pi \left[\left(174\mathrm{A}^2 + 374\mathrm{AB}\right)\lambda + \left(519\mathrm{A}^2 + 2094\mathrm{AB} - 25\mathrm{B}^2\right)\mu\right]}$$

and also

$$\int_{-\pi}^{\pi} P_y \left(\frac{\partial u_y}{\partial x}\right) d\Gamma = \int_{-\pi}^{\pi} P_y \left(\frac{\partial u_y}{\partial x}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} (+225/32) c\, r^{-1} \sin^2(\theta/2) \left\{ \begin{array}{l} \left[\begin{array}{l} \left(5200A^2 - 74800AB\right)\lambda + \\ +\left(9828A^2 - 118544AB - 328372B^2\right)\mu \end{array} \right] + \\ + \left[\begin{array}{l} \left(17800A^2 - 62200AB\right)\lambda + \\ +\left(26856A^2 + 24112AB + 217256B^2\right)\mu \end{array} \right] \cos\theta + \\ + \left[\begin{array}{l} \left(15000A^2 + 75000AB\right)\lambda + \\ +\left(15300A^2 + 76800AB\mu + 1500B^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} r\, d\theta \Rightarrow$$

$$\Rightarrow \int_{-\pi}^{\pi} P_y \left(\frac{\partial u_y}{\partial x}\right) d\Gamma = \frac{5625}{16} c\pi \left[-8725B^2\mu + A^2\left(76\lambda + 81\mu\right) - 4AB\left(31\lambda + 461\mu\right) \right] \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} P_y \left(\frac{\partial u_y}{\partial x}\right) d\Gamma = \frac{5625}{16} c\pi \left[\left(76A^2 - 124AB\right)\lambda + \left(81A^2 - 1844AB - 8725B^2\right)\mu \right]}$$

The total sum for J_1 is:

$$J_1 = \int_\Gamma \left[W dy - P_q \frac{\partial u_q}{\partial x} d\Gamma - R_z D \left(\frac{\partial u_z}{\partial x}\right) d\Gamma \right] =$$

$$= \int_\Gamma W dy - \int_\Gamma \left[P_x \frac{\partial u_x}{\partial x} d\Gamma + P_y \frac{\partial u_y}{\partial x} \right] - \int_\Gamma \left[R_x D \left(\frac{\partial u_x}{\partial x}\right) d\Gamma + R_y D \left(\frac{\partial u_y}{\partial x}\right) d\Gamma \right] =$$

$$= 0 - \left\{ \begin{array}{l} \frac{5625}{16} c\pi \left[\left(174A^2 + 374AB\right)\lambda + \left(519A^2 + 2094AB - 25B^2\right)\mu \right] + \\ + \frac{5625}{16} c\pi \left[\left(76A^2 - 124AB\right)\lambda + \left(81A^2 - 1844AB - 8725B^2\right)\mu \right] \end{array} \right\} -$$

$$- \left\{ \begin{array}{l} \frac{5625}{16} c\pi \left[124\left(A^2 + B\right)\lambda + \left(173A^2 + 368A - 5B^2\right)\mu \right] + \\ + \frac{5625}{16} c\pi \left[\left(26A^2 - 374AB\right)\lambda + \left(27A^2 - 618AB - 1245B^2\right)\mu \right] \end{array} \right\} \Rightarrow$$

$$\Rightarrow J_1 = -140625c\pi\left[-25B^2\mu + A^2(\lambda + 2\mu)\right] =$$

$$= -140625c\pi\left[A^2\lambda + \left(2A^2 - 25B^2\right)\mu\right] \Rightarrow$$

$$\Rightarrow \boxed{J_1 = -140625c\pi\mu\left[A^2\frac{\lambda}{\mu} + \left(2A^2 - 25B^2\right)\right]}$$

APPENDIX 3

EXPLICIT CALCULATIONS FOR THE J_2-INTEGRAL

2.1 Terms W in J_2

Calculation of terms W

W is the same for J_1 and J_2. The only difference is that W is multiplied by cosθ in J_1 and by sinθ in J_2.

2.2 Terms R_q in J_2

Calculation of terms $\mathrm{R}_q\, D\left(\frac{\partial u_q}{\partial y}\right)$

$\mathrm{R}_q\, D\left(\frac{\partial u_q}{\partial y}\right)$ is literally:

$$\mathrm{R}_q\, D\left(\frac{\partial u_q}{\partial y}\right) = \mathrm{R}_x\, D\left(\frac{\partial u_x}{\partial y}\right) + \mathrm{R}_y\, D\left(\frac{\partial u_y}{\partial y}\right)$$

The terms $\mathrm{R}_x, \mathrm{R}_y$ are identical for J_1 and J_2.

Additionally,

$$D\left(\frac{\partial u_x}{\partial y}\right) = \mathrm{n}_r\, \partial_r\left(\frac{\partial u_x}{\partial y}\right) = \mathrm{n}_x\, \partial_x\left(\frac{\partial u_x}{\partial y}\right) + \mathrm{n}_y\, \partial_y\left(\frac{\partial u_x}{\partial y}\right) =$$
$$= \mathrm{n}_x \frac{\partial^2 u_x}{\partial x \partial y} + \mathrm{n}_y \frac{\partial^2 u_x}{\partial y^2} = \cos\theta \frac{\partial^2 u_x}{\partial x \partial y} + \sin\theta \frac{\partial^2 u_x}{\partial y^2}$$

and

$$D\left(\frac{\partial u_y}{\partial y}\right) = \mathrm{n}_r\, \partial_r\left(\frac{\partial u_y}{\partial y}\right) = \mathrm{n}_x\, \partial_x\left(\frac{\partial u_y}{\partial y}\right) + \mathrm{n}_y\, \partial_y\left(\frac{\partial u_y}{\partial y}\right) =$$
$$= \mathrm{n}_x \frac{\partial^2 u_y}{\partial x \partial y} + \mathrm{n}_y \frac{\partial^2 u_y}{\partial y^2} = \cos\theta \frac{\partial^2 u_y}{\partial x \partial y} + \sin\theta \frac{\partial^2 u_y}{\partial y^2}$$

These two expressions can be estimated only after calculating the partial derivatives $\frac{\partial^2 u_x}{\partial x \partial y}$, $\frac{\partial^2 u_x}{\partial y^2}$ (for $D\left(\frac{\partial u_x}{\partial y}\right)$) and $\frac{\partial^2 u_y}{\partial x \partial y}$, $\frac{\partial^2 u_y}{\partial y^2}$ (for $D\left(\frac{\partial u_y}{\partial y}\right)$)

Calculation of the partial derivatives of the displacements

1. $\frac{\partial^2 u_x}{\partial x \partial y}$

The derivative $\frac{\partial^2 u_x}{\partial x \partial y}$ has already been found to be:

$$\boxed{\frac{\partial^2 u_x}{\partial x \partial y} = (-\frac{15}{8})\mathrm{r}^{-1/2}\sin(\theta/2)\left\{(49\mathrm{A}-251\mathrm{B}) - 25(\mathrm{A}+5\mathrm{B})\left[1+2\cos\theta+2\cos(2\theta)\right]\right\}}$$

2. $\frac{\partial u_x}{\partial y}$

$$\frac{\partial u_x}{\partial y} = \frac{\partial}{\partial y}(u_x) = \sin\theta \cdot \frac{\partial}{\partial r}(u_x) + \frac{\cos\theta}{\mathrm{r}} \cdot \frac{\partial}{\partial \theta}(u_x) =$$

$$= \sin\theta \frac{\partial}{\partial r}\left\{(-5/2)\mathrm{r}^{3/2}\left[(49\mathrm{A}-251\mathrm{B})\cos(3\theta/2) + 75(\mathrm{A}+5\mathrm{B})\cos(\theta/2)\right]\right\} +$$

$$+ \frac{\cos\theta}{\mathrm{r}} \frac{\partial}{\partial \theta}\left\{(-5/2)\mathrm{r}^{3/2}\left[(49\mathrm{A}-251\mathrm{B})\cos(3\theta/2) + 75(\mathrm{A}+5\mathrm{B})\cos(\theta/2)\right]\right\} =$$

$$= \sin\theta(-\frac{15}{4})\mathrm{r}^{1/2}\left[(49\mathrm{A}-251\mathrm{B})\cos(3\theta/2) + 75(\mathrm{A}+5\mathrm{B})\cos(\theta/2)\right] +$$

$$+ \cos\theta\, \mathrm{r}^{1/2}(-5/2)\left[(49\mathrm{A}-251\mathrm{B})\left[-\sin(3\theta/2)\right]\frac{3}{2} + 75(\mathrm{A}+5\mathrm{B})\left[-\sin(\theta/2)\right]\frac{1}{2}\right] =$$

$$= (-\frac{15}{4})\mathrm{r}^{1/2}\left\{\begin{matrix}\sin\theta\left[(49\mathrm{A}-251\mathrm{B})\cos(3\theta/2) + 75(\mathrm{A}+5\mathrm{B})\cos(\theta/2)\right] - \\ -\cos\theta\left[(49\mathrm{A}-251\mathrm{B})\sin(3\theta/2) + 25(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\right]\end{matrix}\right\}$$

The bracketed term becomes:

$$\sin\theta\left[(49A\text{-}251B)\cos(3\theta/2)+75(A+5B)\cos(\theta/2)\right]-$$
$$-\cos\theta\left[(49A\text{-}251B)\sin(3\theta/2)+25(A+5B)\sin(\theta/2)\right]=$$

$$=(49A\text{-}251B)\left[\sin\theta\cos(3\theta/2)\text{-}\cos\theta\sin(3\theta/2)\right]+$$
$$+25(A+5B)\left[3\sin\theta\cos(\theta/2)\text{-}\cos\theta\sin(\theta/2)\right]=$$

$$=-(49A\text{-}251B)\sin(\theta/2)+$$
$$+25(A+5B)\left[2\sin\theta\cos(\theta/2)+\sin\theta\cos(\theta/2)\text{-}\cos\theta\sin(\theta/2)\right]=$$

$$=-(49A\text{-}251B)\sin(\theta/2)+25(A+5B)\left[2\sin\theta\cos(\theta/2)+\sin(\theta/2)\right]=$$
$$=\sin(\theta/2)\left[-(49A\text{-}251B)+25(A+5B)\left(2\frac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}+1\right)\right]$$

First, the quotient $\frac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}$ will be simplified.

$$\frac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}=\frac{2\,\cancel{\sin(\theta/2)}\cos(\theta/2)\cos(\theta/2)}{\cancel{\sin(\theta/2)}}=2\cos^2(\theta/2)=1+\cos\theta$$

So,

$$\frac{\partial u_x}{\partial y}=\sin(\theta/2)\left[-(49A\text{-}251B)+25(A+5B)\left(2\frac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}+1\right)\right]=$$
$$=\sin(\theta/2)\left[-(49A\text{-}251B)+25(A+5B)\left(2(1+\cos\theta)+1\right)\right]=$$
$$=\sin(\theta/2)\left[-(49A\text{-}251B)+25(A+5B)(3+2\cos\theta)\right]\Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial u_x}{\partial y}=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[-(49A\text{-}251B)+25(A+5B)(3+2\cos\theta)\right]}$$

3. $\frac{\partial^2 u_x}{\partial y^2}$

The second order partial derivative is:

$$\frac{\partial^2 u_x}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial y}\right)=\sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u_x}{\partial y}\right)+\frac{\cos\theta}{\mathrm{r}}\frac{\partial}{\partial\theta}\left(\frac{\partial u_x}{\partial y}\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left\{(-\frac{15}{4})\mathrm{r}^{1/2}\sin(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]\right\}+$$

$$+\frac{\cos\theta}{\mathrm{r}}\frac{\partial}{\partial\theta}\left\{(-\frac{15}{4})\mathrm{r}^{1/2}\sin(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]\right\}=$$

$$=(-\frac{15}{4})\mathrm{r}^{-1/2}\sin\theta\sin(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]+$$

$$+(-\frac{15}{4})\mathrm{r}^{-1/2}\cos\theta\left\{\begin{array}{l}\cos(\theta/2)\frac{1}{2}\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]+\\+\sin(\theta/2)\left[0+25(\mathrm{A}+5\mathrm{B})(0-2\sin\theta)\right]\end{array}\right\}=$$

$$=(-\frac{15}{4})\mathrm{r}^{-1/2}\sin\theta\sin(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]+$$

$$+(-\frac{15}{4})\mathrm{r}^{-1/2}\cos\theta\left\{\begin{array}{l}\frac{1}{2}\cos(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]+\\+(-50)(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\sin\theta\end{array}\right\}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\left\{\begin{array}{l}\sin\theta\sin(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]+\\+\cos\theta\cos(\theta/2)\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]-\\-100(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\sin\theta\cos\theta\end{array}\right\}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\left\{\begin{array}{l}\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]\left[\sin\theta\sin(\theta/2)+\cos\theta\cos(\theta/2)\right]-\\-100(\mathrm{A}+5\mathrm{B})\sin\theta\sin(\theta/2)\cos\theta\end{array}\right\}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\left\{\begin{array}{l}\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]\cos(\theta/2)-\\-100(\mathrm{A}+5\mathrm{B})\times2\sin(\theta/2)\cos(\theta/2)\sin(\theta/2)\cos\theta\end{array}\right\}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\left\{\begin{array}{l}\left[-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(3+2\cos\theta)\right]-\\-200(\mathrm{A}+5\mathrm{B})\sin^2(\theta/2)\cos\theta\end{array}\right\}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\left\{\begin{array}{l}-(49\mathrm{A}\text{-}251\mathrm{B})+\\+25(\mathrm{A}+5\mathrm{B})\left(3+2\cos\theta-8\sin^2(\theta/2)\cos\theta\right)\end{array}\right\}$$

Simplification of: $\sin^2(\theta/2)\cos\theta$ leads to:

$$\sin^2(\theta/2)\cos\theta=\frac{1-\cos\theta}{2}\cos\theta=\frac{1}{2}\cos\theta-\frac{1}{2}\cos^2\theta=\frac{1}{2}\cos\theta-\frac{1}{2}\frac{1+\cos(2\theta)}{2}=$$

$$=\frac{1}{2}\cos\theta-\frac{1}{4}-\frac{1}{4}\cos(2\theta)=\frac{1}{2}\left(-\frac{1}{2}+\cos\theta-\frac{1}{2}\cos(2\theta)\right)$$

and the simplification already in progress continues this way:

$$\frac{\partial^2 u_x}{\partial y^2}=(-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}-(49\mathrm{A}\text{-}251\mathrm{B})+\\+25(\mathrm{A}+5\mathrm{B})\left(3+2\cos\theta-8\sin^2(\theta/2)\cos\theta\right)\end{Bmatrix}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}-(49\mathrm{A}\text{-}251\mathrm{B})+\\+25(\mathrm{A}+5\mathrm{B})\left[3+2\cos\theta-8\frac{1}{2}\left(-\frac{1}{2}+\cos\theta-\frac{1}{2}\cos(2\theta)\right)\right]\end{Bmatrix}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\begin{Bmatrix}-(49\mathrm{A}\text{-}251\mathrm{B})+\\+25(\mathrm{A}+5\mathrm{B})\left[3+2\cos\theta+2-4\cos\theta+2\cos(2\theta)\right]\end{Bmatrix}=$$

$$=(-\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\left\{-(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left[5-2\cos\theta+2\cos(2\theta)\right]\right\}\Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 u_x}{\partial y^2}=(+\frac{15}{8})\mathrm{r}^{-1/2}\cos(\theta/2)\left[(49\mathrm{A}\text{-}251\mathrm{B})+25(\mathrm{A}+5\mathrm{B})(\text{-}5+2\cos\theta-2\cos(2\theta))\right]}$$

4. $\dfrac{\partial^2 u_y}{\partial x \partial y}$

This partial derivative has already been found to be:

$$\boxed{\frac{\partial^2 u_y}{\partial x\partial y}=\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left\{3(17\mathrm{A}\text{-}83\mathrm{B})+25(\mathrm{A}+5\mathrm{B})\left[1-2\cos\theta+2\cos(2\theta)\right]\right\}}$$

5. $\dfrac{\partial u_y}{\partial y}$

$$\frac{\partial u_y}{\partial y}=\frac{\partial}{\partial y}\left(u_y\right)=\sin\theta\frac{\partial}{\partial r}\left(u_y\right)+\frac{\cos\theta}{\mathrm{r}}\frac{\partial}{\partial\theta}\left(u_y\right)=$$

$$=\sin\theta\frac{\partial}{\partial r}\left\{(-15/2)\mathrm{r}^{3/2}\left[(17\mathrm{A}\text{-}83\mathrm{B})\sin(3\theta/2)+25(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\right]\right\}+$$

$$+\frac{\cos\theta}{\mathrm{r}}\frac{\partial}{\partial\theta}\left\{(-15/2)\mathrm{r}^{3/2}\left[(17\mathrm{A}\text{-}83\mathrm{B})\sin(3\theta/2)+25(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\right]\right\}=$$

$$=(-\frac{15}{2})\frac{3}{2}\mathrm{r}^{1/2}\sin\theta\left[(17\mathrm{A}\text{-}83\mathrm{B})\sin(3\theta/2)+25(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\right]+$$

$$+(-\frac{15}{2})\,\mathrm{r}^{1/2}\cos\theta\left[(17\mathrm{A}\text{-}83\mathrm{B})\cos(3\theta/2)\frac{3}{2}+25(\mathrm{A}+5\mathrm{B})\cos(\theta/2)\frac{1}{2}\right]=$$

$$=(-\frac{15}{4})\,\mathrm{r}^{1/2}\sin\theta\left[3(17\mathrm{A}\text{-}83\mathrm{B})\sin(3\theta/2)+75(\mathrm{A}+5\mathrm{B})\sin(\theta/2)\right]+$$

$$+(-\frac{15}{4})\,\mathrm{r}^{1/2}\cos\theta\left[3(17\mathrm{A}\text{-}83\mathrm{B})\cos(3\theta/2)+25(\mathrm{A}+5\mathrm{B})\cos(\theta/2)\right]=$$

$$=(-\frac{15}{4})\,\mathrm{r}^{1/2}\left\{\begin{matrix}3\left(17\mathrm{A}\text{-}83\mathrm{B}\right)\left[\sin\theta\sin(3\theta/2)+\cos\theta\cos(3\theta/2)\right]+\\+25\left(\mathrm{A}+5\mathrm{B}\right)\left[3\sin\theta\sin(\theta/2)+\cos\theta\cos(\theta/2)\right]\end{matrix}\right\}=$$

$$=(-\frac{15}{4})\,\mathrm{r}^{1/2}\left\{\begin{matrix}3\left(17\mathrm{A}\text{-}83\mathrm{B}\right)\cos(\theta/2)+\\+25\left(\mathrm{A}+5\mathrm{B}\right)\left[\begin{matrix}\sin\theta\sin(\theta/2)+\cos\theta\cos(\theta/2)+\\+2\sin\theta\sin(\theta/2)\end{matrix}\right]\end{matrix}\right\}=$$

$$=(-\frac{15}{4})\,\mathrm{r}^{1/2}\left\{\begin{matrix}3\left(17\mathrm{A}\text{-}83\mathrm{B}\right)\cos(\theta/2)+\\+25\left(\mathrm{A}+5\mathrm{B}\right)\left[\cos(\theta/2)+2\sin\theta\sin(\theta/2)\right]\end{matrix}\right\}=$$

$$=(-\frac{15}{4})\,\mathrm{r}^{1/2}\cos(\theta/2)\left\{\begin{matrix}3\left(17\mathrm{A}\text{-}83\mathrm{B}\right)+\\+25\left(\mathrm{A}+5\mathrm{B}\right)\left[1+2\frac{\sin\theta\sin(\theta/2)}{\cos(\theta/2)}\right]\end{matrix}\right\}$$

But

$$\frac{\sin\theta\sin(\theta/2)}{\cos(\theta/2)}=\frac{2\sin(\theta/2)\cancel{\cos(\theta/2)}\sin(\theta/2)}{\cancel{\cos(\theta/2)}}=2\sin^2(\theta/2)=1\text{-}\cos\theta\,,$$

so:

$$\frac{\partial u_y}{\partial y} = (-\frac{15}{4})\ r^{1/2}\cos(\theta/2)\begin{Bmatrix} 3(17A\text{-}83B) + \\ +25(A+5B)\left[1+2(1\text{-}\cos\theta)\right] \end{Bmatrix} =$$

$$= (-\frac{15}{4})\ r^{1/2}\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right] \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial u_y}{\partial y} = (-\frac{15}{4})\ r^{1/2}\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]}$$

6. $\frac{\partial^2 u_y}{\partial y^2}$

$$\frac{\partial^2 u_y}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u_y}{\partial y}\right) = \sin\theta\,\frac{\partial}{\partial r}\left(\frac{\partial u_y}{\partial y}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u_y}{\partial y}\right) =$$

$$= \sin\theta\,\frac{\partial}{\partial r}\left\{(-\frac{15}{4})\ r^{1/2}\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]\right\} +$$

$$+ \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left\{(-\frac{15}{4})\ r^{1/2}\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]\right\} =$$

$$= (-\frac{15}{8})\ r^{-1/2}\sin\theta\left\{\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]\right\} +$$

$$+(-\frac{15}{4})\ r^{-1/2}\cos\theta\begin{Bmatrix} \text{-}\sin(\theta/2)\frac{1}{2}\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right] + \\ +\cos(\theta/2)\left[0 + 25(A+5B)(0{+}2\sin\theta)\right] \end{Bmatrix} =$$

$$= (-\frac{15}{8})\ r^{-1/2}\sin\theta\left\{\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]\right\} +$$

$$+(-\frac{15}{4})\ r^{-1/2}\cos\theta\begin{Bmatrix} \text{-}\sin(\theta/2)\frac{1}{2}\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right] + \\ +50(A+5B)\cos(\theta/2)\sin\theta \end{Bmatrix} =$$

$$= (-\frac{15}{8})\ r^{-1/2}\sin\theta\left\{\cos(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]\right\} +$$

$$+(-\frac{15}{8})\ r^{-1/2}\cos\theta\begin{Bmatrix} \text{-}\sin(\theta/2)\left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right] + \\ +100(A+5B)\cos(\theta/2)\sin\theta \end{Bmatrix} =$$

$$= (-\frac{15}{8})\ r^{-1/2}\begin{Bmatrix} \left[3(17A\text{-}83B) + 25(A+5B)(3\text{-}2\cos\theta)\right]\left[\sin\theta\cos(\theta/2)\text{-}\cos\theta\sin(\theta/2)\right] + \\ +100(A+5B)\cos\theta\cos(\theta/2)\sin\theta \end{Bmatrix} =$$

$$=(-\frac{15}{8})\, r^{-1/2}\left\{\begin{matrix}\left[3(17A\text{-}83B)+25(A+5B)(3\text{-}2\cos\theta)\right]\sin(\theta/2)+\\ +100(A+5B)\cos\theta\cos(\theta/2)\sin\theta\end{matrix}\right\}=$$

$$=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left\{\begin{matrix}\left[3(17A\text{-}83B)+25(A+5B)(3\text{-}2\cos\theta)\right]+\\ +100(A+5B)\cos\theta\cos(\theta/2)\dfrac{\sin\theta}{\sin(\theta/2)}\end{matrix}\right\}=$$

$$=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left\{\begin{matrix}\left[3(17A\text{-}83B)+25(A+5B)(3\text{-}2\cos\theta)\right]+\\ +100(A+5B)\cos\theta\cos(\theta/2)\dfrac{2\sin(\theta/2)\cos(\theta/2)}{\sin(\theta/2)}\end{matrix}\right\}=$$

$$=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left\{\begin{matrix}\left[3(17A\text{-}83B)+25(A+5B)(3\text{-}2\cos\theta)\right]+\\ +200(A+5B)\cos\theta\cos^2(\theta/2)\end{matrix}\right\}=$$

$$=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left\{3(17A\text{-}83B)+25(A+5B)\left(3\text{-}2\cos\theta+8\cos\theta\cos^2(\theta/2)\right)\right\}$$

where

$$\cos\theta\cos^2(\theta/2)=\cos\theta\frac{1+\cos\theta}{2}=\frac{1}{2}\left(\cos\theta+\cos^2\theta\right)=\frac{1}{2}\left(\cos\theta+\frac{1+\cos(2\theta)}{2}\right)=$$

$$=\frac{1}{2}\left(\frac{1}{2}+\cos\theta+\frac{1}{2}\cos(2\theta)\right)=\frac{1}{4}\left(1+2\cos\theta+\cos(2\theta)\right)$$

and finally,

$$\frac{\partial^2 u_y}{\partial y^2}=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left\{3(17A\text{-}83B)+25(A+5B)\left(3\text{-}2\cos\theta+8\cos\theta\cos^2(\theta/2)\right)\right\}=$$

$$=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(3\text{-}2\cos\theta+8\frac{1}{4}\left(1+2\cos\theta+\cos(2\theta)\right)\right)\right]=$$

$$=(-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)\left(5+2\cos\theta+2\cos(2\theta)\right)\right]\Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial^2 u_y}{\partial y^2} = (-\frac{15}{8})\, r^{-1/2}\sin(\theta/2)\left[3(17A-83B) + 25(A+5B)(5+2\cos\theta + 2\cos(2\theta))\right]}$$

Calculation of $D\left(\frac{\partial u_x}{\partial y}\right) = n_x \frac{\partial^2 u_x}{\partial x \partial y} + n_y \frac{\partial^2 u_x}{\partial y^2} = \cos\theta \frac{\partial^2 u_x}{\partial x \partial y} + \sin\theta \frac{\partial^2 u_x}{\partial y^2}$

$$D\left(\frac{\partial u_x}{\partial y}\right) = n_x \frac{\partial^2 u_x}{\partial x \partial y} + n_y \frac{\partial^2 u_x}{\partial y^2} = \cos\theta \frac{\partial^2 u_x}{\partial x \partial y} + \sin\theta \frac{\partial^2 u_x}{\partial y^2} =$$

$$= \cos\theta(-\frac{15}{8})\, r^{-1/2} \sin(\theta/2)\left\{(49A-251B) - 25(A+5B)\left[1+2\cos\theta + 2\cos(2\theta)\right]\right\} +$$

$$+\sin\theta(\frac{15}{8})\, r^{-1/2}\cos(\theta/2)\left[(49A-251B) + 25(A+5B)(-5+2\cos\theta - 2\cos(2\theta))\right] =$$

$$=(-\frac{15}{8})\, r^{-1/2}\left\{\begin{array}{l}(49A-251B)\left[\cos\theta\sin(\theta/2) - \sin\theta\cos(\theta/2)\right] - \\ -25(A+5B)\left[\begin{array}{l}\cos\theta\sin(\theta/2)\left[1+2\cos\theta+2\cos(2\theta)\right] + \\ +\sin\theta\cos(\theta/2)(-5+2\cos\theta - 2\cos(2\theta))\end{array}\right]\end{array}\right\} =$$

$$= (-\frac{15}{8})\, r^{-1/2}\left\{\begin{array}{l}-(49A-251B)\sin(\theta/2) - \\ -25(A+5B)\left[\begin{array}{l}\cos\theta\sin(\theta/2) + 2\cos^2\theta\sin(\theta/2) + 2\cos\theta\sin(\theta/2)\cos(2\theta) - \\ -5\sin\theta\cos(\theta/2) + 2\sin\theta\cos(\theta/2)\cos\theta - 2\sin\theta\cos(\theta/2)\cos(2\theta)\end{array}\right]\end{array}\right\} =$$

$$= (-\frac{15}{8})\, r^{-1/2}\left\{\begin{array}{l}-(49A-251B)\sin(\theta/2) - \\ -25(A+5B)\left[\begin{array}{l}\cos\theta\sin(\theta/2) - \sin\theta\cos(\theta/2) - 4\sin\theta\cos(\theta/2) + \\ +2\cos\theta\left[\cos\theta\sin(\theta/2) + \sin\theta\cos(\theta/2)\right] + \\ +2\cos(2\theta)\left[\cos\theta\sin(\theta/2) - \sin\theta\cos(\theta/2)\right]\end{array}\right]\end{array}\right\} =$$

$$= (-\frac{15}{8})\, r^{-1/2}\left\{\begin{array}{l}-(49A-251B)\sin(\theta/2) - \\ -25(A+5B)\left[\begin{array}{l}-\sin(\theta/2) - 4\sin\theta\cos(\theta/2) + \\ +2\cos\theta\left[\sin(3\theta/2)\right] + \\ +2\cos(2\theta)\left[-\sin(\theta/2)\right]\end{array}\right]\end{array}\right\} =$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}-\left(49A\text{-}251B\right)\sin(\theta/2)-\\ -25\left(A+5B\right)\sin(\theta/2)\left[\begin{array}{l}-1-4\dfrac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}+\\ +2\cos\theta\dfrac{\sin(3\theta/2)}{\sin(\theta/2)}-\\ -2\cos(2\theta)\end{array}\right]\end{array}\right\},$$

and by replacing:

$$\frac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}=\frac{2\sin(\theta/2)\cos^2(\theta/2)}{\sin(\theta/2)}=2\cos^2(\theta/2),$$

$$\frac{\sin(3\theta/2)}{\sin(\theta/2)}=1+2\cos\theta$$

we get:

$$(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}-\left(49A\text{-}251B\right)\sin(\theta/2)-\\ -25\left(A+5B\right)\sin(\theta/2)\left[\begin{array}{l}-1-4\times 2\cos^2(\theta/2)+\\ +2\cos\theta\left(1+2\cos\theta\right)-\\ -2\cos(2\theta)\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}-\left(49A\text{-}251B\right)\sin(\theta/2)-\\ -25\left(A+5B\right)\sin(\theta/2)\left[\begin{array}{l}-1-4\left(1+\cos\theta\right)+\\ +2\cos\theta+4\cos^2\theta-\\ -2\cos(2\theta)\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}-\left(49A\text{-}251B\right)\sin(\theta/2)-\\ -25\left(A+5B\right)\sin(\theta/2)\left[\begin{array}{l}-1-4\left(1+\cos\theta\right)+\\ +2\cos\theta+4\dfrac{1+\cos(2\theta)}{2}-\\ -2\cos(2\theta)\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}-(49A\text{-}251B)\sin(\theta/2)- \\ -25(A+5B)\sin(\theta/2)\left[\begin{array}{l}-1-4-4\cos\theta+ \\ +2\cos\theta+2+\cancel{2\cos(2\theta)}- \\ -\cancel{2\cos(2\theta)}\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}-(49A\text{-}251B)\sin(\theta/2)- \\ -25(A+5B)\sin(\theta/2)(-3-2\cos\theta)\end{array}\right\}=$$

$$=(+\frac{15}{8})r^{-1/2}\sin(\theta/2)\left[(49A\text{-}251B)+25(A+5B)(-3-2\cos\theta)\right]=$$

$$=(+\frac{15}{8})r^{-1/2}\sin(\theta/2)\left[49A\text{-}251B-75(A+5B)-50(A+5B)\cos\theta\right]=$$

$$=(+\frac{15}{8})r^{-1/2}\sin(\theta/2)\left[-26A-626B-50(A+5B)\cos\theta\right]=$$

$$=(-\frac{15}{4})r^{-1/2}\sin(\theta/2)\left[(13A+313B)+25(A+5B)\cos\theta\right]\Rightarrow$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_x}{\partial y}\right)=(-\frac{15}{4})r^{-1/2}\sin(\theta/2)\left[(13A+313B)+25(A+5B)\cos\theta\right]}$$

Calculation of $D\left(\frac{\partial u_y}{\partial y}\right)=n_x\frac{\partial^2 u_y}{\partial x\partial y}+n_y\frac{\partial^2 u_y}{\partial y^2}=\cos\theta\frac{\partial^2 u_y}{\partial x\partial y}+\sin\theta\frac{\partial^2 u_y}{\partial y^2}$

$$D\left(\frac{\partial u_y}{\partial y}\right)=n_x\frac{\partial^2 u_y}{\partial x\partial y}+n_y\frac{\partial^2 u_y}{\partial y^2}=\cos\theta\frac{\partial^2 u_y}{\partial x\partial y}+\sin\theta\frac{\partial^2 u_y}{\partial y^2}=$$

$$=\cos\theta\left(-\frac{15}{8}\right)r^{-1/2}\cos(\theta/2)\left\{3(17A\text{-}83B)+25(A+5B)\left[1-2\cos\theta+2\cos(2\theta)\right]\right\}+$$

$$+\sin\theta(-\frac{15}{8})\,r^{-1/2}\sin(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(5+2\cos\theta+2\cos(2\theta))\right]=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\left[\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right]+\\+25(A+5B)\left[\begin{array}{l}\cos\theta\cos(\theta/2)(1-2\cos\theta+2\cos(2\theta))+\\+\sin\theta\sin(\theta/2)(5+2\cos\theta+2\cos(2\theta))\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\+25(A+5B)\left[\begin{array}{l}\cos\theta\cos(\theta/2)-2\cos^2\theta\cos(\theta/2)+2\cos\theta\cos(\theta/2)\cos(2\theta)+\\+5\sin\theta\sin(\theta/2)+2\sin\theta\sin(\theta/2)\cos\theta+2\sin\theta\sin(\theta/2)\cos(2\theta)\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\+25(A+5B)\left[\begin{array}{l}\underline{\cos\theta\cos(\theta/2)}-2\cos\theta\left[\cos\theta\cos(\theta/2)-\sin\theta\sin(\theta/2)\right]+\\+\underline{\sin\theta\sin(\theta/2)}+4\sin\theta\sin(\theta/2)+\\+2\cos(2\theta)\left[\cos\theta\cos(\theta/2)+\sin\theta\sin(\theta/2)\right]\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\+25(A+5B)\left[\begin{array}{l}\underline{\cos(\theta/2)}-2\cos\theta\cos(3\theta/2)+\\+4\sin\theta\sin(\theta/2)+\\+2\cos(2\theta)\cos(\theta/2)\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\+25(A+5B)\cos(\theta/2)\left[\begin{array}{l}1-2\dfrac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}+\\+4\dfrac{\sin\theta\sin(\theta/2)}{\cos(\theta/2)}+\\+2\cos(2\theta)\end{array}\right]\end{array}\right\}$$

Though:

$$\frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}=1-\cos\theta+\cos(2\theta),$$

$$\frac{\sin\theta\sin(\theta/2)}{\cos(\theta/2)}=\frac{2\sin^2(\theta/2)\cancel{\cos(\theta/2)}}{\cancel{\cos(\theta/2)}}=2\sin^2(\theta/2)=1-\cos\theta$$

and by virtue of these:

$$D\left(\frac{\partial u_y}{\partial y}\right)=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\ +25(A+5B)\cos(\theta/2)\left[\begin{array}{l}1-2(1-\cos\theta+\cos(2\theta))+\\ +4(1-\cos\theta)+\\ +2\cos(2\theta)\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\ +25(A+5B)\cos(\theta/2)\left[\begin{array}{l}1-2+2\cos\theta-\cancel{2\cos(2\theta)}+\\ +4-4\cos\theta\\ +\cancel{2\cos(2\theta)}\end{array}\right]\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\left\{\begin{array}{l}3(17A\text{-}83B)\cos(\theta/2)+\\ +25(A+5B)\cos(\theta/2)(3-2\cos\theta)\end{array}\right\}=$$

$$=(-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[3(17A\text{-}83B)+25(A+5B)(3-2\cos\theta)\right]=$$

$$=(-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[51A-249B+75A+375B-50(A+5B)\cos\theta\right]=$$

$$=(-\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[126A+126B-50(A+5B)\cos\theta\right]=$$

$$=(-\frac{15}{4})r^{-1/2}\cos(\theta/2)\left[(63A+63B)-25(A+5B)\cos\theta\right]$$

$$\Rightarrow \boxed{D\left(\frac{\partial u_y}{\partial y}\right)=(-\frac{15}{4})r^{-1/2}\cos(\theta/2)\left[(63A+63B)-25(A+5B)\cos\theta\right]}$$

So far, all those terms which contribute to:

$$R_q\,D\left(\frac{\partial u_q}{\partial y}\right)=R_x\,D\left(\frac{\partial u_x}{\partial y}\right)+R_y\,D\left(\frac{\partial u_y}{\partial y}\right)$$

have been obtained. Following these, the products:

$R_x\,D\left(\frac{\partial u_x}{\partial y}\right)$ and

$R_y\,D\left(\frac{\partial u_y}{\partial y}\right)$ will be acquired.

<u>Calculation of the product</u>: $R_x\,D\left(\frac{\partial u_x}{\partial y}\right)$

$$R_x D\left(\frac{\partial u_x}{\partial y}\right) =$$

$$= (-15/2)\ c\ r^{-1/2}\cos(\theta/2)\left[(13A+63B)\mu + (50A\lambda + 49A\mu - B\mu)\cos\theta\right]\cdot$$

$$\cdot(-\frac{15}{4})r^{-1/2}\sin(\theta/2)\left[(13A+313B) + 25(A+5B)\cos\theta\right] =$$

$$= (+\frac{225}{8})cr^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{matrix}(13A+63B)\mu(13A+313B) + (13A+63B)\mu 25(A+5B)\cos\theta + \\ +(50A\lambda + 49A\mu - B\mu)(13A+313B)\cos\theta + \\ +(50A\lambda + 49A\mu - B\mu)25(A+5B)\cos^2\theta\end{matrix}\right\} =$$

$$= (+\frac{225}{8})cr^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{matrix}(13A+63B)(13A+313B)\mu + \\ +\left[\begin{matrix}(13A+63B)25(A+5B)\mu + \\ +(50A\lambda + 49A\mu - B\mu)(13A+313B)\end{matrix}\right]\cos\theta + \\ +(50A\lambda + 49A\mu - B\mu)(25A+125B)\cos^2\theta\end{matrix}\right\}$$

The 3 parts of the term in brackets are:

- the first part:

$$(13A+63B)(13A+313B)\mu = \left(13^2A^2 + 13\cdot 313AB + 63\cdot 13AB + 63\cdot 313B^2\right)\mu =$$

$$= \left(169A^2 + 4888AB + 19719B^2\right)\mu$$

- the second part (multiplied by $\cos\theta$):

$$(13A+63B)(25A+125B)\mu + (50A\lambda + 49A\mu - B\mu)(13A+313B) =$$

$$= \left(13\cdot 25A^2 + 13\cdot 125AB + 63\cdot 25AB + 63\cdot 125B^2\right)\mu +$$
$$+50\cdot 13A^2\lambda + 50\cdot 313AB\lambda + 49\cdot 13AB\mu + 49\cdot 313AB\mu -$$
$$-13AB\mu - 313B^2\mu =$$

$$= \left(325A^2 + 3200AB + 7875B^2\right)\mu + \left(650A^2 + 15650AB\right)\lambda +$$
$$+\left(637A^2 + 15324AB - 313B^2\right)\mu =$$

$$= \left[(325+637)A^2 + (3200+15324)AB + (7875-313)B^2\right]\mu +$$
$$+\left(650A^2 + 15650AB\right)\lambda =$$

$$= \left(9624A^2 + 18524AB + 7562B^2\right)\mu + \left(650A^2 + 15650AB\right)\lambda$$

- the third part (multiplied by $\cos^2\theta$):

$$(50A\lambda + 49A\mu - B\mu)(25A + 125B) =$$

$$= 50 \cdot 25A^2\lambda + 50 \cdot 125AB\lambda + 49 \cdot 25A^2\mu + 49 \cdot 125AB\mu - 25AB\mu - 125B^2\mu =$$

$$= 1250A^2\lambda + 6250AB\lambda + 1225A^2\mu + (6125 - 25)AB\mu - 125B^2\mu =$$

$$= \left(1250A^2 + 6250AB\right)\lambda + \left(1225A^2 + 6100AB - 125B^2\right)\mu$$

The overall $R_x D\left(\frac{\partial u_x}{\partial y}\right)$ is:

$$R_x D\left(\frac{\partial u_x}{\partial y}\right) =$$

$$= (+\frac{225}{8})cr^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{array}{l}\left(169A^2 + 4888AB + 19719B^2\right)\mu + \\ + \left[\begin{array}{l}\left(650A^2 + 15650AB\right)\lambda + \\ + \left(9624A^2 + 18524AB + 7562B^2\right)\mu\end{array}\right]\cos\theta + \\ + \left[\begin{array}{l}\left(1250A^2 + 6250AB\right)\lambda + \\ + \left(1225A^2 + 6100AB - 125B^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}$$

<u>Calculation of the product:</u> $R_y D\left(\frac{\partial u_y}{\partial y}\right)$

$$R_y D\left(\frac{\partial u_y}{\partial y}\right) =$$

$$= (-15/2)cr^{-1/2}\sin(\theta/2)\left\{50A\lambda + 63(A\mu + B\mu) + (50A\lambda + 51A\mu + B\mu)\cos\theta\right\} \cdot$$

$$\cdot(-\frac{15}{4})r^{-1/2}\cos(\theta/2)\left[(63A + 63B) - 25(A + 5B)\cos\theta\right] =$$

$$= (+\frac{225}{8})cr^{-1}\sin(\theta/2)\cos(\theta/2)\left\{\begin{array}{l}50A\lambda(63A + 63B) - 50 \cdot 25A\lambda(A + 5B)\cos\theta + \\ + \mu(63A + 63B)^2 - 63 \cdot 25(A + B)\mu(A + 5B)\cos\theta + \\ + (50A\lambda + 51A\mu + B\mu)\cos\theta(63A + 63B) - \\ - (50A\lambda + 51A\mu + B\mu)25(A + 5B)\cos^2\theta\end{array}\right\} =$$

$$= (+\frac{225}{8}) c r^{-1} \sin(\theta/2)\cos(\theta/2) \begin{Bmatrix} 50A\lambda(63A+63B)+\mu(63A+63B)^2 + \\ + \begin{bmatrix} -50\cdot 25A\lambda(A+5B)- \\ -63\cdot 25(A+B)\mu(A+5B)+ \\ +(50A\lambda+51A\mu+B\mu)(63A+63B) \end{bmatrix} \cos\theta - \\ -(50A\lambda+51A\mu+B\mu)25(A+5B)\cos^2\theta \end{Bmatrix}$$

- The first component of this representation is:

$$50A\lambda(63A+63B)+\mu(63A+63B)^2 =$$
$$=3150(A^2+AB)\lambda+3969(A^2+2AB+B)^2\mu$$

- The second component is:

$$-50\cdot 25A\lambda(A+5B)-63\cdot 25(A+B)\mu(A+5B)+(50A\lambda+51A\mu+B\mu)(63A+63B)=$$

$$=-1250A\lambda(A+5B)-1575(A^2+5AB+AB+5B^2)\mu+$$
$$+50\cdot 63A^2\lambda+50\cdot 63AB\lambda+51\cdot 63A^2\mu+51\cdot 63AB\mu+63AB\mu+63B^2\mu=$$

$$=-1250(A^2+5AB)\lambda-1575(A^2+6AB+5B^2)\mu+$$
$$+3150A^2\lambda+3150AB\lambda+3213A^2\mu+3213AB\mu+63AB\mu+63B^2\mu=$$

$$=\left[(3150-1250)A^2+(3150\text{-}1250\cdot 5)AB\right]\lambda+$$
$$+\left[(-1575+3213)A^2+(-1575\cdot 6+3213+63)AB+(\text{-}1575\cdot 5+63)B^2\right]\mu=$$

$$=(1900A^2-3100AB)\lambda+(1638A^2-6174AB\text{-}7812B^2)\mu$$

- Finally, the third component is:

$$-(50A\lambda+51A\mu+B\mu)25(A+5B)=$$
$$=-25\left[50A^2\lambda+250AB\lambda+51A^2\mu+255AB\mu+AB\mu+5B^2\mu\right]=$$
$$=-25\left[50(A^2+5AB)\lambda+(51A^2+256AB\mu+5B^2)\mu\right]$$

All three components placed together give:

$$R_y D\left(\frac{\partial u_y}{\partial y}\right) =$$

$$= (+\frac{225}{8}) c\, r^{-1} \sin(\theta/2)\cos(\theta/2) \left\{ \begin{array}{l} 3150\left(A^2+AB\right)\lambda + 3969\left(A^2+2AB+B\right)^2 \mu + \\ + \left[\begin{array}{l} \left(1900A^2 - 3100AB\right)\lambda + \\ + \left(1638A^2 - 6174AB\text{-}7812B^2\right)\mu \end{array} \right] \cos\theta - \\ -25\left[50\left(A^2+5AB\right)\lambda + \left(51A^2 + 256AB\mu + 5B^2\right)\mu\right]\cos^2\theta \end{array} \right\}$$

2.3 Terms P_z in J_2

<u>Calculation of terms</u> $P_z\left(\frac{\partial u_z}{\partial y}\right)$

$P_z\left(\frac{\partial u_z}{\partial y}\right)$ can expanded in this manner:

$$P_z\left(\frac{\partial u_z}{\partial y}\right) = P_x\left(\frac{\partial u_x}{\partial y}\right) + P_y\left(\frac{\partial u_y}{\partial y}\right)$$

P_x and P_y are identical for J_1 and J_2.

The components of P_x, P_y that vanish at the crack-tip (r→0), after being integrated, will not be included in the products $P_x\left(\frac{\partial u_x}{\partial y}\right)$ and $P_y\left(\frac{\partial u_y}{\partial y}\right)$.

<u>Calculation of the product:</u> $P_x\left(\frac{\partial u_x}{\partial y}\right)$

It has been calculated that:

$$\frac{\partial u_x}{\partial y} = (-\frac{15}{4}) r^{1/2} \sin(\theta/2)\left[-\left(49A\text{-}251B\right) + 25\left(A+5B\right)\left(3+2\cos\theta\right)\right] =$$

$$=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[-49A+251B+75(A+5B)+50(A+5B)\cos\theta\right]=$$

$$=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[(-49+75)A+(251+375)B+(50A+250B)\cos\theta\right]=$$

$$=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[26A+626B+(50A+250B)\cos\theta\right]\Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial u_x}{\partial y}=(-\frac{15}{4})r^{1/2}\sin(\theta/2)\left[26A+626B+(50A+250B)\cos\theta\right]}$$

Forming the product $P_x\left(\frac{\partial u_x}{\partial y}\right)$, we acquire:

$$P_x\left(\frac{\partial u_x}{\partial y}\right)=$$

$$(-15/8)\,c\,r^{-3/2}\cos(\theta/2)\begin{Bmatrix}-100A\lambda+78A\mu+878B\mu+\\+(300A\lambda+294A\mu-6B\mu)\cos\theta\end{Bmatrix}\cdot$$

$$\cdot(-15/4)r^{1/2}\sin(\theta/2)\left[26A+626B+(50A+250B)\cos\theta\right]=$$

$$=(+\frac{225}{32})c\,r^{-1}\cos(\theta/2)\sin(\theta/2)\begin{Bmatrix}(-100A\lambda+78A\mu+878B\mu)(26A+626B)+\\+(-100A\lambda+78A\mu+878B\mu)(50A+250B)\cos\theta+\\+(300A\lambda+294A\mu-6B\mu)(26A+626B)\cos\theta+\\+(300A\lambda+294A\mu-6B\mu)(50A+250B)\cos^2\theta\end{Bmatrix}=$$

$$=(+\frac{225}{32})c\,r^{-1}\cos(\theta/2)\sin(\theta/2)\begin{Bmatrix}(-100A\lambda+78A\mu+878B\mu)(26A+626B)+\\+\begin{bmatrix}(-100A\lambda+78A\mu+878B\mu)(50A+250B)+\\+(300A\lambda+294A\mu-6B\mu)(26A+626B)\end{bmatrix}\cos\theta+\\+\left[(300A\lambda+294A\mu-6B\mu)(50A+250B)\right]\cos^2\theta\end{Bmatrix}$$

The multiplications are performed separately.

First part

$$(-100A\lambda + 78A\mu + 878B\mu)(26A+626B) =$$

$$= -100 \cdot 26A^2\lambda - 100 \cdot 626AB\lambda + 78 \cdot 26A^2\mu + 78 \cdot 626AB\mu +$$
$$+878 \cdot 26AB\mu + 878 \cdot 626B^2\mu =$$

$$= -2600A^2\lambda - 62600AB\lambda + 2028A^2\mu + (48828 + 22828)AB\mu + 878 \cdot 626B^2\mu =$$
$$= (-2600A^2 - 62600AB)\lambda + (2028A^2 + 71656AB + 549628B^2)\mu$$

<u>Second part (multiplied by cosθ)</u>

$$(-100A\lambda + 78A\mu + 878B\mu)(50A + 250B) + (300A\lambda + 294A\mu - 6B\mu)(26A+626B) =$$
$$= -100 \cdot 50A^2\lambda - 100 \cdot 250AB\lambda + 78 \cdot 50A^2\mu + 78 \cdot 250AB\mu + 878 \cdot 50AB\mu + 878 \cdot 250B^2\mu +$$
$$+300 \cdot 26A^2\lambda + 300 \cdot 626AB\lambda + 294 \cdot 26A^2\mu + 294 \cdot 626AB\mu - 6 \cdot 26AB\mu - 6 \cdot 626B^2\mu =$$

$$= (-5000 + 7800)A^2\lambda + (-25000 + 187800)AB\lambda +$$
$$+(3900 + 7644)A^2\mu + (19500+43900+184044 - 156)AB\mu + (219500 - 3756)B^2\mu =$$

$$= (2800A^2 + 162800AB)\lambda + (11544A^2 + 247288AB + 215744B^2)\mu$$

<u>Third part (multiplied by cos$^2\theta$)</u>

$$(300A\lambda + 294A\mu - 6B\mu)(50A + 250B) =$$
$$= 300 \cdot 50A^2\lambda + 300 \cdot 250AB\lambda + 294 \cdot 50A^2\mu + 294 \cdot 250AB\mu -$$
$$- 6 \cdot 50AB\mu - 6 \cdot 250B^2\mu =$$
$$= (15000A^2 + 75000AB)\lambda + (14700A^2 + 73200AB - 1500B^2)\mu =$$
$$= 15000(A^2 + 5AB)\lambda + (14700A^2 + 73200AB - 1500B^2)\mu$$

Concluding, the product $P_x\left(\frac{\partial u_x}{\partial y}\right)$ is:

$$P_x\left(\frac{\partial u_x}{\partial y}\right)=$$

$$=(+\frac{225}{32})c\,r^{-1}\cos(\theta/2)\sin(\theta/2)\left\{\begin{array}{l}\left(-2600A^2-62600AB\right)\lambda+\\+\left(2028A^2+71656AB+549628B^2\right)\mu+\\+\left[\begin{array}{l}\left(2800A^2+162800AB\right)\lambda+\\+\left(11544A^2+247288AB+215744B^2\right)\mu\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}15000\left(A^2+5AB\right)\lambda+\\+\left(14700A^2+73200AB-1500B^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}$$

<u>**Calculation of the product:**</u> $P_y\left(\frac{\partial u_y}{\partial y}\right)$

The partial derivative $\frac{\partial u_y}{\partial y}$ is:

$$\frac{\partial u_y}{\partial y}=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[3\left(17A\text{-}83B\right)+25\left(A+5B\right)\left(3\text{-}2\cos\theta\right)\right]=$$

$$=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[51A\text{-}249B+75\left(A+5B\right)-50\left(A+5B\right)\cos\theta\right]=$$

$$=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[\left(51+75\right)A+\left(-249+375\right)B-50\left(A+5B\right)\cos\theta\right]=$$

$$=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[126A+126B-50\left(A+5B\right)\cos\theta\right]\Rightarrow$$

$$\Rightarrow\boxed{\frac{\partial u_y}{\partial y}=(-\frac{15}{4})\,r^{1/2}\cos(\theta/2)\left[126\left(A+B\right)-50\left(A+5B\right)\cos\theta\right]}$$

The product $P_y\left(\frac{\partial u_y}{\partial y}\right)$ becomes:

$$P_y\left(\frac{\partial u_y}{\partial y}\right)=$$

$$=(-15/8)c\ r^{-3/2}\sin(\theta/2)\left[(200A\lambda+378A\mu+878B\mu)+(300A\lambda+306A\mu+6B\mu)\cos\theta\right]\cdot$$

$$\cdot(-15/4)\ r^{1/2}\cos(\theta/2)\left[126(A+B)-50(A+5B)\cos\theta\right]=$$

$$=(+\frac{225}{32})c\ r^{-1}\sin(\theta/2)\cos(\theta/2)\left\{\begin{array}{l}126(200A\lambda+378A\mu+878B\mu)(A+B)+\\+\left[\begin{array}{l}-50(200A\lambda+378A\mu+878B\mu)(A+5B)+\\+126(300A\lambda+306A\mu+6B\mu)(A+B)\end{array}\right]\cos\theta+\\+\left[-50(300A\lambda+306A\mu+6B\mu)(A+5B)\right]\cos^2\theta\end{array}\right\}$$

Separate calculations give:

→ For the first multiplication:

$$126(200A\lambda+378A\mu+878B\mu)(A+B)=$$
$$=126\left[200A^2\lambda+200AB\lambda+378A^2\mu+378AB\mu+878AB\mu+878B^2\mu\right]=$$
$$=126\left[200(A^2+AB)\lambda+(378A^2+1256AB+878B^2)\mu\right]=$$
$$=25200(A^2+AB)\lambda+(47628A^2+158256AB+110628B^2)\mu$$

→ For the second multiplication:

$$-50(200A\lambda+378A\mu+878B\mu)(A+5B)+126(300A\lambda+306A\mu+6B\mu)(A+B)=$$

$$=-50(200A^2\lambda+1000AB\lambda+378A^2\mu+1890AB\mu+878AB\mu+4390B^2\mu)+$$
$$+126(300A^2\lambda+300AB\lambda+306A^2\mu+306AB\mu+6AB\mu+6B^2\mu)=$$

$$=-50(200A^2\lambda+1000AB\lambda+378A^2\mu+2768AB\mu+4390B^2\mu)+$$
$$+126(300A^2\lambda+300AB\lambda+306A^2\mu+312AB\mu+6B^2\mu)=$$

$$=\left[(-10000+37800)A^2+(-50000+37800)AB\right]\lambda+$$
$$+\left[(-18900+38856)A^2+(-138400+39312)AB+(-219500+756)B^2\right]\mu=$$
$$=(27800A^2-12200AB)\lambda+(19656A^2-99088AB-218744B^2)\mu$$

→ For the third multiplication:

$$-50\left(300\mathrm{A}\lambda+306\mathrm{A}\mu+6\mathrm{B}\mu\right)\left(\mathrm{A}+5\mathrm{B}\right)=$$
$$=-50\left(300\mathrm{A}^2\lambda+1500\mathrm{AB}\lambda+306\mathrm{A}^2\mu+1530\mathrm{AB}\mu+6\mathrm{AB}\mu+30\mathrm{B}^2\mu\right)=$$
$$=-50\left(300\mathrm{A}^2\lambda+1500\mathrm{AB}\lambda+306\mathrm{A}^2\mu+1536\mathrm{AB}\mu+30\mathrm{B}^2\mu\right)=$$
$$=-15000\left(\mathrm{A}^2+5\mathrm{AB}\right)\lambda-\left(15300\mathrm{A}^2+76800\mathrm{AB}+1500\mathrm{B}^2\right)\mu$$

Combining these three results, the final form for the product into consideration is:

$$\mathrm{P}_y\left(\frac{\partial u_y}{\partial y}\right)=$$
$$=\left(+\frac{225}{32}\right)\mathrm{c}\ \mathrm{r}^{-1}\sin(\theta/2)\cos(\theta/2)\left\{\begin{array}{l}\left[\begin{array}{l}25200\left(\mathrm{A}^2+\mathrm{AB}\right)\lambda+\\+\left(47628\mathrm{A}^2+158256\mathrm{AB}+110628\mathrm{B}^2\right)\mu\end{array}\right]+\\+\left[\begin{array}{l}\left(27800\mathrm{A}^2-12200\mathrm{AB}\right)\lambda+\\+\left(19656\mathrm{A}^2-99088\mathrm{AB}-218744\mathrm{B}^2\right)\mu\end{array}\right]\cos\theta+\\+\left[\begin{array}{l}-15000\left(\mathrm{A}^2+5\mathrm{AB}\right)\lambda-\\-\left(15300\mathrm{A}^2+76800\mathrm{AB}+1500\mathrm{B}^2\right)\mu\end{array}\right]\cos^2\theta\end{array}\right\}\Rightarrow$$

2.4 Integrations in J_2

J_2 can now be calculated, by integrating its parts in the interval [-π, π].

Integration of *W* terms in J_2

The terms consisting *W* are multiplied by sin*θ*.

Term W1

This integral is zero, due to the order of r (vanishes at crack-tip).

Term W2

This integral is zero, too, for the same reason.

Term W3

$$\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]\sin\theta d\Gamma=\int_{-\pi}^{\pi}\left(140625A^2\ r^{-1}\right)r\sin\theta\, d\theta=$$

$$=140625A^2\int_{-\pi}^{\pi}\sin\theta\ d\theta=-140625A^2\left[\cos\theta\right]_{-\pi}^{\pi}=-140625A^2\left[\cos\pi\text{-}\cos(\text{-}\pi)\right]=$$

$$=140625A^2\left[\cos\pi\text{-}\cos\pi\right]=0\Rightarrow$$

$$\Rightarrow\boxed{\int_{-\pi}^{\pi}\left[\left(\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial\varepsilon_{yy}}{\partial x}\right)^2+\left(\frac{\partial\varepsilon_{xx}}{\partial y}+\frac{\partial\varepsilon_{yy}}{\partial y}\right)^2\right]\sin\theta d\Gamma=0}$$

Term W4

Term W4(a)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2 r\sin\theta\, d\theta=$$

$$=\int_{-\pi}^{\pi}\left((\text{-}\frac{15}{8})r^{-1/2}\cos(\theta/2)\left[50\left(A+5B\right)\left(\cos\theta-\cos(2\theta)\right)+124\left(A+B\right)\right]\right)^2 r\ \sin\theta d\theta=0\Rightarrow$$

$$\Rightarrow\boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xx}}{\partial x}\right)^2 r\ \sin\theta d\theta=0}$$

Term W4(b)

$$\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2 r\ \sin\theta d\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(\text{-}\frac{15}{8}\right)r^{-1/2}\sin(\theta/2)\left[\text{-}50\left(A+5B\right)\left(\cos\theta+\cos(2\theta)\right)+\left(24A+124B\right)\right]\right)^2 r\ \sin\theta d\theta=0\Rightarrow$$

$$\Rightarrow\boxed{\int_{-\pi}^{\pi}\left(\frac{\partial\varepsilon_{xy}}{\partial x}\right)^2 r\ \sin\theta d\theta=0}$$

Term W4(c)

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(+\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left[50(\mathrm{A}+5\mathrm{B})(\cos\theta-\cos(2\theta))+(-76\mathrm{A}+124\mathrm{B})\right]\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0}$$

<u>Term W4(d)</u>

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(24\mathrm{A}\text{-}376\mathrm{B})\right]\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0}$$

<u>Term W4(e)</u>

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(-\frac{15}{8}\right)\mathrm{r}^{-1/2}\cos(\theta/2)\left[50(\mathrm{A}+5\mathrm{B})(-\cos\theta+\cos(2\theta))+(76\mathrm{A}+376\mathrm{B})\right]\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{xy}}{\partial y}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0}$$

<u>Term W4(f)</u>

$$\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=$$

$$=\int_{-\pi}^{\pi}\left(\left(\frac{15}{8}\right)\mathrm{r}^{-1/2}\sin(\theta/2)\left[-50(\mathrm{A}+5\mathrm{B})(\cos\theta+\cos(2\theta))+(-176\mathrm{A}\text{-}376\mathrm{B})\right]\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi}\left(\frac{\partial \varepsilon_{yy}}{\partial y}\right)^2 \mathrm{r}\sin\theta \mathrm{d}\theta=0}$$

We observe that *W* consists of zero terms.

Integration of R_g terms in J_2

$$\int_{-\pi}^{\pi} R_x D\left(\frac{\partial u_x}{\partial y}\right) d\Gamma = \int_{-\pi}^{\pi} R_x D\left(\frac{\partial u_x}{\partial y}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} (+\frac{225}{8}) c r^{-1} \cos(\theta/2)\sin(\theta/2) \left\{ \begin{array}{l} \left(169A^2 + 4888AB + 19719B^2\right)\mu + \\ + \left[\begin{array}{l} \left(650A^2 + 15650AB\right)\lambda + \\ + \left(9624A^2 + 18524AB + 7562B^2\right)\mu \end{array}\right] \cos\theta + \\ + \left[\begin{array}{l} \left(1250A^2 + 6250AB\right)\lambda + \\ + \left(1225A^2 + 6100AB - 125B^2\right)\mu \end{array}\right] \cos^2\theta \end{array} \right\} r\, d\theta = 0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} R_x D\left(\frac{\partial u_x}{\partial y}\right) d\Gamma = 0}$$

and

$$\int_{-\pi}^{\pi} R_y D\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = \int_{-\pi}^{\pi} R_y D\left(\frac{\partial u_y}{\partial y}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} (+\frac{225}{8}) c r^{-1} \sin(\theta/2)\cos(\theta/2) \left\{ \begin{array}{l} 3150\left(A^2 + AB\right)\lambda + 3969\left(A^2 + 2AB + B\right)^2 \mu + \\ + \left[\begin{array}{l} \left(1900A^2 - 3100AB\right)\lambda + \\ + \left(1638A^2 - 6174AB - 7812B^2\right)\mu \end{array}\right] \cos\theta - \\ -25\left[50\left(A^2 + 5AB\right)\lambda + \left(51A^2 + 256AB\mu + 5B^2\right)\mu\right] \cos^2\theta \end{array} \right\} r\, d\theta = 0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} R_y D\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = 0}$$

Integration of P_z terms in J_2

$$\int_{-\pi}^{\pi} P_x\left(\frac{\partial u_x}{\partial y}\right) d\Gamma = \int_{-\pi}^{\pi} P_x\left(\frac{\partial u_x}{\partial y}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} \left(+\frac{225}{32}\right) c\, r^{-1} \cos(\theta/2)\sin(\theta/2) \left\{ \begin{array}{l} \left(-2600A^2 - 62600AB\right)\lambda + \\ +\left(2028A^2 + 71656AB + 549628B^2\right)\mu + \\ + \left[\begin{array}{l} \left(2800A^2 + 162800AB\right)\lambda + \\ +\left(11544A^2 + 247288AB + 215744B^2\right)\mu \end{array} \right] \cos\theta + \\ + \left[\begin{array}{l} 15000\left(A^2 + 5AB\right)\lambda + \\ +\left(14700A^2 + 73200AB - 1500B^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} r\, d\theta = 0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} P_x\left(\frac{\partial u_x}{\partial y}\right) d\Gamma = 0}$$

and finally,

$$\int_{-\pi}^{\pi} P_y\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = \int_{-\pi}^{\pi} P_y\left(\frac{\partial u_y}{\partial y}\right) r\, d\theta =$$

$$= \int_{-\pi}^{\pi} \left(+\frac{225}{32}\right) c\, r^{-1} \sin(\theta/2)\cos(\theta/2) \left\{ \begin{array}{l} \left[\begin{array}{l} 25200\left(A^2 + AB\right)\lambda + \\ +\left(47628A^2 + 158256AB + 110628B^2\right)\mu \end{array} \right] + \\ + \left[\begin{array}{l} \left(27800A^2 - 12200AB\right)\lambda + \\ +\left(19656A^2 - 99088AB - 218744B^2\right)\mu \end{array} \right] \cos\theta + \\ + \left[\begin{array}{l} -15000\left(A^2 + 5AB\right)\lambda - \\ -\left(15300A^2 + 76800AB + 1500B^2\right)\mu \end{array} \right] \cos^2\theta \end{array} \right\} r\, d\theta = 0 \Rightarrow$$

$$\Rightarrow \boxed{\int_{-\pi}^{\pi} P_y\left(\frac{\partial u_y}{\partial y}\right) d\Gamma = 0}$$

Noticing these results, we see that all terms in J_2 are zero! This means that there is no energy dissipation in the direction which is perpendicular to the direction of the crack advancement!

$$\boxed{J_2 = 0}$$

APPENDIX 4

TRIGONOMETRIC IDENTITIES AND INTEGRALS

Some useful trigonometric identities and integrals are listed in this Appendix.

Trigonometric identities

Most of these trigonometric identities have already been in the calculations for the J_1 and J_2-Intgrals. Their full proofs are shown in the Appendixes containing the explicit calculations for J_1 and J_2 (Appendixes 2 and 3).

$$\cos(a+b)=\cos a\cos b-\sin a\sin b\,,\;\cos(a-b)=\cos a\cos b+\sin a\sin b$$

$$\sin(a+b)=\sin a\cos b+\cos a\sin b\,,\;\sin(a-b)=\sin a\cos b-\cos a\sin b$$

$$\sin(2\theta)=2\sin\theta\cos\theta\;,\;\cos(2\theta)=\cos^2\theta-\sin^2\theta$$

$$\cos^2\theta=\frac{1+\cos(2\theta)}{2}\,,\;\sin^2\theta=\frac{1-\cos(2\theta)}{2}$$

$$\frac{\sin\theta\sin(3\theta/2)}{\cos(\theta/2)}=\cos\theta-\cos(2\theta)\,,\qquad\frac{\sin\theta\cos(3\theta/2)}{\sin(\theta/2)}=\cos\theta+\cos(2\theta)$$

$$\frac{\sin\theta\sin(\theta/2)}{\cos(\theta/2)}=\sin\theta\tan(\theta/2)=1-\cos\theta\,,\qquad\frac{\cos\theta\cos(3\theta/2)}{\cos(\theta/2)}=1-\cos\theta+\cos(2\theta)$$

$$\frac{\cos\theta\sin(3\theta/2)}{\sin(\theta/2)}=1+\cos\theta+\cos(2\theta)\,,\qquad\frac{\sin\theta\sin(3\theta/2)}{\sin(\theta/2)}=\sin\theta+\sin(2\theta)$$

$$\frac{\sin\theta\cos(\theta/2)}{\sin(\theta/2)}=1+\cos\theta$$

$$\frac{\sin(3\theta/2)}{\sin(\theta/2)} = 1 + 2\cos\theta \ , \qquad \frac{\cos(3\theta/2)}{\cos(\theta/2)} = -1 + 2\cos\theta$$

$$\frac{\sin(5\theta/2)}{\sin(\theta/2)} = 1 + 2\cos\theta + 2\cos(2\theta) , \quad \frac{\cos(5\theta/2)}{\cos(\theta/2)} = 1 - 2\cos\theta + 2\cos(2\theta)$$

$$\sin(\theta/2)\sin(3\theta/2) = \frac{1}{2}\left(\cos\theta - \cos(2\theta)\right), \ \cos(\theta/2)\cos(3\theta/2) = \frac{1}{2}\left(\cos\theta + \cos(2\theta)\right)$$

$$\cos^2(\theta/2)\cos\theta = \frac{1}{4}\left(1 + 2\cos\theta + \cos(2\theta)\right), \ \sin^2(\theta/2)\cos\theta = \frac{1}{4}\left(-1 + 2\cos\theta - \cos(2\theta)\right)$$

$$\cos\theta\cos(2\theta) = \frac{1}{2}\left(\cos\theta + \cos(3\theta)\right), \ \cos(2\theta)\cos(4\theta) = \frac{1}{2}\left(\cos(2\theta) + \cos(6\theta)\right)$$

$$\cos(3\theta/2) + \cos(\theta/2) = 2\cos\theta\cos(\theta/2), \ \cos(3\theta/2) - \cos(\theta/2) = -2\sin\theta\sin(\theta/2)$$

$$\sin(3\theta/2) + \sin(\theta/2) = 2\sin\theta\cos(\theta/2), \ \sin(3\theta/2) - \sin(\theta/2) = 2\cos\theta\sin(\theta/2)$$

$$\cos(3\theta/2) - \cos(5\theta/2) = 2\cos(\theta/2)\left(-1 + 2\cos\theta - \cos(2\theta)\right)$$

Trigonometric integrals

The following trigonometric integrals appear in the calculations for J_1 and J_2. They have been calculated using *Mathematica*.

Integrals for J_1

$$\int_{-\pi}^{\pi} \cos^2(\theta/2)\cos\theta \, d\theta = \frac{\pi}{2} \ , \quad \int_{-\pi}^{\pi} \left[\cos(\theta/2)\left(\cos\theta - \cos(2\theta)\right)\right]^2 \cos\theta \, d\theta = \frac{\pi}{8}$$

$$\int_{-\pi}^{\pi} \sin^2(\theta/2)\cos\theta \, d\theta = -\frac{\pi}{2} \ , \quad \int_{-\pi}^{\pi} \left[\sin(\theta/2)\left(\cos\theta + \cos(2\theta)\right)\right]^2 \cos\theta \, d\theta = -\frac{\pi}{8}$$

$$\int_{-\pi}^{\pi} \left[\cos(\theta/2)\left(-\cos\theta + \cos(2\theta)\right)\right]^2 \cos\theta \, d\theta = \frac{\pi}{8}$$

$$\int_{-\pi}^{\pi} \cos^2(\theta/2)\,d\theta = \frac{\pi}{2}, \quad \int_{-\pi}^{\pi} \cos^2(\theta/2)\cos^2\theta\,d\theta = \frac{\pi}{2}$$

$$\int_{-\pi}^{\pi} \sin^2(\theta/2)\,d\theta = \pi, \quad \int_{-\pi}^{\pi} \sin^2(\theta/2)\cos\theta\,d\theta = -\frac{\pi}{2}, \quad \int_{-\pi}^{\pi} \sin^2(\theta/2)\cos^2\theta\,d\theta = \frac{\pi}{2}$$

Integrals for J_2

$$\int_{-\pi}^{\pi} \cos^2(\theta/2)\sin\theta\,d\theta = 0, \quad \int_{-\pi}^{\pi} \left[\cos(\theta/2)\left(\cos\theta - \cos(2\theta)\right)\right]^2 \sin\theta\,d\theta = 0$$

$$\int_{-\pi}^{\pi} \sin^2(\theta/2)\sin\theta\,d\theta = 0, \quad \int_{-\pi}^{\pi} \left[\sin(\theta/2)\left(\cos\theta + \cos(2\theta)\right)\right]^2 \sin\theta\,d\theta = 0$$

$$\int_{-\pi}^{\pi} \cos(\theta/2)\sin(\theta/2)\,d\theta = \frac{\pi}{2}, \quad \int_{-\pi}^{\pi} \cos(\theta/2)\sin(\theta/2)\cos\theta\,d\theta = 0$$

$$\int_{-\pi}^{\pi} \cos(\theta/2)\sin(\theta/2)\cos^2\theta\,d\theta = 0$$