Throughput of Wireless Relay Networks via Minimum Evacuation Times and Index Coding.

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1 Introduction

Source Coding is the central objective in Information Theory and Coding Theory. Its concern is to code a source of symbols with the minimum number of bits without loss of meaningful information. Shannon proved that the minimum number of bits necessary and sufficient to code a source is equal to the source's Shannon Entropy (up to one bit) in [1]. But this bound can be improved if one takes into account the prior information (a.k.a side information) that already exists in the receivers about the packets. One variant of Source Coding with side information is the Index Coding problem first proposed by Birk and Kol in [2]. In this context, a relay disseminates a set of packets over a broadcast channel to a set of caching receivers. Each receiver has only a subset of the packets cached. It also wants a certain subset of the packets that are not in its cache. The relay knows what packets each receiver has. The objective is to minimize the number of packets needed so that every receiver gets the packets it wants. Another closely related problem is Network Coding, first proposed by Ahlswede, Cai, Li, and Yeung in [3]. Network Coding can be defined as any coding that takes place at a node in a packet network between the contents of packets [4]. Index Coding therefore might seem to be a special case of Network coding, but in reality they are equivalent ([32]). This work therefore can be seen as an extension to both problems, but it is centered around Network Coding models.

Consider the example of Fig. 1 where we show an instance of the Wireless Network Coding problem for $N = 3$ receivers. Three non-symmetric flows are defined, $f_1 : s1 \to r1$, $f_2 : s2 \to r2$ and $f_3 : s3 \to r3$ where $s1, s2, s3$ are the sources and $r1, r2, r3$ are the destination receivers of the flow. All flows use the intermediate node R as a forwarder, which employs interflow network coding by XORing packets from the two flows. The receivers utilize the overhearing erasure channels to obtain side information, i.e. packets destined to the other receivers. For example, $r1$ receives packets destined to $r2$ and $r3$, with probability p_{21} and p_{31}, whenever the $s2$ and $s3$ attempt to upload them to R. We focus on the downlink part which entails the complexity of the problem; node R must make coding and scheduling decisions in order to achieve some objectives, e.g. maximize throughput. This model can be generalized for an arbitrary number of flows. In this context, we will call **2-user ACK** the system where $N = 2$ and node R learns the content of the decoding buffers of $1, 2$ via explicit reports that follow each overhearing event. We will call **2-**

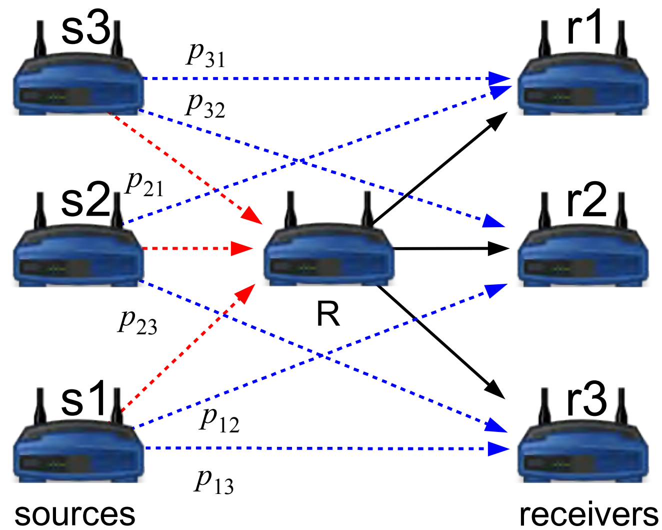

Figure 1: An instance of the Wireless Network Coding problem for $N = 3$. Arrows show wireless links. Solid arrows show the broadcast channel under study; the relay/transmitter must choose a sequence of coded transmissions that guarantees the decoding of all the requested packets. The receivers can use side information obtained through overhearing (long dotted arrows).

user NACK the system where $N = 2$ and the decisions are made based on the probabilities of the overhearing channels and feedback reports following each unsuccessful attempt. Finally, we call **multi-user ACK based on Index Coding** the system where N is arbitrary, the node R learns the content of the decoding buffers of the receivers via explicit reports that follow each overhearing event and the relay can only transmit one packet per timeslot. This formulation is called 1-hop model.

2 Contribution

We study the case of one node broadcasting coded transmissions and a number of receivers having side information. We allow the receivers to store any received or overheard packet (either native or coded) and use it in the future for decoding purposes.

(i) We give an outer bound for the throughput region of the 2-user ACK system assuming general coding (including non-linear coding)-the equivalent information theoretic capacity region is shown in [17]. We show that this region can be achieved by simple XOR policies which operate without knowledge about arrivals.

(ii) For the 2-user NACK system, we give in closed-form the code-constrained throughput region assuming the use of XOR coding. We propose a simple evacuation coding policy which achieves it. We also find a case were this region is identical to the throughput region of the 2-user ACK system.

(iii) For the case of the multi-user ACK based Index Coding system, we show that there is no loss of optimality if we ignore codes that involve intra-flow coding. We also show that the problem of finding an optimal Linear Coding scheme can be decomposed to a number of subproblems.

3 Related Work

3.1 Practical Work

In [6] the authors introduced the first experimental setup of Wireless Network Coding, called COPE . As an architecture, it employed three modes of functionality. *Opportunistic Listening* where each network node can overhear unrelated packet transmissions and use these packets for future decoding. *Opportunistic Coding* where the relay when choosing a transmission control would maximize n such that all n nexthops can immediately decode the packet with simple XOR operations. Finally, *Learning Neighbor State* where the relay would try to predict the states of its nexthops by both direct reception reports and by probability estimations. In [7] XOR_Sym is proposed which employs a more adaptive scheme than the above greedy approach and by constraining the decoding of packets at their destination only, they achieve similar results with COPE but with less complexity in the intermediate nodes. This protocol considers only symmetric flows disregarding opportunistic listening. In [8] a set of algorithms that employ redundant transmissions over lossy environments called CLONE is proposed. MORE [9] introduces intraflow Network Coding in its architecture i.e coding between packets of the same flow. noCoCo [10] integrates per-hop packet scheduling, Network Coding and congestion control in a deterministic packet scheduling scheme within two-way multihop traffic flows. All the above schemes suffer from the problem of keeping the coding nodes informed with the packet indices that have been overheard by its nexthops. NCRAWL [11] addresses that by using reception reports only when a node can't decode a packet. This reduces the amount of reports, but increases the likelihood that the coding node will make a mistake in predicting which packets are overheard by the nexthops, something that inevitably lowers throughput. We use this model of reporting in the 2-user NACK model. [12] introduces I^2NC which combines interflow with intraflow coding to reduce the complexity of acknowledgment messages at the expense of immediate decodability.

3.2 Theoretical Work

Stability in networks with interflow network coding without overhearing is studied in [7] and [13]- [16]. Also, in [17], [18] the studies are extended to capture overhearing with reports, which corresponds to the 2-user ACK system. Note that in these works, the code-constrained stability region is provided, i.e. the stability region under the assumption that XOR coding is used. The 1-hop model is also studied in [19] where the information theoretic capacity is given in the case of overhearing events provided as side information- a model equivalent to the 2-user ACK system. With the exception of [18], all these works do not consider the 2-user NACK system. In [18], the 2-user NACK system with feedback is studied under the assumption that receivers are not allowed to store coded packets and the code-constrained throughput region is provided in parametric form. The obtained throughput region is strictly smaller than that of the 2-user ACK system. In this work, we extend [18] by allowing the storage of coded packets. We show, that if r1 = r2, then the 2-user NACK system can achieve the same throughput as the 2-user ACK one by the use of a simple XOR-based scheduling policy and feedback reports. Thus, the number of reports can be reduced significantly in this case without throughput losses. Studies of the broadcast channel with erasures, i.e. see [20], relate to our work. In these studies, the problem is different since the side information for decoding is obtained from past erased transmissions; however, the techniques used are similar. In [21], the authors show that the capacity can be achieved by XOR coding for the case of 2-4 receivers. A different but related research topic is that of index coding; subsets of information bits are known to subsets of the receivers and we seek the transmission policy that minimizes the time to complete reception by all receivers, [22], [23]. Our work differs from index coding in the fact that the source has partial knowledge of what information each receiver has. Also, for the 2-user ACK system, we extend the index coding problem to variable transmission rates r1; r2. Previous work has shown that in practical wireless networks, where the locations of the nodes are random, the vast majority of interflow coding opportunities involve a small number of nodes, [16], [24], [25]. This motivates the study of simple schemes with a small number of receivers, which can be solved efficiently. In this spirit, we provide optimal solutions that utilize simple XOR operations, require minimal information about system state, are oblivious to arrivals and can be embraced by resource limited wireless devices.

3.3 Index Coding

In [2] the Index Coding problem is first introduced. In this context, a relay disseminates a set of packets over a broadcast channel to a set of caching receivers. Each receiver has only a subset of the packets cached. It also wants a certain subset of the packets that are not in its cache. The relay knows what packets each receiver has. The objective is to minimize the number of packets needed so that every receiver gets the packets it wants. The authors propose a number of algorithms that offer suboptimal solutions to the problem, like finding the minimum clique cover or the maximum matching of the side information graph. Also they find a bounds on the maximum number of matchings and how many transmissions are saved by a matching algorithm relative to no coding. In [5], the authors find the optimal number of transmissions if Linear Index Coding(LNC) is used given by $minrk_2(G) \equiv min\{rk_2(A)$: matrix A fits $G\}$ where G is the side information graph and $rk_2(A)$ is the rank of matrix A over $GF(2)$. Also they prove this number is optimal for *arbitrary* index codes if the graph is a DAG, a perfect graph, an odd-hole or an odd anti-hole. In [22], the authors reduce the problem to SAT and use Tstein transformation to minimize the number of variables in the subsequent CNF. They also propose a number of heuristics for suboptimal but fast solutions for a large number of receivers. In [27], the authors try to combine the bandwith performance of LNC with the fast decoding performance of XOR coding by using a scheme where linear codes are sent but are decoded with simple backward substitution. This technique is called Triangular Network Coding and to be successful it uses a number of header bits equal to $M + Mlog_2(M)$ where M is the number of packets to be disseminated. Also, their model presupposes that the packets in the relay are required by all receivers. In [28], it was shown that the Index Coding problem is essentially an Interference Alignment problem, i.e. the coded transmissions must align with the side information on each receiver so that the decoding of the particular packet is guaranteed. Moreover, they use it to prove that the Multiple Unicast Index Coding problem (each packet in the relay is required by exactly one receiver) and Multiple Groupcast Index Coding problem (each packet may be required by multiple receivers) are equivalent. In [30], an achievable LNC bound is found based on the local chromatic number, a graph property defined in [31]. In [32] it is found that the Index Coding, the Network Coding (for Directed Acyclic Graphs) and the Matroid Representation problems are equivalent. Although there exist several results in the literature dealing with special cases, the general solution of the problem is not known and moreover it has been shown in [29] that linear coding is not in general sufficient for achieving the capacity.

4 Model description

4.1 Abstract Network Model

We will first describe the part of the model that is common to all three systems we study. Consider a broadcast network with one transmitting node called the **relay** (coding node) and n **receivers**. The time is slotted, where slot t occupies the time interval $[t, t+1)$. At the beginning of each slot, packets arrive at the relay, each one belonging to a network **flow**. The packets of a single flow are all destined to a single receiver i.e each receiver is the destination of a single flow. We assume that all packets consist of L bits. The bits are i.i.d. with uniform distribution.

Packets **arrive** with the following property: whenever a packet of flow i destined to receiver i arrives at the relay, a copy of it arrives at another receiver $j \in [n] - \{i\}$ according to a probability distribution P_{ij}. This probability corresponds to random **overhearing** events which are independent from one another. The packets arrive according to a stochastic arrival process with rate $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_n\}, \;\; \lambda_i \;\; \forall i \in [n]$. We assume i.i.d. packet arrivals within each slot.

The relay stores arriving packets in the input **queues**, while the receivers store packets useful for decoding in the decoding **buffers**. It also has some information (definite or statistical) about the packets that the other nodes have in their buffers.

At the beginning of each slot t the relay chooses a **rate** of transmission r^t between a set of integer **rates** $r^t \in R \equiv \{r_1 \dots r_n\}$ and a **control** decision c^t, i.e a choice of a XOR combination of the packets of W^t to be sent as a single packet during each transmission in the slot. This means that during the slot, r^t packets will be transmitted. If there are not enough packets to be transmitted, **dummy** packets are used to fill in this number. Moreover, each receiver i successfully retrieves the packets if $r^t \leq r_i$, otherwise they are discarded.

In order to study the stability of the described model, we consider an operation of the system, which is based on evacuating system **snapshots**. We assume an initial snapshot with $\boldsymbol{k} = \{k_1, \dots, k_n\}$ packets in each queue, where the packets have arrived following the rules explained above regarding overhearing. We define an admissible **evacuation policy** π as a sequence of eligible control actions at the end of which all packets in the queues have been decoded by their respective receivers. Until this happens, no new arrivals enter the queues and their corresponding overhearing events do not enter in the buffers of the receivers. The time interval until the successful decoding of all packets in queues by their destination receiver is called an **epoch**. After the first epoch, all new arrivals and overhearings that happened during the epoch enter the system in the queues and buffers respectively whereas all native and coded packets of the first epoch **depart** the system. Then the process begins again in a new snapsot defined by the new arrivals. Note, that each evacuation policy can be mapped to an epoch-based policy $\sigma(\pi)$, which is admissible in the system with arrivals and evacuates all packets present in the system at the beginning of each epoch using π, see [34]. Next, we follow the steps of [34].

4.2 Stability Considerations

Consider the set of queues at the coding node, denoted Q. Denote the sum of backlogs of queues in Q under policy σ at the end of time slot t as $X_i^{\sigma}(t)$. As in [34], we say that the system is stable if

$$\lim_{q\to\infty} \limsup_{t\to\infty} \Pr\left(X_i^{\sigma}(t) > q\right) = 0.$$

Note that the definition of stability does not include the buffers. Due to the definition of departures, though, stability of queues implies stability for the buffers.

Consider the set of all vectors $\boldsymbol{\lambda}$ for which the system is stable under policy σ; the closure of this set denoted by Λ^{σ} is called the **stability region** of the policy σ. The region $\Lambda \triangleq \cup_{\sigma}\Lambda^{\sigma}$ characterizes the system and is called the **throughput region**. In case we constrain the allowable set of codes (e.g. to XOR only) we will refer to the corresponding region as the **code-constraint throughput region**, see [23].

We expect the code-constraint region of the 2-user NACK system to be a subset of the throughput region of the 2-user ACK due to the partial information available at the coding node and the restriction to XORing.

Let Π be the set of all evacuation policies. We denote with $T^{\pi}(\boldsymbol{k})$ the **evacuation time** of policy $\pi \in \Pi$, which is the minimum number of slots required to empty the system queues under policy π. We denote with $\overline{T}^{\pi}(\boldsymbol{k}) \triangleq \mathbb{E}[T^{\pi}(\boldsymbol{k})]$ the average evacuation time of this policy over the number and 'kind' of the random overhearing events and with $\overline{T}^{\star}(\boldsymbol{k}) \triangleq \inf_{\pi\in\Pi}\{\overline{T}^{\pi}(\boldsymbol{k})\}$ the minimum average evacuation time over all the policies. By 'kind' here we mean any element of $e \in \mathcal{P}([n])$ where e denotes the set of the indices of receivers that overheard a packet.

Lemma 1 [Subadditivity and linear growth]: *The function $\overline{T}^{\star}(\boldsymbol{k})$ is subadditive, is upper bounded by a linear function and the following limit exists*

$$\hat{T}(\boldsymbol{\lambda}) = \lim_{t\to\infty} \frac{\overline{T}^{\star}(\lceil t\boldsymbol{\lambda}\rceil)}{t}.$$

Proof. In [34], Lemma 1 is shown under a general class of policies, provided that these policies have certain Features and under some Assumptions on System operation, all of which hold trivially in our problem.

Most of them hold trivially in our system. Assumption 5) in [34] can be verified by considering a simple policy that evacuates all packets in the system in a random order using native transmissions. This policy evacuates the system in exactly $\left\lceil \frac{k_1}{r_1} \right\rceil + \left\lceil \frac{k_2}{r_2} \right\rceil < \frac{k_1}{r_1} + \frac{k_2}{r_2} + 2$ slots, thus 5) is satisfied. The same policy can be used to show 6).Then Lemma 1 follows from Lem. 1 and Th. 2 in [34]. □

Proposition 2 [Throughput region via evacuation times from [34]]: *The throughput region of the system is the set of rates $\boldsymbol{\lambda} \geq 0$ satisfying*

$$\hat{T}(\boldsymbol{\lambda}) \leq 1.$$

Continuing on, by choosing the number of receivers, the rates and the information that is available to the relay about the buffers of each receiver, we differentiate between the three particular models below.

4.3 2-user ACK model

Here we have $n = 2$ with rates $R = \{r_1, r_2\}$ and the relay knows exactly the buffers of each receiver. This knowledge is achieved through ACK reports sent by the receiver to the relay when the overhearing event took place through a separate channel. This method of overhearing allows the relay to make completely informed decisions. On the other hand, such reports are costly to throughput and its not scalable to the number of receivers, where the number of such reports becomes immense. In this model, we assume that a packet and all coded packets that contain it depart the system when the packet is successfully decoded by its destination receiver. This is the model of feedback used in [6].

All packets that are both in the relay's queue and in the opposite receiver's buffer (i.e those not coded) are called **good** packets due to their ability to be efficiently combined. All other packets (i.e those not overheard) are called **bad**. Note, that since overhearing takes place only upon arrival, the categorization of good/bad does not change during the lifetime of a packet. Classifying the packets of the relay according to which receiver they are destined, and whether they are good/bad, we use four queues to classify them upon arrival, named $\mathrm{g}_1, \mathrm{b}_1, \mathrm{g}_2, \mathrm{b}_2$. The control set is then defined as:

$$\mathcal{C}_{det} \triangleq \{g_1, b_1, g_2, b_2, g_1 \oplus g_2\},$$

where, for example, control g_1 denotes the transmission of r_1 packets from queue g_1. The control $\{g_1 \oplus g_2\}$ is directed to both receivers (sent at rate $\min\{r_1, r_2\}$) and the controls $\{g_i, b_i\}$ are directed to receiver i (sent at rate r_i), $i = 1, 2$. Note, that we omit controls that apply XORs on bad packets. Although this is a constrained control set, we will show that optimal performance can be achieved using this set.

The state of the system at time slot t is $S_{det}(t) = (k_1, k_2, n_1, n_2)$, where k_1 (k_2) is the number of packets destined to receiver 1 (2), and n_1 (n_2) is the number of packets in $H_1^t(H_2^t)$.

The above formulation allows us to give a more precise definition for the term **policy** for this system. A policy is a mapping from system state at the beginning of slot t to a control $c^t \in \mathcal{C}_{det}$, which corresponds to $r^t \in \{r_1, r_2\}$ transmissions, where r^t is determined by the chosen control. It is convenient to denote with $\mathcal{C}_{det}(t) \subseteq \mathcal{C}_{det}$ a subset of the control set with the property that the member controls correspond to non-empty queues. If for some t we have $\mathcal{C}_{det}(t) = \emptyset$, then clearly the system queues are empty.

4.4 2-user NACK model

Here we have $n = 2$ with rates $R = \{r_1, r_2\}$ and for both queues in the relay, there is an associated **probability** p_1 and p_2 respectively. p_1 is the probability that a packet of the first flow is in the buffers of the receiver of the second flow and likewise for p_2. These probabilities are known to the relay. This knowledge is achieved through statistical analysis. Like in the ACK case, we distinguish between **good** and **bad** packets. Moreover, if the receiver fails to decode a sent packet, a NACK packet is sent back to the relay which in turn can determine with certainty whether a constituent packet of the combination is in the receiver's buffer or not. In this model, we assume that a packet and all coded packets that contain it depart the system when the packet is successfully decoded by its destination receiver. This method of overhearing allows the relay to make partially informed decisions. On the other hand, the number of reports needed are reduced. This is the model of feedback used in [11].

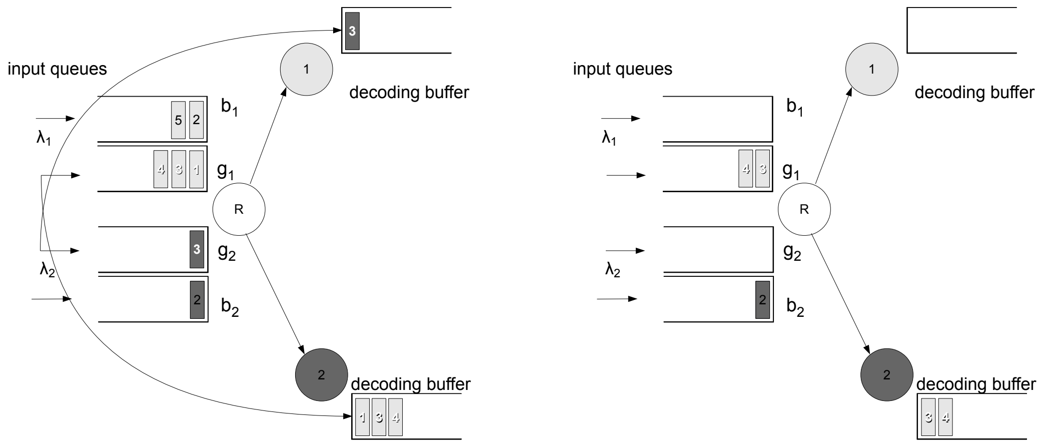

Figure 2: Two snapshots of the 2-User ACK system. (left) Packets from two unicast flows arrive at the coding node R and are destined to two different receivers. Due to side overhearing channels, a copy of the arriving packet, destined to one receiver, also arrives at the other with a probability. The packets are classified by the coding node as good (overheard) or bad (not overheard) at the time they arrive. (right) System after the controls $g_1 \oplus g_2$ and b_1 are used with $r = 2$. During the first control, a dummy packet was used for the second transmission.

To give an example, for a transmitted packet $x_1 \oplus x_2$, the mechanism used by the relay is as follows; if no NACKs are received, then both packets were decoded. If x_1 is NACKed but x_2 not, then the latter is decoded and the relay obtains the information that receiver 2 has x_1 and receiver 1 has $x_1 \oplus x_2$. In this case, x_1 is put in the g_1 queue, while the coded packet is not stored since it is a function of the departed packet. The symmetric case where x_2 is NACKed but x_1 not, is obtained by exchanging 1 and 2. Finally, if both packets are NACKed, then both receivers have $x_1 \oplus x_2$ and both packets are stored in the corresponding queues b_1, b_2. It should be noted that all packets in bad queues are associated with the knowledge that a XOR function is stored in the buffers.

Upon arrival, the packets are classified as **unknown** since the coding node only possesses stochastic knowledge about the corresponding overhearing events. For this reason, queue u_i for unknown packets is introduced and all arrivals enter the coding node at these queues in the begining of an epoch. The packets may leave this queue when they depart the system or if moved to another queue according to the above-described mechanism.

The system state is $S_{sto}(t) = (k_1, k_2, n_1, n_2, m_1, m_2)$, where k_i is the total number of packets of flow i in the queues that are not yet decoded in their respective receivers, n_i the number of packets in g_i and m_i the number of packets in b_i. We define the control set as:

$$\mathcal{C}_{sto} \triangleq \{g_1, b_1, u_1, g_2, b_2, u_2, g_1 \oplus g_2, g_1 \oplus u_2, u_1 \oplus g_2, u_1 \oplus u_2\}.$$

The set is again constrained to exclude XOR controls involving packets from the bad queues. This happens without loss of optimality. To see this, first notice that all good packets are evacuated either singleton, or after being XORed with another good packet. A XOR control between a known good and a known bad packet only evacuates the bad packet. Therefore this control

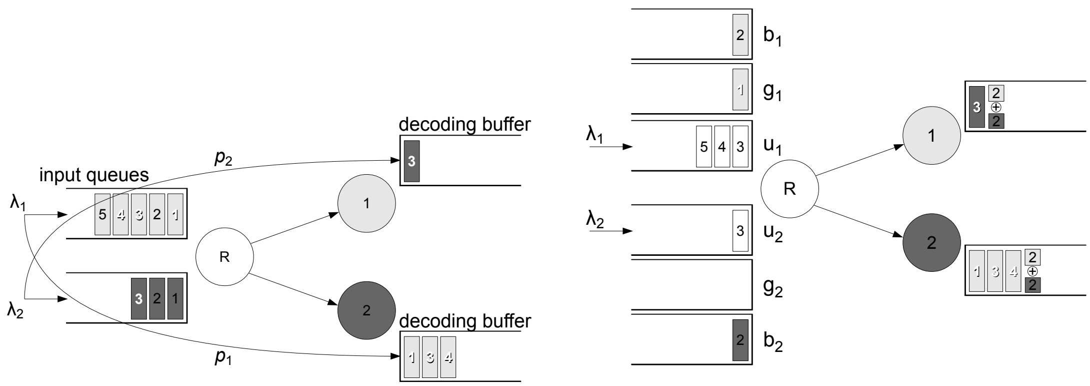

Figure 3: Two snapshots of the 2-User NACK system. (left) Packets from two unicast flows arrive at the coding node R and are destined to two different receivers. Due to side overhearing channels, a copy of the arriving packet, destined to one receiver, also arrives at the other with a probability. (right) The packets are classified by the coding node as unknown (no knowledge of overhearing event), good (overheard) or bad (not overheard), after the control $u_1 \oplus u_2$ is used with $r = 2$.

could just be replaced with the bad packet control. Controls $\{g_i, u_i\}$ are directed to receiver i (as before), while the rest of the controls are directed to both receivers. The policies and set $\mathcal{C}_{sto}(t)$ are defined as in the 2-user ACK case. We will refer to controls $\{g_1, b_1, \dots\}$ as *single* controls and to $\{g_1 \oplus g_2, \dots\}$ as *XOR* controls, denoting the corresponding sets with $\mathcal{C}^{\mathrm{s}}$, $\mathcal{C}^{\mathrm{x}}$.

4.5 Multi-user ACK Model based on Index Coding

Here $n \in \mathbb{N}$ with rates $R = \{1\}$ that is, the relay can only send one packet per timeslot. Again the relay has full knowledge of the buffers of each receiver and this is also achieved by ACK reports. This model is similar to the index coding model proposed in [5]. The work in [5], among other things, proposes how we can find the optimal policy, but computing it is NP-hard on the total number of overhearing events. For this particular problem, we will provide a set of theorems that expand on [5].

The set of packets in the queue of flow i at the beginning of an epoch is called the "Wants" set of receiver i and is denoted by $W_i, i \in [n]$. The set of packets that are in the buffers of receiver i at the beginning of an epoch is called the "Has" set of receiver i and is denoted by $H_i \in [n]$. For those sets, it holds:

$$H_i \cap W_i = \emptyset,\ \forall i \in [n]$$
$$W_i \cap W_j = \emptyset\ \forall i, j \in [n] \text{ and } i \neq j$$
$$\cup_{i=1}^{n} W_i = W$$

i.e all packets are destined to exactly one receiver and no packet that is destined to a receiver is in the buffers of that receiver. A packet being destined to multiple receivers is out of the scope of this work and is studied in [23].

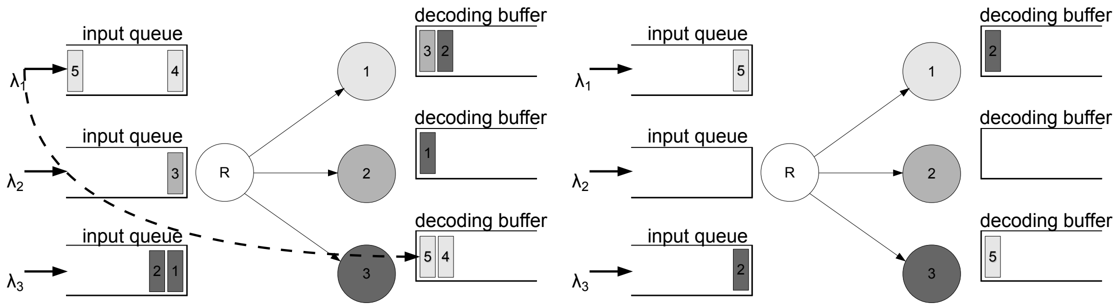

Figure 4: The system under consideration for the case of three receivers. (left) Packets from three unicast flows arrive at the coding node R and are destined to three different receivers. Due to side overhearing channels, a copy of the arriving packet, destined to one receiver, may also arrive at another with a probability. We illustrate this for an arrival destined to receiver 1 that is also overheard by receiver 3. (right) The system after controls $1 \oplus 3$ and $3 \oplus 4$.

5 2-User ACK Throughput Region

For the purposes of this section, we will allow arbitrary coding functions (including non-linear coding) on any subset of packets, relaxing the restriction of XORing only two packets from different flows. This way, we provide a lower bound on the minimal evacuation time $\overline{T}^{\star}(k_1, k_2)$ and correspondingly, its linear growth. Then, we show that simple XOR-based online policies, which operate agnostically to arrival rates, can be used to evacuate the system with the same growth. This in turn establishes the throughput region for the 2-user ACK system, which is given in a closed-form expression.

5.1 Lower bound on evacuation time under general coding

The development of the lower bound is based on a preliminary result which we present next. Let $\mathcal{X}, \mathcal{Y}, \mathcal{M}_1, \mathcal{M}_2$ be finite sets. Consider sequences $X_l \in \mathcal{X}$, $l = 1, \ldots, k_1$ and $Y_l \in \mathcal{Y}$, $l = 1, \ldots, k_2$. Denote $A^K \triangleq (A_1, \ldots, A_k)$. We also consider two coding functions $\Phi_1 : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_1$, $\Phi_2 : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_2$ and two decoding functions $g_1 : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{Y}^{n_2} \to \mathcal{X}^{k_1}$ and $g_2 : \mathcal{M}_1 \times \mathcal{X}^{n_1} \to \mathcal{Y}^{k_2}$, where $0 \leq n_i \leq k_i$, $i = 1, 2$. We impose error-free decoding:

Condition 1 [Decoding]: *For any* (X^{k_1}, Y^{k_2})

(i) $g_1\left(\Phi_1(X^{k_1}, Y^{k_2}), \Phi_2(X^{k_1}, Y^{k_2}), Y^{n_2}\right) = X^{k_1}$.

(ii) $g_2\left(\Phi_1(X^{k_1}, Y^{k_2}), X^{n_1}\right) = Y^{k_2}$.

Fix Y^{n_2} and define the mapping $\Psi : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2 - n_2} \to \mathcal{M}_1 \times \mathcal{M}_2$, where

$$\Psi\left(X^{k_1}, Z^{k_2-n_2}\right) = \left(\Psi_1\left(X^{k_1}, Z^{k_2-n_2}\right), \Psi_2\left(X^{k_1}, Z^{k_2-n_2}\right)\right),$$

$$\Psi_l\left(X^{k_1}, Z^{k_2-n_2}\right) = \Phi_l\left(X^{k_1}, Y^{n_2} || Z^{k_2-n_2}\right),\ l = 1, 2,$$

$Y^{n_2} || Z^{k_2-n_2}$ is the concatenation of sequences Y^{n_2}, $Z^{k_2-n_2}$.

Similarly fix X^{n_1} and define the mapping $\Theta : \mathcal{X}^{k_1-n_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_1 \times \mathcal{M}_2$, with

$$\Theta_l \left(Z^{k_1-n_1}, Y^{k_2}\right) = \Phi_l \left(X^{n_1}||Z^{k_1-n_1}, Y^{k_2}\right),\ l = 1, 2.$$

We also define the mapping $\Phi : \mathcal{X}^{k_1} \times \mathcal{Y}^{k_2} \to \mathcal{M}_1 \times \mathcal{M}_2$ as,

$$\Phi \left(X^{k_1}, Y^{k_2}\right) = \left(\Phi_1 \left(X^{k_1}, Y^{k_2}\right), \Phi_2 \left(X^{k_1}, Y^{k_2}\right)\right).$$

LEMMA 3: *Under condition 1, for any fixed Y^{n_2} (fixed X^{n_1}) the mapping Ψ (Θ) is injective. Hence it holds,*

$$|\mathcal{R}(\Psi)| = |\mathcal{X}|^{k_1}|\mathcal{Y}|^{k_2-n_2}, \quad |\mathcal{R}(\Theta)| = |\mathcal{X}|^{k_1-n_1}|\mathcal{Y}|^{k_2}$$

where $\mathcal{R}(\Phi)$ denotes the range of a mapping Φ. Moreover, for any fixed X^{k_1}, the mapping $\tilde{\Phi}_1 : |\mathcal{Y}|^{k_2} \to \mathcal{M}_1$ defined by $\tilde{\Phi}_1 \left(Y^{k_2}\right) = \Phi_1 \left(X^{k_1}, Y^{k_2}\right)$ is injective, hence

$$\left|\mathcal{R}(\tilde{\Phi}_1)\right| = |\mathcal{Y}|^{k_2} \tag{1}$$

Proof. To show that Ψ is injective, it suffices to show that if

$$\Psi_l \left(X^{k_1}, Z^{k_2-n_2}\right) = \Psi_l \left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), l = 1, 2,$$

then $X^{k_1} = \hat{X}^{k_1}$, and $Z^{k_2-n_2} = \hat{Z}^{k_2-n_2}$. We write

$$\begin{aligned}
X^{k_1} &= g_1 \left(\Phi_1(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}), \Phi_2(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}), Y^{n_2}\right) \\
&= g_1 \left(\Psi_1(X^{k_1}, Z^{k_2-n_2}), \Psi_2(X^{k_1}, Z^{k_2-n_2}), Y^{n_2}\right) \\
&= g_1 \left(\Psi_1 \left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), \Psi_2 \left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), Y^{n_2}\right) \\
&= g_1 \left(\Phi_1(\hat{X}^{k_1}, Y^{n_2}||\hat{Z}^{k_2-n_2}), \Phi_2(\hat{X}^{k_1}, Y^{n_2}||\hat{Z}^{k_2-n_2}), Y^{n_2}\right) \\
&= \hat{X}^{k_1}.
\end{aligned}$$

$$\begin{aligned}
Y^{n_2}||Z^{k_2-n_2} &= g_2 \left(\Phi_1(X^{k_1}, Y^{n_2}||Z^{k_2-n_2}), X^{n_1}\right) \\
&= g_2 \left(\Psi_1(X^{k_1}, Z^{k_2-n_2}), X^{n_1}\right) \\
&= g_2 \left(\Psi_1 \left(\hat{X}^{k_1}, \hat{Z}^{k_2-n_2}\right), X^{n_1}\right) \\
&= g_2 \left(\Psi_1 \left(X^{k_1}, \hat{Z}^{k_2-n_2}\right), X^{n_1}\right) \\
&= g_2 \left(\Phi_1 \left(X^{k_1}, Y^{n_2}||\hat{Z}^{k_2-n_2}\right), X^{n_1}\right) \\
&= Y^{n_2}||\hat{Z}^{k_2-n_2}.
\end{aligned}$$

Hence, $Z^{k_2-n_2} = \hat{Z}^{k_2-n_2}$. To prove Θ is injective, we argue similarly starting from the decoding function g_2. Finally, (1) follows by the fact that for any X^{k_1}, if $\Phi_1(X^{k_1}, Y^{k_2}) = \Phi_1(X^{k_1}, \hat{Y}^{k_2})$, then $Y^{k_2} = \hat{Y}^{k_2}$, which again follows from

$$g_2 \left(\Phi_1(X^{k_1}, Y^{k_2}), X^{n_1}\right) = g_2 \left(\Phi_1(X^{k_1}, \hat{Y}^{k_2}), X^{n_1}\right).$$

□

COROLLARY 4: *Assume that $X^{k_1}$and Y^{k_2} consist of independent identically distributed random variables and are independent of each other. Then the mappings Φ_1, Φ_2 are random variables and it holds,*

$$H\left(\Phi_1\right) \geq k_2 H\left(Y\right) \tag{2}$$

$$\begin{aligned} H\left(\Phi\right) \geq \max\{&k_1 H(X) + \left(k_2 - n_2\right) H\left(Y\right), \\ &\left(k_1 - n_1\right) H\left(X\right) + k_2 H\left(Y\right)\} \end{aligned} \tag{3}$$

Proof. Since for fixed X^{k_1} the mapping $\tilde{\Phi}_1$ is injective

$$\begin{aligned} H\left(\Phi_1 \middle| X^{k_1} = x^{k_1}\right) &= H\left(\tilde{\Phi}_1 \middle| X^{k_1} = x^{k_1}\right) \\ &= H\left(Y^{k_2}\right) = k_2 H(Y). \end{aligned}$$

Hence, $H\left(\Phi_1\right) \geq H\left(\Phi_1 \left| X^{k_1}\right.\right) = k_2 H\left(Y\right)$. Also

$$H\left(\Phi\right) \geq H\left(\Phi \left| Y^{n_2}\right.\right) = k_1 H(X) + \left(k_2 - n_2\right) H\left(Y\right)$$
$$H\left(\Phi\right) \geq H\left(\Phi \left| X^{n_1}\right.\right) = \left(k_1 - n_1\right) H\left(X\right) + k_2 H\left(Y\right)$$

are derived in a similar fashion. □

The interpretation of this formulation in our current context is the following: $\mathcal{X}$ and $\mathcal{Y}$ are all possible L-bit sequences that can be contained in a packet, $|\mathcal{X}| = |\mathcal{Y}| = 2^L$. The sequence X^{k_1} represents the k_1 packets at the transmitter that are destined to receiver 1, while the subsequence X^{n_1} represents the packets at the transmitter that are destined for receiver 1 and have been overheard by receiver 2. The interpretation of sequence Y^{k_2} and its subsequence Y^{n_2} is similar. Since bit sequences are assumed i.i.d with uniform distribution, we have $H\left(X\right) = H\left(Y\right) = L$.

For the rest of the discussion we assume that $r_1 \geq r_2$, hence receiver 1 observes all slots, while receiver 2 observes only slots at which packets are transmitted at rate r_2. The set $\mathcal{R}(\Phi_1)$ represents the values of the mapping which must be known to receiver 2 so that together with X^{n_1} successful decoding is effected at this receiver. Therefore, the values of $\mathcal{R}(\Phi_1)$ must be transmitted during slots at which the rate is r_2. We denote by ξ_2 the (random) number of packets transmitted during these slots, and by $\bar{\xi}_2$ its average value. Hence, the average number of bits transmitted in slots with rate r_2 is $\bar{\xi}_2 L$. Similarly, for the set $\mathcal{R}(\Phi)$ and receiver 1. We denote by ξ the (random) number of slots used in the transmission of all the packets to both receivers, and by $\bar{\xi}$ its average value.

In order for the receivers to obtain the values of the sets $\mathcal{R}(\Phi_1)$, $\mathcal{R}(\Phi)$ these values must be source-coded and transferred through the channel using packets of L bits. We use uniquely decodable codes and hence the average number of bits that need to be transmitted is bounded from below as follows.

To transfer the values of $\mathcal{R}(\Phi_1)$, using (2)

$$\bar{\xi}_2 L \geq H\left(\Phi_1\right) \geq k_2 H(Y) = k_2 L. \tag{4}$$

Similarly, to transfer the values of $\mathcal{R}(\Phi)$, using (3)

$$\begin{aligned} \bar{\xi} L \geq H\left(\Phi\right) &\geq \max\left\{H(\Psi), H(\Theta)\right\} \\ &\geq \max\{k_1 H(X) + \left(k_2 - n_2\right) H\left(Y\right), \\ &\qquad \left(k_1 - n_1\right) H\left(X\right) + k_2 H\left(Y\right)\} \\ &= L \max\left\{k_1 + \left(k_2 - n_2\right), \left(k_1 - n_1\right) + k_2\right\} \\ &= L\left(k_1 + k_2 - \min\left\{n_1, n_2\right\}\right). \end{aligned} \tag{5}$$

THEOREM 5 [LOWER BOUND WITH ARBITRARY CODING]: *The 2-user ACK system satisfies under any $\pi \in \Pi$:*

$$\overline{T}^{\pi}(k_1,k_2) \geq T^{bdet}(k_1,k_2),$$

where $T^{bdet}(k_1,k_2) \triangleq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[\min\{N_1,N_2\}]}{\max\{r_1,r_2\}}$.

Proof. Assume without loss of generality $r_1 \geq r_2$. Also, let $\xi_1 \triangleq \xi - \xi_2$ be the number of packets transmitted during slots where rate r_1 is used, so that only receiver 1 observes them. For any policy π we have

$$T^{\pi}(k_1,k_2,n_1,n_2) \geq \left\lceil \frac{\xi_1}{r_1} \right\rceil + \left\lceil \frac{\xi_2}{r_2} \right\rceil \geq \frac{\xi}{r_1} + \frac{\xi_2}{r_2} - \frac{\xi_2}{r_1}.$$

Taking into account (4), (5) we then have

$$\begin{aligned} \overline{T}^{\pi}(k_1,k_2,N_1,N_2) &\geq \frac{\overline{\xi}}{r_1} + \frac{\overline{\xi}_2}{r_2} - \frac{\overline{\xi}_2}{r_1} \\ &\geq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[\min\{N_1,N_2\}]}{r_1}, \end{aligned}$$

where $N_i, i = 1,2$ are binomial RVs with k_i, p_i. The result follows by using the same methodology for the case of $r_2 > r_1$. □

5.2 A class of simple XOR-based policies

DEFINITION 1 [CLASS Π_{DET}]: *At slot t the control is chosen according to the following two steps:*

(i) If $\{g_1 \oplus g_2\} \in \mathcal{C}_{det}(t)$, choose control $\{g_1 \oplus g_2\}$.

(ii) Else, choose any single control (each policy in the class defines a different order).

When $\mathcal{C}_{det}(t) = \emptyset$, stop.

Let $\overline{r} \triangleq r_1$ if $n_1 \geq n_2$ and $\overline{r} \triangleq r_2$, otherwise. Notice, that since the policies in Π_{det} do not depend on the values of the bits in the packets, their evacuation times are deterministic. By enumerating the two above steps, we have

$$\begin{aligned} T^{\pi}(k_1,k_2,n_1,n_2) \leq \left\lceil \frac{\min\{n_1,n_2\}}{\min\{r_1,r_2\}} \right\rceil + \\ + \left\lceil \frac{\max\{n_1,n_2\} - \min\{n_1,n_2\}}{\overline{r}} \right\rceil + \left\lceil \frac{k_1 - n_1}{r_1} \right\rceil + \left\lceil \frac{k_2 - n_2}{r_2} \right\rceil \\ \leq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\min\{n_1,n_2\}}{\max\{r_1,r_2\}} + 4, \quad \text{for all } \pi \in \Pi_{\text{det}}. \end{aligned} \tag{6}$$

THEOREM 6 [THROUGHPUT REGION]: *The throughput region of the 2-user ACK system is the area defined by $(\lambda_1,\lambda_2) \geq (0,0)$, and the following inequality:*

$$\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min\{p_{12}\lambda_1, p_{12}\lambda_2\}}{\max\{r_1,r_2\}} \leq 1. \tag{7}$$

Proof. Consider $k_i = \lceil \lambda_i t \rceil$, $i = 1,2$ packets to be evacuated and note that the number of good packets per flow are binomial random variables, denoted by $N_1(k_1), N_2(k_2)$ correspondingly. Since the status of arriving packets (good, bad) is an i.i.d. process, we have,

$$\lim_{t\to\infty} \mathbb{E}\left[\frac{N_i(\lceil \lambda_i t \rceil)}{t}\right] = p_i\lambda_i, \tag{8}$$

and also, by the strong law of large numbers,

$$\lim_{t\to\infty} N_i(\lceil \lambda_i t \rceil, \omega)/t = p_i\lambda_i \quad \text{w.p.1.} \tag{9}$$

Recall that $\overline{T}^{\star}(k_1,k_2)$ is the minimum average evacuation time over all policies, hence a lower bound of $\overline{T}^{\pi}$. By Theorem 5

$$\mathbb{E}\left[T^{bdet}(k_1,k_2,N_1,N_2)\right] \le \overline{T}^{*}(k_1,k_2) \le \mathbb{E}[T^{\pi}(k_1,k_2,N_1,N_2)] \tag{10}$$

We calculate the limit of the upper bound of $\overline{T}^{\pi}(k_1,k_2)$ using the the RHS of (6)

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[\frac{\frac{k_1 t}{r_1} + \frac{k_1 t}{r_1} - \frac{\min\{N_1(k_1 t,\omega),N_2(k_2 t,\omega)\}}{\max\{r_1,r_2\}} + 4}{t}\right] = \\
&= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \lim_{t\to\infty} \mathbb{E}\left[\frac{\min\{N_1(k_1 t,\omega),N_2(k_2 t,\omega)\}}{t\max\{r_1,r_2\}}\right] \\
&= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\min\{p_1 k_1, p_2 k_2\}}{\max\{r_1,r_2\}}, \quad \text{w.p.1,}
\end{aligned}$$

where in the last step, we exchange the order of limit expectation and min function due to uniform integrability which follows from convergence in expectation (8) and almost everywhere convergence (9) of the involved sequences, see [35] Th. 16.14. We can repeat the limit derivation for the case of T^{bdet} and derive the same limit, hence from (10) we conclude

$$\hat{T}(\lambda_1,\lambda_2) = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min\{p_1\lambda_1, p_2\lambda_2\}}{\max\{r_1,r_2\}}$$

and the result follows by invoking Proposition 2. □

6 2-User NACK Code Constrained Throughput Region

In this section, we study a set of evacuation policies Π for the 2-user NACK system, constrained to the use of XORs (i.e. general coding is not considered) and derive the corresponding code-constrained throughput region in closed-form.

6.1 Treating packets in queues $\mathbf{b}_1$, $\mathbf{b}_2$

We focus on a special control sequence. Following a control $u_1 \oplus u_2$, and two NACK messages from the receivers, the corresponding transmitted packets x_1, x_2 are characterized as bad and put in the corresponding bad queues. Assume, that in a succeeding time slot, one of the two packets is transmitted using a single control, say b_1, directed to both receivers (i.e. at rate $r =$

$\min\{r_1, r_2\}$). Evidently receiver 1 will obtain x_1, which departs the system. Since receiver 2 has previously obtained the coded packet $x_1 \oplus x_2$ from the NACKed broadcast transmission, receiver 2 can combine it with x_1 and obtain x_2. In a total of two transmissions, both bad packets are obtained.

Due to the control set $\mathcal{C}_{\text{sto}}$, the packets in the bad queues can only be evacuated by single controls. Since, for each bad packet in the queue b_1 there is a bad packet in queue b_2 (the one with which it was coded), for efficiency reasons we will assume that they are always directed to both receivers. Finally, since the evolution of the system state is not affected, we will constrain the set of evacuation policies to those that choose controls b_1, b_2 last.

6.2 Lower bound on the code-constrained growth rate

Let $(f, s) = (1, 2)$ if $r_1 \geq r_2$ and $(f, s) = (2, 1)$ otherwise, where f=fast and s=slow. Also, let $(.)^+ \triangleq \max\{., 0\}$ and

$$B^{req} \triangleq \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min(\lambda_1 p_1, \lambda_2 p_2)}{p_f} \left[\frac{1}{r_f} - \frac{1 - p_f}{r_s} \right]^+ .$$

THEOREM 7 [LOWER BOUND ON THE GROWTH RATE]: *For the 2-user NACK system, constrained to the use of XOR coding, it holds*

$$\liminf_{t \to \infty} \frac{\overline{T}^{\pi} \left(\lceil t\lambda_1 \rceil , \lceil t\lambda_2 \rceil \right)}{t} \geq B^{req}, \textit{ for all } \pi \in \Pi. \tag{11}$$

The proof is in the Appendix.

6.3 An optimal evacuation policy

DEFINITION 2 [POLICY π^*]: *Policy $\pi^* \in \Pi$ operates as follows. If $1 - p_f > \frac{\min(r_1, r_2)}{\max(r_1, r_2)}$ is true, controls from $\mathcal{C}^s$ are chosen in arbitrary order. Else, at slot t*

- *If $\{u_1 \oplus g_2\} \in \mathcal{C}_{sto}(t)$ or $\{u_2 \oplus g_1\} \in \mathcal{C}_{sto}(t)$, then select the corresponding control*
- *elseif $\{u_1 \oplus u_2\} \in \mathcal{C}_{sto}(t)$ select this control*
- *else select any control from the set $\mathcal{C}^s$. During this step, controls b_1 and b_2 are used in the way explained in subsection 6.1.*

When $\mathcal{C}_{sto}(t) = \emptyset$ stop.

THEOREM 8 [ASYMPTOTIC OPTIMALITY OF POLICY π^*]: *For the 2-user NACK system operating under policy π^* we have*

$$\limsup_{t \to \infty} \frac{\overline{T}^{\pi^*} \left(\lceil t\lambda_1 \rceil , \lceil t\lambda_2 \rceil \right)}{t} \leq B^{req}. \tag{12}$$

The proof of Theorem 8 is in the Appendix. Combining (11) and (12), we conclude $\hat{T}^\star(\lambda_1, \lambda_2) = B^{req}$.

COROLLARY 9: *The code-constrained region of the 2-user NACK system is the set of rates $(\lambda_1, \lambda_2) \geq (0, 0)$ satisfying*

$$\frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2} - \frac{\min\{\lambda_1 p_1, \lambda_2 p_2\}}{p_f} \left[\frac{1}{r_f} - \frac{1 - p_f}{r_s} \right]^+ \leq 1, \tag{13}$$

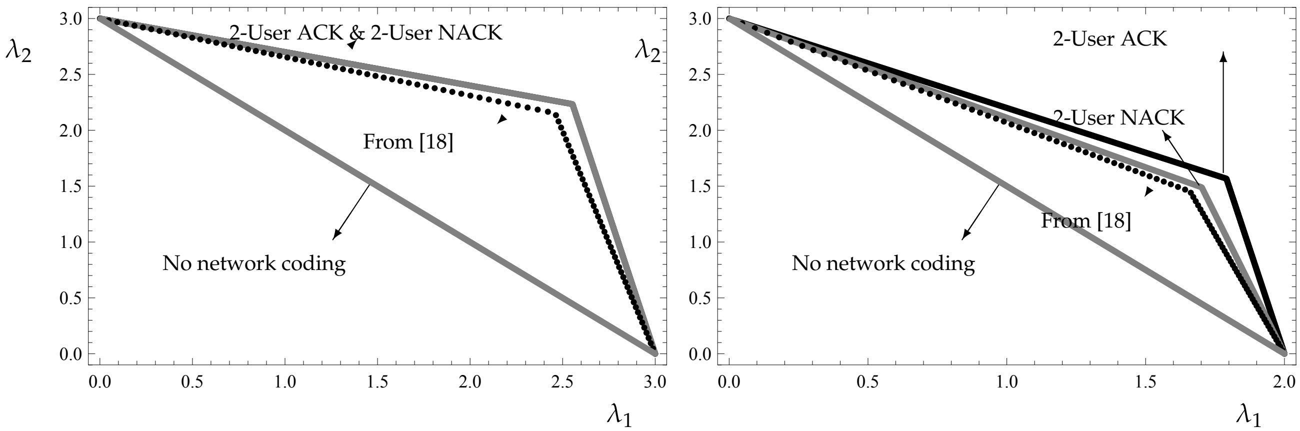

Figure 5: Throughput regions of no network coding and 2-user ACK system and code-constrained regions of the 2-user NACK system with or without storing XORs (from [18]). Parameters: $p_1 = 0.7, p_2 = 0.8, r_2 = 3$ and $r_1 = 3$ (left), $r_1 = 2$ (right).

where $r_s = \min\{r_1, r_2\}$ and $r_f = \max\{r_1, r_2\}$. Whenever the term in the brackets is negative, network coding is not beneficial, and the maximum throughput is achieved without coding. If $r_1 = r_2$, the terms cancel out and (13) equals (7), therefore the code-constrained region of the 2-user NACK system and the throughput region of the 2-user ACK are equal. In Fig. 5 we plot the regions for two different settings.

7 Multi-User ACK based on Index Coding Results

7.1 Theoretical Results

In this section we will prove some useful properties on the model proposed in [5]. First, we will introduce the notion of the side information graph.

DEFINITION 3: *Consider a system with a relay that contains a set W of packets and n receivers with $H_1, ..., H_n$ and $W_1, ..., W_n$ the "Has" and "Wants" sets of each receiver respectively that satisfy these conditions:*

$$H_i \cap W_i = \emptyset, \ \forall i \in [n]$$
$$W_i \cap W_j = \emptyset \ \forall i, j \in [n] \ and \ i \neq j$$
$$\cup_{i=1}^{n} W_i = W$$

*The **side information graph** (SIG) $G \equiv (V, E)$ is defined as the multipartite digraph which is constructed in this way:*
If $w \in W$ we construct the vertex $v \in V$.
*If $h \in H_i$ and v is the corresponding vertex of h, we construct the set of edges $O_v^i \equiv \{(w, v) : w \in W_i\}$. We call this set an **overhearing hyperedge**, i.e the set of edges that correspond to a single overhearing event.*

Notice that each partition is composed of the vertices that represent the packets that belong to the "Wants" set of a certain receiver. It might also seem that the definition given here is different than that in [5] but in reality they are equivalent. That is because, in [5], each receiver has only one packet in its "Wants" set. However, if a receiver i has a "Wants" set $|W_i| = z > 1$ where

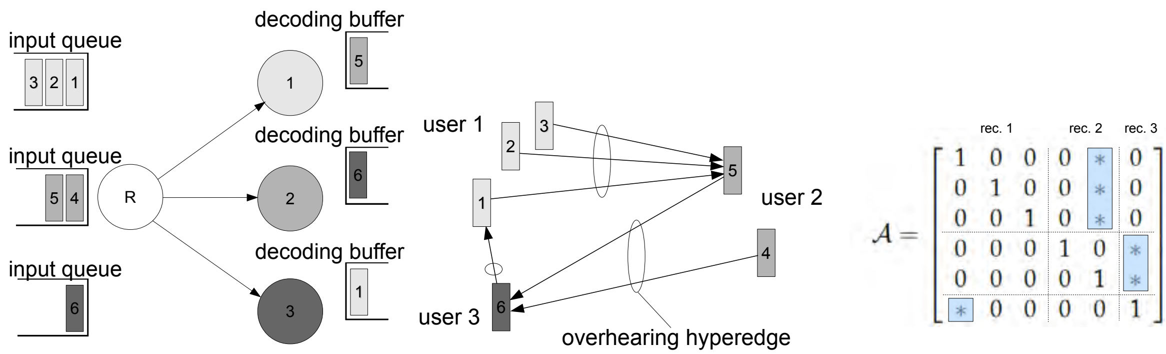

Figure 6: An example with three receivers and six packets, where three packets are overheard and they reside on the corresponding decoding buffers (left). The corresponding overhearing SIG is shown (middle) and the set of matrices that fit the SIG (right). '$*$' indicate entries to be completed with elements of the binary field. The sub-columns highlighted in blue indicate overhearing sub-columns

packets $w_1 \dots w_z \in W_i$ and a "Has" set H_i then this receiver can be decomposed to z receivers were receiver $j \in [z]$ has a "Wants" set $W_j = \{w_j\}$ and a "Has" set H_i. See Figures 6 and 7 for an example.

Secondly, we define what it means for a matrix to fit a side information graph G:

Definition 4: *Let $G \equiv (V, E)$ be any SIG. We say that the 0-1 matrix $A \in \mathbb{F}_2^{|V| \times |V|}$ **fits** G if for all i and j:*

$$A_{ij} = \begin{cases} 1 & i = j \\ 0 & (i, j) \notin E \end{cases}$$

*The sub-columns of A that correspond to an overhearing hyperedge are called **overhearing sub-columns**.*

Finally, we define the quantity of $minrk_2(G)$ that stands for the minimum rank attained by all matrices that fit G:

Definition 5: *Let G be a SIG. We define:*

$$minrk_2(G) \equiv \min\{rk_2(A) : A \text{ fits } G\}$$

A useful observation we will need later on is that, if G is a SIG of disconnected components $G_1, ..., G_n$ then:

$$minrk_2(G) = \sum_{i=1}^{m} minrk_2(G_i) \tag{14}$$

Denote by $\mathcal{I}_G$ the set of all index codes and by $\mathcal{I}_G^L$ the set of all *linear* index codes that evacuate the packets of a SIG G respectively. Let $len(C),\ C \in \mathcal{I}_G$ be the length of code C and MAIS(G) the size of the maximal acyclic induced subgraph of G. In [5] (theorems 1 and 3) it is proved that:

$$MAIS(G) \leq \min_{C \in \mathcal{I}_G} len(C) \leq \min_{C \in \mathcal{I}_G^L} len(C) = minrk_2(G) = \overline{T}^{\star}(\boldsymbol{k}) \tag{15}$$

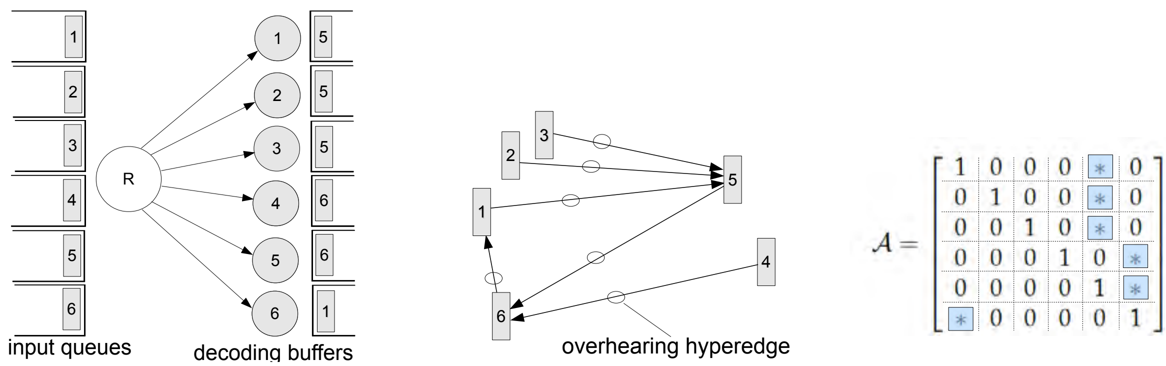

Figure 7: An example with six receivers and six packets, where three packets are overheard and they reside on the corresponding decoding buffers (left). The corresponding overhearing SIG is shown (middle) and the set of matrices that fit the SIG (right). '$*$' indicate entries to be completed with elements of the binary field. The sub-columns highlighted in blue indicate overhearing sub-columns. Notice that the resulting SIGs in both Figures 6 and 7 are the same, as well as the matrices that fit them.

were $\boldsymbol{k}$ is the vector of the number of packets belonging to each flow in the system represented by the SIG. Moreover, $minrk_2(G)$ is the optimal length for all $C \in \mathcal{I}_G^L$ and for some special classes of SIG, it is the optimal length for all $C \in \mathcal{I}_G$. Finally, for $C \in \mathcal{I}_G^L$, $\min_{C \in \mathcal{I}_G^L} len(C) = minrk_2(G)$ is achievable by taking any $A \in \mathbb{F}_2$ that satisfies its definition, finding a maximal independent set of rows and XORing the packets that correspond to the columns where the '1' are.

We will now prove a series of theorems based on those results. First, we show that the optimal evacuation policy for a system that has a SIG which is a directed acyclic graph (DAG), is to send all of the packets singleton (i.e one at a time):

Theorem 10: *Let $G = (V, E)$ be a SIG with $|V| = n$ and $V = \{v_1, ...v_n\}$ which corresponds to the set of packets $W = \{w_1, ..., w_n\}$. If G is a DAG, then:*

$$\min_{C \in \mathcal{I}_G}(len(C)) = n$$

Proof. Since G is a DAG, we have

$$MAIS(G) = n$$

From (15) we have:

$$n \leq \min_{C \in \mathcal{I}_G} len(C)$$

But there is at least one index code for G with length n and that is: $\{w_1, w_2, ..., w_n\}$, i.e we send all packets singleton. Therefore the optimal minimum in the above inequality is achieved with this code.

□

Next, we will show that the problem of finding the $minrk_2(G)$ for any SIG can be reduced to a problem of finding the $minrk_2$ of a number of subgraphs that are strongly connected.

THEOREM 11: *Let $G = (V, E)$ be a SIG with $|V| = n$ and $V = \{v_1, ..., v_n\}$ the set of all vertices. Also, denote by $G_1, ..., G_m$ the strongly connected components of G. Then, the optimal length for linear index codes that evacuate the system corresponding to G is:*

$$\min_{C \in \mathcal{I}_G^L} len(C) = minrk_2(G) = \sum_{i=1}^{m} minrk_2(G_i)$$

Proof. The equality is given by [5]. To prove the equality, we first impose a topological ordering on the condensation of G (the graph that is made by having all the vertices of a strongly connected component represented by a single vertex. This graph is a DAG therefore the topological ordering is applicable. These are well known results of graph theory). Without loss of generality, let $(G_1, ..., G_n)$ be this order, where $G_i = (V_i, E_i)$ is the i-th strongly connected component of the topological order.

Let A be any matrix that fits G. We can write A in such a way that the vertices of V_1 correspond to the first $|V_1|$ rows of A, the vertices of V_2 correspond to the next $|V_2|$ rows of A and so on and so forth. Due to the topological ordering, there are no edges (v_i, v_j) such that $i > j$. Therefore, A' is block upper triangular.

We can write A as:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ 0 & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{mm} \end{pmatrix}$$

where A_{ii} is a matrix that fits G_i' and $A_{ij} \ \ , i \neq j$ is a matrix that fits the induced subgraph of the neighboring vertices of G_i and G_j.

We have:

$$\begin{aligned} minrk_2(G) &\geq rk_2 A \\ &\overset{(26)}{\geq} \sum_{i=1}^{n} rk_2(A_{ii}) \\ &\geq \sum_{i=1}^{n} minrk_2(G_i) \end{aligned} \tag{16}$$

where the inequality 26 is proved in the Appendix (Basic Theorems). Also, if we let:

$$A_d = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{mm} \end{pmatrix}$$

be the diagonal blocks of A. This matrix fits the graph G_d, i.e the graph with no edges between the strong connected components of G. This graph is comprised of the disconnected graphs $G_1 ... G_n$. We have:

$$\begin{aligned} minrk_2(G) &\leq minrk(G_d) \\ &\stackrel{(14)}{=} \sum_{i=1}^{n} minrk_2(G_i) \end{aligned} \tag{17}$$

where the first inequality is due to the fact that all matrices that fit G_d also fit G. Intuitively, this means that the evacuation time is larger if we choose to omit some overhearings that occured.

By combining (16) and (17) the proof is complete. □

We continue with a useful definition. The intuition behind this definition is that some overhearing events are needed in the system (essential) if we can evacuate the system faster with a XORing scheme than if they hadn't occurred:

DEFINITION 6: *Let $G = (V, E)$ be any SIG. We say that an overhearing hyperedge $O \subseteq E$ is **essential** to G iff, for $G' = (V, E - O)$ it holds that:*

$$minrk_2(G) < minrk_2(G')$$

*Otherwise, we say it is **non-essential** to G.*

From the above definition, we prove the following lemma. The intuition behind this lemma is that if an overhearing event is essential, then it must be used in the XORing scheme:

LEMMA 12: *Let $G = (V, E)$ be any SIG. An overhearing hyperedge $O \subseteq E$ is essential to G iff all matrices that are a solution to $minrk(G)$ have '1' in at least one entry of the overhearing sub-column C that corresponds to O.*

Proof. Let $G' = (V, E - O)$. The statement 'O is essential to G' is equivalent to

$$minrk_2(G) < minrk_2(G')$$

P1: All matrices that fit G' are all the matrices that fit G and have '0' to all entries of C.
P2: None of the matrices that fit G' are a solution to $minrk_2(G)$ due to the above inequality
C1: Due to P1 and P2, all solutions to $minrk_2(G)$ have '1' in at least one entry of C. □

REMARK 1: *An overhearing hyperedge O with a corresponding overhearing sub-column C is non-essential iff there exists at least one solution where $C = 0$.*

The above concepts of essentiality, overhearing hyperedge and overhearing sub-column are necessary for the proof of the next theorem, where we establish that there is no loss of optimality if we constrain the set of XOR policies to those that have no intra-flow coding.

First lets clarify what we mean by intra-flow coding. A coding scheme contains intra-flow codewords when two or more packets of the same flow are coded together in the same codeword. To translate it to the formalism we described above, we could say that in the matrix solution to the problem, there are rows that contain 2 or more '1' that correspond to packets of the same flow.

Let's proceed with our proof:

THEOREM 13: *Let $G = (V, E)$ be any SIG. There exists at least one solution to $minrk_2(G)$ such that there is no intra-flow XOR coding.*

Proof. We will use induction on the number of overhearing events for this proof. If a solution to $minrk_2(G)$ has no intra-flow XOR coding, we will say that it satisfies the NIC property. Suppose that G has a constant number of vertices and n overhearing hyperedges

(Base case) Denote N the maximum number of overhearing events (i.e all possible overhearing events took place or equivalently, G is a complete multipartite graph). In Appendix 10.2 we prove that for the base case:

- P1: for all non-essential overhearing hyperedges there is a solution where their corresponding overhearing sub-columns are 0 and satisfies the NIC property.

- P2: for all essential overhearing hyperedges there is a solution where their corresponding overhearing sub-columns is a column of an identity matrix (i.e it has exactly one '1') and it satisfies the NIC property.

therefore the Theorem is satisfied for the base case.

(Inductive hypothesis) Let the theorem hold for $n = N - k$ overhearing events and for every SIG G with this number of overhearing hyperedges, P1 and P2 hold.

(Inductive step) For $n = N - (k+1)$ overhearing events: Take any overhearing event that has not happened and suppose that it happened and let O and C be its corresponding overhearing hyperedge and sub-column respectively in graph $G' = (V, E \cup O)$. Distinguish between two cases

- O is non-essential to G':
  Due to the inductive hypothesis, P1 holds. Hence, there exists a solution S' for $minrk_2(G')$ such that $C = 0$ and that satisfies the NIC property. S' is also a solution to G, therefore we are done.

- O is essential to G'. Due to the inductive hypothesis, P2 holds. Hence there exists a solution S' for $minrk_2(G')$ such that C has exactly one '1' and it satisfies the NIC property. Let $C(i, j)$ be the non-zero entry of C. We construct a matrix S such that:

  $$S(k, l) = \begin{cases} 0 & , k = i \text{ and } l = j \\ S'(k, l) & , \text{otherwise} \end{cases}$$

  i.e we remove the essential '1'. Due to (27) (see Appendix 10.3):

  $$rk_2(S') - 1 \leq rk_2(S) \leq rk_2(S') + 1$$

  or

  $$minrk_2(G') - 1 \leq rk_2(S) \leq minrk_2(G') + 1$$

All solutions to $minrk_2(G')$ have non-zero C. Since $C = 0$ for S, S is not a solution to $minrk_2(G')$ therefore:

$$rk_2(S) > minrk_2(G')$$

from the two above inequalities we conclude:

$$rk_2(S) = minrk_2(G') + 1$$

Moreover O is essential therefore:

$$minrk_2(G') < minrk_2(G)$$

Since the next integer after $minrk_2(G')$ is $minrk_2(G') + 1$, and because S fits G, S is a solution to $minrk_2(G)$ and has the required properties. That concludes the proof.

□

7.2 A Proposed Heuristic

Theorem 13 drastically reduces the search space for finding an optimal solution to the Index Coding problem, but computing $minrk_2$ is still NP-hard. This motivates a textsfHeuristic were we decompose the problem into smaller problems that can be solved offline and stored on a lookup table. The codes we use do not include intra-flow coding. We construct a lookup table containing all instances of the problem such that each receiver requests *at most one packet*. Due to different possible overhearing event combinations, there exist $\sum_{i=1}^{N} 2^{i(i-1)} \binom{N}{i}$ such instances, many of which are, however, homomorphic. Also, note that this is exponential to the number of receivers but not the number of packets (as before). We compute the $minrnk_2$ for each of these. Then we order the elements in this table according to the efficiency metric $\frac{\#decoded\ packets}{minrnk_2}$. In case of tied metric, priority is given to the packets with the smaller out degree in the overhearing SIG. Yet another tie is solved arbitrarily. The proposed policy simply chooses the top element of the lookup table for which all involved packets appear in the input queues at least as many times as the lowest rate of those queues.

8 Comparison with State-of-the-art Policies

In this section we consider two important state-of-the-art schemes that are used as solutions to the WNC problem and compare their performance to our 2-user ACK, 2-user NACK and Heuristic policy by use of simulation.

IDNC: we have in mind greedy immediately decodable policies like the one proposed in [6]; form the XOR sum of packets such that each packet can be immediately decoded at the corresponding intended receiver and choose the largest such sum (ties solved randomly). We expect this policy to be inefficient compared to the optimal because it fails to solve cycles in G, for example consider the example in figure 10. Also, because it solves the ties randomly instead of sending the packets that have been least overheard, it misses opportunities for coding.

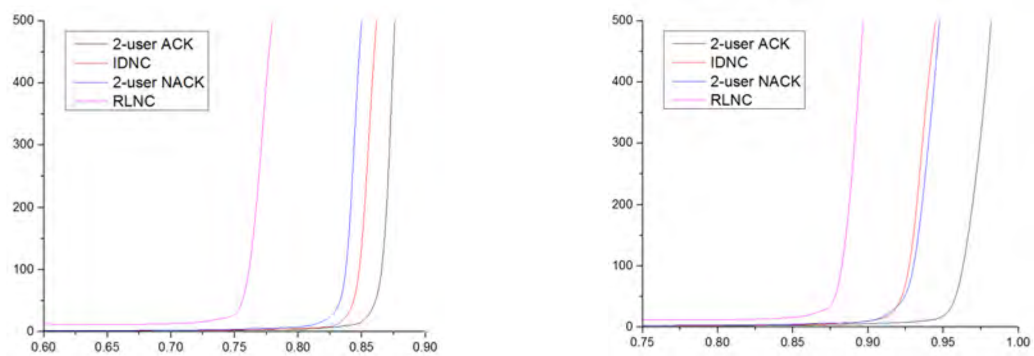

Figure 8: Average delay performance for a system with 2 receivers, in two chosen directions $\boldsymbol{\lambda} = (\lambda, \lambda)$(left) and $\boldsymbol{\lambda} = (\lambda, .8\lambda)$ (right). The overhearing probabilities are: $p_1 = .9, p_2 = .7$. For both cases, the rates are $r_1 = 2, r_2 = 1$

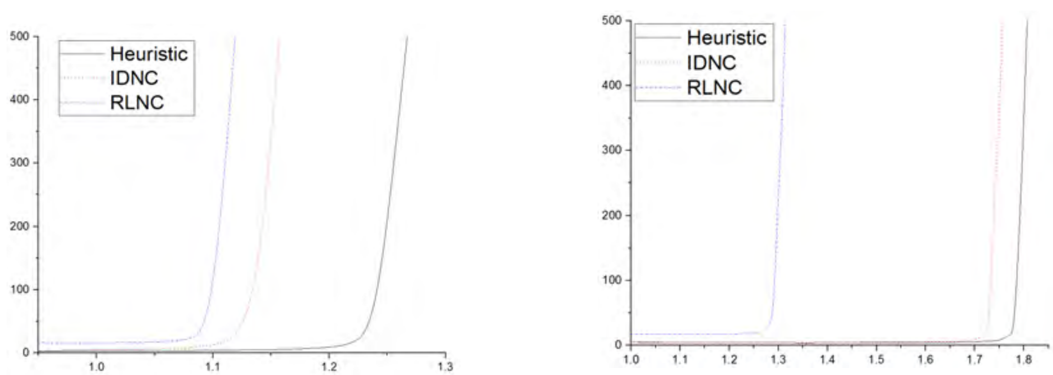

Figure 9: Average delay performance for a system with 2 receivers, in two chosen directions $\boldsymbol{\lambda} = (\lambda, \lambda, \lambda)$, $r_1 = 2, r_2 = 2, r_3 = 2$ (left) and $\boldsymbol{\lambda} = (\lambda, .8\lambda, .6\lambda)$, $r_1 = 4, r_2 = 3, r_3 = 2$(right). The overhearing probabilities are: $p_{12} = p_{23} = p_{31} = .8$, $p_{13} = p_{21} = p_{32} = .5$.

$\mathsf{RLNC} - g$: this policy considers the packets at the input in different generations of size g. In the each generation, g packets from each receiver are coded together forming Ng equations with randomly drawn coefficients. In some cases, some receivers do not participate if they do not have any packets. We assume an idealized version of the policy where the coefficients are pseudo-random and they result in linearly independent coded packets. These equations are transmitted until all receivers have decoded all Ng packets. Side information packets are also linearly independent equations that can be used to accelerate the decoding. However, the required transmissions are calculated based on the receiver with the smallest number of side information packets. When the first generation is decoded, the transmissions stop and we proceed to the second generation. When all the packets of the generation are decoded we move to the next generation. We expect this policy to be inefficient compared to the optimal because it requires all the receivers to decode all packets. On the contrary, optimal index codes transmit only the amount of information that

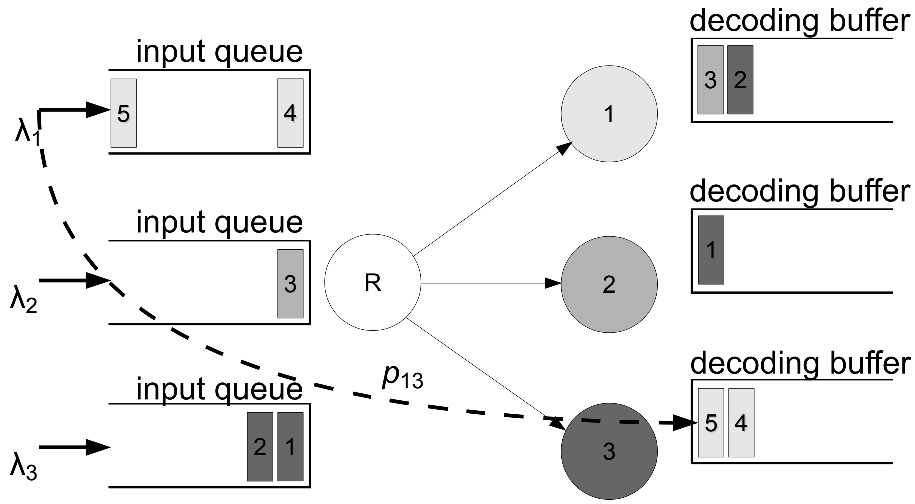

Figure 10: The system under consideration for the case of three receivers were both IDNC and RLNC have lower performance than the proposed Heuristic. Rates are $R = \{1, 1, 1\}$ The Heuristic will manage to evacuate the queues in three timeslots with the controls $2 \oplus 5$ which corresponds to a 2-cycle in G, $3 \oplus 1$ and $3 \oplus 4$ where the last two controls correspond to a 3-cycle in G. IDN will fail to solve the 3-cycle and will instead sent packets $3, 1, 5$ singleton, taking 4 timeslots. RLNC will need to send 4 linearly independent equations, therefore also taking 4 slots.

is required so that each receiver obtains the packets it is interested in. Moreover, all transmissions are done in the slowest possible rate, so that all receivers can listen. This puts RLNC in a significant disadvantage if we diversify the rates, as shown in the figures.

We compare the dynamic policies $\mathsf{IDNC}, \mathsf{RLNC} - 16$ to our proposed 2-user ACK and **2-user NACK** policies in Figure 8. g was chosen to maximize the throughput of RLNC in the setup. The Figure shows the average packet delay of the compared policies. In two chosen directions $\boldsymbol{\lambda} = (\lambda, \lambda)$, $\boldsymbol{\lambda} = (.7\lambda, \lambda)$ and for a specific overhearing probability matrix given in the caption. As we already proved, the 2-user ACK policy is optimal and therefore outperforms all others in delay. The 2-user NACK algorithm has a better performance than RLNC and can reach or surpass IDNC. This happens because when there are no opportunities for XORing, IDNC will blindly send singleton packets whereas the 2-user NACK algorithm will conserve those packets that have been overheard so as not to miss on coding opportunities later on. Moreover, when singleton bad packets are sent in the slowest rate, two packets per transmission evacuate the system as we explained. The asymmetric example on the right of figure 8 shows a case where this algorithm and IDNC have about the same delay performance, even though the 2-user NACK algorithm admits no explicit acknowledgements.

We also compare the dynamic policies $\mathsf{IDNC}, \mathsf{RLNC} - 36$ to our proposed Heuristic in Figure 9. g was chosen to maximize the throughput of RLNC in the setup. The Figure shows the average packet delay of the compared policies. In two chosen directions $\boldsymbol{\lambda} = (\lambda, \lambda, \lambda)$, $\boldsymbol{\lambda} = (\lambda, .8\lambda, .6\lambda)$ and for a specific overhearing probability matrix given in the caption, Heuristic outperforms prior approaches, showing a throughput increase.

9 Conclusions and Extensions

In the problem of reporting overhearing events in wireless network coding, we study the 2-user ACK system, the 2-user NACK system and the Index Coding system. For the 2-user ACK system we derive analytical expressions for its throughput region. For the 2-user NACK we derive the code-constraint region and we show that its equal to the throughput region of the first system when $r_1 = r_2$. When $r_1 \neq r_2$, we analyze the throughput-overhead tradeoff and conclude that this system is a very efficient approach when the overhearing probabilities are sufficiently high. For the case of the Index Coding problem, we show that there is no loss of optimality if we ignore codes that involve intra-flow coding. We also show that the problem of finding an optimal Linear Coding scheme can be decomposed to a number of subproblems equal to the number of strongly connected components of the SIG proposed in [5]. Alongside with the theoretical results, we propose simple and efficient evacuation policies which can be used in practice to achieve optimal throughput for the case of two receivers and a heuristic algorithm for more than two receivers. Future avenues of research could be the expansion of the 2-user NACK and ACK cases for an arbitrary number of users, possibly with the help of Index Coding techniques.

10 Appendix

10.1 Proofs of Theorems 7 and 8 for the 2-user NACK Code Constrained Throughput Region

Proof of Theorem 7. We assume that the packets are served from the queues in a FCFS manner, since all packets in a given queue are statistically equivalent and thus reordering them does not change the expected outcome.

We partition the set of policies Π to three sets, the subset of policies using only single controls Π_{sin}, the subset of policies using always XOR controls if $\mathcal{C}^x \cap \mathcal{C}_{sto}(t) \neq \emptyset$, called Π_{xor} and the rest Π_{mix}. We immediately get

$$\overline{T}^{\pi}(k_1, k_2) \geq \frac{k_1}{r_1} + \frac{k_2}{r_2}, \text{ for all } \pi \in \Pi_{sin}. \tag{18}$$

Next we will find a bound for policies in Π_{xor} and ultimately we will show that the policies in Π_{mix} are outperformed (in asymptotic sense) by those in $\Pi_{sin} \cup \Pi_{xor}$.

Let $N_{\min} \triangleq \min\{N_1, N_2\}$ and recall $(f, s) = (1, 2)$ if $r_1 \geq r_2$ and $(f, s) = (2, 1)$ otherwise. Observe that the following hold under any policy in Π_{xor}:

(i) While XOR controls are still available (i.e. $\mathcal{C}^x \cap \mathcal{C}_{sto}(t) \neq \emptyset$), a good packet departs only if coded with another good packet independently of the XOR control used.

(ii) At the end of the slot that the packets from one flow are all evacuated for the first time, it holds: exactly $N_{\min}$ good packets of both flows have departed.

We make the following helpful conventions:

(i) In case of a $\{u_1 \oplus u_2\}$ control involving two bad packets followed by a single control of one of the two bad packets (these two transmissions evacuate both packets), we assign one evacuated packet to each control.

(ii) Then, all XOR controls evacuate exactly one packet with the exception of the control $\{g_1 \oplus g_2\}$, which evacuates two packets. We make the convention that the first $N_{\min}$ good packets of the fast flow take up zero transmissions (the corresponding transmissions are counted for the first $N_{\min}$ good packets of the slow flow).

Let $J(i) - 1, i = 0, 1$ be the number of packets in front of the $N_{\min} + i$-th good packet in the unknown queue of the fast flow at time 0. Using the law of iterative expectations we get $\mathbb{E}[J(0)] = \mathbb{E}[N_{\min}] / p_f$ and $\mathbb{E}[J(1)] = (\mathbb{E}[N_{\min}] + 1) / p_f$.

All packets of the slow flow plus the bad packets of fast flow of at least up to $J(0)$ are evacuated in slots of r_s packets requiring one transmission each. Then the remaining $k_f - J(0)$ packets of the fast flow are evacuated in slots of r_f packets. Thus, for any $\pi \in \Pi_{XOR}$

$$\begin{aligned} \overline{T}^{\pi} &\geq \mathbb{E}\left[\left\lceil \frac{k_s + J(0) - N_{\min}}{r_s} \right\rceil\right] + \mathbb{E}\left[\left\lceil \frac{k_f - J(0)}{r_f} \right\rceil\right] \\ &\geq \mathbb{E}\left[\frac{k_s + J(0) - N_{\min}}{r_s}\right] + \mathbb{E}\left[\frac{k_f - J(0)}{r_f}\right] \\ &= \frac{k_s}{r_s} + \frac{(1-p_f)\mathbb{E}[N_{\min}]}{p_f r_s} + \frac{k_f}{r_f} - \frac{\mathbb{E}[N_{\min}]}{p_f r_f} \\ &= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[N_{\min}]}{p_f}\left[\frac{1}{\max\{r_1, r_2\}} - \frac{1-p_f}{\min\{r_1, r_2\}}\right], \end{aligned} \tag{19}$$

which combined with (18) yields

$$\overline{T}^{\pi}(k_1, k_2) \geq \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[N_{\min}]}{p_f}\left[\frac{1}{r_f} - \frac{1-p_f}{r_s}\right]^+,$$

for all $\pi \in \Pi_{sin} \cup \Pi_{xor}$. Using $\lim_{t\to\infty} \frac{\mathbb{E}[N_{\min}]}{t} = \min\{k_1 p_1, k_2 p_2\}$ found above, we conclude that

$$\liminf_{t\to\infty} \frac{\overline{T}^{\pi}(\lceil t\lambda_1 \rceil, \lceil t\lambda_2 \rceil)}{t} \geq B^{req}, \quad \pi \in \Pi_{sin} \cup \Pi_{xor},$$

where B^{req} is the requested limit. Next, we consider set Π_{mix}.

Pick a policy $\pi \in \Pi_{mix}$. Let $L_s(k_s, k_f, \omega), L_f(k_s, k_f, \omega)$ be random variables denoting the number of packets that were evacuated with controls $\{g_s\}, \{u_s\}$ and $\{g_f\}, \{u_f\}$ respectively. We have $l_i \triangleq \mathbb{E}\left[L_i(k_s, k_f, \omega)\right]$ and $0 \leq l_i \leq k_i$, for $i \in \{s, f\}$.

Let $M_s(k_s, k_f, \omega), M_f(k_s, k_f, \omega)$ be the number of good packets that were evacuated with the above controls in the fast and slow flow respectively. Furthermore, let $H_i(k_s, k_f, \omega)$ be the number of good packets evacuated by controls $\{g_i\}$. By the law of large numbers we have w.p.1:

$$\lim_{t\to\infty} \frac{M_i(k_s t, k_f t, \omega)}{t} = \mathbb{E}[H_i] + p_i(\mathbb{E}[L_i] - \mathbb{E}[H_i]) \geq p_i l_i. \tag{20}$$

All k_s packets and the $k_f - L_f$ packets of the fast flow are evacuated with rate r_s. Therefore, the expected number of timeslots needed to evacuate these packets is:

$$\begin{aligned} \overline{T}_1 &\geq \mathbb{E}\left[\frac{k_s}{r_s} + \frac{k_f - L_f}{r_s} - \frac{\min(N_s - M_s, N_f - M_f)}{r_s}\right] \\ &= \frac{k_s}{r_s} + \frac{k_f - l_f}{r_s} - \frac{\mathbb{E}\left[\min(N_s - M_s, N_f - M_f)\right]}{r_s}, \end{aligned} \tag{21}$$

where we have subtracted the time corresponding to XORs between good packets. Also, the inequality is due to the assumption that no dummy packets were used. The rest L_f packets are evacuated with rate r_f thus:

$$\overline{T}_2 \geq \mathbb{E}\left[\left\lceil \frac{L_f}{r_f} \right\rceil\right] \geq \frac{l_f}{r_f}. \tag{22}$$

Therefore, using (21) and (22), we have:

$$\begin{aligned} \overline{T}^{\pi}(k_1, k_2) = \overline{T}_1 + \overline{T}_2 \geq \frac{k_s}{r_s} + \frac{k_f}{r_s} - l_f \left(\frac{1}{r_s} - \frac{1}{r_f} \right) \\ - \frac{\mathbb{E}[\min(N_s - M_s, N_f - M_f)]}{r_s}. \end{aligned} \tag{23}$$

Define $B^{mix} \triangleq \liminf_{t \to \infty} \frac{\overline{T}^{\pi}(\lceil tk_1 \rceil, \lceil tk_2 \rceil)}{t}$, $\pi \in \Pi_{mix}$. Taking the limit in RHS of (23), using uniform integrability of the considered random sequences, we get w.p.1:

$$\begin{aligned} B^{mix} \overset{(20)}{\geq} \frac{k_s}{r_s} + \frac{k_f}{r_s} - l_f \left(\frac{1}{r_s} - \frac{1}{r_f} \right) \\ - \frac{\min\left(p_s(k_s - l_s), p_f(k_f - l_f) \right)}{r_s}. \end{aligned} \tag{24}$$

Next we show that $B^{mix} \geq B^{req}$. Define the conditions:

$$c_1 \equiv p_s k_s \geq p_f k_f \qquad\qquad c_2 \equiv p_s(k_s - l_s) \geq p_f(k_f - l_f)$$

Using $0 \leq l_i \leq k_i$, $i \in \{s, f\}$, for $\overline{p_f} > \frac{r_s}{r_f}$ we have

$$B^{mix} - B^{req} = \begin{cases} (k_f - l_f)(\frac{\overline{p_f}}{r_s} - \frac{1}{r_f}) & , c_2 \\ \frac{p_f(k_f - l_f) - p_s(k_s - l_s)}{r_s} + (k_f - l_f)(\frac{\overline{p_f}}{r_s} - \frac{1}{r_f}) & , \overline{c_2}, \end{cases}$$

while for $\overline{p_f} \leq \frac{r_s}{r_f}$

$$B^{mix} - B^{req} = \begin{cases} l_f(\frac{1}{r_f} - \frac{\overline{p_f}}{r_s}) & , c_1 \text{ and } c_2 \\ \frac{p_f(k_f - l_f) - p_s(k_s - l_s)}{r_s} + l_f(\frac{1}{r_f} - \frac{\overline{p_f}}{r_s}) & , c_1 \text{ and } \overline{c_2} \\ \frac{p_s k_s + p_f l_f - p_f k_f}{p_f}(\frac{1}{r_f} - \frac{\overline{p_f}}{r_s}) & , \overline{c_1} \text{ and } c_2 \\ \frac{p_f(k_f - l_f) - p_s(k_s - l_s)}{p_f}(\frac{1}{r_s} - \frac{1}{r_f}) & \\ + \frac{p_s l_s}{p_f}(\frac{1}{r_f} - \frac{\overline{p_f}}{r_s}) & , \overline{c_1} \text{ and } \overline{c_2} \end{cases}$$

All cases can be verified to be nonnegative. □

Proof of Theorem 8. We follow the steps of the proof of Theorem 7 closely. First, note that if $1 - p_f > \frac{\min(r_1,r_2)}{\max(r_1,r_2)}$ is true, then π^* chooses only single controls and we quickly get

$$\hat{T}^{\pi^*}(\lambda_1, \lambda_2) = \frac{\lambda_1}{r_1} + \frac{\lambda_2}{r_2}.$$

If on the other hand the condition is false, then we have $\pi^* \in \Pi_{XOR}$. The difference from the proof of Theorem 7 is how packets between $J(0)$ and $J(1)$ are treated.

$$\begin{aligned}
\overline{T}^{\pi^*}(k_1,k_2) &\leq \mathbb{E}\left[\left\lceil \frac{k_s + J(1) - N_{\min}}{r_s} \right\rceil\right] + \mathbb{E}\left[\left\lceil \frac{k_f - J(0)}{r_f} \right\rceil\right] \\
&\leq \mathbb{E}\left[\frac{k_s + J(1) - N_{\min}}{r_s}\right] + \mathbb{E}\left[\frac{k_f - J(0)}{r_f}\right] + 2 \\
&= \frac{k_s}{r_s} + \frac{(1-p_f)\mathbb{E}[N_{\min}]}{p_f r_s} + \frac{1}{p_f r_s} + \frac{k_f}{r_f} - \frac{\mathbb{E}[N_{\min}]}{p_f r_f} + 2 \\
&= \frac{k_1}{r_1} + \frac{k_2}{r_2} - \frac{\mathbb{E}[N_{\min}]}{p_f}\left[\frac{1}{\max\{r_1,r_2\}} - \frac{1-p_f}{\min\{r_1,r_2\}}\right] \\
&\quad +2 + \frac{1}{p_f \min\{r_1,r_2\}}.
\end{aligned}$$

Taking the lim sup completes the proof. □

10.2 Proof of Base Case for Theorem 13

We begin by providing some basic lemmas for the case that all possible overhearing events took place. That is, the corresponding SIG G is a complete multipartite graph where each partition corresponds to the packets that belong to a single flow or, more formally, each partition i correlates to the "Wants" set W_i. We will use this case as an inductive basis later for the proof of the Theorem. We will say that a solution to $minrk_2(G)$ has the property **NIC** if it does not involve intra-flow XOR coding.

LEMMA 14: *Let $G = (V,E)$ be a SIG which is a complete multipartite graph were each partition i correlates to the "Wants" set W_i. Then there exists at least one solution to $minrk_2(G)$ with the NIC property.*

Proof. This will be a proof by construction. Suppose that G has n partitions (which translates to flows) named $P_1, \ldots, P_n$ and the number of vertices (packets of the "Wants" set) belonging to each partition respectively is given by the vector $\mathbf{k} = (k_{P_1}, k_{P_2}, ...k_{P_n})$ where $k = \sum_{i=1}^{n} k_{P_i}$. We number each vertex of G according to the partition it belongs as $v_1^{P_i}, v_2^{P_i}, \ldots, v_{k_{P_i}}^{P_i}$ if it belongs to partition P_i. Without loss of generality, suppose that $k_{P_1} \geq k_{P_2} \geq ... \geq k_{P_n}$. We propose the solution:

$$S = \begin{pmatrix} I_{k_{P_1}} & S_{P_1P_2} & \cdots & S_{P_1P_n} \\ S_{P_2P_1} & I_{k_{P_2}} & \cdots & S_{P_2P_n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{P_nP_1} & S_{P_nP_2} & \cdots & I_{k_{P_n}} \end{pmatrix} \tag{25}$$

where

$$S_{P_iP_j} = \begin{cases} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} & k_{P_i} > k_{P_j} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} & k_{P_i} = k_{P_j} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} & k_{P_i} < k_{P_j} \end{cases}$$

or more compactly:

$$S_{P_iP_j} = \begin{cases} \begin{pmatrix} I_{k_{P_j}} \\ 0 \end{pmatrix} & k_{P_i} > k_{P_j} \\ I_{k_{P_i}} & k_{P_i} = k_{P_j} \\ \begin{pmatrix} I_{k_{P_i}} & 0 \end{pmatrix} & k_{P_i} < k_{P_j} \end{cases}$$

and the first k_{P_1} rows correspond to the vertices $v_1^{P_1}, v_2^{P_1}, \ldots, v_{k_{P_1}}^{P_1}$, the next k_{P_2} rows correspond to the vertices $v_1^{P_2}, v_2^{P_2}, \ldots, v_{k_{P_2}}^{P_2}$ and so on.

In order for S to be a solution to $minrk_2(G)$ it has to fit G first. This it does, because since G is a complete multipartite graph, the submatrices $S_{P_iP_j}$ can be chosen arbitrarily. Also, it must achieve the $minrk_2(G)$. This is shown by noticing that the row that corresponds to $v_1^{P_1}$ is the same as the row that corresponds to $v_1^{P_2}$ etc. This also holds for $v_2^{P_1}$, $v_2^{P_2}$ etc whose corresponding rows are all equal as well. This means we have k_{P_1} classes that contain rows which are equal to each other. Therefore it follows that:

$$rk_2(S) = k_{P_1}$$

Moreover, there is at least one non-singular submatrix of size k_{P_1}, namely $I_{k_{P_1}}$, in all possible solutions. Therefore:

$$minrk_2(A) \geq k_{P_1}$$

From the two above relations, the proof that S is a solution is complete. It is also evident that it has the NIC property, since every sub-row of the submatrices $S_{P_iP_j}$ has at most one '1'.

An example solution with 4 flows with $\mathbf{k} = (3,2,1,1)$ is as follows:

$$S = \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 & 1 & | & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 & 0 & | & 0 & | & 0 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 & 1 & | & 0 & | & 0 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \end{pmatrix}$$

□

LEMMA 15: *Let $G = (V, E)$ be a SIG which is a complete multipartite graph were each partition i correlates to the "Wants" set W_i. Let $k_{max} \triangleq \max_i |W_i|$ be the number of packets in the largest flow and let R be the set of indices of the receivers that are the destinations of all flows that have k_{max} packets. Then all overhearing hyperedges that correspond to the overheard packets by the receivers in R are essential.*

Proof. We will prove this by contradiction. Suppose that there exists a non-essential overhearing hyperedge $O \subset E$ that corresponds to an overhearing event in a receiver $r \in R$. Let p be the packet that was overheard. Then there exists at least one solution of the form in Figure 11, where C is the overhearing sub-column of O in the solution and $C = 0$.

The row that corresponds to p and all rows that correspond to the packets of r are linearly independent. Therefore:

$$minrk_2(G) \geq k_{max} + 1.$$

But solution S in (25) has rank k_{max} which is a contradiction.

□

If we combine the above lemma with solution (25) then we see that for all essential overhearing hyperedges there is a solution, i.e (25), where their corresponding overhearing sub-columns are a column of the Identity matrix of the same size. That is, P2 is satisfied for the base case in the proof of Theorem 13.

We continue by proving that all other overhearing hyperedges are non-essential:

LEMMA 16: *Let $G = (V, E)$ be a SIG which is a complete multipartite graph were each partition i correlates to the "Wants" set W_i. Let $k_{max} \triangleq \max_i |W_i|$ be the number of packets in the largest flow and let R be the set of indices of the receivers that are the destinations of all flows that have k_{max} packets. Then all overhearing hyperedges that correspond to the overhearing events of the receivers in R^c (the complement of R) are non-essential.*

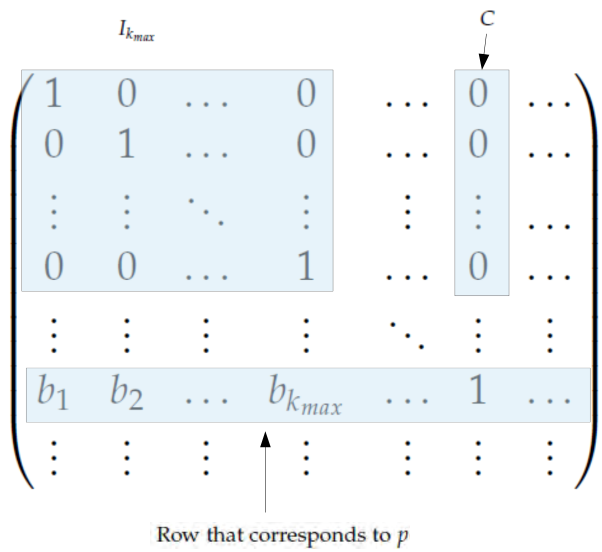

Figure 11: The set of rows that are highlighted in blue is linearly independent, but solution (25) has rank lesser than the number of rows of this set.

Proof. This will be a proof by construction. For every overhearing sub-column C that corresponds to an overhearing hyperedge O, we will use the solution in S to construct a new solution S' that has the NIC property and $C = 0$. Since $C = 0$, O is non-essential according to Remark 1.

Let S_r be the r-th row of S and $k_{max} \triangleq k_{P_1}$. Take any row j that corresponds to a packet $p \in W_i, i \in R^c$. We will essentially make row S_j and all rows that are equal to $S_{k_{max}}$ equal. Create the row:

$$u(a) = \begin{cases} 1 & a = j \\ S_{k_{max}}(a) & \text{otherwise} \end{cases}$$

Finally we construct S' such that:

$$S'_r = \begin{cases} u & S_r = S_{k_{max}} \text{ or } r = j \\ S_r & \text{otherwise} \end{cases}$$

S' is also a solution and has the exact same properties of S. That is, the rank is conserved since there are still k_{P_1} classes of equal rows and the NIC property still holds. As an example, we will try to replace a '1' in the last row of S (underlined):

$$S = \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 & 1 & | & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 & 0 & | & 0 & | & 0 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 & 1 & | & 0 & | & 0 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & \underline{1} & | & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 & 1 & | & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 & 0 & | & 0 & | & 1 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ 0 & 1 & 0 & | & 0 & 1 & | & 0 & | & 1 \\ - & - & - & + & - & - & + & - & + & - \\ 1 & 0 & 0 & | & 1 & 0 & | & 1 & | & 1 \\ - & - & - & + & - & - & + & - & + & - \\ 0 & 0 & 1 & | & 0 & 0 & | & 0 & | & 1 \end{pmatrix} = S'$$

□

Notice that, for each non-essential overhearing hyperedge there exists a solution with the NIC property that has a zero corresponding overhearing sub-column. This is due to the way we constructed the new solutions S' in the above proof. Therefore, P1 for the base case of Theorem 13 is also satisfied.

10.3 Basic Theorems

THEOREM 17: *Let $A \in \mathbb{F}_2^{n\times n}$ be any block triangular 0-1 matrix with diagonal blocks $D_1 \in \mathbb{F}_2^{n_1\times n_1}, ..., D_m \in \mathbb{F}_2^{n_m\times n_m}$, $n = \sum_{i=1}^{m} n_i$. Then:*

$$rk_2(A) \geq \sum_{i=1}^{m} rk_2(D_i) \tag{26}$$

Proof. Without loss of generality, suppose that A is block upper triangular. Let M_i be a maximal linearly independent set of rows of D_i. Let M_i' be the rows of A that correspond to the rows of M_i. The set $M = \cup_{i=1}^{m} M_i'$ is also linearly independent. To prove that, we prove that a linear combination of the rows of M produces the zero vector iff the linear coefficients are zero. Indeed, to produce the first n_1 elements of the zero vector then we have to choose linear coefficients equal to 0 for the vectors of $M_1' \in M$, otherwise this would imply that the vectors of M_1 are not linearly independent. Continuing, to produce the next n_2 elements of the zero vector, we first observe that:

1) The rows of M_1' were "zeroed out" as described above, and therefore they contribute a zero for each entry.
2) All other rows of M are below the rows of M_2' and have zero between columns $n_1 + 1$ and n_2 due to the structure of the matrix.

Therefore only the columns of M_2' can contribute and again we have to choose linear coefficients equal to 0 for the rows of $M_2' \in M$ since otherwise it would imply that the rows of M_2 are linearly dependent. Continuing the above process, all linear coefficients are chosen to be zero.

The set M is linearly independent, but has not been proven to be necessarily maximal. Moreover, $|M_i'| = |M_i| = rk_2(D_i)$ from the definition of matrix rank, and therefore $|M| = \sum_{i=1}^{n} rk_2(D_i)$. Now, let J be a maximal linear independent set of A. We have

$$rk_2(A) = |J| \geq |M| = \sum_{i=1}^{n} rk_2(D_i)$$

which proves the theorem. □

A useful property is that by reversing any entry of a matrix in $\mathbb{F}_2$, we get another matrix that has a rank difference from the first of at most one.

THEOREM 18: *Let $A \in \mathbb{F}_2^{n \times m}$ be any 0-1 matrix and $A' \equiv A + E_{ij}$ where $E_{ij} \in \mathbb{F}_2^{n \times m}$ is a matrix with '1' in entry (i, j) and '0' otherwise. Then:*

$$rk_2(A) - 1 \leq rk_2(A') \leq rk_2(A) + 1 \tag{27}$$

Proof. This will be a direct proof, using the subadditivity property of matrix rank of any field ([33]). For the right part of the inequality, we have

$$\begin{aligned} rk_2(A') &= rk_2(A + E_{ij}) \\ &\overset{\text{subadditivity}}{\leq} rk_2(A) + rk_2(E_{ij}) \\ &= rk_2(A) + 1 \end{aligned}$$

The left part of the inequality is proved if one notices that $A = A' + E_{ij}$ also and repeating the above. □

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