Inventory policies in continuous review systems: evaluation and estimation

by

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A thesis submitted for the degree of Doctor of Philosophy

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May, 2016

Preface

Due to the rapid growth of the manufacturing industries during the last centuries, a lot of managerial problems have been developed. Among them, the problem of inventory management is one of the most significant areas which requires efficient solutions. Recognizing that there is no approach or technique that is applicable to all firms, we present various inventory models and describe what techniques should be suitable. Knowing that the goal of managers is the development of inventory models to obtain solutions for real world problems, it is also a fact that the literature for inventory models also presents interesting theoretically mathematical problems.

Thus, in this thesis, we explore the conditions under which the continuous review inventory model has its optimal solution and we try to examine many different variations of this inventory model in order to study their theoretically mathematical properties which may be helpful in practice at some future time. Specially, assuming fixed lead-time and different lead-time demand distributions, in the first chapters, under known demand parameters, our concern is to study only theoretically this inventory model in order to determine the minimum of the total cost function for either an approximate or an exact expression of the expected on-hand inventory level at any point in time. Extending the analysis, in the next chapters, for the case in which demand estimation is required, we develop confidence intervals for the minimum of the total cost and we present a theoretical work which may be helpful in practice, as this work gives the ability to managers to know with certainty under a specific nominal confidence level the range of the minimum total cost.

The entire text has been divided into six chapters. All the chapters have been written in a lucid and illustrative manner. Chapter 1 discusses classification of various types of inventory models and includes the analytical structure and the objectives of the thesis. The other five chapters deal to different variations of the continuous review (Q,R) inventory model. More specific, in Chapter 2, assuming known demand parameters, backorders, fixed lead-time and an exact expression for the expected on-hand inventory level at any point in time, the objective is to investigate the minimization process of the total cost function for J-shaped or unimodal lead-time demand distributions. In Chapter 3, the same inventory model as in chapter 2 is studied but with an approximate expression for the expected on-hand inventory level at any point in time. Unmet demand is, again, fully backlogged and orders arrive after a lead-time. The objective is to find an inventory replenishment policy so as to achieve target order quantities and reorder points with minimum total cost. Chapter 4 addresses the issue of calculating optimal inventory policies when unknown independent demand parameters are estimated from a sample of generated demand observations. Using the total cost function of chapter 3, firstly we develop estimator for the minimum total cost and secondly we derive its confidence interval whose performance is evaluated through Monte-Carlo simulations. Chapter 5 is related to the investigation of the continuous review inventory model when correlated demand exists. In fact, assuming the same total cost function as in chapter 3, we develop a procedure to determine the target inventory measures when the issue of both correlated demand and unknown demand parameters is addressed. Using the maximum likelihood estimators for the stationary mean, the stationary variance and the theoretical autocorrelation coefficient at lag one, we develop estimators for the optimal order quantity, optimal reorder point and minimum total cost. Further, we study under which conditions the independent demand model can be a good approximation to handle the autocorrelated demand. Finally, in Chapter 6 conclusions and suggestions for further research are presented.

Acknowledgements

I would like to express my wholehearted gratitude to my supervisor Dr. Ilias Kevork for his patience and his continuous support and guidance throughout the period of my Ph.D study.

I am also grateful to my co-advisors Prof. Christos Kollias and Dr. Nickolaos Tzeremes for their comments and suggestions during the period of this research.

I would like to thank Prof. Ioannis Halikias, Prof. Jenny Pange, Dr. Helen Kandilorou and Dr. Stephanos Papadamou for their participation in the reading committee of my Ph.D thesis.

Finally, I would like to thank my father Nikolaos Tziourtzioumis, my mother Despoina Papathanasiou and my brother Dimitrios Tziourtzioumis for all their love, encouragement and for always supporting my choices.

Christos Tziourtzioumis

May 2016

Contents

List of symbols

D_t	:	Demand size occurred during time unit t.
μ_t	:	Expected value of D_t.
σ_t^2	:	Variance of D_t.
X	:	Total demand in the lead-time, $X = \sum_{t=1}^{L} D_t$.
μ_L	:	Expected demand in the lead-time.
σ_L^2	:	Variance of the lead-time demand.
D	:	Expected demand in the reference period.
$f(x)$	:	Probability density function of x.
$F(x)$	:	Cumulative distribution function of x.
Q	:	Order quantity.
R	:	Reorder point.
C	:	Total cost function.
L	:	Lead-time.
A	:	Fixed ordering cost.
h	:	Holding cost per unit of product per reference period.
s	:	Shortage cost per unit of product backordered.
s'	:	Shortage cost per unit of product backordered per reference period.
s_o	:	Profit per unit.
c	:	Purchasing cost.
B	:	Cost per stockout occasion.
T	:	Review period.
J	:	Cost of making a review.
I	:	Expected on-hand inventory level at any point in time.
Y	:	Inventory position.
$R(L)$	:	Lead-time crashing cost.
Q_W	:	Wilson economic order quantity.
$S(R)$	:	Expected size of backorders per inventory cycle.
S_{tm}	:	Expected size of backorders using s'.
z	:	Safety factor, $z = (R - \mu_L)/\sigma_L$.
$\varphi(z)$	:	Probability density function of the standard Normal evaluated at z.
$\Phi(z)$	:	Cumulative density function of the standard Normal evaluated at z.
z_x	:	The x^{th} percentile of the standard Normal, or alternatively, the inverse cumulative distribution function of the standard Normal evaluated at x.
P	:	Fixed cycle service level, $P = \Pr\left(\sum_{t=1}^{L} D_t \le R\right) = \Phi(z_P)$.
$B(P)$	:	Safety stock, $B(P) = R - \mu_L = z_P \cdot \sigma_L$.

Chapter 1

Introduction

1.1 Business context

The inventory policy constitutes an important part to all enterprises in any sector of the economy. The determination of the optimal reorder point, inventory level as well as optimal order quantity are a direct interest of organizations in order to take right decisions for maximizing (minimizing) the profit (cost). Today, as the inventory control can give a significant competitive advantage, the control of the flow of goods from suppliers to final customers is fully organized by top managers.

There are many reasons why organizations should carry inventories of goods. The most important reason is the customer service because, as it is well known, without inventories customers need to wait until their order quantities are received. In fact, however, the customers do not want to wait for long periods of time and they are usually satisfied when organizations have available goods on time when demands for them occur. Therefore, the mismatch between demand and supply is the main reason of maintaining inventories of goods. Nonetheless, there are other reasons for holding inventories which according to Waters (2008) are:

(a) to act as a buffer between different production operations,

(b) to allow for demands that are larger than expected or come at unexpected times,

(c) to allow for deliveries that are delayed or too small,

(d) to take advantage of price discounts on large orders,

(e) to buy items when the price is low and expected to rise,

(f) to buy items that are going out of production or are difficult to find,

(g) to make full loads and reduce transport costs,

(h) to give cover for emergencies.

Furthermore, many times, organizations need to maintain either finished or semi-finished goods where for the last it holds that there is a time-lag before a semi-finished product can be used from one stage of production to the next stage of production. Hence, as many different independently demand items are held in inventory, especially in manufacturing, it is very difficult for the company to make different inventory control for each item of them. For this problem, a solution has been given by Vilfredo Pareto by developing the ABC classification system which divides inventory into three categories based on the fact that a relatively small number of products consists the greatest part of the total dollar value of inventory. Specially, class A includes a small number of products which holds in inventory but they have big dollar value, namely this class represents 20 percent of total inventory units with about 80 percent of total inventory dollar value. Class B represents approximately 30 percent of total inventory units but only about 15 percent of total inventory dollar value. Finally, class C accounts for 50 percent of all inventory units but represents only 5 percent of total dollar value (e.g. Silver et al., 1998).

Summarizing, inventory management must be designed to meet the requirements of the market and to provide information to the other departments of the company (marketing, finance, etc) about customer’s behavior. For this reason, the inventory control of any physical good requires some crucial decisions to be taken by managers. According to Mansfield (1996), the decision making process both for enterprises and for non-profit organizations can be separated into five main stages which are:

Stage 1: Establishing objective targets

When companies take a decision must specify the objectives. In particular, if they do not know what is trying to implement then there is no reasonable way to get the right decision.

Stage 2: Defining the problem

One of the most difficult parts of the decision making is the definition of the problem. Often, top managers face a situation which is unsatisfactory.

Stage 3: Identification of feasible solutions

After the time where the problem has been identified, the company must find possible solutions.

Stage 4: Choosing the best feasible solution

When companies identify all the feasible alternative solutions, then they should evaluate each of them and determine the best, based on their objectives.

Stage 5: Implementation of the decision

When a specific solution has been selected, then it must be implemented in order to be effective. This phase of the decision-making process is very important because the best decisions have no value without implementation.

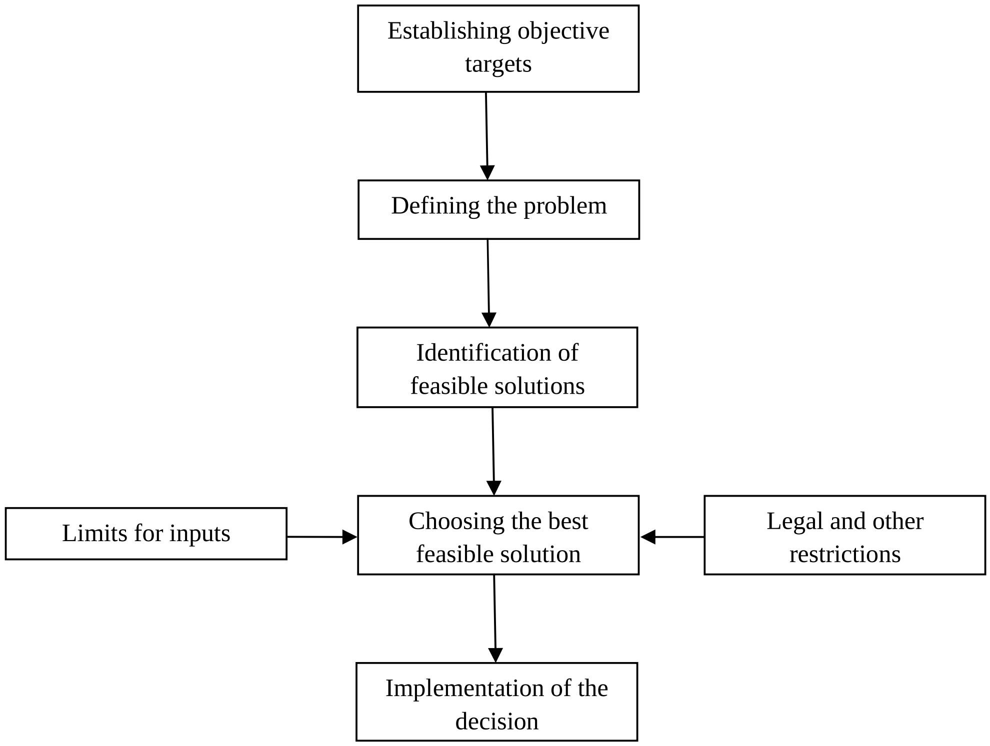


Figure 1.1 Basic decision process.

1.2 Inventory systems

Every decision which is made by managers in order to control inventories is associated with two crucial questions which are:

(a) how much quantity should be ordered for replenishment, and

(b) when the inventory should be replaced.

In real life, there are many types of inventory problems which require answers for these two fundamental questions. For those problems, many studies have been conducted in the inventory literature which show how a mathematical model can be used for controlling inventory systems. However, in a recent paper, Silver (2008) mentioned that there is a gap between theory and practice and therefore it is impossible the mathematical model to represent the real world problems. So, for this reason many assumptions must be made in order the real world inventory problem to be approached with accuracy.

Based on the widely cited book of Hadley and Whitin (1963), we can find many inventory systems which differ according to the number of items they carry, the review interval, the nature of demand parameters, the echelon structure of inventory system and the costs associated with the system. A classification of inventory systems is given in Figure 1.2 and will be explained analytically below. Similar classifications can be found in Bijvank (2009). Among them, in this thesis we develop mathematical models only for the continuous review (Q,R) inventory system. For these models we derive procedures in order to compute the optimal pair order quantity (Q) - reorder point (R) and to find the minimum cost. More details about the scope of this thesis are discussed in section 1.3.

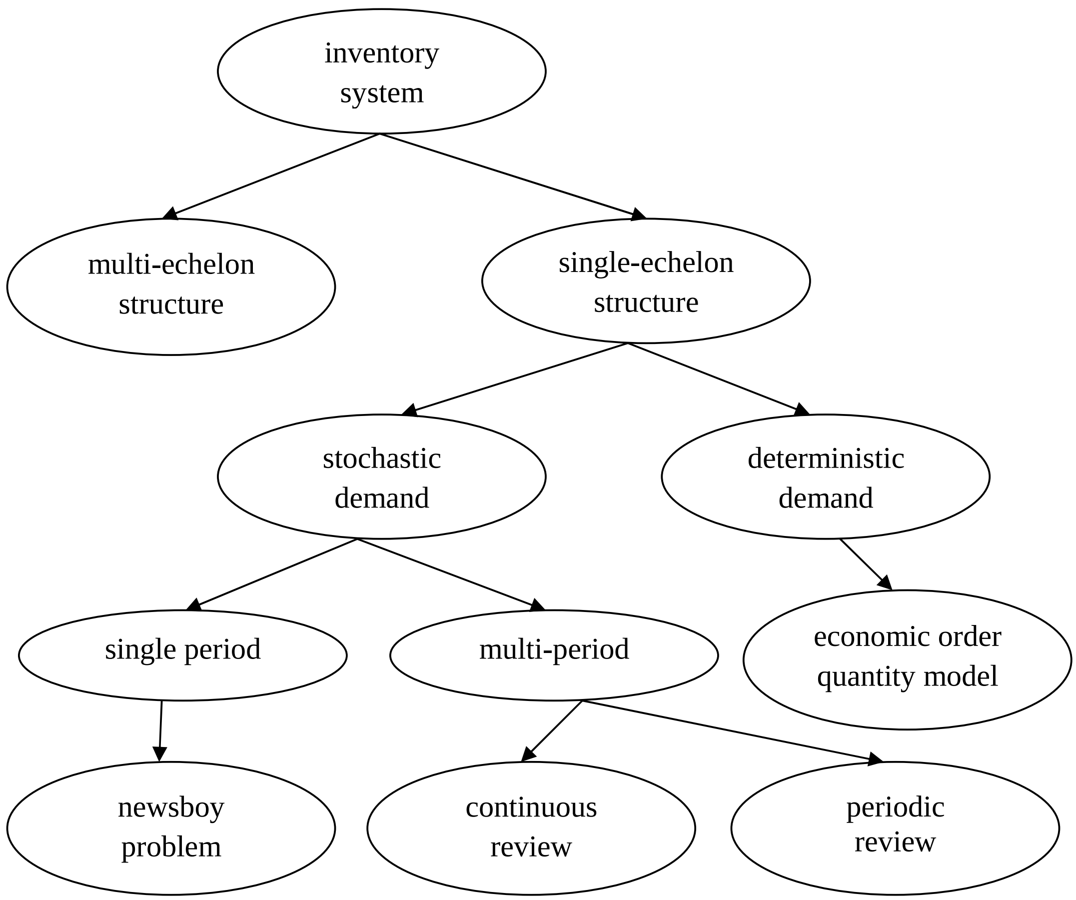


Figure 1.2 Classification of inventory systems.

When in an inventory system an item may be stocked in more than one stocking point then this leads to a multiechelon inventory system. Specifically, a type of multiechelon inventory system is given in Figure 1.3 for which there are three levels. For this system we consider that customers' demand can occur only at level 1 where retailers exist and sell the item. These retailers receive their stocks from warehouses at level 2 where the warehouses have their stocks from the factory warehouse at level 3. On the other hand when there is a single stocking point for example only at level 1 (a retail store, for example) then the system is called as single echelon. For more details about echelon inventory systems see Hadley and Whitin (1963).

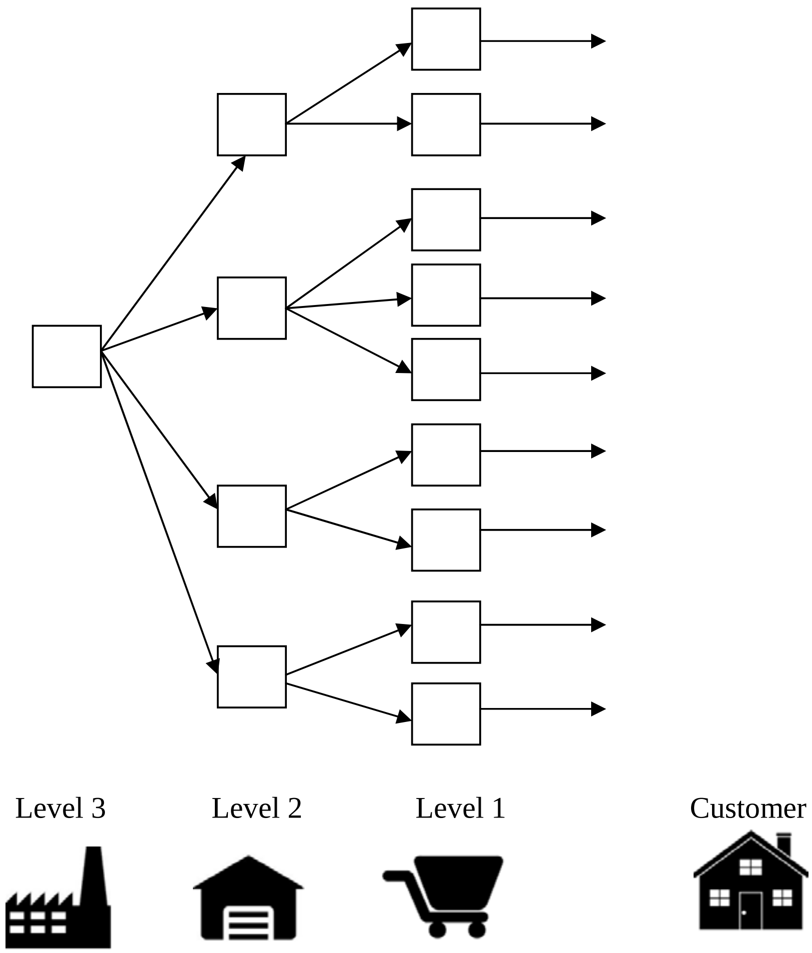


Figure 1.3 Echelon structure of inventory system.

According to the nature of demand pattern for an item which is carried by an inventory system, we can find in the literature two different inventory models. When the demand parameters are known (may be constant), and give us the opportunity to forecast the future demand periods with accuracy, then the inventory system is called as deterministic. For the deterministic models, it is possible to determine for all future times precisely what the state of the system will be when the state is known at a given time and the quantity to be ordered and the reorder point are specified. For this category the most popular inventory system is that one of Harris's 1913 Economic Order Quantity model. On the other hand, if randomness is introduced into the demand pattern making impossible its future predictions with certainty, then it becomes necessary to use stochastic systems where demands are described probabilistically. Therefore, one cannot know the state of the system at each point in time unless each transaction (demand or placement of order) is recorded and reported as it occurs. In this category, for which demand is assumed as random variable we can find inventory systems for which products either have a limited purchasing period (single period systems) or they can be sold at any point in time (multi period systems). For single period inventory systems we can find products for example Christmas tree or newspaper and the essential characteristic of this category is the required determination only of the order quantity for a single time period. The most popular inventory model of this category is the newsboy (or newsvendor) problem. On the other hand, in multi period inventory systems except the order size, the reorder point also should be determined. Such systems are characterized by the review interval. The most common inventory models in business and industry are reviewed either continuously (continuous review) or at regular period intervals of specified length (period review). Finally, in the inventory literature we can find different replenishment policies according to the costs associated with the system. The most important costs among others are: (a) the ordering cost which incurred each time an order is placed, (b) the holding cost for maintaining inventory and (c) the shortage cost for unsatisfied demands when the system is out of stock.

1.2.1 Economic Order Quantity Model

Even though inventory control is a natural situation that everyone undertakes either at home or at work, the study of inventory problems almost started at the beginnings of the 20th century. The earliest study has been conducted by Ford Whitman Harris (1913), who presented the familiar economic order quantity (EOQ) model in a paper published in *Factory, The Magazine of Management*. This model, according to the work of Erlenkotter (1990), was unnoticed for many years before its rediscovery in 1988 (Erlenkotter, 1989). The annual total cost function of Harris's EOQ model, which is illustrated graphically in Figure 1.4, is based on the following assumptions:

- Deterministic demand (demand is known with certainty and constant over time)
- Fixed lead-time
- There is no shortage in inventory

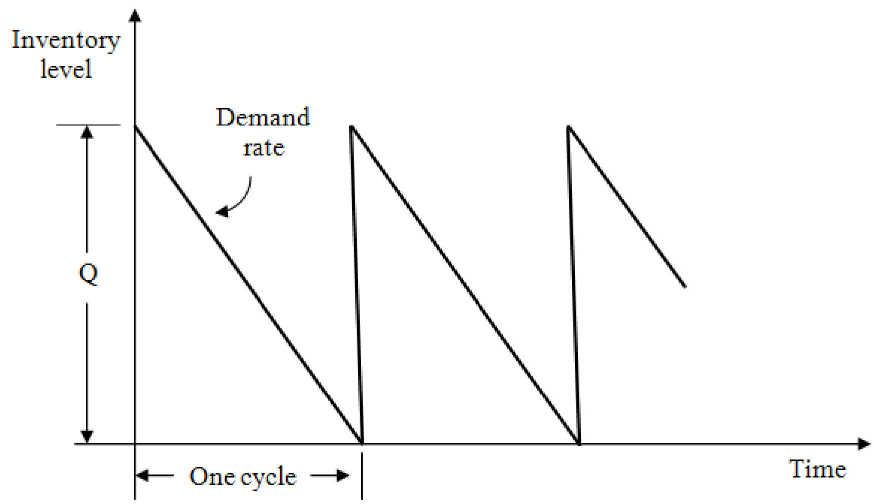


Figure 1.4 Economic order quantity model.

The economic order quantity is the order size that minimizes the sum of purchasing costs, holding costs and ordering costs. Specifically, if the order quantity is Q then for the annual demand, D, which is assumed to be known and constant the number of cycles per year must be $\frac{D}{Q}$. Therefore, since the purchasing cost per order is $c \times Q$ and the number of cycles per year is $\frac{D}{Q}$ then the average annual purchasing cost is $c \times D$:

$$\text{Annual purchasing cost} = c \times Q \times \frac{D}{Q} = c \times D.$$

Further, the average annual ordering cost is computed by multiplying the fixed cost per order, A, with the number of cycles per year, $\frac{D}{Q}$:

$$\text{Annual ordering cost} = A \times \frac{D}{Q}.$$

Finally, since the inventory holding cost per unit depends on the length of time where the unit remains in inventory, then the average annual inventory holding cost must be h times the average inventory which is determined by dividing the maximum inventory Q and the minimum inventory 0, by two:

$$\text{Annual holding cost} = h \times \frac{Q+0}{2} = h \times \frac{Q}{2}.$$

Consequently, the total average annual cost is the sum only of the ordering and holding costs because the decision variable which is the order quantity is not included in the purchasing cost. Then the cost function is written as:

$$C(Q) = \frac{A \times D}{Q} + \frac{h \times Q}{2}. \tag{1.1}$$

Taking the first order condition with respect to Q, $dC/dQ = 0$, we get the optimal order quantity which is given by (1.3)

$$\frac{dC}{dQ} = -\frac{A \times D}{Q^2} + \frac{h}{2}, \tag{1.2}$$

or

$$EOQ = Q_W = Q^* = \sqrt{2\frac{A \times D}{h}}. \tag{1.3}$$

The total minimum cost is determined by substituting the value of the optimal order size, Q^*, into the total cost equation (1.1):

$$C(Q^*) = \frac{A \times D}{Q^*} + \frac{h \times Q^*}{2}. \tag{1.4}$$

It will be noted that (1.3) gives the Q which yields the unique minimum of the cost function (1.1) since $dC/dQ = 0$ and $d^2C/dQ^2 > 0$ for all $Q > 0$:

$$\frac{d^2C}{dQ^2} = 2\frac{A \times D}{Q^3} > 0. \tag{1.5}$$

In the case where the company orders the optimal quantity (1.3) then the holding cost equals to the ordering cost (see Figure 1.5). More specific,

$$\text{Holding cost} = h\frac{Q^*}{2} = \frac{h}{2}\sqrt{2\frac{A \times D}{h}} = \sqrt{\frac{A \times D \times h}{2}} \tag{1.6}$$

$$\text{Ordering cost} = \frac{A \times D}{Q^*} = \frac{A \times D}{\sqrt{2\frac{A \times D}{h}}} = \sqrt{\frac{A \times D \times h}{2}}. \tag{1.7}$$

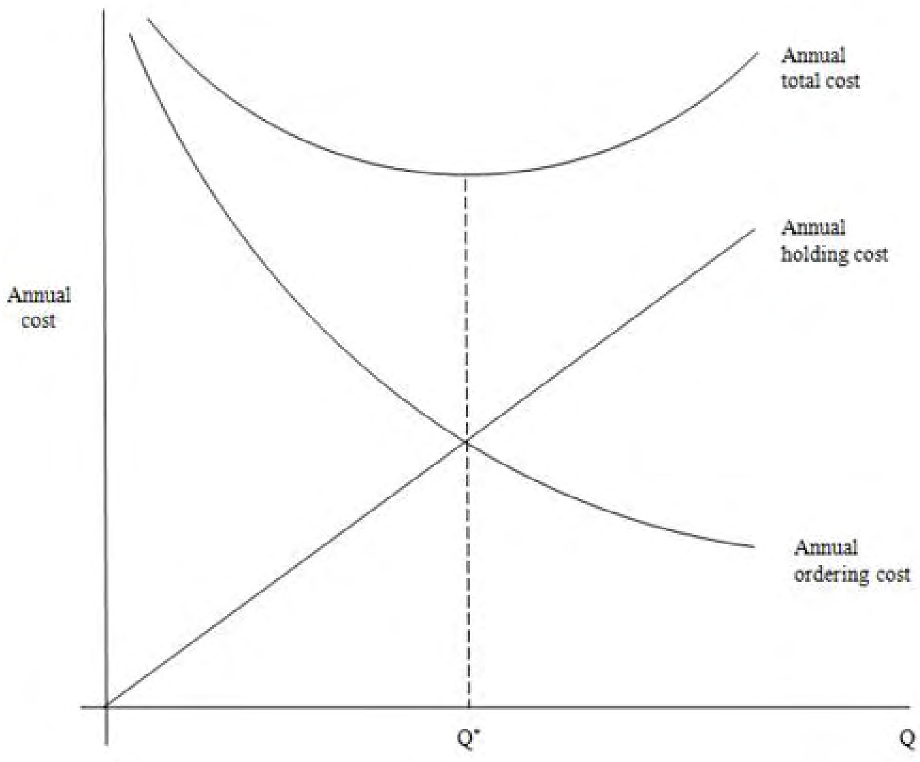


Figure 1.5 Optimal order quantity and minimum total cost.

Concluding, the EOQ model is often referred to the literature as the Wilson formula since R.H. Wilson extended and studied in depth Harris's EOQ model. The first full length book which examines inventory problems was that of Raymond (1931). At the present time, there are many books which deal in any detail with stochastic inventory models such as the most cited Hadley and Whitin (1963) and Silver, Pyke and Peterson (1998).

An example

We consider a store with an item which wants to determine the optimal order quantity and the minimum total cost by using the EOQ model, which is illustrated graphically in Figure 1.5, given the following parameters:

D (average annual demand) = 500 units per year,

A (ordering cost) = \$20,

h (holding cost per unit per year) = \$0.5.

The optimal order quantity is

$$Q^* = \sqrt{2\frac{A \times D}{h}} = \sqrt{2\frac{20 \times 500}{0.5}} = 200 \text{ units.}$$

The length of a cycle and the number of orders per year are respectively

length of a cycle = $\frac{Q^*}{D} = \frac{200}{500} = 0.4$ or $365\frac{Q^*}{D} = 365 \times 0.4 = 146$ days,

number of orders per year = $\frac{D}{Q^*} = \frac{500}{200} = 2.5$ orders per year.

The minimum total average annual cost is

$$C\left(Q^*\right) = \frac{A \times D}{Q^*} + \frac{h \times Q^*}{2} = \frac{20 \times 500}{200} + \frac{0.5 \times 200}{2} = 50 + 50 = \$100 .$$

1.2.2 Types of inventory systems

Although many inventory systems are met in the literature, in this chapter we describe the four most common ones, that is, two for the continuous review inventory model and two for the periodic. Before we begin the discussion of these systems, it is important to note the different definitions of the inventory level (see Figure 1.6).

On-hand inventory level: This is the amount of physical inventory which is immediately available to satisfy customer's demand.

Net inventory level: This is the on-hand inventory minus backorders.

Inventory on order: This is the item which is ordered but it is not delivered yet because of the lead-time.

Inventory position: This is the sum of the inventory on-hand plus the inventory on order minus backorders.

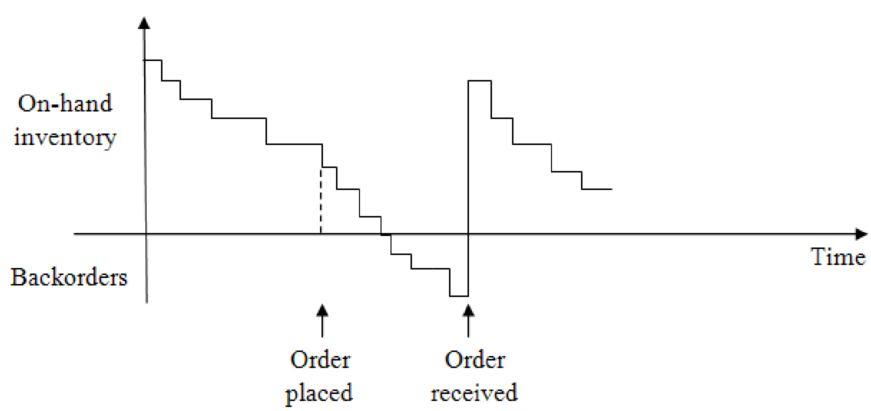


Figure 1.6 Inventory level.

1.2.2.1 Continuous review inventory model

The continuous review inventory model has been studied extensively in the area of inventory control. For this type of review policy, when the inventory position (on-hand plus on order minus backorders) drops to or falls below the reorder point R then an order of size Q is placed and is delivered after a fixed or variable period of time (lead-time) has elapsed. For this category, Silver et al. (1998) state that there are two inventory systems according to the order size. If the order quantity Q is fixed then the system is called as (R,Q) model. On the other hand, when a variable order quantity exists then the order size is such that the inventory level is increased to an order-up-to level S. This model is called as (R,S) model. Figure 1.7 and Figure 1.8 illustrate the (R,Q) and (R,S) inventory policies, respectively.

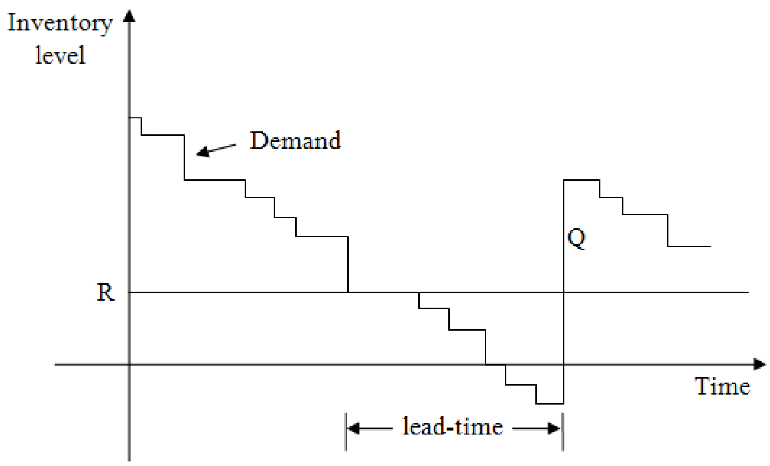


Figure 1.7 Continuous review (Q,R) inventory model.

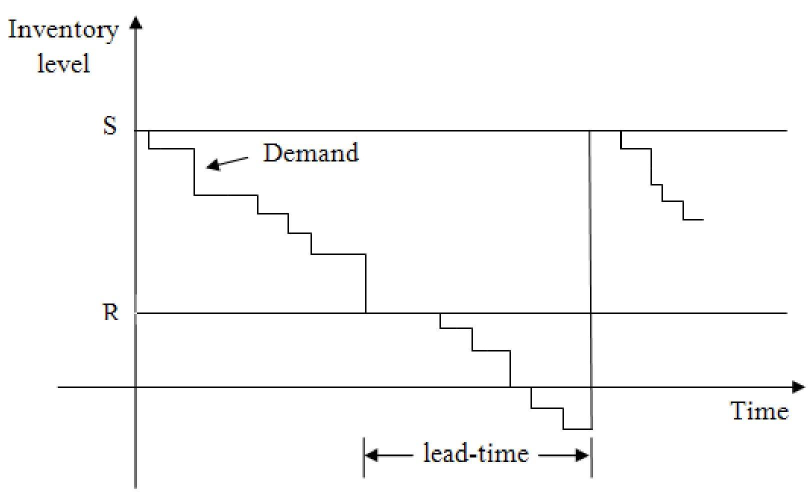


Figure 1.8 Continuous review (R,S) inventory model.

The book by Hadley and Whitin (1963) is a standard reference on mathematical inventory models. They proposed for the continuous review (Q,R) inventory model under fixed lead-time and backorders an expected annual total cost function which is constituted of a sum of three expected annual cost expressions, the first representing an ordering cost, the second an inventory carrying cost and the third a backordering cost. This average annual total cost function as mentioned by the authors is convex in the variables Q and R and therefore any solution obtained by setting the partial derivatives of the objective function equal to zero will determine a unique and global minimum cost. Specifically, the authors developed an iterative procedure and they said that this function is strictly convex when lead-time demand is normally distributed. These results are unproven and the reader is directed to solve a series of exercises. The first exercise asks the reader to show that the average number of backorders incurred per year is convex. Unfortunately, this function while convex in Q and convex in R, is not in general convex in both Q and R. It was pointed out by Veinott (1964) that this nonconvexity can lead to a failure of the optimization technique. Brooks and Lu (1969) addressed the same problem and showed, for normally distributed lead-time demand, that the average number of backorders incurred per year is convex when the reorder point is greater than the mean of the lead-time demand. Minh (1975) investigated the solutions of the continuous review model (Q,R) and showed that the total cost function is not convex and if a solution exists its partial derivatives equated to zero have exactly two solutions. His proof is given by explicitly expressing the ordering cost in terms of R and taking second partial derivatives. So, the author succeeded in classifying one of the solutions as a relative minimum and the other as a saddle point. Especially, he showed that when the reorder point is lower than the mean of the lead-time demand then feasible minimum solutions can be obtained.

Further, more recently, Lau et al. (2002a) examined the degeneracy problem in the order quantity-reorder point system when backordering is allowed and the excessive stockouts are presented as a cost per unit short, a cost per stockout occasion and a target fill rate. The authors have shown the reasons why and when the cost function breaks down. Chung et al. (2009) determined the optimal reorder point R and the optimal reorder quantity Q for the (Q,R) inventory system with a specified cost per stockout occasion by assuming that the lead-time demand is normally distributed. They showed that the total relevant cost function is not convex in general and proposed an alternative method to locate the optimal solution of the total relevant cost function because the convergence of the solution procedure described in Silver et al. (1998) is not necessarily true. In 1988, Das observed that the global minimum of the Hadley and Whitin model can be identified by means of the first order conditions and the shape of the lead-time demand distribution. In particular, for this model when lead-time demand distribution is either unimodal or J-shaped, the author found that the local minimum is also the global minimum.

Many studies in the literature, developed approximate but simpler alternatives to the Hadley and Whitin's (1963) iterative procedure for determining the reorder point and the order quantity when lead-time demand is normally distributed. Parker (1964) first introduced an Exponential approximation of the probability density function of the lead-time demand in the face of uncertain demand and indicated that this method gives approximately a 5%-7% reduction in the total costs when compared with the use of the Wilson economic order quantity. Related studies have been done by Presutti and Trepp (1970), Schroeder (1974) and Byrkett (1981). Further, Das (1976a) developed an iterative technique for the (Q,R) model and showed that the solution can be obtained algebraically by using an explicit formula.

Finally, in the literature, an amount of research has been conducted for the (R,S) policy. Scarf (1959a) has shown the optimality of this model for a class of discrete review dynamic nonstationary inventory models under the assumption that the one period expected costs are convex. Veinott (1966) extended Scarf's results and replaced his assumption by the principle that the negatives of these expected costs are unimodal. Archibald and Silver (1978) studied the (R,S) model with a discrete compound Poisson demand process and developed a formulae to calculate the cost for any pair (R,S). Concluding, several algorithms and approximations have been developed to determine the optimal values of the policy parameters. Such algorithms, are presented in Zheng and Federgruen (1990) and Federgruen and Zipkin (1984).

1.2.2.2 Periodic review inventory model

For this type of review policy, there are two models according to the existence or not of the reorder point R. If R does not exist then the basic idea is that every T units of time, we check the inventory position and we order enough to raise it to an order- up-to level S. This system is called as (T,S) model. On the other hand, when $R \neq 0$ then every T units of time we check the inventory position and if it is at or below the reorder point R, we order enough to raise it to S. However, if the inventory level is above R, nothing is done until the next review. This system is called as (T,R,S) model. Figure 1.9 and Figure 1.10 illustrate the (T,S) and (T,R,S) inventory policies, respectively.

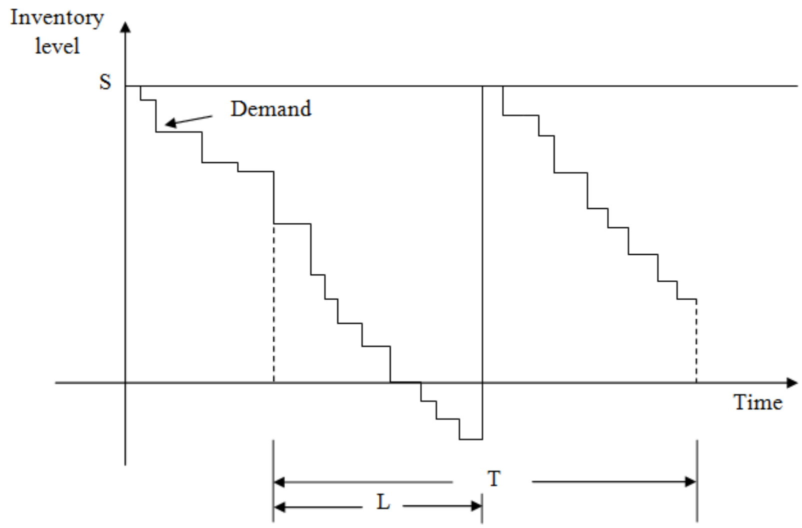


Figure 1.9 Periodic review (T,S) inventory model.

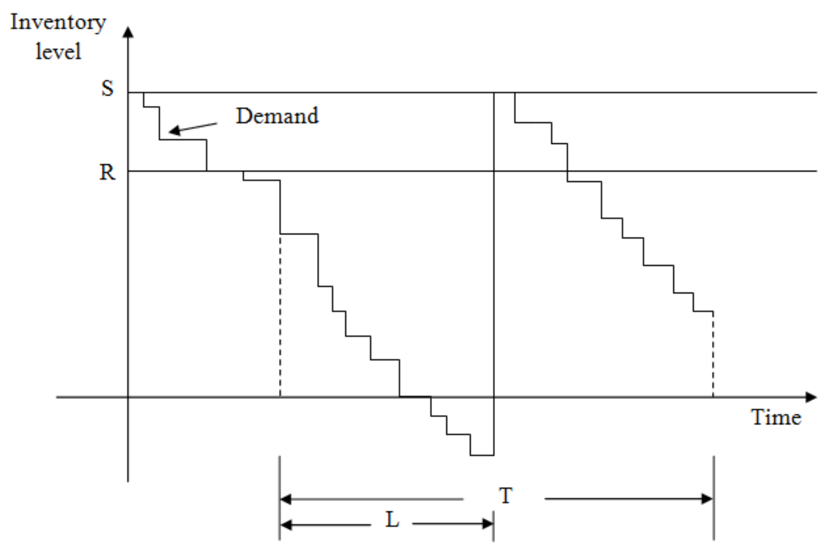


Figure 1.10 Periodic review (T,R,S) inventory model.

In real life conditions, the most widely used periodic review inventory system is the (T,S) model. For this model, one of the first studies has been conducted by Bellman et al. (1955). The authors examined the lost-sales case with non-zero lead-times. Under the restriction that the lead-time equals one review period (L=T), they minimized the total average ordering and shortage costs. Karlin and Scarf (1958) extended Bellman et al.'s (1955) model to the case where the holding cost is included and they provided a process in order to find the optimal policy. Morton (1969) extended Karlin and Scarf's (1958) model and by assuming linear and proportional ordering, holding and shortage costs he derived bounds on the optimal order sizes. Recently, his results are strengthened by Zipkin (2008). Other studies, under the assumption that the lead-time is an integral multiple of the review period length (L=nT) have been conducted by e.g., Morton (1971), Janakiraman et al. (2007) and Johansen and Thorstenson (2008).

Hadley and Whitin (1963) examined the (T,S) inventory model and developed approximate average annual total cost functions for the backorders and lost-sales cases. Under general assumptions, and by assuming that the expected net inventory just prior to the arrival of the next order is $R-\mu_L-DT$ while the expected net inventory immediately after the arrival of a procurement is $R-\mu_L$, the authors developed the following cost functions for stochastic demands which are given by:

Backorders case: $$C(S,T)=\frac{A+J}{T}+h\left(R-\mu_L-\frac{DT}{2}\right)+s\cdot E(S,T) \tag{1.8}$$

Lost-sales case: $$C(S,T)=\frac{J}{T}+h\left(R-\mu_L-\frac{DT}{2}+T\cdot E(S,T)\right)+s\cdot E(S,T) \tag{1.9}$$

where, $E(S,T)=\frac{1}{T}\int_S^\infty (x-S)h(x,T)dx$ is the average number of backorders incurred per year (for more details about h(x,T) see chapter 5 of the Hadley and Whitin's book) and J is the cost of making a review.

On the other hand, it will be recalled above that if at a review time T the inventory position is less than or equal to R, then an order quantity is placed which is sufficient to bring the inventory position up to S. For the (T,R,S) inventory model under constant or variable lead-time, Nahmias (1979) extended the model of Morton (1971) to include a fixed cost of placing an order. With constant lead-time, Hill and Johansen (2006) proposed an iteration algorithm in order to find the optimal order quantity under the assumption that no more than one order may be outstanding. Without the assumption for the number of outstanding orders, Bijvank et al. (2010) developed mathematical models for replenishment policies with fixed order sizes. Tijms and Groenevelt (1984) examined a service model for the periodic review (T,R,S) inventory system. Under the assumption that any excess demand is backordered, they set the optimal value of the reorder point R based on renewal theory. Bijvank and Vis (2012) developed a service model for a periodic review inventory system with lost-sales and compared the optimal replenishment policy to the (T,R,S) policy. Under numerical investigation they found that the (T,R,S) policy is not optimal but it results in an average cost increase only of 1.1%. Further, they proposed efficient approximation procedure in order to find the optimal values of the control parameters for the (T,R,S) system. Finally, they extended the numerical results of Tijms and Groenevelt's (1984) study to the lost-sales case and they found that Tijms and Groenevelt's procedure results in a cost increase of 1.3% compared to the best (T,R,S) policy.

1.2.3 Backorders and Lost-Sales inventory models

When the system is out of stock then there are two extreme cases to deal with the excess demand. First, if customers wait for a new delivery to replenish the inventory then this is referred as a backordering model (see Figure 1.11). On the contrary, when customers will either buy different product or visit another store or do not buy any product at all then this model is called as lost-sales (see Figure 1.12).

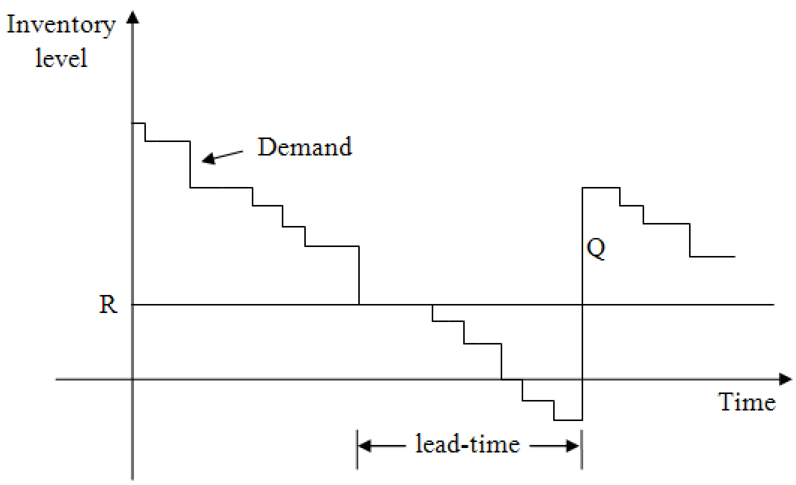


Figure 1.11 Continuous review (Q,R) inventory model with backorders.

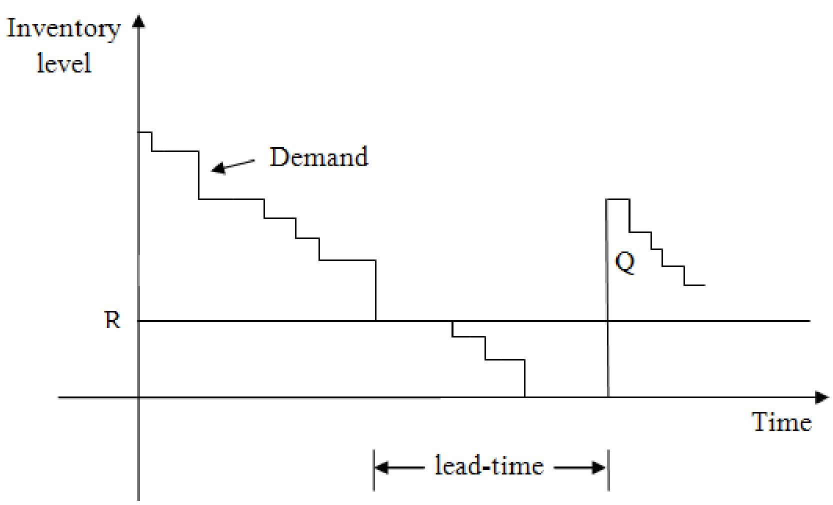


Figure 1.12 Continuous review (Q,R) inventory model with lost-sales.

Backorders or lost-sales

In the inventory literature, many authors studied the continuous review inventory model with either backorders or lost-sales. The earliest work for the (Q,R) inventory model has been conducted by Hadley and Whitin (1963) where the authors examined not only the backorder case but also the lost-sales case when the demand follows either Poisson or Normal distribution and the lead-time is constant. Under the restriction that only one order can be outstanding at a given time and the reorder point needs to be always positive the approximate cost functions for the backorder and lost-sales cases are respectively:

$$\mathrm{C}(\mathrm{Q},\mathrm{R})=\frac{\mathrm{A}\cdot\mathrm{D}}{\mathrm{Q}}+\mathrm{h}\cdot\mathrm{I}+\mathrm{s}\cdot\frac{\mathrm{D}}{\mathrm{Q}}\cdot\mathrm{S}(\mathrm{R}) \tag{1.10}$$

$$\mathrm{C}(\mathrm{Q},\mathrm{R})=\frac{\mathrm{A}\cdot\mathrm{D}}{\mathrm{Q}}+\mathrm{h}\cdot\mathrm{I}+\left(\mathrm{h}+\mathrm{s}\cdot\frac{\mathrm{D}}{\mathrm{Q}}\right)\cdot\mathrm{S}(\mathrm{R}) \tag{1.11}$$

where, A is the fixed ordering cost, I the expected on-hand inventory level at any point in time, h the holding cost per unit per year, s the shortage cost per unit backordered, D the annual expected demand, and $\mathrm{S}(\mathrm{R})$ the expected number of backorders $\left[\mathrm{S}(\mathrm{R})=\int_{\mathrm{R}}^{\infty}(\mathrm{x}-\mathrm{R})\,f(\mathrm{x})\mathrm{dx}\right]$.

Since μ_L is the mean of the lead-time demand, Hadley and Whitin proposed for the calculation of the expected on-hand inventory level at any point in time an approximate method which is based on the average of the inventory level at the beginning of the cycle, $\mathrm{Q}+\mathrm{R}-\mu_L$, and at the end of the cycle, $\mathrm{R}-\mu_L$:

$$\mathrm{I}_{\mathrm{HW}}=\frac{\mathrm{Q}}{2}+\mathrm{R}-\mu_L\,. \tag{1.12}$$

But, according to Lau and Lau (2002) the Hadley and Whitin’s approximate cost function is accurate only when the stockout probability is sufficiently small. However, for the backorders case and when the service level is not so high many studies have been conducted in order to find alternative approximate methods for the expected on-hand inventory level at any point in time. Among others are:

$$\mathrm{I}_{\mathrm{LV}}=\frac{\mathrm{Q}}{2}+\mathrm{R}-\mu_L+\frac{1}{2}\mathrm{S}(\mathrm{R}) \quad \text{(Love, 1979),} \tag{1.13}$$

$$\mathrm{I}_{\mathrm{W}}=\frac{\mathrm{Q}}{2}+\mathrm{R}-\mu_L+\frac{1}{2\mathrm{Q}}\mu_L\cdot\mathrm{S}(\mathrm{R}) \quad \text{(Wagner, 1975),} \tag{1.14}$$

$$I_{LAU} = \frac{Q}{2} + R - \mu_L + \frac{1}{2Q}\left(R - \mu_L\right)^2 \text{ (Lau and Lau, 2002),} \tag{1.15}$$

$$I_{YN} = \frac{Q}{2} - S\left(R\right) + \frac{1}{2Q}S\left(R\right)^2 \quad \text{(Yano, 1985).} \tag{1.16}$$

Concluding, it is important to note that, except the approximate methods in the literature, one can find an exact I-expression (see Zheng, 1992 and Platt et al., 1997) which is given by:

$$I_{ex} = \frac{1}{Q}\int_{R}^{Q+R}\left\{\int_{0}^{Y}\left(Y - x\right) f\left(x\right) dx\right\} dY \tag{1.17}$$

where Y is the inventory position and X is a random variable which represents the lead-time demand with probability density function $f(x)$ and cumulative distribution function $F(x)$.

The previous models discussed so far assume fixed lead-time. However, for the continuous review (Q,R) inventory model under backorders, but with variable lead-time, many studies have been also conducted. Among others, Ben-Daya and Rauf (1994) examined the determination of the order quantity under variable lead-time. Specially, using Hadley and Whitin's approximate method for the expected on-hand inventory level at any point in time and extending the classical model of Liao and Shyu (1991), they determine the average annual total cost function and they found the optimal order size. More specific, the cost function is given by:

$$C\left(Q,R\right) = \frac{A \cdot D}{Q} + h \cdot \left(\frac{Q}{2} + R - \mu_L\right) + \frac{D}{Q} \cdot R\left(L\right) \tag{1.18}$$

where, assuming that the lead-time, L, has n components and the *i*th has a minimum duration a_i and maximum duration b_i and a crashing cost per unit time c_i then the lead-time crashing cost is $R\left(L\right) = c_i\left(L_{i-1} - L\right) + \sum_{j=1}^{i-1} c_j\left(b_j - a_j\right)$.

Taking the first order conditions the optimal order size is

$$Q^* = \sqrt{2\frac{AD}{h} + 2\frac{D}{h}R\left(L\right)}\,. \tag{1.19}$$

Further, the authors proposed a new method in which the total crashing cost is related to the lead-time by a function which is given by:

$$R\left(L\right) = \alpha e^{-\beta L} \tag{1.20}$$

where, α and β are the minimum and maximum values of the lead-time respectively.

Then the new optimal order size is

$$Q^* = \sqrt{2\frac{AD}{h} + 2\frac{D}{h}\alpha e^{-\beta L}} \; . \qquad (1.21)$$

Comparing the two methods with the same parameter values they found that their method gives lower value to the total crashing cost.

Additionally, for the continuous review (Q,R) inventory model with lost-sales, Hadley and Whitin (1963) derived also an expression for the average annual total cost function under the assumption that there is only one order can be outstanding at a given time. For this method, by considering stochastic lead-times and using fixed (Q,R- model) or variable (R,S - model) order size, many studies have been conducted. Among others are those of Ravichandran (1984), Buchanan and Love (1985), Bechmann and Srinivasan (1987) and Johansen and Thorstenson (1993). Particularly, under Poisson demand process, these authors investigated the Hadley and Whitin's method for Phase type, Erlang and Exponential lead-time distributions, respectively. Archibald (1981) derived and solved an exact model for the (R,S) policy, assuming compound Poisson demand. For the case in which the demand is assumed to follow stochastic distributions, a large amount of studies have been conducted in the literature. Beyer (1994) derived the exact cost under the assumption that accumulated demand following a Wiener process. Kalpakam and Arivarignan (1988, 1989) examined the case in which the demand is generated from a renewal process and lead-time is assumed to be Exponential distributed. Mohebbi and Posner (1998) investigated the minimization of the total cost when the demand and the lead-time follow the compound Poisson and Erlang distributions, respectively. Finally, Rosling (1998) developed a general model in which any lead-time distribution can be used and demand is assumed to be continuous or Poisson distributed.

Mixture of backorders and lost-sales

Except the above inventory systems in which excess demand is either lost or backordered, there are also inventory models in which during the stockout period a fraction *b* of the excess demand is backordered and the remaining fraction *1-b* is lost forever. Montgomery et al. (1973) are the first who analyzed inventory systems with mixture of backorders and lost-sales under fixed lead-time and proposed average annual total cost functions with stochastic or deterministic demand for the continuous review (Q,R) or periodic review (T,S) inventory models. For instance, for the (Q,R) inventory model under stochastic demand, by assuming that the expected net inventory at the beginning of the cycle is given by

$$Q + R - \mu_L + (1-b)S(R) \tag{1.22}$$

while the expected net inventory at the end of the cycle is given by

$$R - \mu_L + (1-b)S(R) \tag{1.23}$$

the authors found that the expected on-hand inventory level at any point in time is:

$$\frac{Q}{2} + R - \mu_L + (1-b)S(R). \tag{1.24}$$

Using (1.24) and assuming that the annual shortage cost and the annual lost profit are $s\frac{D}{Q}S(R)$ and $s_o\frac{D}{Q}(1-b)S(R)$ (s_o is the profit per unit), respectively, they developed the average annual total cost function for the partial backordering model

$$C(Q,R) = \frac{AD}{Q} + h\left(\frac{Q}{2} + R - \mu_L + (1-b)S(R)\right) + s\frac{D}{Q}S(R) + s_o\frac{D}{Q}(1-b)S(R). \tag{1.25}$$

Taking the first partial derivatives with respect to Q and R and equating to zero they get

$$Q^* = \sqrt{2\frac{AD}{h} + 2\frac{sD}{h}S(R) + 2\frac{s_oD}{h}(1-b)S(R)}, \tag{1.26}$$

$$1 - F(R) = \frac{hQ}{sD + s_oD(1-b) + hQ(1-b)}. \tag{1.27}$$

On the other hand, continuous review (Q,R) models in which demand is stochastic but the lead-time is considered as a decision variable are also developed by many authors, e.g., Moon and Choi (1998), Hariga and Ben-Daya (1999) and Ouyang and Chang (2001). Using Montgomery et al’s (1973) average on-hand inventory level and extending Ben-Daya and Rauf’s (1994) model, Ouyang et al. (1996) defined the average annual total cost function and the optimal order quantity under variable lead-time with stochastic demand as:

$$C(Q,R) = \frac{AD}{Q} + h\left(\frac{Q}{2} + R - \mu_L + (1-b)S(R)\right) + \frac{D}{Q}R(L) + s\frac{D}{Q}S(R) + s_o\frac{D}{Q}(1-b)S(R) \tag{1.28}$$

$$Q^* = \sqrt{2\frac{AD}{h} + 2\frac{sD}{h}S(R) + 2\frac{s_oD}{h}(1-b)S(R) + 2\frac{D}{h}R(L)}. \tag{1.29}$$

The authors examined the effects of b on the minimum average annual total cost and optimal order quantity and they found that $Q^*_{b=0} > Q^*_{b=1}$ and $C(Q,R)_{b=0} > C(Q,R)_{b=1}$.

For the periodic review (T,S) inventory model, as mentioned before, Montgomery et al. (1973) proposed also an annual total cost function under stochastic demand. Using this review interval and assuming that the expected net inventory at the beginning of the cycle is given by

$$R - \mu_L + (1-b)S(R) \tag{1.30}$$

while the expected net inventory at the end of the cycle is given by

$$R - \mu_L - DT + (1-b)S(R) \tag{1.31}$$

the authors found that the expected on-hand inventory level at any point in time is:

$$R - \mu_L - \frac{DT}{2} + (1-b)S(R)\,. \tag{1.32}$$

Then the average annual total cost function can be written as

$$C(Q,R) = \frac{L}{T} + h\left(R - \mu_L - \frac{DT}{2} + (1-b)S(R)\right) + s\frac{D}{T}S(R) + s_o\frac{D}{T}(1-b)S(R)\,. \tag{1.33}$$

Related studies for the periodic review (T,S) inventory model have been also developed by e.g., Ouyang and Chuang (1999), Chuang et al. (2004) and Ouyang et al. (2007). Finally, for the continuous or periodic review inventory models but with deterministic demand, except Montgomery et al.’s (1973) paper, other studies have been conducted by e.g., Rosenberg (1979), Park (1983) and Pentico and Drake (2009). Specially, Pentico and Drake (2009) have found that at the extremes b=0 and b=1 the presented model reduces to the usual backorders and lost-sales cases, respectively.

1.2.4 Single period system

In this category we can find inventory systems for which products have a limited purchasing period. Specially, the essential characteristic of these models is the required determination only of the order quantity for a single time period. The most popular inventory model of this category is the newsboy (newsvendor) problem (how many newspapers a boy should buy on a given day?) where in a given day the boy has only one opportunity to place an order. Thus the inventory decision aims at determining a single order quantity that maximizes the expected profit in a single period probabilistic demand framework.

In the inventory literature many alternative forms of newsvendor models exist. The classical version refers to the purchasing inventory problem where newsvendors decide for products whose life cycle of demand lasts a single relatively short period and their decisions are followed by a stochastic sales outcome. In such cases, for the newsboy model it is necessary to predict the order quantity in the beginning of each inventory cycle. So, if the actual demand is greater than the order quantity, products cannot be sold in the next time period, as any excess inventory is disposed of by buyback arrangements. On the contrary, there is an opportunity cost of lost profit (Chen and Chen, 2009), where at the end of the inventory cycle an excess demand is observed.

During the last decades, a number of studies have been developed in order to explore the issue of the optimal order quantity for cases of uncertainty in demand. In the literature, alternative forms of newsboy models have been published, with the paper of Khouja (1999) to provide an extensive search of these works till 1999. Since then, many studies have explored the newsboy inventory problem. Among them are: Schweitzer and Cachon (2000), Casimir (2002), Dutta et al. (2005), Salazar-Ibarra (2005), Matsuyama (2006), Benzion et al. (2008), Wang and Webster (2009), Chen and Chen (2010), Huang et al. (2011), Lee and Hsu (2011), and Jiang et al. (2011). However, the most important condition for the application of these models in real life is that parameters of demand distributions should be known. But, unfortunately, many times this condition does not hold and in order to apply newsvendor models in inventory management to determine the level of customer service we need to estimate the demand parameters.

There is a limited research on studying the effects of demand estimation on optimal inventory policies (Conrad 1976; Nahmias, 1994; Agrawal and Smith, 1996; Hill, 1997; Bell, 2000). None of these works addressed the problem of how sampling variability of estimated values of demand parameters influences the estimated optimal ordering policy. Kevork (2010) by assuming that demand follows the Normal distribution, for the classical newsvendor model, developed appropriate estimators to explore the variability of estimates for the optimal order quantity and the maximum expected profit. The statistical properties of the two estimators are explored for both small and large samples, analytically and through Monte-Carlo simulations. The results have shown that when high shortage costs occur there is a weak point of applying this model to real life situations due to the significant reductions in precision and stability of confidence intervals for the true maximum expected profit. Su and Pearn (2011), in order to compare two newsboy-type products, developed a statistical hypothesis testing methodology. They selected the product which has a higher probability of achieving a target profit under the optimal ordering policy. The authors, in order to attain the designated type I and II errors, provided tables with critical values of the test and the sample sizes. Finally, Olivares et al. (2008) presented an estimation framework to disentangle whether specific factors affect the observed order quantity either through the distribution of demand or through the overage/underage cost ratio.

1.2.5 Lead-time demand distribution

A variety of distributions for the lead-time demand have been tested for their usefulness in inventory theory. These include the Normal (see Hadley and Whitin, 1963, Peterson and Silver, 1979 and Lau et al., 2002b), the Gamma (see Burgin, 1975, Murphy, 1975, Das, 1976b and Schneider, 1978), the Lognormal (see Crouch and Oglesby, 1978, Tadikamalla, 1979, Silver, 1980 and Das, 1983a), the Weibull (see Tadikamalla, 1978 and Lau and Lau, 1993), the Logistic (see Vanbeek, 1978 and Fortuin, 1980), the Tukey's lamba (see Silver, 1977) and the Poisson (see Gross and Ince, 1975 and Archibald and Silver, 1978) distributions. Further, systems of distributions have also been used to approximate the lead-time demand such as Pearson system (1895) and Johnson system (1949). Ramberg and Schmeiser (1974) and Schmeiser and Deutsch (1977) have developed versatile systems of four parameter distributions that are defined in terms of the inverse cumulative distribution function (see Kottas and Lau, 1979). But the use of these systems is limited because of the computational complexity in the determination of the reorder point and the order quantity.

1.3 Thesis objective

From the aforementioned analysis, it is verified that the continuous review (Q,R) inventory model with backorders and fixed lead-time has been studied extensively in the area of inventory literature. However, most of the papers that appear in this area focus on the development of theoretically mathematical models for cost functions with either exact or approximate expressions for the expected on-hand inventory level at any point in time. To the extent of our knowledge, little though has been written on examining (a) the convexity and the existence of a unique minimum of these cost functions and (b) at what extent the values of the cost parameters affect the optimal solution. Furthermore, recognizing that most of the research on continuous review inventory systems focus on the description of decision rules which are either understood or accepted by management, for a model which is realistic for practical purposes, in this thesis we develop new approaches and solution techniques in order to apply inventory theory in real world inventory control systems.

Further, a problem that has received very limited attention in the relevant literature is that one of obtaining optimal solutions for this (Q,R) inventory model when the estimation of unknown demand parameters is required. It is common practice in the inventory literature to assume that parameters of the demand distribution are known. But, in real life conditions, neither the process of generating demand data nor the values of demand parameters are known. Therefore, pointing out that the estimation process of this thesis is classified in the area of the Frequentist approach, for the first time we develop appropriate estimators for the optimal order quantity and optimal reorder point which enable us to produce estimates for the minimum cost of the reference period. Particularly, assuming that demand is fully observed in each period, we adopt this approach because we consider the situation in which the decision maker knows the class of the random demand distribution but does not know the actual values of some or all the stationary parameters of such a distribution.

Finally, for the (Q,R) inventory system, we observe also in the inventory literature that a limited amount of research has been conducted in the area of determining the target inventory measures when the demand is generated by an autocorrelated process. Despite the existence of some works in this area, we can see that they focus mainly on the investigation of the effect of the autocorrelation parameter on the reorder point and safety stock when the demand parameters and the autocorrelation coefficient are known. Extending these studies, in this thesis we go further the analysis in order to investigate the effect of a serially correlated demand on the behavior of the order quantity, reorder point and minimum cost when estimators for the demand parameters and the autocorrelation coefficient are used.

Given the above discussion, these special problems seem to have been underestimated or ignored in the past by both practitioners and academicians. Consequently, as many important practical problems for the continuous review (Q,R) inventory system still remain unsolved today due to the complexity of the mathematical models, the aim of this thesis is to study the continuous review (Q,R) inventory model with backorders, fixed lead-time and non-negative reorder point and bring solutions for the following issues which are not addressed in the literature satisfactorily:

1. To examine the convexity problem and to identify the unique minimum for a cost function with exact or approximate expressions for the expected on-hand inventory level at any point in time,
2. To develop algorithms in order the optimal solution to be attained,
3. To derive optimal policies for unimodal and J-shaped lead-time demand distributions,
4. To investigate how the values of the cost parameters affect the optimal solution,
5. To develop estimators for target inventory measures,
6. To derive asymptotic confidence interval for a cost function,

7. To test the validity of the theoretical results on a set of generated data through Monte-Carlo simulation,
8. To identify optimal solutions for correlated demand.

The conclusions of our research are discussed in chapter 6 of this thesis.

1.4 Structure of the thesis

For the continuous review (Q, R) inventory model, the order quantity, Q, and the reorder point, R, are determined by minimizing a cost function resulting from the sum of the annual expected costs of ordering, inventory carrying and shortage. For evaluating the annual expected inventory carrying cost, we consider either approximate or exact expressions for the expected on-hand inventory level at any point in time, while for the calculation of the annual expected shortage cost the cost per unit backordered is used. Therefore, this thesis is structured as follows:

In chapter 2, we investigate the minimization process of the exact cost function (exact expression for the expected on-hand inventory level at any point in time) with known demand parameters. Provided that the lead-time demand has J-shaped or unimodal distributions satisfying specific assumptions we derive the general condition when the minimum cost is attained at a positive reorder point or at a reorder point equal to zero. Based on this condition we obtain the range of the cost parameters values in order the optimal reorder point to be equal to zero and we develop a general algorithm. Some numerical experimentation using parameter values from the relevant literature indicates that with large demand variability measured by the standard deviation of the lead-time demand the optimal inventory policies lead to excessively large orders and zero reorder points.

In chapter 3, we investigate the minimization process of the Hadley & Whitin's cost function (approximate expression for the expected on-hand inventory level at any point in time) with known demand parameters. Assuming that the lead-time demand has unimodal distributions satisfying specific assumptions, we derive the necessary conditions for three mutually exclusive events which should be examined at the process of determining the minimum of the approximate cost function. Next, these three cases are integrated to a general algorithm, and its application is illustrated when the lead-time demand has the Normal and the Log-Normal distribution. Through a numerical experimentation, we investigate the managerial implications of changing the values of cost parameters on the optimal sizes of Q and R, as well as, on the minimum cost. Further, we obtain the range of the cost parameters values in order the optimal reorder point to be equal to zero. Finally, we give the results from a comparative study for the target inventory measures taken after minimizing the Hadley & Whitin and the exact cost functions in order to investigate the validity of the Hadley & Whitin's approximate expression.

In chapter 4, considering normally distributed lead-time demand and by assuming that the values of demand parameters are unknown we address for the first time the issue of the Hadley & Whitin's cost function estimation. Making the assumption that the cycle service level is constant and using maximum likelihood estimators for the parameters of the Normal distribution we develop estimators for the optimal order quantity, optimal reorder point and minimum cost of the reference period. Based on the asymptotic distribution of the estimators, confidence interval for the minimum cost is derived whose validity is tested through Monte-Carlo simulations in different sample sizes.

In chapter 5, for the Hadley & Whitin's cost function we develop a procedure to determine the target inventory measures when the issue of both correlated demand and unknown demand parameters is addressed. Particularly, for the first time, we develop appropriate estimators for the optimal order quantity, optimal reorder point and minimum total cost as functions of the maximum likelihood estimators of the stationary mean, the stationary variance and the theoretical autocorrelation coefficient at lag one of an AR(1) demand process. Furthermore, through Monte-Carlo simulations in different sample sizes we study under which conditions the independent demand model can be a good approximation to handle the autocorrelated demand.

Finally, in chapter 6, the conclusions, recommendations and possible extensions of the thesis are discussed.

Each appendix appears at the end of each chapter.

Chapter 2

The (Q,R) inventory system with an exact cost function

2.1 Introduction

As mentioned before in section 1.2.2.1 the continuous review (Q,R) inventory model has been studied extensively in the area of inventory control. According to Silver et al. (1998) the validity of this review policy relies on the assumption that any undershoot of the reorder point is negligible compared to the magnitude of the total lead-time demand. Further, provided that there is never more than one order outstanding at any point in time, which this in turn means that the lead-time demand never exceeds the order quantity, it has been noted earlier that the values of the decision variables (Q,R) are determined by minimizing the annual total cost function (the sum of the annual expected ordering, holding and shortage costs). Given, however, that different ways to compute the holding and shortage costs have been suggested in the relevant literature, this type of continuous review policy can be differentiated according to the form of the annual total cost objective function.

In the current chapter we investigate the annual cost function in which the annual expected holding cost is computed using an exact expression for the expected on-hand inventory at any point in time. Further, assuming that "best customer communication policies" have been developed, which do not allow dissatisfaction to be expressed by customers who, not finding the product, should wait till its delivery, the shortage cost per inventory cycle is derived multiplying the shortage cost per unit backordered by the expected size of backorders. Particularly, for this exact annual cost function, we study its convexity when the lead-time demand is a non-negative continuous random variable which has a unimodal distribution satisfying specific assumptions or J-shaped distribution with decreasing probability density function.

Under these two types of distributions, for the first time we state when the unique minimum cost is determined through mathematical optimization. Further, provided that the reorder point is non-negative, we explain why, when the degeneracy problem occurs, the unique minimum of the exact annual cost function is attained at zero reorder point. Particularly, taking the first order conditions from this minimization with respect to Q and R and following an analogous approach to that of Das (1988) and Chung et al. (2009), we rewrite the cost function in terms only of R. Transferring in that way the analysis from the three dimensional to two dimensional space, we derive a general condition which identifies the following three cases: (a) the cost function is convex and has a unique minimum determined through mathematical optimization, (b) the cost function is not convex but it has a unique minimum determined through mathematical optimization, and (c) the cost function is increasing at an increasing rate in the entire domain of R in which case the minimum cost occurs at the smallest value of R, that is, at $R = 0$. The relevance of the general condition in determining whether or not the minimum cost will be obtained through mathematical optimization lies in the fact that this condition does not depend on the form of the lead-time demand distribution as it is expressed in terms of the annual expected demand, the variance of the lead-time demand and the three cost parameters, A, h and s.

The three cases stated above, which should be examined for determining the minimum of the exact annual cost function, are incorporated into a "*general algorithm*" which constitutes a further contribution of the current work. Although today the finding of the minimum of the exact annual cost function can be performed through the use of spreadsheets software, the implementation of this general algorithm may have numerous benefits. At first, a key-element of the algorithm is the aforementioned general condition, which, when it does not hold, the use of spreadsheets software is not recommended. As we shall show in a next section, this happens because the minimization process of the exact annual cost function, with for instance the Excel Solver under various lead-time unimodal demand distributions (e.g., Log-Normal, Gamma, etc.), breaks down and solution does not exist. The general algorithm can be also implemented in cases of lead-time demand distributions for which explicit expressions to determine the optimal Q and R values can be derived in which case the use of any spreadsheet software is not necessary. We shall illustrate that such a case is met when lead-time demands are described by the exponential distribution. Lastly, we feel that the algorithm is also useful in the domain of specialists who can incorporate it into OR/MS software packages using various programming languages.

Based on the aforementioned discussion and remarks the rest of the chapter is organized as follows. Section 2.2 contains a literature review for the exact annual cost functions. In Section 2.3 we write the cost function as function only of R and derive analytic forms for the linear and quadratic loss functions when lead-time demand distribution is Gamma, Log-Normal and Weibull. These analytic forms are obtained from the *N*th truncated moment expressions given by Jawitz (2004). In Section 2.4, under unimodal and J-shaped lead-time demand distributions satisfying certain assumptions we obtain the general condition to have a unique minimum after solving the first order conditions from the minimization of the cost function. In the same section, from that condition we obtain the range of the cost parameters values in order the optimal reorder point to be equal to zero. In Section 2.5, we present a general algorithm for the minimization process of the cost function and applying this algorithm to a set of parameter values used in the inventory literature we investigate the managerial implications of increasing lead-time demand variability on the optimal target inventory measures. Finally, the last section concludes chapter 2 summarizing the most important findings.

2.2 Relevant literature review

For the calculation of the annual expected holding cost, Hadley & Whitin (1963) were the first who derived an exact expression for the expected on-hand inventory at any point in time. Their analysis was based on the assumptions that the lead-time demand has the Poisson distribution and each order delivery brings the on-hand inventory level above the non-negative reorder point. Extending the original Poisson results of Hadley & Whitin, among others Zheng (1992), Platt et al. (1997), and Agrawal & Seshadri (2000) gave the following exact expression

$$\mathrm{I}_{\mathrm{ex}} = \frac{1}{\mathrm{Q}} \int_{\mathrm{R}}^{\mathrm{Q+R}} \left\{ \int_{0}^{\mathrm{Y}} (\mathrm{Y} - \mathrm{x})\, f(\mathrm{x})\, \mathrm{dx} \right\} \mathrm{dY} \tag{2.1}$$

for the expected on-hand inventory at any point in time when the lead-time demand is a random variable, X, with probability density function $f(\mathrm{x})$ and cumulative distribution function $F(\mathrm{x})$. The derivation of (2.1) was made under the conditions that the lead-time demand never exceeds Q and the lead-time is constant. These conditions ensure that the inventory position Y is uniformly distributed between R and $\mathrm{Q}+\mathrm{R}$ (see Serfozo & Stidham, 1978; Browne & Zipkin, 1991; Zipkin, 1986b for more discussion). In the meantime, however, due to the complexity of (2.1), approximations for I_{ex} have been suggested (see Holt et al., 1960; Hadley & Whitin, 1963; Wagner, 1975; Love, 1979; Yano, 1985; Lau & Lau, 2002).

The second factor which differentiates the form of the annual cost function is the way of computing the expected shortage cost. Three shortage cost models are available in the literature (e.g., Silver et al., 1998; Lau et al., 2002a; Lau & Lau, 2008), where each one of them has its own way to evaluate the expected size of backorders incurred per year. In specific terms, the first model assumes that only a fixed cost per stockout occasion is known, in which case the annual expected size of backorders is given by the product of the expected number of cycles per year and the stockout probability. On the contrary, the second model considers a shortage cost per unit backordered and the resulting expected size of backorders is

$$\mathrm{S}(\mathrm{R}) = \int_{\mathrm{R}}^{\infty} (\mathrm{x} - \mathrm{R}) f(\mathrm{x}) \mathrm{dx}\,.$$

Finally, the third model, taking into account the time factor in evaluating the shortage cost, uses the shortage cost per unit backordered per year. In this case the expression which gives the annual expected size of backorders becomes

$$\mathrm{S}_{\mathrm{tm}} = \frac{1}{\mathrm{Q}} \int_{\mathrm{R}}^{\mathrm{Q+R}} \left\{ \int_{\mathrm{Y}}^{\infty} (\mathrm{x} - \mathrm{Y})\, f(\mathrm{x})\, \mathrm{dx} \right\} \mathrm{dY}\,.$$

Combining the exact expression (2.1) with each one of the above models evaluating the annual expected shortage cost, the following three alternative *exact annual cost functions* are produced according to the terminology established by Lau et al. (2002a):

$$\mathrm{C}_{\mathrm{ex}}(\mathrm{Q},\mathrm{R}) = \frac{\mathrm{A} \cdot \mathrm{D}}{\mathrm{Q}} + \mathrm{h} \cdot \mathrm{I}_{\mathrm{ex}} + \mathrm{B} \cdot \frac{\mathrm{D}}{\mathrm{Q}} \left[1 - F(\mathrm{R}) \right], \tag{2.2a}$$

$$\mathrm{C}_{\mathrm{ex}}(\mathrm{Q},\mathrm{R}) = \frac{\mathrm{A} \cdot \mathrm{D}}{\mathrm{Q}} + \mathrm{h} \cdot \mathrm{I}_{\mathrm{ex}} + \mathrm{s} \cdot \frac{\mathrm{D}}{\mathrm{Q}} \cdot \mathrm{S}(\mathrm{R}), \tag{2.2b}$$

$$\mathrm{C}_{\mathrm{ex}}(\mathrm{Q},\mathrm{R}) = \frac{\mathrm{A} \cdot \mathrm{D}}{\mathrm{Q}} + \mathrm{h} \cdot \mathrm{I}_{\mathrm{ex}} + \mathrm{s}' \cdot \mathrm{S}_{\mathrm{tm}}\,. \tag{2.2c}$$

where, A is the fixed ordering cost, B the cost per stockout occasion, h the holding cost per unit per year, s the shortage cost per unit backordered (namely, a "*direct cost*" related to labor and special delivery costs incurred from the handling of backorders), $s' = s + s_{tm}$ the shortage cost per unit per year with s_{tm} to be an "*indirect*" cost element associated to the time a customer has to wait to receive his order creating in that way dissatisfaction to him (loss of goodwill), and D the annual expected demand.

Assuming that "best customer communication policies" have been developed which do not allow dissatisfaction to be expressed by customers who, not finding the product, should wait till its delivery, namely $s_{tm} = 0$, (e.g., Kevork, 2010), in this chapter we investigate the exact annual cost function which is given in (2.2b). In the past decades, before Lau et al. (2002b) offered simple computational expressions for I_{ex} for various continuous lead-time demand distributions, it was extremely rare the use of (2.2b). This happened because there were considerable difficulties for implementing the exact expression for the expected on-hand inventory at any point in time due to its original complicated expression involving a double integral (Lau & Lau, 2002). Today although the minimization of the exact annual cost function can be easily performed using various spreadsheets software, two issues are raised concerning the optimization process itself. The first issue is related to the choice of initial Q and R values before starting the minimization process. To the extent of our knowledge, until today there has been no attempt to study the convexity of (2.2b), namely, to investigate whether different pairs of initial values (Q, R) result in a global minimum or in different local minima. The second issue refers to the so called "degeneracy problem" representing a situation in which the minimization process of the exact annual cost function through some spreadsheet software breaks down and does not give a solution. This happens because at some stage of the iterative procedure the cycle service level (representing the probability for not observing stock-outs during the lead-time) takes on negative values. Again, to the extent of our knowledge, reasons for this degeneracy occurred at minimizing the exact annual cost function when the reorder point is non-negative have not been given yet.

In the inventory literature, there has been little research on studying the convexity of the cost functions (2.2a) and (2.2b), focused mainly on problems encountered during the minimization process. Specifically, for Normally distributed lead-time demands and through the use of numerical examples, Lau et al. (2002b) found that, solving iteratively the first-order conditions from the minimization of (2.2b) with respect to Q and R and using relatively low values for s, at some stages the iterative procedure led to negative cycle service levels. This had as a result the procedure to break down and nonsensical solutions to be obtained. To resolve the so called "degeneracy problem", the authors derived an altered form of (2.2b) to include also the case of negative reordered points and presented the minimization of the new cost function through the Excel's Solver under specific parameter values. Although some explanations have been given by several authors (e.g., Lau & Lau, 2002; Lau et al., 2002a), in the current work we overcome the "degeneracy problem" considering that R takes on only non-negative values.

On the other side, the convexity of the exact cost function (2.2c) has been studied extensively in the inventory literature. Zheng (1992) proved the convexity of (2.2c) based on the results of Zipkin (1986a) who showed that the expected size of backorders is a jointly convex function of Q and R. Under discrete demand, Federgruen & Zheng (1992) developed a surprisingly simple and efficient algorithm to reach the minimum cost. But according not only to these authors but also to Platt et al. (1997), the algorithm is valid provided that $-\left(h \cdot I_{ex} + s' \cdot S_{tm}\right)$ is a unimodal function. The same algorithm was used by Zhao et al. (2012) to find the minimum cost in a single-item system with limited resource for goods in on-hand inventory and outstanding orders. For Poisson distributed lead-time demand, Guan & Zhao (2011) proved the convexity of (2.2c) for any given Q and R, noting, however, the computational difficulties in determining the minimum cost.

2.3 The cost function in the two dimensional space

Let X be a continuous non-negative random variable representing the demand in the lead-time with mean μ_L and variance σ_L^2. Lau et al. (2002b) converted the double integration of (2.1) into a single integration and for the case of $R \geq 0$ they resulted in the following simplified expression

$$I_{ex} = \frac{Q}{2} + R - \mu_L + \frac{\Theta(R) - \Theta(Q+R)}{2Q}$$

where $\Theta(z) = \int_z^\infty (x-z)^2 f(x)dx$ and z stands for either Q or $Q+R$. Further with $R \geq 0$ the authors stated that the assumption "there is more than one order outstanding at any point in time" implies $\Theta(Q+R) = 0$. So, the exact expression for the expected on-hand inventory at any point in time takes the form

$$I_{ex} = \frac{Q}{2} + R - \mu_L + \frac{\Theta(R)}{2Q}, \tag{2.3}$$

where $\Theta(R) = \int_R^\infty (x-R)^2 f(x)dx = \int_R^\infty x^2 f(x)dx - 2R\int_R^\infty x f(x)dx + R^2\left[1 - F(R)\right],$ (2.4)

and $F(R)$ is the cumulative distribution function of X evaluated at R. Replacing (2.3) in (2.2b) and taking $\partial C_{ex}(Q,R)/\partial Q = 0$ and $\partial C_{ex}(Q,R)/\partial R = 0$, the solution of first-order conditions for the minimization of the exact annual cost function gives

$$Q(R) = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S(R) + \Theta(R)}, \tag{2.5a}$$

and

$$F(\mathrm{R}) = 1 - \frac{\mathrm{h}[\mathrm{Q}(\mathrm{R}) - \mathrm{S}(\mathrm{R})]}{\mathrm{s} \cdot \mathrm{D}}, \tag{2.5b}$$

where $\mathrm{S}(\mathrm{R}) = \int_{\mathrm{R}}^{\infty} (\mathrm{x} - \mathrm{R}) f(\mathrm{x}) \mathrm{dx} = \int_{\mathrm{R}}^{\infty} \mathrm{x} f(\mathrm{x}) \mathrm{dx} - \mathrm{R}[1 - F(\mathrm{R})].$ (2.6)

Solving iteratively (2.5a) and (2.5b) until convergence is achieved (e.g., Hadley & Whitin, 1963; Silver et al., 1998; Nahmias, 1976; Lau et al., 2002b), the optimal pair of values $(\mathrm{Q}^*, \mathrm{R}^*)$ is obtained. Regarding the order quantity, we distinguish between the notation Q (or Q^*) meaning a given number and the notation $\mathrm{Q}(\mathrm{R})$ which illustrates a function of R derived after solving the first-order conditions for a minimum of the cost function (2.2b).

Substituting $\mathrm{Q}(\mathrm{R})$ for Q first in (2.3) and then in (2.2b), and performing some algebraic manipulation in the resulting expression of the cost function, (2.2b) is transformed to a function only of R, and is written as

$$\mathrm{C}(\mathrm{R}) = \mathrm{h}\left[\mathrm{Q}(\mathrm{R}) + \mathrm{R} - \mu_{\mathrm{L}}\right]. \tag{2.7}$$

The stated assumption in the introductory section that *"there is never more than one order outstanding at any point in time"* is true only if at each delivery the lead-time demand never exceeds the order quantity (Lau et al., 2002b). This also means that $\mathrm{Q}(\mathrm{R}) > \mu_{\mathrm{L}}$, which in turn leads to a positive $\mathrm{C}(\mathrm{R})$ for any $\mathrm{R} \geq 0$.

To compute $\mathrm{Q}(\mathrm{R})$ and $\mathrm{C}(\mathrm{R})$ we need analytic expressions for the general functions $\Theta(\mathrm{R})$ and $\mathrm{S}(\mathrm{R})$ which are given in (2.4) and (2.6) respectively. To the extent of our knowledge such analytic expressions are not directly available in the literature. Assuming that X has some specific probability distribution (e.g., Gamma, Log-Normal etc), these analytic expressions are obtained having available the solutions of integrals $\mathrm{m}_1 = \int_{\mathrm{R}}^{\infty} \mathrm{x} f(\mathrm{x}) \mathrm{dx}$ and $\mathrm{m}_2 = \int_{\mathrm{R}}^{\infty} \mathrm{x}^2 f(\mathrm{x}) \mathrm{dx}$. For instance, these solutions can be obtained directly from the formulae reported in table 1 of Jawitz (2004), and in particular, for the probability distributions which are considered in the current work, setting $\mathrm{u} = \infty$ and $\ell = \mathrm{R}$ to these formulae we take the following results:

Gamma $(\boldsymbol{\alpha}, \boldsymbol{\beta})$: $\mathrm{m}_{\mathrm{N}} = \frac{\beta^{\mathrm{N}}}{\Gamma(\alpha)} \{\Gamma(\alpha + \mathrm{N}) - \gamma(\alpha + \mathrm{N}, \mathrm{R}/\beta)\},$ (2.8a)

Log-Normal $(\boldsymbol{\lambda}, \boldsymbol{\theta})$: $\mathrm{m}_{\mathrm{N}} = \mathrm{e}^{\mathrm{N}\lambda + \mathrm{N}^2\theta^2/2} \{1 - \Phi(\mathrm{r} - \mathrm{N}\theta)\}, \mathrm{r} = (\ln \mathrm{R} - \lambda)/\theta,$ (2.8b)

$$\textbf{Weibull}(\boldsymbol{\alpha},\boldsymbol{\beta}): m_N = \beta^N \left\{ \Gamma\left(1+\frac{N}{\alpha}\right) - \gamma\left(1+\frac{N}{\alpha}, \left[\frac{R}{\beta}\right]^{\alpha}\right) \right\}, \tag{2.8c}$$

where $\Gamma(\alpha)$ and $\gamma(\alpha, R/\beta)$ are the complete and the lower incomplete Gamma function respectively. To obtain from the initial formulae reported by Jawitz the final forms of m_N in (2.8) we proceeded as follows:

(a) for the Weibull distribution first we expanded the summation and then we set $c = 0$,

(b) for the Log-Normal, we used the transformation $\mathrm{erf}\left(x/\sqrt{2}\right) = [\Phi(x) - 0.5]/0.5$, where $\Phi(x)$ is the cumulative distribution function of the standard Normal evaluated at x, and

(c) for both Gamma and Weibull distributions we used the limiting result $\lim_{x\to\infty} \gamma(\kappa, x) = \Gamma(\kappa)$.

The corresponding integrals for the Exponential and Rayleigh distributions can be obtained setting $\alpha = 1$ in the integral (2.8a) and $\alpha = 2$ in the integral (2.8c). Furthermore, for the Rayleigh distribution we used the identity $\gamma(0.5, x) = \sqrt{\pi}\,\mathrm{erf}\left(\sqrt{x}\right)$, while for the Gamma and Weibull distributions, to evaluate the lower incomplete gamma function we used the recursive equation $\gamma(\kappa, x) = (\kappa - 1)\gamma(\kappa - 1, x) - x^{\kappa-1}e^{-x}$ and the identity $\gamma(1, x) = 1 - e^{-x}$. Following the aforementioned discussion, the analytic forms of $S(R)$ and $\Theta(R)$ are given in Table 2.1. In the Appendix at the end of Chapter 2 we offer the analytical derivations of these forms in Proof 2.1.

From (2.4) and (2.6), it is deduced that when $R \to 0$ we have $S(R) \to \mu_L$ and $\Theta(R) \to \mu_L^2 + \sigma_L^2$, while if $R \to \infty$ then $S(R)$ and $\Theta(R)$ tend to zero. These limits are also justified as follows. Since $S(R)$ expresses the expected shortage (or backorders size) per inventory cycle, $\Theta(R)$ equals to the sum of the squared expected shortage plus the variance of the shortage. So, when $R \to \infty$ the lead-time becomes infinity, the shortage goes to zero and its mean and variance tend also to zero. On the contrary, the fact that $R \to 0$ implies that shortage tends to be identical with the lead-time demand verifying in that way the aforementioned limits of $S(R)$ and $\Theta(R)$.

Table 2.1 Analytic forms of the functions $\Theta(R)$ and $S(R)$ defined respectively in (2.4) and (2.6).

Gamma $(\boldsymbol{\alpha},\boldsymbol{\beta})$: $f(R)=\beta^{-\alpha}R^{\alpha-1}e^{-R/\beta}/\Gamma(\alpha)$, $F(R)=\gamma(\alpha,R/\beta)/\Gamma(\alpha)$, $\mu_L=\alpha\beta$, $\sigma_L^2=\alpha\beta^2$ $m_1=\int_R^\infty xf(x)dx=\mu_L\left[1-F(R)\right]+R\beta f(R)$, and $m_2=\int_R^\infty x^2 f(x)dx=\left(\mu_L^2+\sigma_L^2\right)\left[1-F(R)\right]+R\beta f(R)\left(\mu_L+\beta+R\right)$ $S(R)=\left(\mu_L-R\right)\left[1-F(R)\right]+R\beta f(R)$ and $\Theta(R)=\left\{\left(\mu_L-R\right)^2+\sigma_L^2\right\}\left[1-F(R)\right]+R\beta f(R)\left(\mu_L-R+\beta\right)$
Exponential $(\boldsymbol{\beta})$: $f(R)=e^{-R/\beta}/\beta$, $F(R)=1-e^{-R/\beta}$, $\mu_L=\beta$, $\sigma_L^2=\beta^2$ $m_1=\left(\mu_L+R\right)e^{-R/\mu_L}$ and $m_2=\left\{2\mu_L\left(\mu_L+R\right)+R^2\right\}e^{-R/\mu_L}$ $S(R)=\mu_L e^{-R/\mu_L}$ and $\Theta(R)=2\mu_L^2 e^{-R/\mu_L}$
Log-Normal $(\boldsymbol{\lambda},\boldsymbol{\theta})$: $f(R)=\left(\theta R\sqrt{2\pi}\right)^{-1}e^{-[(\ln R-\lambda)/\theta]^2/2}$, $F(R)=\Phi(r)$, $\mu_L=e^{\lambda+\theta^2/2}$, $\sigma_L^2=\mu_L^2\left(e^{\theta^2}-1\right)$, $r=(\ln R-\lambda)/\theta$ $m_1=\mu_L\left\{1-\Phi(r-\theta)\right\}$ and $m_2=\left(\mu_L^2+\sigma_L^2\right)\left\{1-\Phi(r-2\theta)\right\}$ $S(R)=\mu_L\Phi(\theta-r)-R\Phi(-r)$ and $\Theta(R)=\left(\mu_L^2+\sigma_L^2\right)\Phi(2\theta-r)-2R\mu_L\Phi(\theta-r)+R^2\Phi(-r)$
Weibull $(\boldsymbol{\alpha},\boldsymbol{\beta})$: $f(R)=\alpha\beta^{-\alpha}R^{\alpha-1}e^{-(R/\beta)^\alpha}$, $F(R)=1-e^{-(R/\beta)^\alpha}$, $\mu_L=\alpha^{-1}\beta\Gamma(1/\alpha)$, $\sigma_L^2=\alpha^{-1}\beta^2\left\{2\Gamma(2/\alpha)-\left[\Gamma(1/\alpha)\right]^2/\alpha\right\}$ $m_1=\mu_L-\alpha^{-1}\beta\cdot\gamma\left(1/\alpha,[R/\beta]^\alpha\right)+Re^{-(R/\beta)^\alpha}$ and $m_2=\mu_L^2+\sigma_L^2-2\alpha^{-1}\beta^2\cdot\gamma\left(2/\alpha,[R/\beta]^\alpha\right)+R^2e^{-(R/\beta)^\alpha}$ $S(R)=\mu_L-\alpha^{-1}\beta\gamma\left(1/\alpha,[R/\beta]^\alpha\right)$ and $\Theta(R)=\mu_L^2+\sigma_L^2-2R\cdot S(R)-\frac{2\beta^2}{\alpha}\gamma\left(2/\alpha,[R/\beta]^\alpha\right)$
Rayleigh $(\boldsymbol{\beta})$: $f(R)=2\beta^{-2}Re^{-(R/\beta)^2}$, $F(R)=1-e^{-(R/\beta)^2}$, $\mu_L=\beta\sqrt{\pi}/2$, $\sigma_L^2=\beta^2(4-\pi)/4$ $m_1=2\mu_L\left\{1-\Phi\left(R\sqrt{2}/\beta\right)\right\}+Re^{-(R/\beta)^2}$ and $m_2=\left(\mu_L^2+\sigma_L^2+R^2\right)e^{-(R/\beta)^2}$ $S(R)=2\mu_L\left\{1-\Phi\left(R\sqrt{2}/\beta\right)\right\}$ and $\Theta(R)=\left(\mu_L^2+\sigma_L^2\right)e^{-(R/\beta)^2}-4R\mu_L\left\{1-\Phi\left(R\sqrt{2}/\beta\right)\right\}$

2.4 Minimization process of the cost function

From (2.7) and using the derivatives $dS(R)/dR=-[1-F(R)]$ and $d\Theta(R)/dR=-2S(R)$, we obtain $C'(R)=-h\cdot V(R)$ and $C''(R)=h\cdot g(R)/[Q(R)]^3$, where

$$V(R)=-\frac{dQ(R)}{dR}-1, \tag{2.9}$$

$$g(R)=\left\{\frac{s}{h}Df(R)+[1-F(R)]-\left[\frac{dQ(R)}{dR}\right]^2\right\}[Q(R)]^2, \tag{2.10}$$

and

$$\frac{dQ(R)}{dR}=-\frac{s\cdot D[1-F(R)]/h+S(R)}{Q(R)}. \tag{2.11}$$

From the forms of $C'(R)$ and $C''(R)$ we deduce that (a) the range of function $g(R)$ determines whether $C(R)$ is convex or not, and (b) provided that $C''(R) > 0$, namely $C(R)$ is convex, the range of function $V(R)$ determines whether or not there exists a single value $R^* > 0$ for which $C'(R^*) = 0$.

The range of $g(R)$ and $V(R)$ is determined by investigating how the first derivatives $V'(R) = -g(R)/[Q(R)]^3$ and $g'(R) = u(R)[Q(R)]^2$ respond to changes of R, that is, when R increases from zero to infinity, given that

$$u(R) = \left\{ \frac{s}{h} D f'(R) - f(R) \right\}. \tag{2.12}$$

(**Proof 2.2:** See in the Appendix at the end of the chapter)

This investigation is carried out in the remaining of this section when the lead-time demand distribution belongs to one of the following two types of skewed distributions: (a) J-shaped with $f'(R) < 0$, and (b) unimodal satisfying the following two assumptions:

Assumption 1: $\lim_{R \to 0} f(R) = \lim_{R \to \infty} f(R) = 0$.

Assumption 2: Given that the mode of distribution occurs at R_m, there is only one value $R_o < R_m$ for which $u(R_o) = 0$, with $u(R) > 0$ for $R < R_o$, and $u(R) < 0$ for $R > R_o$.

(**Proof 2.3:** See in the Appendix at the end of the chapter)

Regarding the five distributions of Table 2.1, in the Appendix we show that assumption 2 is true for the unimodal Gamma(α, β) and Weibull(α, β), with $\alpha > 1$, and for the Log-Normal (λ, θ) as the latter one is unimodal for any λ and θ.

At this point it is important to mention that we have chosen Gamma, Weibull and Log-Normal because, under these distributions we can handle large demand variability when R is always positive. According to Gallego et al. (2007) when the demand coefficient of variation (CV) is large, it is preferable to describe the demand by non-negative skewed distributions instead of the Normal. This is one reason why the Normal distribution has not been included in our analysis as this distribution offers tractable results and good approximations for target inventory measures only when the demand has relatively low coefficient of variation, preferably below 0.3 (e.g., Lau, 1997; Syntetos & Boylan, 2008; Janssen et al., 2009; Kevork, 2010). The second reason is that, according to Lau & Lau (2002), under Normally distributed lead-time demand with low CV the Hadley & Whitin approximation behaves well even when cycle service levels are not large.

Given the above analysis, the next Lemmas 2.1 and 2.2 investigate the range of functions $g(R)$ and $V(R)$ for J-shaped and unimodal distributions respectively.

Lemma 2.1: When X has a J-shaped distribution with $f'(R) < 0$ then:

(a) $g(R)$ is positive for any $R \geq 0$ and

(b) when it holds

$$(s/h)^2 D^2 - (2A/h) D - \sigma_L^2 > 0 \tag{2.13}$$

there is a unique $R^* > 0$ for which $V(R^*) = 0$.

Proof 2.4: See in the Appendix at the end of the chapter.

From part (a) of Lemma 2.1, we conclude that when the lead-time demand is J-shaped distributed with $f'(R) < 0$ then the cost function is always convex since $C''(R) = h \cdot g(R) / [Q(R)]^3 > 0$. Further, from part (b) of Lemma 2.1 we observe that condition (2.13) ensures a positive R^* for which $C'(R^*) = -h \cdot V(R^*) = 0$. The value of R^* is obtained after solving either the equation $\left(dQ(R)/dR\right) + 1 = 0$ or the system of the first order conditions (2.5a) and (2.5b). Both ways lead to the same equation which is

$$\frac{s}{h} D\left[1 - F(R^*)\right] + S(R^*) = Q(R^*). \tag{2.14}$$

Lemma 2.2: If X has a unimodal distribution satisfying assumptions 1, 2 and condition (2.13), then it holds:

(a) $g(R)$ intersects the horizontal axis at a unique positive value R_1 so that $g(R) < 0$ for $0 \leq R < R_1$ and $g(R) > 0$ for $R > R_1$,

(b) $V(R)$ intersects the horizontal axis at $R^* > R_1$ for which $V(R^*) = 0$, $V(R) > 0$ for $R < R^*$ and $V(R) < 0$ for $R > R^*$.

Proof 2.5: See in the Appendix at the end of the chapter.

From Lemma 2.2 it is deduced that although $C(R)$ is not convex as $C''(R) = h \cdot g(R) / [Q(R)]^3 < 0$ for $0 \le R < R_1$, it will have a unique minimum at $R^* > 0$ since $C'(R^*) = 0$ and $C''(R^*) > 0$. The value of R^* is obtained by solving again equation (2.14).

From the two Lemmas it is also realized that, for both types of distributions, when $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 < 0$ then for $R \ge 0$ the function $g(R)$ is always positive while $V(R)$ is always negative. This means that when R increases on the interval $(0, \infty)$ then $C(R)$ increases at an increasing rate, as $C'(R) = -h \cdot V(R) > 0$ and $C''(R) = h \cdot g(R) / [Q(R)]^3 > 0$. Hence, $C(R)$ is convex but an extreme value does not exist under a strict mathematical framework. In this case, however, we shall consider as minimum the lowest point of the $C(R)$ curve which is located at $R^* = 0$. Then the minimum cost is given by

$$C(0) = h\left(\lim_{R^* \to 0} Q(R^*) - \mu_L\right) = h\left(\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu_L + \mu_L^2 + \sigma_L^2} - \mu_L\right). \tag{2.15}$$

Finally, when $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 = 0$ it holds $g(R) > 0$ and $V(R) < 0$ meaning that for J-shaped distributions $\lim_{R \to 0} g(R) = +\infty$, $\lim_{R \to 0} V(R) = 0$ and for unimodal distributions $\lim_{R \to 0} g(R) = \lim_{R \to 0} V(R) = 0$. Therefore for both types of distributions $C(R)$ is flat at $R = 0$ and starts to increase at an increasing rate for $R > 0$. In this case the lowest point of the $C(R)$ curve occurs at $R^* = 0$ and the minimum cost is given again by (2.15).

The results of the above analysis, concerning the convexity of $C(R)$ and the existence of a unique minimum are summarized in Table 2.2. Further, Figures 2.1-2.3 illustrate graphically g(R), V(R) and C(R) when the expression $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2$ is positive, negative or zero.

Table 2.2 Summarized results for the convexity of $C(R)$ and the existence of a unique minimum under two types of lead-time demand distribution.

	$(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 > 0$
J-shaped:	$C(R)$ is strictly convex and has a unique minimum at $R^* > 0$.
unimodal:	$C(R)$ is not convex and has a unique minimum at $R^* > 0$.
	$(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 < 0$
J-shaped, unimodal:	$C(R)$ is increasing at an increasing rate for any $R \geq 0$. The minimum cost occurs at $R^* = 0$.
	$(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 = 0$
J-shaped:	$C(R)$ is strictly convex and has a unique minimum at $R^* = 0$.
unimodal:	$C(R)$ is flat at $R = 0$ and starts to increase at an increasing rate for any $R > 0$. The minimum cost occurs at $R^* = 0$.

(a) J-shaped **(b)** unimodal

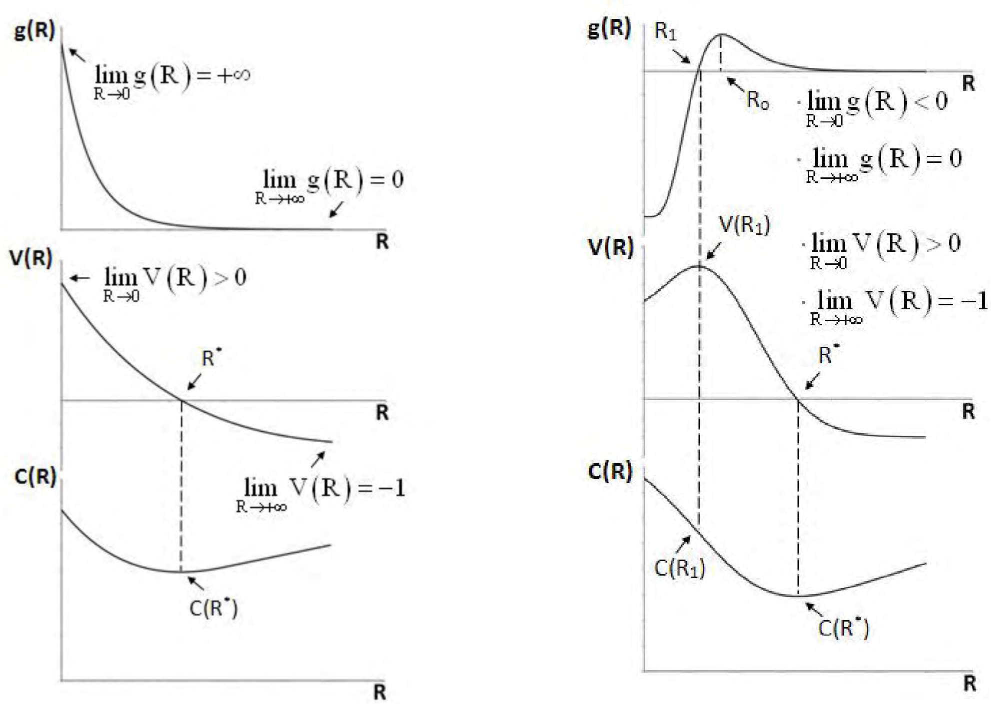


Figure 2.1 Graphs of g(R), V(R) and C(R) functions when $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 > 0$.

(a) J-shaped **(b)** unimodal

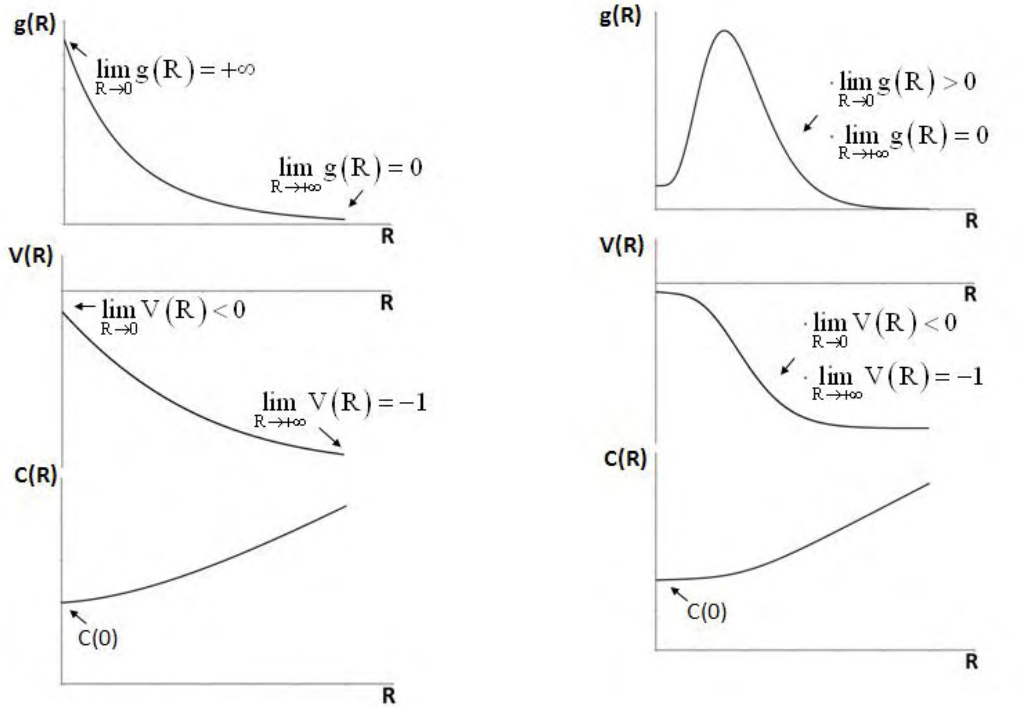


Figure 2.2 Graphs of g(R), V(R) and C(R) functions when $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 < 0$.

(a) J-shaped **(b)** unimodal

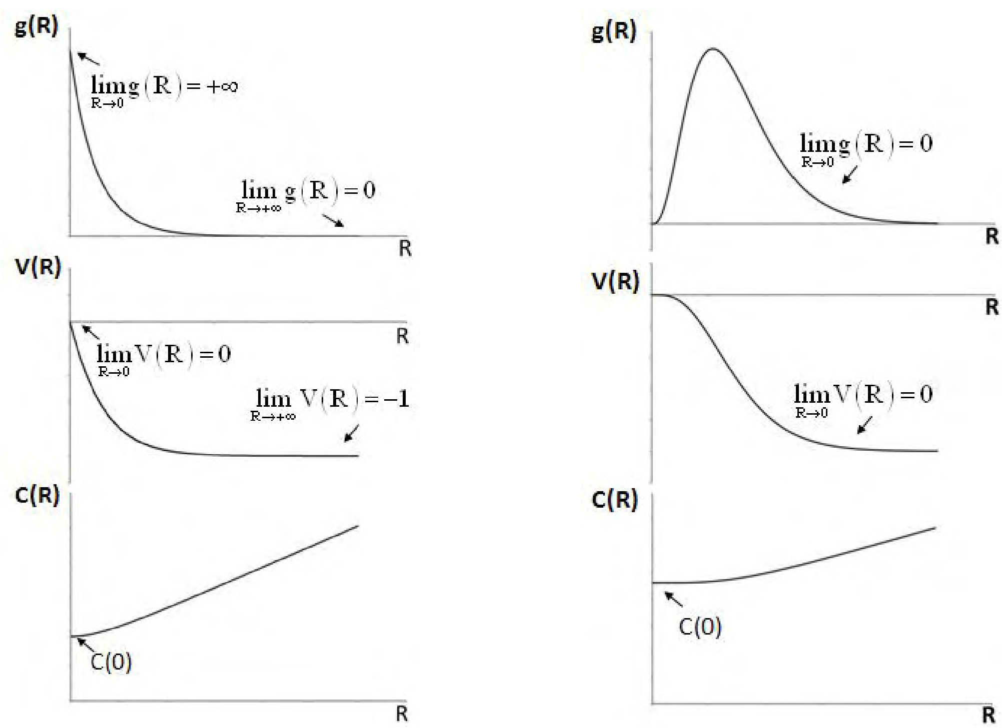


Figure 2.3 Graphs of g(R), V(R) and C(R) functions when $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 = 0$.

Closing this section, we note that the usefulness of condition (2.13) is twofold. First, solving the inequality with respect to one of the cost parameters keeping the other two fixed we obtain threshold values which determine the range values of the cost parameters in order the unique minimum to be attained for $R^* > 0$ or $R^* = 0$. Second, these threshold values are independent of the form of the lead-time demand distribution and to compute them we need to know only the mean and the variance of the lead-time demand. In Table 2.3 we give the range values of s, A and h in order the minimum cost to occur at a positive R value. When this happens the threshold value is the minimum for s and the maximum for A and h.

Table 2.3 Interval values of the cost parameters for a minimum cost at a positive reorder point.

Cost		
shortage	**ordering**	**holding**
$\sqrt{2\frac{A}{D}h + \frac{h^2}{D^2}\sigma_L^2} \le s < +\infty$	$0 \le A \le \dfrac{\frac{s^2D^2}{h} - h\sigma_L^2}{2D}$	$0 \le h \le \dfrac{-AD + \sqrt{A^2D^2 + \sigma_L^2 s^2 D^2}}{\sigma_L^2}$

Proof 2.6: See in the Appendix at the end of the chapter.

2.5 An algorithm for the solution approach

The solution steps for finding the minimum of the cost function $C(R)$ defined in (2.7) when the lead-time demand has a J-shaped distribution with $f'(R) < 0$ or a unimodal distribution satisfying assumptions 1 and 2 described in section 2.3, are summarized into the following general algorithm:

Step 1: Give values to the parameters: s, A, h, D, μ_L and σ_L^2.

Step 2: If $(s/h)^2 D^2 - 2(A/h)D - \sigma_L^2 > 0$ then go to Step 3, otherwise go to Step 6.

Step 3: Find analytic forms for the functions $F(R)$, $f(R)$, $S(R)$ and $\Theta(R)$ (for the distributions Gamma, Weibull, and Log-Normal such analytic forms are offered in Table 1) and go to step 4.

Step 4: Find the optimum reorder point, R^*, by solving the equation

$$\frac{s\cdot D\left[1-F(R^*)\right]/h + S(R^*)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D\cdot S(R^*) + \Theta(R^*)}} - 1 = 0, \tag{2.16a}$$

and go to Step 5.

Step 5: Compute the optimal order quantity and the minimum total cost respectively from

$$Q^* = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}\cdot D\cdot S\left(R^*\right) + \Theta\left(R^*\right)}\,, \tag{2.16b}$$

$$C_{ex}\left(Q^*, R^*\right) = h\left(Q^* + R^* - \mu_L\right), \tag{2.16c}$$

and go to Step 7.

Step 6: Set $R^* = 0$ and compute the optimal order quantity and the minimum total cost from

$$Q^* = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D\cdot\mu_L + \mu_L^2 + \sigma_L^2}\,, \tag{2.16d}$$

$$C_{ex}\left(Q^*, 0\right) = h\left(Q^* - \mu_L\right). \tag{2.16e}$$

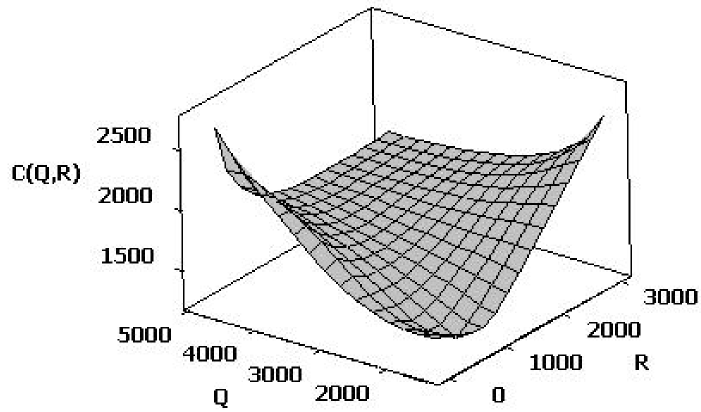
Step 7: End of algorithm.

Numerical example

To illustrate the application of the algorithm, we have chosen the Exponential distribution, and the following cost parameter values which are suggested by Zhao et al. (2012): $A = 60$, $h = 0.6$, $s = 3$, and $D = 10000$, under which condition (2.13) is true. Setting also $\mu_L = \sigma_L = \beta = 200$ the analytic forms of $f(R)$, $F(R)$, $S(R)$, $\Theta(R)$ for the exponential distribution are from Table 2.1: $f(R) = e^{-0.005R^*}$, $F(R) = 1 - e^{-0.005R^*}$, $S(R) = 0.02\cdot 10^4 e^{-0.005R^*}$, $\Theta(R) = 8\cdot 10^4 e^{-0.005R^*}$. Then substituting into (2.16a) we take the quadratic equation $252004y^2 - 2008y - 200 = 0$ where $y = \exp\left(-0.005R^*\right)$. From the two roots of the equation we keep the positive one $y = 0.0324$, and solving with respect to R^* we obtain the optimal reorder point $R^* = 685.70$. Substituting $R^* = 685.70$, $S\left(R^*\right) = 6.49$, and $\Theta\left(R^*\right) = 2594.88$ into (2.16b) we take the optimal order quantity $Q^* = 1628.29$. Finally, from (2.16c) the minimum cost is $C_{ex}\left(Q^*, R^*\right) = 1268.39$ (see graph (a) of Figure 2.4).

If on the other hand we used the same parameter values as above with the exception that instead of $A = 60$, we set $A = 75000$, we would find that condition (2.13) is negative. In this case from Step 2 of the algorithm we have to go to Step 6 and to set $R^* = 0$. Finally, from (2.16d) and (2.16e) we compute respectively $Q^* = 50200.40$ and $C_{ex}\left(Q^*, 0\right) = 30000.24$ (see graph (b) of Figure 2.4).

(a) $A = 60$, $h = 0.6$, $D = 10000$, $\mu_L = \sigma_L = 200$ and $s = 3$



(b) $A = 75000$, $h = 0.6$, $D = 10000$, $\mu_L = \sigma_L = 200$ and $s = 3$

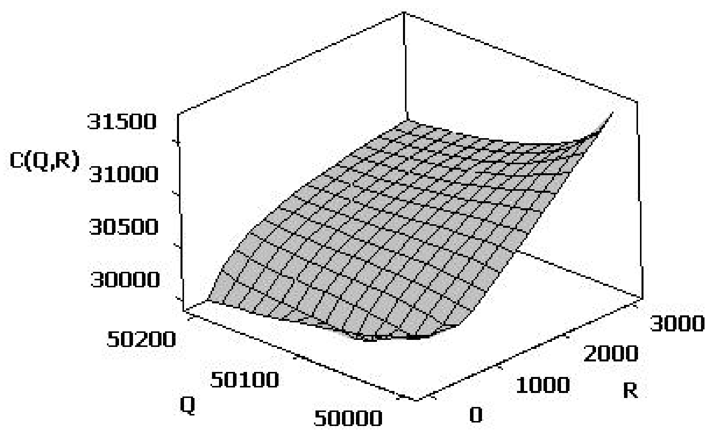


Figure 2.4 Graph of the cost function $C_{ex}(Q,R)$ under Exponential lead-time demand.

If instead of the Exponential distribution we used as lead-time demand distribution one of the Gamma, Log-Normal or Weibull then the equation (2.16a) of step 4 could not be solved analytically. In this case an iterative procedure should be used such as the Newton-Raphson method which first should have been coded to some programming language (e.g., FORTRAN or C). Furthermore, an alternative approach to obtain the optimal pair (Q^*, R^*) when condition (2.13) holds is by using the Excel's Solver. For instance, in Figure 2.5 we give an example for minimizing the cost function when the lead-time demand has the Gamma distribution with parameters α=25 and β=12. In this Figure, column A contains the symbols, column B the values and column C the Excel's expressions. Setting as initial values $Q = \sqrt{2\frac{A}{h}D}$ and $R = \mu_L$ respectively in cells B11 and B12, a few mouse-clicks with the

Excel's Solver minimizing cell B18 by changing the cells B11 and B12 produce the optimal values:

$$Q^* = 1558.262 \text{ (cell B11)}, \; R^* = 421.291 \text{ (cell B12) and}$$

$$C_{ex}\left(Q^*, R^*\right) = 1007.732 \text{ (cell B18).}$$

Similar Excel spreadsheets as that one of Figure 2.5 can be developed for the remaining distributions of Table 2.1.

	A	B	C
1	μ_L	300	
2	σ_L	60	
3			
4	D	10000	
5	A	70	
6	s	3	
7	h	0.6	
8	α	25	=B1*B1/B2/B2
9	β	12	=B2*B2/B1
10			
11	**Q**	**1558.262**	
12	**R**	**421.291**	
13			
14	f(R)	0.000935	=GAMMADIST(B12;B8;B9;FALSE)
15	F(R)	0.968854	=GAMMADIST(B12;B8;B9;TRUE)
16	S(R)	0.947924	=(B1-B12)*(1-B15)+B12*B9*B14
17	Θ(R)	53.86033	=((B1-B12)^2+B2^2)*(1-B15)+B12*B9*B14*(B1-B12+B9)
18	**C_{ex}(Q,R)**	**1007.732**	=B5*B4/B11+B7*(B11/2+(B12-B1)+0.5*B17/B11)+B6*B4/B11*B16

Figure 2.5 Excel spreadsheet for the exact cost function.

Using for the holding cost, h and shortage cost per unit backordered, s the values which are suggested by Zhao et al. (2012), namely h=[0.1-3] and s=[5h-15h], and assigning to D and μ_L the values 3000 and 300 respectively, we give in Tables 2.4-2.7 the optimal target inventory measures under different combinations of sizes for the coefficient of variation (CV) and the ordering cost, A. Especially for A, we selected both larger and smaller sizes than its threshold value which determines the range where the optimal reorder point is positive or zero. From Tables 2.4-2.7 we observe that, given the CV size, as A raises then the order quantity and the minimum cost increase, while the reorder point and the cycle service level decline. Furthermore, increasing the size of CV, keeping all the other parameter values fixed, results in (a) larger order quantities and minimum costs, and (b) smaller reorder points and cycle service levels. Therefore, optimal inventory policies with large lead-time demand variability expressed by the size of CV lead to excessively large orders, zero reorder points and higher minimum costs.

Table 2.4 Optimal target inventory measures for Rayleigh distributed lead-time demand when s=3, h=0.6, D=3000 and μ_L=300.

CV		A	Exact CSL	Exact (Q^*,R^*)	Exact $C_{ex}(Q^*,R^*)$
0.52	Case 1	30	0.96	(638.53,601.97)	564.30
		200	0.90	(1513.04,513.41)	1035.87
		900	0.79	(3111.81,425.52)	1942.40
		8000	0.40	(9101.44,241.74)	5425.91
		14500	0.19	(12232.68,157.52)	7254.12
		18000	0.10	(13631.33,112.20)	8066.11
	Case 2	23000	0	(15463.32,0)	9097.99
		25000	0	(16097.03,0)	9478.22
		27000	0	(16706.72,0)	9844.03

Table 2.5 Optimal target inventory measures for Exponential distributed lead-time demand when s=3, h=0.6, D=3000 and μ_L=300.

CV		A	Exact CSL	Exact (Q^*,R^*)	Exact $C_{ex}(Q^*,R^*)$
1	Case 1	30	0.94	(924.50,841.91)	879.84
		200	0.89	(1745.68,651.21)	1258.14
		900	0.78	(3314.96,458.82)	2084.27
		8000	0.40	(9249.30,150.99)	5460.18
		14500	0.19	(12345.33,64.37)	7265.82
		18000	0.10	(13719.76,32.70)	8071.48
	Case 2	23000	0	(15465.45,0)	9099.27
		25000	0	(16099.07,0)	9479.44
		27000	0	(16708.68,0)	9845.21

Table 2.6 Optimal target inventory measures for Log-Normal distributed lead-time demand when s=3, h=0.6, D=3000 and μ_L=300.

CV		A	Exact		
			CSL	(Q^*,R^*)	$C_{ex}(Q^*,R^*)$
0.2	Case 1	30	0.96	(584.64,417.16)	421.08
		200	0.90	(1453.30,380.62)	920.36
		900	0.80	(3042.72,346.97)	1853.82
		8000	0.40	(9001.61,280.09)	5389.02
		14500	0.20	(12112.07,248.36)	7236.26
		18000	0.10	(13497.65,229.50)	8056.29
	Case 2	23000	0	(15462.65,0)	9097.59
		25000	0	(16096.38,0)	9477.83
		27000	0	(16706.09,0)	9843.66
0.52	Case 1	30	0.95	(716.43,603.01)	611.66
		200	0.90	(1565.44,493.63)	1055.44
		900	0.79	(3145.00,396.45)	1944.86
		8000	0.40	(9095.93,235.03)	5418.57
		14500	0.20	(12207.38,174.80)	7249.30
		18000	0.10	(13594.83,143.87)	8063.22
	Case 2	23000	0	(15463.32,0)	9097.99
		25000	0	(16097.03,0)	9478.22
		27000	0	(16706.72,0)	9844.03
1	Case 1	30	0.93	(1067.31,727.73)	897.03
		200	0.88	(1812.21,568.86)	1248.64
		900	0.78	(3330.16,406.34)	2061.90
		8000	0.40	(9207.95,170.51)	5447.08
		14500	0.19	(12297.39,103.30)	7260.42
		18000	0.10	(13674.12,74.22)	8069.01
	Case 2	23000	0	(15465.45,0)	9099.27
		25000	0	(16099.07,0)	9479.44
		27000	0	(16708.68,0)	9845.21
2	Case 1	30	0.89	(1772.97,625.59)	1259.13
		200	0.85	(2306.35,507.94)	1508.57
		900	0.77	(3648.90,336.36)	2211.16
		8000	0.39	(9333.57,94.97)	5477.13
		14500	0.19	(12377.44,44.49)	7273.15
		18000	0.10	(13735.36,26.89)	8077.35
	Case 2	23000	0	(15474.17,0)	9104.50
		25000	0	(16107.45,0)	9484.47
		27000	0	(16716.76,0)	9850.05

Table 2.7 Optimal target inventory measures for Gamma distributed lead-time demand when s=3, h=0.6, D=3000 and μ_L=300.

			Exact		
CV		**A**	**CSL**	(Q^*, R^*)	$C_{ex}(Q^*, R^*)$
0.2	**Case 1**	30	0.96	(579.68,414.08)	416.25
		200	0.90	(1449.32,380.46)	917.87
		900	0.80	(3039.64,348.48)	1852.87
		8000	0.40	(9000.88,281.51)	5389.43
		14500	0.20	(12113.12,247.99)	7236.67
		18000	0.10	(13500.14,227.52)	8056.59
	Case 2	23000	0	(15462.65,0)	9097.59
		25000	0	(16096.38,0)	9477.83
		27000	0	(16706.09,0)	9843.66
0.52	**Case 1**	30	0.96	(670.58,607.97)	587.12
		200	0.90	(1536.71,507.08)	1046.28
		900	0.79	(3128.34,412.73)	1944.65
		8000	0.40	(9100.15,237.56)	5422.62
		14500	0.19	(12221.04,165.40)	7251.87
		18000	0.10	(13613.92,127.36)	8064.77
	Case 2	23000	0	(15463.32,0)	9097.99
		25000	0	(16097.03,0)	9478.22
		27000	0	(16706.72,0)	9844.03
1	**Case 1**	30	0.94	(924.50,841.91)	879.84
		200	0.89	(1745.68,651.21)	1258.14
		900	0.78	(3314.96,458.82)	2084.27
		8000	0.40	(9249.30,150.99)	5460.18
		14500	0.19	(12345.33,64.37)	7265.82
		18000	0.10	(13719.76,32.70)	8071.48
	Case 2	23000	0	(15465.45,0)	9099.27
		25000	0	(16099.07,0)	9479.44
		27000	0	(16708.68,0)	9845.21
2	**Case 1**	30	0.88	(1844.35,789.38)	1400.24
		200	0.85	(2431.34,598.40)	1637.85
		900	0.76	(3815.50,328.19)	2306.22
		8000	0.39	(9436.49,18.98)	5493.28
		14500	0.19	(12426.65,1.09)	7276.64
		18000	0.10	(13763.99,0.09)	8078.45
	Case 2	23000	0	(15474.17,0)	9104.50
		25000	0	(16107.45,0)	9484.47
		27000	0	(16716.76,0)	9850.05

2.6 Summary

In this chapter, for the continuous review (Q,R) inventory model with backorders and fixed lead-time we examined the minimization process of the exact annual cost function. This function is the sum of the annual expected ordering, holding and shortage costs. For the calculation of the expected annual holding cost we used the exact expression for the expected on-hand inventory at any point in time. Further, the shortage cost per unit backordered and the resulting size of backorders were used for the determination of the annual expected shortage cost. The investigation of the minimization process was carried out under J-shaped and unimodal distributions satisfying specific assumptions.

Expressing the cost function in terms only of the reorder point we derived a general condition to identify when the minimum of the cost function (a) is obtained through mathematical optimization and b) occurs when the reorder point takes on the value zero. The usefulness of this analysis relies on the fact that interval values of the cost parameters are obtained from the general condition in order the minimum cost to occur at zero reorder point. Further, the limits of these intervals are independent of the form of the lead-time demand distribution and to compute them we need, apart from the cost parameter values, the annual expected demand and the variance of the lead-time demand.

Finally, based on this general condition we offer an algorithm for finding the minimum of the cost function. After some numerical experimentation applying parameter values taken from the inventory literature to this algorithm, we observed that as the ordering cost increases we move from a situation where the unique minimum cost is attained at a positive reorder point to a situation where the minimum cost occurs at zero reorder point. Furthermore, as CV raises with fixed cost parameter values we result in larger optimal order quantities and larger minimum costs while the reorder points and cycle service levels decline. From the managerial aspects of inventory this means that as lead-time demand variability grows the optimal policies lead to excessively large orders, zero reorder points and higher minimum costs.

Appendix

Proof 2.1:

(A) Gamma(α,β) distribution

- Probability density function: $f\left(\mathrm{R}\right)=\beta^{-\alpha}\mathrm{R}^{\alpha-1}e^{-\mathrm{R}/\beta}\big/\Gamma\left(\alpha\right)$
- Cumulative distribution function: $F\left(\mathrm{R}\right)=\gamma\left(\alpha,\mathrm{R}/\beta\right)\big/\Gamma\left(\alpha\right)$
- Mean $\mu_{\mathrm{L}}=\alpha\beta$ and variance $\sigma_{\mathrm{L}}^{2}=\alpha\beta^{2}$.

From Jawitz (2004) the *N*th absolute truncated moment expression of a distribution $f\left(x\right)$ with lower and upper bounds *l* and *u* is defined as:

$$\mathrm{m}_{\mathrm{N}}=\int_{\ell}^{u}x^{\mathrm{N}}f\left(x\right)dx$$

where for the Gamma distribution the incomplete moment is

$$\mathrm{m}_{\mathrm{N}}=\frac{\beta^{\mathrm{N}}}{\Gamma\left(\alpha\right)}\left[\gamma\left(\alpha+\mathrm{N},\frac{u}{\beta}\right)-\gamma\left(\alpha+\mathrm{N},\frac{\ell}{\beta}\right)\right].$$

1) For N=1 and setting $u=\infty$ and $\ell=\mathrm{R}$ the first moment will be

$$\mathrm{m}_{1}=\int_{\mathrm{R}}^{\infty}x f\left(x\right)dx=\frac{\beta}{\Gamma\left(\alpha\right)}\left[\gamma\left(\alpha+1,\frac{\infty}{\beta}\right)-\gamma\left(\alpha+1,\frac{\mathrm{R}}{\beta}\right)\right]=$$

$$=\beta\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha\right)}-\beta\frac{\gamma\left(\alpha+1,\frac{\mathrm{R}}{\beta}\right)}{\Gamma\left(\alpha\right)},$$

where $\gamma\left(\alpha+1,\frac{\infty}{\beta}\right)=\int_{0}^{\infty}t^{\alpha}e^{-t}dt=\Gamma\left(\alpha+1\right)$.

Further, using the recursive equation

$$\gamma\left(\kappa,x\right)=\left(\kappa-1\right)\gamma\left(\kappa-1,x\right)-x^{\kappa-1}e^{-x} \tag{A.1}$$

we take

$$\mathrm{m}_{1}=\int_{\mathrm{R}}^{\infty}x f\left(x\right)dx=\beta\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(\alpha\right)}-\frac{\beta}{\Gamma\left(\alpha\right)}\left[\alpha\cdot\gamma\left(\alpha,\frac{\mathrm{R}}{\beta}\right)-\left(\frac{\mathrm{R}}{\beta}\right)^{\alpha}e^{-\frac{\mathrm{R}}{\beta}}\right]=$$

$$=\beta\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}-\mu_L\cdot F(R)+\frac{\beta}{\beta}\frac{R}{R}\frac{\beta}{\Gamma(\alpha)}R^{\alpha}\beta^{-\alpha}e^{-\frac{R}{\beta}}=$$

$$=\alpha\cdot\beta-\mu_L\cdot F(R)+R\cdot\beta\cdot f(R)=$$

$$=\mu_L\left[1-F(R)\right]+R\cdot\beta\cdot f(R),$$

where $\gamma\left(\alpha+1,\frac{R}{\beta}\right)=\alpha\cdot\gamma\left(\alpha,\frac{R}{\beta}\right)-\left(\frac{R}{\beta}\right)^{\alpha}e^{-\frac{R}{\beta}}$. (A.2)

Hence the expected size of backorders in each inventory cycle is

$$S(R)=\int_R^{\infty}xf(x)dx-R\left[1-F(R)\right]=m_1-R\left[1-F(R)\right]=$$

$$=\mu_L\left[1-F(R)\right]+R\cdot\beta\cdot f(R)-R\left[1-F(R)\right]=$$

$$=(\mu_L-R)\left[1-F(R)\right]+R\cdot\beta\cdot f(R).$$

2) For N=2 and setting $u=\infty$ and $\ell=R$ the second moment will be

$$m_2=\int_R^{\infty}x^2f(x)dx=\frac{\beta^2}{\Gamma(\alpha)}\left[\gamma\left(\alpha+2,\frac{\infty}{\beta}\right)-\gamma\left(\alpha+2,\frac{R}{\beta}\right)\right]=$$

$$=\beta^2\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}-\beta^2\frac{\gamma\left(\alpha+2,\frac{R}{\beta}\right)}{\Gamma(\alpha)},$$

where $\gamma\left(\alpha+2,\frac{\infty}{\beta}\right)=\int_0^{\infty}t^{\alpha+1}e^{-t}dt=\Gamma(\alpha+2)$.

Further, using (A.1) and (A.2) we take

$$m_2=\int_R^{\infty}x^2f(x)dx=$$

$$=\beta^2\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}-\frac{\beta^2}{\Gamma(\alpha)}\left[(\alpha+1)\cdot\alpha\cdot\gamma\left(\alpha,\frac{R}{\beta}\right)-\left(\frac{R}{\beta}\right)^{\alpha}e^{-\frac{R}{\beta}}(\alpha+1)-\left(\frac{R}{\beta}\right)^{\alpha+1}e^{-\frac{R}{\beta}}\right]=$$

$$=\beta^2\frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)}-\sigma_L^2(\alpha+1)\cdot F(R)+\frac{\beta^2\cdot R^{\alpha}\cdot\beta^{-\alpha}\cdot(\alpha+1)\cdot e^{-\frac{R}{\beta}}}{\Gamma(\alpha)}\frac{R}{R}+$$

$$+\frac{\beta^2\cdot R^{\alpha}\cdot R\cdot\beta^{-\alpha}\cdot\beta^{-1}\cdot e^{-\frac{R}{\beta}}}{\Gamma(\alpha)}\frac{R}{R}=$$

$$=\beta^2\cdot\alpha\cdot(\alpha+1)-\sigma_L^2(\alpha+1)\cdot F(R)+R\cdot\beta^2\cdot(\alpha+1)\cdot f(R)+R^2\cdot\beta\cdot f(R)=$$

$$=\sigma_L^2(\alpha+1)\cdot\left[1-F(R)\right]+R\cdot f(R)\left(\sigma_L^2+\beta^2\right)+R^2\cdot\beta\cdot f(R)=$$

$$=\alpha\cdot\sigma_L^2\cdot\left[1-F(R)\right]+\sigma_L^2\cdot\left[1-F(R)\right]+R\cdot f(R)\cdot\sigma_L^2+R\cdot f(R)\cdot\beta^2+R^2\cdot\beta\cdot f(R)=$$

$$=\frac{\mu_L^2}{\sigma_L^2}\sigma_L^2\cdot\left[1-F(R)\right]+\sigma_L^2\cdot\left[1-F(R)\right]+R\cdot f(R)\cdot\frac{\sigma_L^2}{\beta}\beta+R\cdot f(R)\cdot\beta^2+R^2\cdot\beta\cdot f(R)=$$

$$=\left(\mu_L^2+\sigma_L^2\right)\cdot\left[1-F(R)\right]+R\cdot f(R)\cdot\beta\cdot\left(\mu_L+\beta+R\right),$$

where $\gamma\left(\alpha+2,\frac{R}{\beta}\right)=(\alpha+1)\cdot\alpha\cdot\gamma\left(\alpha,\frac{R}{\beta}\right)-\left(\frac{R}{\beta}\right)^{\alpha}e^{-\frac{R}{\beta}}(\alpha+1)-\left(\frac{R}{\beta}\right)^{\alpha+1}e^{-\frac{R}{\beta}}$.

Therefore,

$$\Theta(R)=\int_R^\infty (x-R)^2 f(x)dx=\int_R^\infty x^2 f(x)dx-2R\int_R^\infty xf(x)dx+R^2\left[1-F(R)\right]=$$

$$=m_2-2\cdot R\cdot m_1+R^2\left[1-F(R)\right]=$$

$$=\left(\mu_L^2+\sigma_L^2\right)\cdot\left[1-F(R)\right]+R\cdot f(R)\cdot\beta\cdot\left(\mu_L+\beta+R\right)-2R\left\{\mu_L\left[1-F(R)\right]+R\cdot\beta\cdot f(R)\right\}$$

$$+R^2\left[1-F(R)\right]=$$

$$=\left[\left(\mu_L-R\right)^2+\sigma_L^2\right]\cdot\left[1-F(R)\right]+R\cdot f(R)\cdot\beta\cdot\left(\mu_L+\beta-R\right).$$

(B) Exponential (β) distribution

- Probability density function: $f(R)=\frac{1}{\beta}e^{-\frac{R}{\beta}}$
- Cumulative distribution function: $F(R)=1-e^{-\frac{R}{\beta}}$
- Mean $\mu_L=\beta$ and variance $\sigma_L^2=\beta^2$.

From Jawitz (2004) the incomplete moment for the Exponential distribution is

$$m_N = \int_{\ell}^{u} x^N f(x)\,dx = \sum_{\kappa=0}^{N} \binom{N}{\kappa} \beta^{\kappa} c^{N-\kappa} \left[\gamma\left(\kappa+1, \frac{u-c}{\beta}\right) - \gamma\left(\kappa+1, \frac{\ell-c}{\beta}\right) \right].$$

1) For N=1 and setting $u=\infty$, $\ell = R$ and $c=0$ the first moment will be

$$m_1 = \sum_{\kappa=0}^{1} \binom{1}{\kappa} \beta^{\kappa} 0^{1-\kappa} \left[\gamma\left(\kappa+1, \frac{\infty-0}{\beta}\right) - \gamma\left(\kappa+1, \frac{R-0}{\beta}\right) \right] =$$

$$= \binom{1}{0} \beta^{0} 0^{1-0} \left[\gamma\left(0+1, \frac{\infty-0}{\beta}\right) - \gamma\left(0+1, \frac{R-0}{\beta}\right) \right]$$

$$+ \binom{1}{1} \beta^{1} 0^{1-1} \left[\gamma\left(1+1, \frac{\infty-0}{\beta}\right) - \gamma\left(1+1, \frac{R-0}{\beta}\right) \right] =$$

$$= \frac{1!}{(1-0)!0!} \beta^{0} 0^{1-0} \left[\gamma\left(1, \frac{\infty}{\beta}\right) - \gamma\left(1, \frac{R}{\beta}\right) \right] + \frac{1!}{(1-1)!1!} \beta^{1} 0^{1-1} \left[\gamma\left(2, \frac{\infty}{\beta}\right) - \gamma\left(2, \frac{R}{\beta}\right) \right] =$$

$$= 0 + \beta \cdot 0^{0} \left[\gamma\left(2, \frac{\infty}{\beta}\right) - \gamma\left(2, \frac{R}{\beta}\right) \right] = \beta \left[\gamma\left(2, \frac{\infty}{\beta}\right) - \gamma\left(2, \frac{R}{\beta}\right) \right] =$$

$$= \beta \left[\Gamma(2) - \gamma\left(2, \frac{R}{\beta}\right) \right],$$

where $\gamma\left(2, \frac{\infty}{\beta}\right) = \int_{0}^{\infty} t^{2-1} e^{-t}\,dt = \Gamma(2)$.

Further, using (A.1) we take

$$m_1 = \int_{R}^{\infty} x f(x)\,dx = \beta \left[1 - \left\{ 1 - e^{-\frac{R}{\beta}} - \left(\frac{R}{\beta}\right) e^{-\frac{R}{\beta}} \right\} \right] = (\beta + R) \cdot e^{-\frac{R}{\beta}},$$

where $\gamma\left(2, \frac{R}{\beta}\right) = (2-1)\gamma\left(2-1, \frac{R}{\beta}\right) - \left(\frac{R}{\beta}\right)^{2-1} e^{-\frac{R}{\beta}} = 1 - e^{-\frac{R}{\beta}} - \left(\frac{R}{\beta}\right) e^{-\frac{R}{\beta}}$.

Hence the expected size of backorders in each inventory cycle is

$$S(R) = \int_{R}^{\infty} x f(x)\,dx - R\left[1 - F(R)\right] = m_1 - R\left[1 - F(R)\right] =$$

$$= (\beta + R) \cdot e^{-\frac{R}{\beta}} - R\left[1 - F(R)\right] =$$

$$=(\beta+R)\cdot e^{-\frac{R}{\beta}}-R\left[1-1+e^{-\frac{R}{\beta}}\right]=\beta\cdot e^{-\frac{R}{\beta}}.$$

2) For N=2 and setting $u=\infty$, $\ell=R$ and $c=0$ the second moment will be

$$m_2=\int_R^{\infty}x^2 f(x)dx \Leftrightarrow$$

$$m_2=\sum_{\kappa=0}^{2}\binom{2}{\kappa}\beta^{\kappa}0^{2-\kappa}\left[\gamma\left(\kappa+1,\frac{\infty-0}{\beta}\right)-\gamma\left(\kappa+1,\frac{R-0}{\beta}\right)\right]=$$

$$=\binom{2}{0}\beta^{0}0^{2-0}\left[\gamma\left(0+1,\frac{\infty-0}{\beta}\right)-\gamma\left(0+1,\frac{R-0}{\beta}\right)\right]$$

$$+\binom{2}{1}\beta^{1}0^{2-1}\left[\gamma\left(1+1,\frac{\infty-0}{\beta}\right)-\gamma\left(1+1,\frac{R-0}{\beta}\right)\right]$$

$$+\binom{2}{2}\beta^{2}0^{2-2}\left[\gamma\left(2+1,\frac{\infty-0}{\beta}\right)-\gamma\left(2+1,\frac{R-0}{\beta}\right)\right]=$$

$$=\frac{2!}{(2-0)!0!}\beta^{0}0^{2}\left[\gamma\left(1,\frac{\infty}{\beta}\right)-\gamma\left(1,\frac{R}{\beta}\right)\right]$$

$$+\frac{2!}{(2-1)!1!}\beta^{1}0^{1}\left[\gamma\left(2,\frac{\infty}{\beta}\right)-\gamma\left(2,\frac{R}{\beta}\right)\right]$$

$$+\frac{2!}{(2-2)!2!}\beta^{2}0^{0}\left[\gamma\left(3,\frac{\infty}{\beta}\right)-\gamma\left(3,\frac{R}{\beta}\right)\right]=$$

$$=0+0+\beta^{2}0^{0}\left[\gamma\left(3,\frac{\infty}{\beta}\right)-\gamma\left(3,\frac{R}{\beta}\right)\right]=$$

$$=\beta^{2}\left[\gamma\left(3,\frac{\infty}{\beta}\right)-\gamma\left(3,\frac{R}{\beta}\right)\right]=\beta^{2}\left[\Gamma(3)-\gamma\left(3,\frac{R}{\beta}\right)\right],$$

where $\gamma\left(3,\frac{\infty}{\beta}\right)=\int_0^{\infty}t^{3-1}e^{-t}dt=\Gamma(3)$.

Further, using (A.1) we take

$$m_2=\int_R^{\infty}x^2 f(x)dx=\beta^{2}\left[2-\left\{2-\frac{1}{\beta^{2}}\left\{2\cdot\beta\cdot(R+\beta)+R^{2}\right\}e^{-\frac{R}{\beta}}\right\}\right]=$$

$$=e^{-\frac{R}{\beta}}\left[2\cdot\beta\cdot(R+\beta)+R^2\right],$$

where $\gamma\left(3,\frac{R}{\beta}\right)=(3-1)\gamma\left(3-1,\frac{R}{\beta}\right)-\left(\frac{R}{\beta}\right)^{3-1}e^{-\frac{R}{\beta}}=2\cdot\gamma\left(2,\frac{R}{\beta}\right)-\left(\frac{R}{\beta}\right)^{2}e^{-\frac{R}{\beta}}=$

$$=2\cdot\left[1-e^{-\frac{R}{\beta}}-\left(\frac{R}{\beta}\right)e^{-\frac{R}{\beta}}\right]-\left(\frac{R}{\beta}\right)^{2}e^{-\frac{R}{\beta}}=$$

$$=2\cdot\left[1-e^{-\frac{R}{\beta}}\left(\frac{R+\beta}{\beta}\right)\right]-\left(\frac{R}{\beta}\right)^{2}e^{-\frac{R}{\beta}}=$$

$$=2-\frac{1}{\beta^2}\left\{2\cdot\beta\cdot(R+\beta)+R^2\right\}e^{-\frac{R}{\beta}}.$$

Therefore,

$$\Theta(R)=\int_R^\infty(x-R)^2 f(x)dx=\int_R^\infty x^2 f(x)dx-2R\int_R^\infty xf(x)dx+R^2\left[1-F(R)\right]=$$

$$=m_2-2\cdot R\cdot m_1+R^2\left[1-F(R)\right]=$$

$$=e^{-\frac{R}{\beta}}\left[2\cdot\beta\cdot(R+\beta)+R^2\right]-2R(\beta+R)\cdot e^{-\frac{R}{\beta}}+R^2\left[1-F(R)\right]=$$

$$=2\cdot\beta\cdot R\cdot e^{-\frac{R}{\beta}}+2\cdot\beta^2\cdot e^{-\frac{R}{\beta}}+R^2\cdot e^{-\frac{R}{\beta}}-2\cdot R^2\cdot e^{-\frac{R}{\beta}}-2\cdot R\cdot\beta\cdot e^{-\frac{R}{\beta}}+R^2\cdot e^{-\frac{R}{\beta}}=$$

$$=2\cdot\beta^2\cdot e^{-\frac{R}{\beta}}.$$

(C) Log-Normal (λ, θ) distribution

- Probability density function: $f(R)=\frac{1}{\theta R\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln R-\lambda}{\theta}\right)^2}$
- Cumulative distribution function: $F(R)=\Phi(r)$
- Mean $\mu_L=e^{\lambda+\theta^2/2}$ and variance $\sigma_L^2=\mu_L^2\left(e^{\theta^2}-1\right)$

where $r=(\ln R-\lambda)/\theta$.

From Jawitz (2004) the incomplete moment for the Log-Normal distribution is

$$m_N=\int_\ell^u x^N f(x)dx \Leftrightarrow$$

$$m_N=\frac{1}{2}\exp\left(N\cdot\lambda+\frac{N^2\theta^2}{2}\right)\left[\operatorname{erf}\left(\frac{\ln(u)-\lambda}{\theta\sqrt{2}}-\frac{N\cdot\theta}{\sqrt{2}}\right)-\operatorname{erf}\left(\frac{\ln(\ell)-\lambda}{\theta\sqrt{2}}-\frac{N\cdot\theta}{\sqrt{2}}\right)\right].$$

1) For N=1 and setting $u=\infty$ and $\ell = R$ the first moment will be

$$m_1 = \int_R^\infty x f(x)dx =$$

$$= \frac{1}{2}\exp\left(\lambda + \frac{\theta^2}{2}\right)\left[\operatorname{erf}\left\{\frac{\left(\frac{\ln(\infty)-\lambda}{\theta} - \theta\right)}{\sqrt{2}}\right\} - \operatorname{erf}\left\{\frac{\left(\frac{\ln(R)-\lambda}{\theta} - \theta\right)}{\sqrt{2}}\right\}\right].$$

Further, using the transformation

$$\Phi(x) = 0.5 + 0.5\cdot\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \Leftrightarrow \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{\Phi(x)-0.5}{0.5} \Leftrightarrow \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = 2\Phi(x)-1, \qquad \text{(A.3)}$$

and setting respectively $x = \frac{\ln(\infty)-\lambda}{\theta} - \theta$ and $x = \frac{\ln(R)-\lambda}{\theta} - \theta$ we take

$$(A)\operatorname{erf}\left(\frac{\frac{\ln(\infty)-\lambda}{\theta} - \theta}{\sqrt{2}}\right) = \operatorname{erf}\left(\frac{\frac{\infty-\lambda}{\theta} - \theta}{\sqrt{2}}\right) = \operatorname{erf}\left(\frac{\infty}{\sqrt{2}}\right) = 2\Phi(\infty) - 1 = 1$$

and

$$(B)\operatorname{erf}\left(\frac{\frac{\ln(R)-\lambda}{\theta} - \theta}{\sqrt{2}}\right) = 2\Phi\left(\frac{\ln(R)-\lambda}{\theta} - \theta\right) - 1.$$

Therefore,

$$m_1 = \int_R^\infty x f(x)dx =$$

$$= \frac{1}{2}\exp\left(\lambda + \frac{\theta^2}{2}\right)\left[1 - 2\Phi\left(\frac{\ln(R)-\lambda}{\theta} - \theta\right) + 1\right] =$$

$$= \exp\left(\lambda + \frac{\theta^2}{2}\right)\left[1 - \Phi\left(\frac{\ln(R)-\lambda}{\theta} - \theta\right)\right],$$

and the expected size of backorders in each inventory cycle is

$$S(R) = \int_R^\infty x f(x)dx - R\left[1 - F(R)\right] =$$

$$= m_1 - R\left[1 - F(R)\right] =$$

$$= \exp\left(\lambda + \frac{\theta^2}{2}\right)\left[1 - \Phi\left(\frac{\ln(R) - \lambda}{\theta} - \theta\right)\right] - R\left[1 - F(R)\right] =$$

$$= e^{\lambda + \frac{\theta^2}{2}}\left[\Phi\left(\theta - \frac{\ln(R) - \lambda}{\theta}\right)\right] - R\left[1 - F(R)\right].$$

Setting $r = (\ln(R) - \lambda)/\theta$, the function $S(R)$ for the Log-Normal lead-time demand takes the form

$$S(R) = \mu_L \Phi(\theta - r) - e^{\lambda + r\cdot\theta}\Phi(-r)$$

where $r = (\ln(R) - \lambda)/\theta \Leftrightarrow \ln(R) = r\cdot\theta + \lambda \Leftrightarrow R = e^{\lambda + r\cdot\theta}$.

2) For N=2 and setting $u = \infty$ and $\ell = R$ the second moment will be

$$m_2 = \int_R^\infty x^2 f(x)\,dx =$$

$$= \frac{1}{2}\exp\left(2\lambda + 2\theta^2\right)\left[\operatorname{erf}\left\{\frac{\left(\frac{\ln(\infty) - \lambda}{\theta} - 2\theta\right)}{\sqrt{2}}\right\} - \operatorname{erf}\left\{\frac{\left(\frac{\ln(R) - \lambda}{\theta} - 2\theta\right)}{\sqrt{2}}\right\}\right].$$

Further, using (A.3) and setting respectively $x = \frac{\ln(\infty) - \lambda}{\theta} - 2\theta$ and $x = \frac{\ln(R) - \lambda}{\theta} - 2\theta$ we take

$$(A)\operatorname{erf}\left(\frac{\frac{\ln(\infty) - \lambda}{\theta} - 2\theta}{\sqrt{2}}\right) = \operatorname{erf}\left(\frac{\frac{\infty - \lambda}{\theta} - 2\theta}{\sqrt{2}}\right) = \operatorname{erf}\left(\frac{\infty}{\sqrt{2}}\right) = 2\Phi(\infty) - 1 = 1$$

and

$$(B)\operatorname{erf}\left(\frac{\frac{\ln(R) - \lambda}{\theta} - 2\theta}{\sqrt{2}}\right) = 2\Phi\left(\frac{\ln(R) - \lambda}{\theta} - 2\theta\right) - 1.$$

Therefore,

$$m_2 = \int_R^\infty x^2 f(x)\,dx = \exp\left(2\lambda + 2\theta^2\right)\left[1 - \Phi\left(\frac{\ln(R) - \lambda}{\theta} - 2\theta\right)\right].$$

Hence,

$$\Theta(R)=\int_R^{\infty}(x-R)^2 f(x)dx=\int_R^{\infty}x^2 f(x)dx-2R\int_R^{\infty}xf(x)dx+R^2\left[1-F(R)\right]=$$

$$=m_2-2\cdot R\cdot m_1+R^2\left[1-F(R)\right]=$$

$$=\exp\left(2\lambda+2\theta^2\right)\left[1-\Phi\left(\frac{\ln(R)-\lambda}{\theta}-2\theta\right)\right]-2\cdot R\cdot\exp\left(\lambda+\frac{\theta^2}{2}\right)\left[1-\Phi\left(\frac{\ln(R)-\lambda}{\theta}-\theta\right)\right]$$

$$+R^2\left[1-\Phi\left(\frac{\ln(R)-\lambda}{\theta}\right)\right].$$

Setting $r=(\ln(R)-\lambda)/\theta$, the function $\Theta(R)$ for the Log-Normal lead-time demand takes the form

$$\Theta(R)=\left(\mu_L^2+\sigma_L^2\right)\Phi(2\theta-r)-2\cdot R\cdot\mu_L\cdot\Phi(\theta-r)+R^2\Phi(-r)$$

where $\exp\left(2\lambda+2\theta^2\right)=\exp\left(2\lambda+\theta^2\right)\exp\left(\theta^2\right)=\mu_L^2\exp\left(\theta^2\right)=\mu_L^2+\sigma_L^2$.

(D) Weibull (α, β) distribution

- Probability density function: $f(R)=\alpha\beta^{-\alpha}R^{\alpha-1}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}$
- Cumulative distribution function: $F(R)=1-e^{-\left(\frac{R}{\beta}\right)^{\alpha}}$
- Mean $\mu_L=\alpha^{-1}\beta\Gamma\left(\frac{1}{\alpha}\right)$ and variance $\sigma_L^2=\alpha^{-1}\beta^2\left\{2\Gamma\left(\frac{2}{\alpha}\right)-\frac{1}{\alpha}\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^2\right\}$.

From Jawitz (2004) the incomplete moment for the Weibull distribution is

$$m_N=\int_{\ell}^{u}x^N f(x)dx\Leftrightarrow$$

$$m_N=\sum_{\kappa=0}^{N}\binom{N}{\kappa}(\beta-c)^{\kappa}c^{N-\kappa}\left[\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{u-c}{\beta-c}\right)^{\alpha}\right)-\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{\ell-c}{\beta-c}\right)^{\alpha}\right)\right].$$

1) For N=1 and setting $u=\infty$, $\ell=R$ and $c=0$ the first moment will be

$$m_1=\int_R^{\infty}x f(x)dx=$$

$$=\sum_{\kappa=0}^{1}\binom{1}{\kappa}(\beta-0)^{\kappa}0^{1-\kappa}\left[\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{\infty-0}{\beta-0}\right)^{\alpha}\right)-\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{R-0}{\beta-0}\right)^{\alpha}\right)\right]=$$

$$=\binom{1}{0}\beta^{0}0^{1-0}\left[\gamma\left(1+\frac{0}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{0}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]$$

$$+\binom{1}{1}\beta^{1}0^{1-1}\left[\gamma\left(1+\frac{1}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$=\frac{1!}{(1-0)!0!}\beta^{0}0^{1-0}\left[\gamma\left(1,\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1,\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]$$

$$+\frac{1!}{(1-1)!1!}\beta^{1}0^{1-1}\left[\gamma\left(1+\frac{1}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$=0+\beta\cdot 0^{0}\left[\gamma\left(1+\frac{1}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$=\beta\left[\Gamma\left(1+\frac{1}{\alpha}\right)-\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$=\beta\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)-\beta\frac{1}{\alpha}\gamma\left(\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)+\beta\frac{R}{\beta}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}=$$

$$=\mu_L-\beta\alpha^{-1}\gamma\left(\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)+Re^{-\left(\frac{R}{\beta}\right)^{\alpha}},$$

where $\gamma\left(1+\frac{1}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)=\Gamma\left(1+\frac{1}{\alpha}\right)$ and from (A.1) it holds

$$\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)=\left(1+\frac{1}{\alpha}-1\right)\gamma\left(1+\frac{1}{\alpha}-1,\left(\frac{R}{\beta}\right)^{\alpha}\right)-\left[\left(\frac{R}{\beta}\right)^{\alpha}\right]^{1+\frac{1}{\alpha}-1}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}=$$

$$=\frac{1}{\alpha}\gamma\left(\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)-\frac{R}{\beta}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}.$$

Hence the expected size of backorders in each inventory cycle is

$$S(R)=\int_{R}^{\infty}xf(x)dx-R\left[1-F(R)\right]=m_1-R\left[1-F(R)\right]=$$

$$= \mu_L - \beta\alpha^{-1}\gamma\left(\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right) + \mathrm{R}e^{-\left(\frac{R}{\beta}\right)^{\alpha}} - \mathrm{R}e^{-\left(\frac{R}{\beta}\right)^{\alpha}} =$$

$$= \mu_L - \beta\alpha^{-1}\gamma\left(\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right).$$

2) For N=2 and setting $u=\infty$, $\ell = R$ and $c=0$ the second moment will be

$$m_2 = \int_R^{\infty} x^2 f(x)\,dx =$$

$$= \sum_{\kappa=0}^{2}\binom{2}{\kappa}(\beta-0)^{\kappa}\,0^{2-\kappa}\left[\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{\infty-0}{\beta-0}\right)^{\alpha}\right)-\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{R-0}{\beta-0}\right)^{\alpha}\right)\right]=$$

$$= \binom{2}{0}\beta^{0}0^{2-0}\left[\gamma\left(1+\frac{0}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{0}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]$$

$$+\binom{2}{1}\beta^{1}0^{2-1}\left[\gamma\left(1+\frac{1}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]$$

$$+\binom{2}{2}\beta^{1}0^{2-2}\left[\gamma\left(1+\frac{2}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$= \frac{2!}{(2-0)!0!}\beta^{0}0^{2-0}\left[\gamma\left(1,\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1,\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]$$

$$+\frac{2!}{(2-1)!1!}\beta^{1}0^{2-1}\left[\gamma\left(1+\frac{1}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]$$

$$+\frac{2!}{(2-2)!2!}\beta^{2}0^{2-2}\left[\gamma\left(1+\frac{2}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$= 0+0+\beta^{2}0^{0}\left[\gamma\left(1+\frac{2}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$= \beta^{2}\left[\gamma\left(1+\frac{2}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)-\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)\right]=$$

$$=\beta^2\Gamma\left(1+\frac{2}{\alpha}\right)-\beta^2\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)=$$

$$=\beta^2\frac{2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)-\beta^2\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)=$$

$$=\beta^2\cdot 2\cdot\alpha^{-1}\cdot\Gamma\left(\frac{2}{\alpha}\right)-\beta^2\left[\frac{2}{\alpha}\gamma\left(\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)-\left(\frac{R}{\beta}\right)^{2}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}\right]=$$

$$=\mu_L^2+\sigma_L^2-\beta^2\cdot 2\cdot\alpha^{-1}\cdot\gamma\left(\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)+R^2e^{-\left(\frac{R}{\beta}\right)^{\alpha}},$$

where (a) $\gamma\left(1+\frac{2}{\alpha},\left(\frac{\infty}{\beta}\right)^{\alpha}\right)=\Gamma\left(1+\frac{2}{\alpha}\right)$,

(b) $\gamma\left(1+\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)=\left(1+\frac{2}{\alpha}-1\right)\gamma\left(1+\frac{2}{\alpha}-1,\left(\frac{R}{\beta}\right)^{\alpha}\right)-\left[\left(\frac{R}{\beta}\right)^{\alpha}\right]^{1+\frac{2}{\alpha}-1}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}=$

$$=\frac{2}{\alpha}\gamma\left(\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)-\left(\frac{R}{\beta}\right)^{2}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}\text{, and}$$

(c) $\mu_L^2+\sigma_L^2=\left[\alpha^{-1}\cdot\beta\cdot\Gamma\left(\frac{1}{\alpha}\right)\right]^2+\alpha^{-1}\cdot\beta^2\cdot 2\cdot\Gamma\left(\frac{2}{\alpha}\right)-\frac{\alpha^{-1}\cdot\beta^2\cdot\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^2}{\alpha}=$

$$=\alpha^{-2}\cdot\beta^2\cdot\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^2+\alpha^{-1}\cdot\beta^2\cdot 2\cdot\Gamma\left(\frac{2}{\alpha}\right)-\alpha^{-2}\cdot\beta^2\cdot\left[\Gamma\left(\frac{1}{\alpha}\right)\right]^2=\alpha^{-1}\cdot\beta^2\cdot 2\cdot\Gamma\left(\frac{2}{\alpha}\right).$$

Therefore,

$$\Theta(R)=\int_R^{\infty}(x-R)^2 f(x)dx=\int_R^{\infty}x^2 f(x)dx-2R\int_R^{\infty}xf(x)dx+R^2\left[1-F(R)\right]=$$

$$=m_2-2\cdot R\cdot m_1+R^2\left[1-F(R)\right]=$$

$$=\mu_L^2+\sigma_L^2-\beta^2\cdot 2\cdot\alpha^{-1}\cdot\gamma\left(\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)+R^2e^{-\left(\frac{R}{\beta}\right)^{\alpha}}$$

$$-2\cdot R\cdot\left[\mu_L-\beta\alpha^{-1}\gamma\left(\frac{1}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)+Re^{-\left(\frac{R}{\beta}\right)^{\alpha}}\right]+R^2e^{-\left(\frac{R}{\beta}\right)^{\alpha}}=$$

$$=\mu_L^2+\sigma_L^2-\beta^2\cdot 2\cdot\alpha^{-1}\cdot\gamma\left(\frac{2}{\alpha},\left(\frac{R}{\beta}\right)^{\alpha}\right)-2\cdot R\cdot S(R).$$

(E) Rayleigh (α) distribution

- Probability density function: $f(R)=2\cdot\beta^{-2}\cdot R\cdot e^{-\left(\frac{R}{\beta}\right)^2}$
- Cumulative distribution function: $F(R)=1-e^{-\left(\frac{R}{\beta}\right)^2}$
- Mean $\mu_L=\beta\frac{\sqrt{\pi}}{2}$ and variance $\sigma_L^2=\beta^2\frac{4-\pi}{\pi}$.

From Jawitz (2004) the incomplete moment for the Weibull distribution is

$$m_N=\int_{\ell}^{u}x^N f(x)\,dx \Leftrightarrow$$

$$m_N=\sum_{\kappa=0}^{N}\binom{N}{\kappa}(\beta-c)^{\kappa}c^{N-\kappa}\left[\gamma\left(1+\frac{\kappa}{\alpha},\left(\frac{u-c}{\beta-c}\right)^{\alpha}\right)-\gamma\left(1+\frac{\kappa}{\alpha},\frac{\ell-c}{\beta-c}\right)^{\alpha}\right].$$

1) For N=1 and setting $u=\infty$, $\ell=R$, $c=0$ and $\alpha=2$ the first moment will be

$$m_1=\int_{R}^{\infty}x f(x)\,dx=$$

$$=\sum_{\kappa=0}^{1}\binom{1}{\kappa}(\beta-0)^{\kappa}0^{1-\kappa}\left[\gamma\left(1+\frac{\kappa}{2},\left(\frac{\infty-0}{\beta-0}\right)^2\right)-\gamma\left(1+\frac{\kappa}{2},\left(\frac{R-0}{\beta-0}\right)^2\right)\right]=$$

$$=\binom{1}{0}\beta^0 0^{1-0}\left[\gamma\left(1+\frac{0}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{0}{2},\left(\frac{R}{\beta}\right)^2\right)\right]$$

$$+\binom{1}{1}\beta^1 0^{1-1}\left[\gamma\left(1+\frac{1}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{1}{2},\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=\frac{1!}{(1-0)!0!}\beta^0 0^{1-0}\left[\gamma\left(1,\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1,\left(\frac{R}{\beta}\right)^2\right)\right]$$

$$+\frac{1!}{(1-1)!1!}\beta^1 0^{1-1}\left[\gamma\left(1+\frac{1}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{1}{2},\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=0+\beta\cdot 0^{0}\left[\gamma\left(1+\frac{1}{2},\left(\frac{\infty}{\beta}\right)^{2}\right)-\gamma\left(1+\frac{1}{2},\left(\frac{R}{\beta}\right)^{2}\right)\right]=$$

$$=\beta\left[\gamma\left(\frac{3}{2},\infty\right)-\gamma\left(\frac{3}{2},\left(\frac{R}{\beta}\right)^{2}\right)\right]=$$

$$=\beta\left[\frac{\sqrt{\pi}}{2}-\frac{1}{2}\gamma\left(\frac{1}{2},\left(\frac{R}{\beta}\right)^{2}\right)+\left(\frac{R}{\beta}\right)e^{-\left(\frac{R}{\beta}\right)^{2}}\right]=$$

$$=\beta\left[\frac{\sqrt{\pi}}{2}-\frac{1}{2}\sqrt{\pi}\cdot\frac{\Phi\left(\frac{R\sqrt{2}}{\beta}\right)-\frac{1}{2}}{\frac{1}{2}}+\left(\frac{R}{\beta}\right)e^{-\left(\frac{R}{\beta}\right)^{2}}\right]=$$

$$=\beta\left[\frac{\sqrt{\pi}}{2}-\sqrt{\pi}\cdot\Phi\left(\frac{R\sqrt{2}}{\beta}\right)+\frac{\sqrt{\pi}}{2}+\left(\frac{R}{\beta}\right)e^{-\left(\frac{R}{\beta}\right)^{2}}\right]=\beta\sqrt{\pi}\cdot\left\{1-\Phi\left(\frac{R\sqrt{2}}{\beta}\right)\right\}+Re^{-\left(\frac{R}{\beta}\right)^{2}},$$

where

(a) $\gamma\left(\frac{3}{2},\infty\right)=\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}}{2}$,

(b) from (A.1) it holds $\gamma\left(\frac{3}{2},\left(\frac{R}{\beta}\right)^{2}\right)=\left(1+\frac{1}{2}-1\right)\gamma\left(1+\frac{1}{2}-1,\left(\frac{R}{\beta}\right)^{2}\right)-\left[\left(\frac{R}{\beta}\right)^{2}\right]^{1+\frac{1}{2}-1}e^{-\left(\frac{R}{\beta}\right)^{2}}=$

$$=\frac{1}{2}\gamma\left(\frac{1}{2},\left(\frac{R}{\beta}\right)^{2}\right)-\left(\frac{R}{\beta}\right)e^{-\left(\frac{R}{\beta}\right)^{2}},$$

(c) $\gamma\left(\frac{1}{2},x\right)=\sqrt{\pi}\cdot\mathrm{erf}\left(\sqrt{x}\right)$ and

(d) from (A.3) it holds $\gamma\left(\frac{1}{2},\left(\frac{R}{\beta}\right)^{2}\right)=\sqrt{\pi}\cdot\mathrm{erf}\left(\frac{R}{\beta}\right)=\sqrt{\pi}\cdot\mathrm{erf}\left(\frac{R\sqrt{2}}{\beta\sqrt{2}}\right)=\sqrt{\pi}\cdot\frac{\Phi\left(\frac{R\sqrt{2}}{\beta}\right)-\frac{1}{2}}{\frac{1}{2}}$.

Hence, the expected size of backorders in each inventory cycle is

$$S(R)=\int_{R}^{\infty}xf(x)dx-R\left[1-F(R)\right]=m_{1}-R\left[1-F(R)\right]=$$

$$=\beta\sqrt{\pi}\cdot\left\{1-\Phi\left(\frac{R\sqrt{2}}{\beta}\right)\right\}+Re^{-\left(\frac{R}{\beta}\right)^2}-Re^{-\left(\frac{R}{\beta}\right)^2}=$$

$$=2\mu_L\left\{1-\Phi\left(\frac{R\sqrt{2}}{\beta}\right)\right\}.$$

2) For N=2 and setting $u=\infty$, $\ell=R$, $c=0$ and $\alpha=2$ the second moment will be

$$m_2=\int_R^\infty x^2 f(x)\,dx=$$

$$=\sum_{\kappa=0}^{2}\binom{2}{\kappa}(\beta-0)^\kappa 0^{2-\kappa}\left[\gamma\left(1+\frac{\kappa}{2},\left(\frac{\infty-0}{\beta-0}\right)^2\right)-\gamma\left(1+\frac{\kappa}{2},\left(\frac{R-0}{\beta-0}\right)^2\right)\right]=$$

$$=\binom{2}{0}\beta^0 0^{2-0}\left[\gamma\left(1+\frac{0}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{0}{2},\left(\frac{R}{\beta}\right)^2\right)\right]$$

$$+\binom{2}{1}\beta^1 0^{2-1}\left[\gamma\left(1+\frac{1}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{1}{2},\left(\frac{R}{\beta}\right)^2\right)\right]$$

$$+\binom{2}{2}\beta^1 0^{2-2}\left[\gamma\left(1+\frac{2}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{2}{2},\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=\frac{2!}{(2-0)!0!}\beta^0 0^{2-0}\left[\gamma\left(1,\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1,\left(\frac{R}{\beta}\right)^2\right)\right]$$

$$+\frac{2!}{(2-1)!1!}\beta^1 0^{2-1}\left[\gamma\left(1+\frac{1}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{1}{2},\left(\frac{R}{\beta}\right)^2\right)\right]$$

$$+\frac{2!}{(2-2)!2!}\beta^2 0^{2-2}\left[\gamma\left(1+\frac{2}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{2}{2},\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=0+0+\beta^2 0^0\left[\gamma\left(1+\frac{2}{2},\left(\frac{\infty}{\beta}\right)^2\right)-\gamma\left(1+\frac{2}{2},\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=\beta^2\left[\gamma(2,\infty)-\gamma\left(2,\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=\beta^2\left[\Gamma(2)-\gamma\left(2,\left(\frac{R}{\beta}\right)^2\right)\right]=$$

$$=\beta^2\left[1-1+e^{-\left(\frac{R}{\beta}\right)^2}+\left(\frac{R}{\beta}\right)^2 e^{-\left(\frac{R}{\beta}\right)^2}\right]=$$

$$=\beta^2 e^{-\left(\frac{R}{\beta}\right)^2}+R^2 e^{-\left(\frac{R}{\beta}\right)^2}$$

$$=\left(\mu_L^2+\sigma_L^2\right)e^{-\left(\frac{R}{\beta}\right)^2}+R^2 e^{-\left(\frac{R}{\beta}\right)^2},$$

where (a) $\gamma(2,\infty)=\Gamma(2)$,

(b) using (A.3) it holds $\gamma\left(2,\left(\frac{R}{\beta}\right)^2\right)=(2-1)\gamma\left(2-1,\left(\frac{R}{\beta}\right)^2\right)-\left[\left(\frac{R}{\beta}\right)^2\right]^{2-1} e^{-\left(\frac{R}{\beta}\right)^2}=$

$$=1-e^{-\left(\frac{R}{\beta}\right)^2}-\left(\frac{R}{\beta}\right)^2 e^{-\left(\frac{R}{\beta}\right)^2} \text{ and}$$

(c) $\mu_L^2+\sigma_L^2=\frac{\pi\cdot\beta^2}{4}+\frac{\beta^2(4-\pi)}{4}=\frac{\pi\cdot\beta^2}{4}+\frac{4\beta^2}{4}-\frac{\pi\cdot\beta^2}{4}=\beta^2$.

Therefore,

$$\Theta(R)=\int_R^\infty (x-R)^2 f(x)dx=\int_R^\infty x^2 f(x)dx-2R\int_R^\infty xf(x)dx+R^2\left[1-F(R)\right]=$$

$$=m_2-2\cdot R\cdot m_1+R^2\left[1-F(R)\right]=$$

$$=\left(\mu_L^2+\sigma_L^2\right)e^{-\left(\frac{R}{\beta}\right)^2}+R^2 e^{-\left(\frac{R}{\beta}\right)^2}-2R\left[\beta\sqrt{\pi}\cdot\left\{1-\Phi\left(\frac{R\sqrt{2}}{\beta}\right)\right\}+Re^{-\left(\frac{R}{\beta}\right)^2}\right]+R^2 e^{-\left(\frac{R}{\beta}\right)^2}$$

$$=\left(\mu_L^2+\sigma_L^2\right)e^{-\left(\frac{R}{\beta}\right)^2}-4R\mu_L\left\{1-\Phi\left(\frac{R\sqrt{2}}{\beta}\right)\right\}.$$

Proof 2.2:

$$C(R)=h\left(\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot S(R)+\Theta(R)}+R-\mu_L\right),$$

whose first derivative is

$$C'(R)=\frac{dC(R)}{dR}=\frac{d}{dR}\left[h\left(\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot S(R)+\Theta(R)}+R-\mu_L\right)\right]=$$

$$=h\frac{dQ(R)}{dR}+h=h\left(\frac{dQ(R)}{dR}+1\right).$$

But we have,

$$\frac{dQ(R)}{dR}=\frac{d}{dR}\left[2\frac{A}{h}D+2\frac{s}{h}D\cdot S(R)+\Theta(R)\right]^{\frac{1}{2}}=$$

$$=\frac{1}{2}Q(R)^{-1}\left[2\frac{s}{h}D\cdot\frac{d}{dR}S(R)+\frac{d}{dR}\Theta(R)\right]=$$

$$=\frac{1}{2Q(R)}\left[2\frac{s}{h}D\cdot\left\{-\left[1-F(R)\right]\right\}-2S(R)\right]=$$

$$=-\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}.$$

Thus,

$$C'(R)=\frac{dC(R)}{dR}=h\left(\frac{dQ(R)}{dR}+1\right)=$$

$$=h\left[-\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}+1\right]\Leftrightarrow$$

$$=-hV(R),$$

where

$$V(R)=\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1.$$

The second derivative of $C(R)$ is

$$C''(R)=\frac{d^2C(R)}{dR^2}=-hV'(R)$$

where

$$V'(R)=\frac{dV(R)}{dR}=\frac{d}{dR}\left(\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1\right)=$$

$$=\left\{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right\}\frac{dQ(R)^{-1}}{dR}+Q(R)^{-1}\frac{d}{dR}\left\{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right\}=$$

$$=\left\{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right\}\frac{dQ(R)^{-1}}{dQ(R)}\frac{dQ(R)}{dR}+Q(R)^{-1}\left\{-\frac{s}{h}D\cdot f(R)-\left[1-F(R)\right]\right\}=$$

$$=-\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)^2}\left\{-\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}\right\}-\frac{\frac{s}{h}D\cdot f(R)+\left[1-F(R)\right]}{Q(R)}=$$

$$=\frac{\left[\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right]^2}{Q(R)^3}-\frac{\frac{s}{h}D\cdot f(R)+\left[1-F(R)\right]}{Q(R)}=$$

$$=-\frac{1}{Q(R)^3}\left\{Q(R)^2\left(\frac{s}{h}D\cdot f(R)+\left[1-F(R)\right]\right)-\left[\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right]^2\right\}=$$

$$=-\frac{g(R)}{Q(R)^3},$$

where

$$g(R)=Q(R)^2\left(\frac{s}{h}D\cdot f(R)+\left[1-F(R)\right]\right)-\left[\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right]^2=$$

$$=\left\{\left(\frac{s}{h}Df(R)+\left[1-F(R)\right]\right)\left[Q(R)\right]^2-\left[Q(R)\frac{dQ(R)}{dR}\right]^2\right\}=$$

$$=\left\{\frac{s}{h}Df(R)+\left[1-F(R)\right]-\left[\frac{dQ(R)}{dR}\right]^2\right\}\left[Q(R)\right]^2$$

and

$$\frac{dQ(R)}{dR}=-\frac{s\cdot D[1-F(R)]/h+S(R)}{Q(R)}.$$

To determine the sign of $\mathrm{g}(\mathrm{R})$, we take its derivative

$$\mathrm{g}'(\mathrm{R})=\frac{\mathrm{dg}(\mathrm{R})}{\mathrm{dR}}=$$

$$=\frac{\mathrm{d}}{\mathrm{dR}}\left\{\mathrm{Q}(\mathrm{R})^2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})+\left[1-F(\mathrm{R})\right]\right)-\left[\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\left[1-F(\mathrm{R})\right]+\mathrm{S}(\mathrm{R})\right]^2\right\}=$$

$$=\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})+\left[1-F(\mathrm{R})\right]\right)\frac{\mathrm{d}\left[\mathrm{Q}(\mathrm{R})\right]^2}{\mathrm{dQ}(\mathrm{R})}\frac{\mathrm{dQ}(\mathrm{R})}{\mathrm{dR}}$$

$$+\mathrm{Q}(\mathrm{R})^2\frac{\mathrm{d}}{\mathrm{dR}}\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})+\left[1-F(\mathrm{R})\right]\right)-\frac{\mathrm{d}}{\mathrm{dR}}\left[\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\left[1-F(\mathrm{R})\right]+\mathrm{S}(\mathrm{R})\right]^2=$$

$$=2\mathrm{Q}(\mathrm{R})\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})+\left[1-F(\mathrm{R})\right]\right)\left[-\frac{\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\left[1-F(\mathrm{R})\right]+\mathrm{S}(\mathrm{R})}{\mathrm{Q}(\mathrm{R})}\right]$$

$$+\mathrm{Q}(\mathrm{R})^2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\frac{\mathrm{d}f(\mathrm{R})}{\mathrm{dR}}-f(\mathrm{R})\right)-2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\left[1-F(\mathrm{R})\right]+\mathrm{S}(\mathrm{R})\right)\left(-\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})-\left[1-F(\mathrm{R})\right]\right)=$$

$$=-2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})+\left[1-F(\mathrm{R})\right]\right)\left[\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\left[1-F(\mathrm{R})\right]+\mathrm{S}(\mathrm{R})\right]$$

$$+2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot f(\mathrm{R})+\left[1-F(\mathrm{R})\right]\right)\left[\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\left[1-F(\mathrm{R})\right]+\mathrm{S}(\mathrm{R})\right]$$

$$+\mathrm{Q}(\mathrm{R})^2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\frac{\mathrm{d}f(\mathrm{R})}{\mathrm{dR}}-f(\mathrm{R})\right)\Leftrightarrow$$

$$\mathrm{g}'(\mathrm{R})=\frac{\mathrm{dg}(\mathrm{R})}{\mathrm{dR}}=\mathrm{Q}(\mathrm{R})^2\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\frac{\mathrm{d}f(\mathrm{R})}{\mathrm{dR}}-f(\mathrm{R})\right).$$

Proof 2.3:

Validity of Assumption 2

Using the formulae of Table 2.1, we prove below that Assumption 2 is true for the unimodal distributions Gamma(α,β), Weibull(α,β) for $\alpha>1$ and Log-Normal (λ,θ) for any values of λ and θ.

Gamma(α,β): Substituting $f'(R)=\frac{df(R)}{dR}=\frac{d}{dR}\left[\frac{\beta^{-\alpha}R^{\alpha-1}e^{-\frac{R}{\beta}}}{\Gamma(\alpha)}\right]=$

$$=\left[\frac{R^{\alpha-2}e^{-\frac{R}{\beta}}(\alpha-1)}{\beta^{\alpha}\Gamma(\alpha)}-\frac{R^{\alpha-1}e^{-\frac{R}{\beta}}\beta^{-1}}{\beta^{\alpha}\Gamma(\alpha)}\right]=f(R)\left(\frac{\alpha-1}{R}-\frac{1}{\beta}\right)$$

into (2.12) we take

$$u(R)=f(R)\left\{\frac{s}{h}D\left[\frac{\alpha-1}{R}-\frac{1}{\beta}\right]-1\right\}. \tag{A.4}$$

Setting $R=R_o$ in (A.4),

where

$$\frac{s}{h}D\left[\frac{\alpha-1}{R_o}-\frac{1}{\beta}\right]-1=0\Leftrightarrow$$

$$\frac{s}{h}D(\alpha-1)=R_o\left(\frac{s}{h}D\frac{1}{\beta}+1\right)\Leftrightarrow$$

$$R_o=\frac{\frac{s}{h}D(\alpha-1)\beta}{\frac{s}{h}D+\beta}\Leftrightarrow$$

$$R_o=R_m\frac{s\cdot D/h}{(s\cdot D/h)+\beta}<R_m$$

with $R_m=\beta(\alpha-1)$, we take $u(R_o)=0$.

Further, it holds

(a) $u(R)>0$ for $R<R_o$ and

(b) $u(R)<0$ for $R>R_o$.

Log-Normal (λ,θ): With

$$f'(R)=-\frac{1}{\theta R^2\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\ln R-\lambda}{\theta}\right]^2}-\frac{1}{\theta R\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{\ln R-\lambda}{\theta}\right]^2}\frac{\ln R-\lambda}{\theta}\frac{1}{R\theta}=$$

$$=-\frac{f(R)}{R}-\frac{f(R)}{R\theta}\frac{\ln R-\lambda}{\theta}=-\frac{f(R)}{R}\left[1+\frac{1}{\theta}\frac{\ln R-\lambda}{\theta}\right],$$

we obtain

$$u(R) = -\frac{f(R)}{R}\left\{\frac{sD}{h}\left(1+\frac{\ln R-\lambda}{\theta^2}\right)+R\right\}. \tag{A.5}$$

In (A.5) the function inside the brackets is strictly increasing on the interval $(0,+\infty)$ with range $(-\infty,+\infty)$,

$$\lim_{R\to 0}\left\{\frac{sD}{h}+\frac{sD}{h}\frac{\ln R-\lambda}{\theta^2}+R\right\} = -\infty$$

$$\lim_{R\to \infty}\left\{\frac{sD}{h}+\frac{sD}{h}\frac{\ln R-\lambda}{\theta^2}+R\right\} = +\infty .$$

Hence, there is a single $R_o > 0$ for which $u(R_o) = 0$. Solving the equation $\frac{sD}{h}\left(1+\frac{\ln R_o-\lambda}{\theta^2}\right)+R_o = 0$ with respect to R_o, we take

$$\frac{sD}{h}\left(1+\frac{\ln R_o-\lambda}{\theta^2}\right)+R_o = 0 \Leftrightarrow$$

$$\frac{sD}{h}\theta^2+\frac{sD}{h}\ln R_o-\frac{sD}{h}\lambda+R_o\theta^2 = 0 \Leftrightarrow$$

$$\frac{sD}{h}\ln R_o = \frac{sD}{h}\lambda - R_o\theta^2 - \frac{sD}{h}\theta^2 \Leftrightarrow$$

$$R_o = e^{\lambda-\theta^2-R_o\frac{h\theta^2}{sD}} \Leftrightarrow$$

$$R_o = e^{\lambda-\theta^2}e^{-R_o\frac{h\theta^2}{sD}} \Leftrightarrow$$

$$R_o = R_m e^{-R_o\frac{h\theta^2}{sD}}$$

with $R_m = e^{\lambda-\theta^2}$ and $R_o\frac{h\theta^2}{sD} > 0$. Hence $e^{-R_o\frac{h\theta^2}{sD}} < 1$ from which it follows that $R_o < R_m$.

Further, it is easily deduced that $\frac{sD}{h}\left(1+\frac{\ln R-\lambda}{\theta^2}\right)+R$ is negative (positive) and $u(R)$ is positive (negative) for $R < R_o$ ($R > R_o$).

Weibull$(\boldsymbol{\alpha},\boldsymbol{\beta})$: In this case we have

$$f'(R) = \frac{d}{dR}f(R) = \frac{d}{dR}\left[\alpha\beta^{-\alpha}R^{\alpha-1}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}\right] =$$

$$=\alpha\cdot\beta^{-\alpha}(\alpha-1)R^{\alpha-2}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}-\alpha\cdot\beta^{-\alpha}R^{\alpha-1}e^{-\left(\frac{R}{\beta}\right)^{\alpha}}\frac{\alpha}{\beta}\left(\frac{R}{\beta}\right)^{\alpha-1}=$$

$$=\frac{(\alpha-1)}{R}f(R)-f(R)\frac{\alpha}{\beta}\left(\frac{R}{\beta}\right)^{\alpha-1}=$$

$$=f(R)\left\{\frac{\alpha-1}{R}-\frac{\alpha}{\beta}\left(\frac{R}{\beta}\right)^{\alpha-1}\right\}$$

and

$$u(R)=f(R)\left\{\frac{s}{h}D\left[\frac{\alpha-1}{R}-\frac{\alpha}{\beta}\left(\frac{R}{\beta}\right)^{\alpha-1}\right]-1\right\}\ . \tag{A.6}$$

In (A.6), for $R>0$, the function within the brackets is strictly decreasing with range $(-\infty,+\infty)$,

$$\lim_{R\to\infty}\left\{\frac{s}{h}D\frac{\alpha-1}{R}-\frac{s}{h}D\frac{\alpha}{\beta}\left(\frac{R}{\beta}\right)^{\alpha-1}-1\right\}=-\infty$$

$$\lim_{R\to 0}\left\{\frac{s}{h}D\frac{\alpha-1}{R}-\frac{s}{h}D\frac{\alpha}{\beta}\left(\frac{R}{\beta}\right)^{\alpha-1}-1\right\}=+\infty\,.$$

Therefore, there exists a single $R_o>0$ for which $u(R_o)=0$. Given that $R_m=\beta\left(\frac{\alpha-1}{\alpha}\right)^{1/\alpha}$, solving the equation $\frac{s}{h}D\left[\frac{\alpha-1}{R_o}-\frac{\alpha}{\beta}\left(\frac{R_o}{\beta}\right)^{\alpha-1}\right]-1=0$ with respect to R_m we take

$$\frac{s}{h}D\frac{\alpha-1}{R_o}-\frac{s}{h}D\frac{\alpha}{\beta}\left(\frac{R_o}{\beta}\right)^{\alpha-1}-1=0\Leftrightarrow$$

$$\beta^{\alpha}\frac{\alpha-1}{\alpha}\frac{1}{R_o}-R_o^{\alpha-1}-\beta^{\alpha}\frac{h}{\alpha\cdot s\cdot D}=0\Leftrightarrow$$

$$\beta^{\alpha}\frac{\alpha-1}{\alpha}\frac{1}{R_o^{\alpha}}-1-\beta^{\alpha}\frac{h\cdot R_o^{1-\alpha}}{\alpha\cdot s\cdot D}=0\Leftrightarrow$$

$$R_o^{\alpha}=\frac{\beta^{\alpha}\frac{\alpha-1}{\alpha}}{1+\beta^{\alpha}\frac{h\cdot R_o^{1-\alpha}}{\alpha\cdot s\cdot D}}\Leftrightarrow$$

$$R_o = \frac{\beta\left[\frac{\alpha-1}{\alpha}\right]^{\frac{1}{\alpha}}}{\left[1+\beta^{\alpha}\frac{h\cdot R_o^{1-\alpha}}{\alpha\cdot s\cdot D}\right]^{\frac{1}{\alpha}}} \Leftrightarrow$$

$$R_o = \frac{R_m}{\left[1+\beta^{\alpha}\frac{h\cdot R_o^{1-\alpha}}{\alpha\cdot s\cdot D}\right]^{\frac{1}{\alpha}}} \Leftrightarrow$$

$$R_m = R_o\left(1+\frac{\beta^{\alpha}h\left(R_o\right)^{1-\alpha}}{\alpha\cdot s\cdot D}\right)^{1/\alpha}$$

from where it is easily concluded that $R_o < R_m$. Further it holds that

(a) $u(R)$ is positive for $R < R_o$ and

(b) $u(R)$ is negative for $R > R_o$.

Proof 2.4:

Proof of Lemma 2.1

(a) Given that $C''(R) = h\cdot g(R)/[Q(R)]^3$ and $g'(R) = u(R)[Q(R)]^2$, where $Q(R)$, $g(R)$ and $u(R)$ are defined in (2.5a), (2.10) and (2.12) respectively, we find from (2.12) that for any $R > 0$ it holds $u(R) < 0$ and $g'(R) < 0$. Using also the limits

$$\lim_{R\to 0} f(R) = \infty,\ \lim_{R\to\infty} f(R) = 0 \tag{A.7}$$

$$\lim_{R\to 0} S(R) = \mu_L,\ \lim_{R\to 0}\Theta(R) = \mu_L^2 + \sigma_L^2, \tag{A.8}$$

$$\lim_{R\to\infty} S(R) = \lim_{R\to\infty}\Theta(R) = 0, \tag{A.9}$$

we take $\lim_{R\to 0} g(R) = \infty$ and $\lim_{R\to\infty} g(R) = 0$,

$$\bullet \lim_{R\to 0} g(R) = \lim_{R\to 0}\left\{Q(R)^2\left(\frac{s}{h}D\cdot f(R) + \left[1-F(R)\right]\right) - \left[\frac{s}{h}D\cdot\left[1-F(R)\right] + S(R)\right]^2\right\} =$$

$$= \lim_{R\to 0}\left[\sqrt{2(A/h)D + 2(s/h)D\cdot S(R) + \Theta(R)}\right]^2 \cdot \left(\lim_{R\to 0}\left[\frac{s}{h}D\cdot f(R)\right] + \lim_{R\to 0}\left[1-F(R)\right]\right)$$

$$-\left[\frac{s}{h}D\cdot\lim_{R\to 0}\left[1-F(R)\right] + \lim_{R\to 0}S(R)\right]^2 =$$

$$=\left[2(A/h)D+2(s/h)D\mu_L+\mu_L^2+\sigma_L^2\right]\cdot\left(\frac{s}{h}D\cdot\infty+1\right)-\left[\frac{s}{h}D+\mu_L\right]^2=$$

$$=\infty-\left[\frac{s}{h}D+\mu_L\right]^2=\infty$$

$$\bullet\lim_{R\to\infty}g(R)=\lim_{R\to\infty}\left\{Q(R)^2\left(\frac{s}{h}D\cdot f(R)+\left[1-F(R)\right]\right)-\left[\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)\right]^2\right\}=$$

$$=\lim_{R\to\infty}\left[\sqrt{2(A/h)D+2(s/h)D\cdot S(R)+\Theta(R)}\right]^2\cdot\left(\lim_{R\to\infty}\left[\frac{s}{h}D\cdot f(R)\right]+\lim_{R\to\infty}\left[1-F(R)\right]\right)$$

$$-\left[\frac{s}{h}D\cdot\lim_{R\to\infty}\left[1-F(R)\right]+\lim_{R\to\infty}S(R)\right]^2=$$

$$=\left(\frac{s}{h}D\cdot 0+0\right)2\frac{A}{h}D-0^2=0.$$

It is evident, therefore, that when R increases from zero to $+\infty$, $g(R)$ is strictly decreasing with range $(0,+\infty)$. This means that for any $R>0$ the function $g(R)$ is positive and hence $C''(R)>0$. This completes the proof of part (a) of Lemma 2.1.

(b) In part (a) of Lemma 2.1 we found that $g(R)>0$ for any $R>0$. This leads to $V'(R)=-g(R)\big/\left[Q(R)\right]^3<0$. Using also the limits given in (A.7), (A.8) and (A.9), we take

$\lim_{R\to 0}V(R)=\dfrac{(s/h)D+\mu_L}{\lim_{R\to 0}Q(R)}-1$ and $\lim_{R\to\infty}V(R)=-1$,

$$\bullet\lim_{R\to 0}V(R)=\lim_{R\to 0}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1\right]=\frac{\frac{s}{h}D+\mu_L}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2}}-1$$

$$\bullet\lim_{R\to\infty}V(R)=\lim_{R\to\infty}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1\right]=\frac{0}{\sqrt{2\frac{A}{h}D}}-1=-1$$

where

$$\lim_{R\to 0}Q(R)=\sqrt{2(A/h)D+2(s/h)D\mu_L+\mu_L^2+\sigma_L^2}\,. \tag{A.10}$$

Then, setting $\lim_{R\to 0} V(R) = 0$, we take the equation

$$\frac{\frac{s}{h}D + \mu_L}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D\cdot\mu_L + \mu_L^2 + \sigma_L^2}} - 1 = 0 \Leftrightarrow$$

$$\left[\frac{s}{h}D + \mu_L\right]^2 = 2\frac{A}{h}D + 2\frac{s}{h}D\cdot\mu_L + \mu_L^2 + \sigma_L^2 \Leftrightarrow$$

$$\left[\frac{s}{h}D\right]^2 = 2\frac{A}{h}D + \sigma_L^2 \Leftrightarrow$$

$$(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 = 0$$

with roots

$$D_{1,2} = \frac{2\frac{A}{h} \pm \sqrt{4\frac{A^2}{h^2} + 4\frac{s^2}{h^2}\sigma_L^2}}{2\frac{s^2}{h^2}} =$$

$$= \frac{2\frac{A}{h} \pm 2\frac{s}{h}\sqrt{\frac{A^2}{s^2} + \sigma_L^2}}{2\frac{s^2}{h^2}} =$$

$$= \frac{2\frac{s}{h}\left(\frac{A}{s} \pm \sqrt{\frac{A^2}{s^2} + \sigma_L^2}\right)}{2\frac{s^2}{h^2}} \Leftrightarrow$$

$$D_{1,2} = \frac{h}{s}\left\{\frac{A}{s} \pm \sqrt{\frac{A^2}{s^2} + \sigma_L^2}\right\} .$$

From the two roots it is easily deduced that the first one with the plus sign is positive, $D_1 = \frac{h}{s}\left\{\frac{A}{s} + \sqrt{\frac{A^2}{s^2} + \sigma_L^2}\right\} > 0$ and the second one with the minus sign is negative, $D_2 = \frac{h}{s}\left\{\frac{A}{s} - \sqrt{\frac{A^2}{s^2} + \sigma_L^2}\right\} < 0$. We conclude, therefore, that $\lim_{R\to 0} V(R)$ is positive when $D > (h/s)\left\{(A/s) + \sqrt{(A/s)^2 + \sigma_L^2}\right\}$ or equivalently when the condition (2.13) is true. In this case, $V(R)$ is strictly decreasing with positive limit when $R \to 0$ and negative limit when $R \to \infty$. Hence, there exists a single $R^* > 0$ for which $V(R^*) = 0$.

Proof 2.5:

Proof of Lemma 2.2

(a) Using Assumption 1 and the limits in (A.8), (A.9) and (A.10) we find $\lim_{R\to 0} g(R) = \left(\lim_{R\to 0} Q(R)\right)^2 - \left[(s \cdot D/h) + \mu_L\right]^2$ and $\lim_{R\to\infty} g(R) = 0$,

$$\bullet \lim_{R\to 0} g(R) = \lim_{R\to 0}\left\{Q(R)^2\left(\frac{s}{h}D\cdot f(R) + \left[1-F(R)\right]\right) - \left[\frac{s}{h}D\cdot\left[1-F(R)\right] + S(R)\right]^2\right\} =$$

$$= \lim_{R\to 0}\left[\sqrt{2(A/h)D + 2(s/h)D\cdot S(R) + \Theta(R)}\right]^2 \cdot \left(\lim_{R\to 0}\left[\frac{s}{h}D\cdot f(R)\right] + \lim_{R\to 0}\left[1-F(R)\right]\right)$$

$$- \left[\frac{s}{h}D\cdot\lim_{R\to 0}\left[1-F(R)\right] + \lim_{R\to 0} S(R)\right]^2 =$$

$$= \left[2(A/h)D + 2(s/h)D\mu_L + \mu_L^2 + \sigma_L^2\right]\cdot\left(\frac{s}{h}D\cdot 0 + 1\right) - \left[\frac{s}{h}D + \mu_L\right]^2 =$$

$$= \left[2(A/h)D + 2(s/h)D\mu_L + \mu_L^2 + \sigma_L^2\right] - \left[\frac{s}{h}D + \mu_L\right]^2$$

$$\bullet \lim_{R\to\infty} g(R) = \lim_{R\to\infty}\left\{Q(R)^2\left(\frac{s}{h}D\cdot f(R) + \left[1-F(R)\right]\right) - \left[\frac{s}{h}D\cdot\left[1-F(R)\right] + S(R)\right]^2\right\} =$$

$$= \lim_{R\to\infty}\left[\sqrt{2(A/h)D + 2(s/h)D\cdot S(R) + \Theta(R)}\right]^2 \cdot \left(\lim_{R\to\infty}\left[\frac{s}{h}D\cdot f(R)\right] + \lim_{R\to\infty}\left[1-F(R)\right]\right)$$

$$- \left[\frac{s}{h}D\cdot\lim_{R\to\infty}\left[1-F(R)\right] + \lim_{R\to\infty} S(R)\right]^2 =$$

$$= \left(\frac{s}{h}D\cdot 0 + 0\right)2\frac{A}{h}D - 0^2 = 0.$$

Further, from Assumption 2, we deduce that the first derivative $g'(R) = u(R)[Q(R)]^2$ is

(a) zero for $R = R_o$,

(b) positive for $R < R_o$ and

(c) negative for $R > R_o$.

Thus, $g(R)$ has a positive maximum at R_o, and it is strictly increasing on $(0, R_o)$ and strictly decreasing on $(R_o, +\infty)$. Thus, to prove part (a) of Lemma 2.2, we need to show that $\lim_{R\to 0} g(R) < 0$ in which case there will exist a single $R_1 < R_o$ for which it holds $g(R_1) = 0$, $g(R) < 0$ for $R < R_1$ and $g(R) > 0$ for $R > R_1$. But the inequality $\lim_{R\to 0} g(R) < 0$ is true since from condition (2.13) and (A.10) it holds $(s\cdot D/h) + \mu_L > \lim_{R\to 0} Q(R)$.

(b) In part (a) of Lemma 2.2 we found that $\lim_{R\to 0} g(R) < 0$, $g(R_1) = 0$, $g(R) < 0$ for $R < R_1$ and $g(R) > 0$ for $R > R_1$. From these results it follows that $V'(R_1) = -g(R_1) \big/ \left[Q(R_1)\right]^3 = 0$ while

(a) $V'(R) > 0$ for $R < R_1$ and

(b) $V'(R) < 0$ for $R > R_1$.

Additionally, using Assumption 1, the limits in (A.8), (A.9) and (A.10), and the inequality $(s\cdot D/h) + \mu_L > \lim_{R\to 0} Q(R)$ which holds when condition (2.13) is true, we obtain

$\lim_{R\to 0} V(R) = \dfrac{(s\cdot D/h) + \mu_L}{\lim_{R\to 0} Q(R)} - 1 > 0$ and $\lim_{R\to\infty} V(R) = -1$,

$$\bullet \lim_{R\to 0} V(R) = \lim_{R\to 0}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right] + S(R)}{Q(R)} - 1\right] = \frac{\frac{s}{h}D + \mu_L}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D\cdot\mu_L + \mu_L^2 + \sigma_L^2}} - 1$$

$$\bullet \lim_{R\to\infty} V(R) = \lim_{R\to\infty}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right] + S(R)}{Q(R)} - 1\right] = \frac{0}{\sqrt{2\frac{A}{h}D}} - 1 = -1.$$

These findings indicate that the function V(R) attains its positive maximum at $R = R_1$, while it is strictly increasing for $0 < R < R_1$ (with positive limit when $R \to 0$) and strictly decreasing for $R_1 < R < +\infty$. Hence there is a single $R^* > R_1$ for which $V(R^*) = 0$ with $V(R) > 0$ for $0 < R < R^*$ and $V(R) < 0$ for $R^* < R < +\infty$.

Proof 2.6:

When it holds $\lim_{R\to 0} g(R)<0 \Leftrightarrow \lim_{R\to 0} V(R)>0$ then the minimum of the cost function is obtained after solving the first order conditions. So, solving the inequality (2.13), $(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 > 0$, with respect to one of the cost parameters keeping the other two fixed we obtain the range values of s, A and h in order the minimum cost to occur at a positive R value:

(A) threshold value for the shortage cost

$$\lim_{R\to 0} V(R) \geq 0 \Leftrightarrow \lim_{R\to 0}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1\right]\geq 0 \Leftrightarrow$$

$$\frac{\frac{s}{h}D+\mu_L}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2}}-1\geq 0\Leftrightarrow$$

$$\frac{s}{h}D+\mu_L \geq \sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2} \Leftrightarrow$$

$$\left[\frac{s}{h}D+\mu_L\right]^2 \geq 2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2 \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2 - 2\frac{A}{h}D - \sigma_L^2 \geq 0 \Leftrightarrow$$

$$s^2D^2 \geq 2\cdot A\cdot D\cdot h + \sigma_L^2 h^2 \Leftrightarrow$$

$$s^2 \geq 2\frac{A}{D}h + \frac{h^2}{D^2}\sigma_L^2 \Leftrightarrow$$

$$s \geq \sqrt{2\frac{A}{D}h + \frac{h^2}{D^2}\sigma_L^2}\ .$$

Thus, the interval value for the shortage cost is

$$\sqrt{2\frac{A}{D}h + \frac{h^2}{D^2}\sigma_L^2} \leq s < +\infty\ .$$

(B) threshold value for the ordering cost

$$\lim_{R\to 0} V(R) \geq 0 \Leftrightarrow \lim_{R\to 0}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1\right] \geq 0 \Leftrightarrow$$

$$\frac{\frac{s}{h}D+\mu_L}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2}}-1 \geq 0 \Leftrightarrow$$

$$\frac{s}{h}D+\mu_L \geq \sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2} \Leftrightarrow$$

$$\left[\frac{s}{h}D+\mu_L\right]^2 \geq 2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2 \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2 \geq 2\frac{A}{h}D+\sigma_L^2 \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2 - 2\frac{A}{h}D-\sigma_L^2 \geq 0 \Leftrightarrow$$

$$\frac{s^2}{h}D^2 - 2\cdot A\cdot D-\sigma_L^2 h \geq 0 \Leftrightarrow$$

$$A \leq \frac{1}{2D}\left[\frac{s^2}{h}D^2-\sigma_L^2 h\right].$$

Thus, the interval value for the ordering cost is

$$0 \leq A \leq \frac{\frac{s^2D^2}{h}-h\sigma_L^2}{2D}.$$

(C) threshold value for the holding cost

$$\lim_{R\to 0} V(R) \geq 0 \Leftrightarrow \lim_{R\to 0}\left[\frac{\frac{s}{h}D\cdot\left[1-F(R)\right]+S(R)}{Q(R)}-1\right] \geq 0 \Leftrightarrow$$

$$\frac{\frac{s}{h}D+\mu_L}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2}}-1 \geq 0 \Leftrightarrow$$

$$\left[\frac{s}{h}D+\mu_L\right]^2 \ge 2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L+\mu_L^2+\sigma_L^2 \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2 \ge 2\frac{A}{h}D+\sigma_L^2 \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2\cdot h^2-2\frac{A}{h}D\cdot h^2-\sigma_L^2h^2 \ge 0 \Leftrightarrow$$

$$s^2D^2-2\cdot A\cdot D\cdot h-\sigma_L^2h^2 \ge 0 \Leftrightarrow$$

$$\sigma_L^2h^2+2\cdot A\cdot D\cdot h-s^2D^2 \le 0.$$

Solving the quadratic equation and using the quadratic formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ the two roots are:

$$h_{1,2}=\frac{-2AD\pm\sqrt{(2AD)^2-4\sigma_L^2\left(-s^2D^2\right)}}{2\sigma_L^2}=$$

$$=\frac{-2AD\pm\sqrt{4A^2D^2\left[1+\frac{s^2D^2\sigma_L^2}{A^2D^2}\right]}}{2\sigma_L^2}\Leftrightarrow$$

$$=\frac{-AD\pm AD\sqrt{1+\frac{s^2D^2\sigma_L^2}{A^2D^2}}}{\sigma_L^2}.$$

From the two roots it is easily deduced that the first one with the plus sign is positive, and the second one with the minus sign is negative. Hence, we take

$$h_1=\frac{-AD+AD\sqrt{1+\frac{s^2D^2\sigma_L^2}{A^2D^2}}}{\sigma_L^2}.$$

Thus, the interval value for the holding cost is

$$0 \le h \le \frac{-AD+\sqrt{A^2D^2+\sigma_L^2s^2D^2}}{\sigma_L^2}.$$

Chapter 3

The (Q,R) inventory system with an approximate treatment for the cost function

3.1 Introduction

Due to the complexity of the exact cost function, approximations for the expected on-hand inventory level at any point in time have been suggested in the inventory literature, as mentioned before in chapter 2. Among them, the most common approximation is the Hadley & Whitin's (H-W) expression. Therefore, in the current chapter, we consider the Hadley & Whitin's (1963) cost function (hereafter "HWCF"). Rewriting the bivariate HWCF as a univariate function of R, we show that its convexity depends upon the sign of the first derivative of the lead-time demand probability density function. Then, representing the lead-time demand by the class of unimodal distributions for which the probability density function vanishes at $\mathrm{R}=0$ and when $\mathrm{R}\to\infty$, for the first time we derive the conditions which identify the following three cases: (a) the HWCF has a unique minimum at the optimal solution which is obtained through the H-W iterative procedure, (b) the minimum of HWCF is taken after comparing the cost at $\mathrm{R}=0$ with the "local" minimum cost computed from the Q and R values which are obtained from the H-W iterative procedure, and (c) the transformed HWCF in the two-dimensional space is either an increasing or non-decreasing function of R in which case the unique minimum cost occurs at $\mathrm{R}=0$. Next, these three cases are integrated to a general algorithm, and its application is illustrated when the lead-time demand has the Normal and the Log-Normal distribution (e.g. Viswanathan et al., 2008; Walsh et al., 2008).

The added value of the general algorithm in the relevant inventory literature is illustrated by comparing the minimum of HWCF taken after following the algorithm with the corresponding minimum of the exact cost function. The latter one is obtained when the expression $\left(\mathrm{Q}/2+\text{safety stock}\right)$ in HWCF is replaced by the exact expression for the expected on-hand inventory at any point in time. Contrary to what is believed that HWCF should be used only when the resulting cycle service level from the optimal R is sufficiently large, the results from the comparative study indicate that acceptable approximations using the HWCF can be taken even in the case where $R = 0$. Especially, for the Log-Normal distribution with relatively high coefficient of variation, we illustrate that accurate approximations can be observed even at large sizes of the fixed ordering cost where zero cycle service levels occur.

Based on the aforementioned discussion and remarks the rest of the chapter is organized as follows. Section 3.2, contains a literature review for the HWCF. In Section 3.3, we provide the necessary theoretical background in order to address the value and contribution of the current study. In Section 3.4, we present the general algorithm, while its application for Normal and Log-Normal lead-time demands is described in Section 3.5. In Section 3.6, through a numerical experimentation, we investigate the managerial implications of changing the values of cost parameters on the optimal sizes of Q and R, as well as, on the minimum cost. In Section 3.7, we obtain the range of the cost parameters values in order the optimal reorder point to be equal to zero. Section 3.8 gives and discusses the results from the comparative study for the target inventory measures taken after minimizing the H-W and the exact cost functions. Finally, Section 3.9 concludes the chapter summarizing the most important findings.

3.2 Relevant literature review

Continuous review inventory systems have been studied extensively in inventory literature and many works have been published optimizing their operation (e.g. Silver et al., 1998; Dohi et al., 1999; Betts & Johnston, 2005; Cobb et al., 2013). A number of these works have focused on a special class of continuous review (Q, R) models with fixed lead-time and backorders where the aim is the determination of the order quantity, Q, and the reorder point, R, through the minimization of a cost function proposed by Hadley & Whitin (1963). This cost function results from the sum of the annual expected costs of ordering, inventory carrying and shortage. For evaluating the annual expected inventory carrying cost, the authors, by assuming that the expected size of backorders per inventory cycle is negligible, approximated the expected on-hand inventory at any point in time with the expression $(Q/2 + \text{safety stock})$, while for the calculation of the annual expected shortage cost they used the cost per unit backordered. Setting the partial derivatives of HWCF with respect to Q and R equal to zero, the authors developed an iterative procedure to determine an optimal solution in terms of Q and R values which ensure minimum cost, claiming also that any time the cost per unit backordered is relatively high this solution will be unique.

In the years since Hadley & Whitin presented their seminal work, a number of authors has questioned the convexity of HWCF. Recognizing that the convexity depends upon the mathematical behavior of the related function concerning the expected backorders, Veinott (1964) stated that the latter function in Q and R jointly is convex only under restrictive conditions such as when the probability density of lead-time demand distribution is non-increasing given that demand is positive. A few years later, Brooks & Lu (1969) proved that the expected backorders function, although convex in Q given R and convex in R given Q, is not in general convex in both Q and R. Especially, for Normal lead-time demand, the authors showed that convexity exists only when the cycle service level is above 0.50. So, the research had already started to shift on developing general conditions which ensure the existence of a unique minimum for HWCF.

When the lead-time demand is Normally distributed, Minh (1975) gave the condition of a unique optimal solution which can be obtained solving iteratively the two equations derived from the first-order conditions for minimizing HWCF. Unfortunately, however, it can be verified numerically that even when this condition holds, there are cases where the application of the H-W iterative scheme for solving the two equations breaks down and no solution is available. At the same time, such a condition was also given by Gross & Ince (1975) when the lead-time demand is Poisson. The authors illustrated numerically that when this condition is violated there are two alternative situations. In the first, HWCF has a local minimum and a local maximum, and the application of H-W iterative procedure always converges to the local minimum. In the second situation, they found that the application of H-W iterative scheme breaks down and proposed a zero reorder point as optimal, without giving any further justification. Besides, the lack of a proper mathematical framework in their analysis did not allow the authors to derive the corresponding condition(s) for distinguishing the two aforementioned situations.

Many years later, Das (1988) was the first who rewrote the bivariate HWCF as a univariate function of R. Then deriving the necessary conditions for the existence of an optimal solution in terms of R only, the author suggested an approach for determining the optimal R, stating also the importance of probability density function of the lead-time demand in the process of finding the minimum of HWCF. Especially, the author gave the condition when no solution exists at the process of solving the equations resulted from the first-order conditions for minimizing either the bivariate or the univariate cost functions, emphasizing the fact that this situation occurs when the fixed ordering cost is relatively high. Finally, the author stated that when an optimal solution exists, the local minimum is also the global minimum for the case of unimodal or J-shaped lead-time demand distributions. However, in his analysis, Das allowed R to take any value on the real numbers line, and today it is known (e.g. Lau et al., 2002b) that when R takes on negative values a new modified expression should be applied for the expected backorders function. Unfortunately, Das used HWCF without taking into account this correction.

Apart from the aforementioned works, the convexity problem of the HWCF has been also investigated under different shortage cost models for calculating the annual expected shortage cost. Particularly, using a fixed cost per stock-out occasion or a fractional charge per unit short per unit time, works which have investigated the convexity of the cost function are those ones of Das (1983b), Silver et al. (1998), Lau et al. (2002a) and Chung et al. (2009). However, at this point it is worthwhile to mention that the examination of the convexity problem in all the aforementioned papers takes place under the assumption that the parameters of lead-time demand distribution are known. For cases of unknown demand parameters, there are works which have suggested methods to estimate these parameters aiming mainly to study the behavior of cycle service level (e.g. Syntetos & Boylan, 2008).

3.3 Theoretical Background

For the continuous review (Q,R) inventory system with fixed order quantity-reorder point and the demand to be backordered when the system is out of stock, we provide below the required assumptions held when stating the H-W cost function

$$C_{HW}\left(Q,R\right)=\frac{A\cdot D}{Q}+h\left(\frac{Q}{2}+R-\mu_L\right)+\frac{s\cdot D}{Q}S\left(R\right), \tag{3.1}$$

where, X is a continuous non-negative random variable representing the demand in the lead-time with mean μ_L and variance σ_L^2, and

$$S(R)=\int_R^{\infty} x f(x)dx - R\left[1-F(R)\right]. \tag{3.2}$$

Assumptions

(a) Lead-time demand distribution has the same mean and the same standard deviation at any inventory cycle.

(b) The reorder point is nonnegative $(R \geq 0)$ and kept constant at any inventory cycle.

(c) Lead-time is fixed and remains the same at any inventory cycle.

(d) When the order quantity is received, the inventory level is always raised above the reorder point[1].

As Hadley & Whitin (1963, pp. 166) stated, the condition $R \geq 0$ is of crucial importance for deriving (3.2). In the opposite case where R takes values on $(-\infty, +\infty)$, Lau et al. (2002b) showed that (3.1) holds only when $S(R)$ is replaced by the expression $S(R) - S(Q + R)$.

Taking now the first-order conditions

$$\frac{\partial C_{HW}}{\partial Q} = -\frac{A \cdot D}{Q^2} + \frac{h}{2} - \frac{s \cdot D}{Q^2} \cdot S(R) = 0 \quad \text{and} \quad \frac{\partial C_{HW}}{\partial R} = h - \frac{s \cdot D}{Q}\left[1 - F(R)\right] = 0$$

which minimize (3.1), the optimal Q^*, R^* are determined by solving the pair of equations

$$Q = \sqrt{2 \frac{s}{h} D \left[\frac{A}{s} + S(R)\right]} \tag{3.3}$$

$$R = F^{-1}\left(1 - \frac{h \cdot Q}{s \cdot D}\right) \tag{3.4}$$

where $F^{-1}(x)$ is the inverse cumulative distribution function evaluated at x.

To obtain the optimal (Q^*, R^*) pair, Hadley & Whitin (1963, pp. 170) introduced a numerical iterative procedure which starts by substituting Q in (3.4) with the Wilson Economic Order Quantity, $Q_W = \sqrt{2 \cdot A \cdot D / h}$, taking in that way the first value of R, say $R^{(1)}$. Then $R^{(1)}$ is used in (3.3) to take the second value of Q, say $Q^{(2)}$, which in turn is used in (3.4) to take the second value $R^{(2)}$. This procedure is repeated until the differences in Q and R values between two successive iterations satisfy some pre-specified accuracy. The issue, however, is if the resulting optimal values constitute a global or a local minimum, and in the second case if this local minimum is unique or not. To prove the existence of a single solution, Hadley & Whitin gave a graph of equations (3.3) and (3.4) in the Q-R Cartesian Coordinate system, and this graph is reproduced in Figure 3.1. The staircase line with the arrows illustrates that starting with Q_W, the iterative procedure converges to the intersection of the two curves which represents the optimal solution. The authors, however, claimed that such a graph does not always hold.

[1]This assumption implies that at each inventory cycle the lead-time demand never exceeds the order quantity, ensuring that there is never more than one order outstanding at any point in time.

Apart from the case of Figure 3.1, Gross & Ince (1975) concluded under Poisson demand and performing an extensive numerical experimentation that two more cases can be met [see Figures (3.2) and (3.3)] when the following condition does not hold

$$\frac{s \cdot D}{h} \geq \sqrt{2\frac{s}{h}D\left(\frac{A}{s}+\mu_L\right)}. \tag{3.5}$$

In the case of Figure (3.2) the authors stated that point A constitutes a local minimum and point B a local maximum, thus the optimal (Q^*, R^*) pair is obtained by comparing the cost at point A with the cost at $R^* = 0$. For the case of Figure 3.3, they suggested without any further explanation that the minimum cost is obtained setting $R^* = 0$.

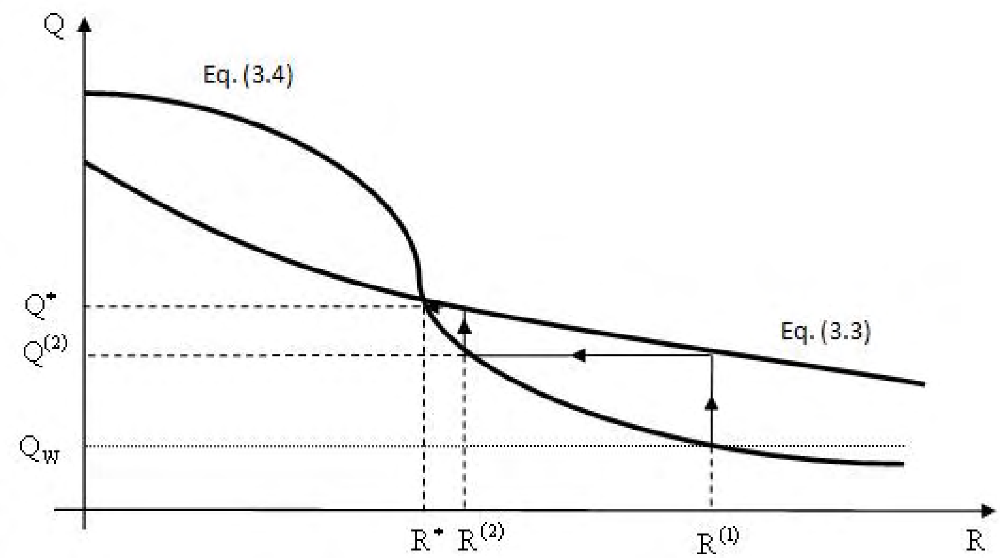


Figure 3.1 Graphical representation of the Hadley-Whitin iterative procedure.

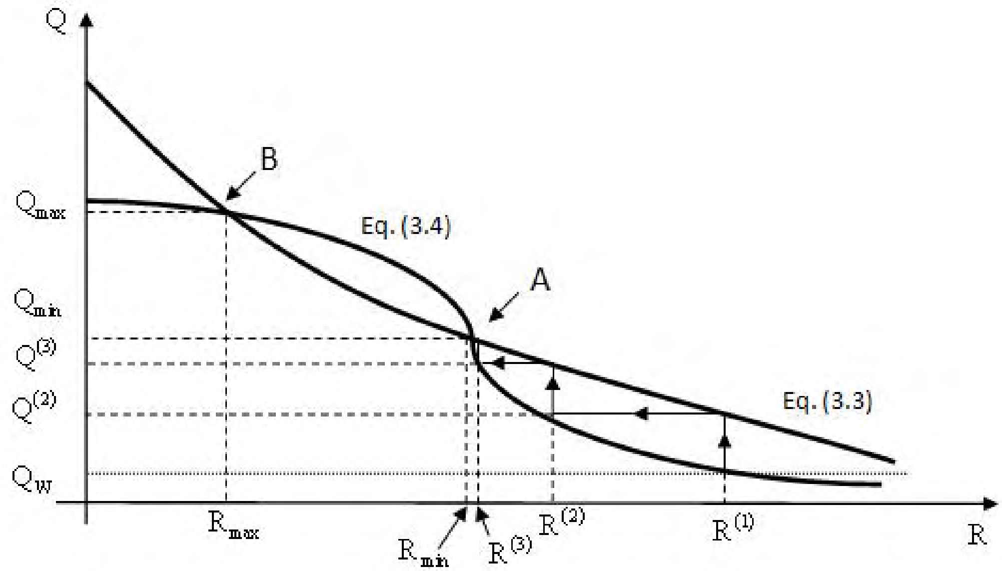


Figure 3.2 The Hadley-Whitin iterative procedure with a local minimum and a local maximum.

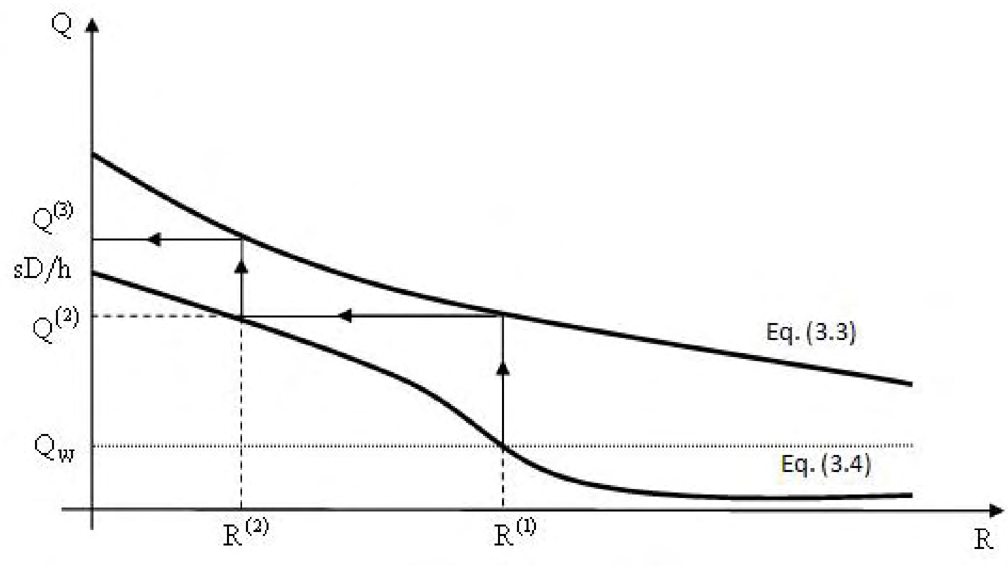


Figure 3.3 Graphical representation of the “Degeneracy Problem”.

The following remarks to the discussion above establish the contribution of the current research. Specifically, our first remark concerns the type of curve of equation (3.3) in Figure 3.1. The shape of the curve holds only if

$$\frac{d^2Q}{dR^2} = -\frac{s \cdot D}{h \cdot Q}\left\{\frac{s \cdot D}{h}\left(\frac{1-F(R)}{Q}\right)^2 - f(R)\right\} > 0 .$$

Proof 3.1: See in the Appendix at the end of chapter 3.

In the opposite case we would observe an S-type of curve for (3.3). In such an event, the two curves might have such locations that three points of intersection would occur instead of one. Our second remark refers to the distinction between Figures (3.2) and (3.3). To the extent of our knowledge no valid condition exists which distinguishes the two cases. The reason is that Gross & Ince (1975) resulted in the two figures after conducting numerical experimentation and not following some formal mathematical approach which would establish such a condition. Furthermore, under Normal lead-time demand, Minh (1975) tried to give a condition for the case of Figure 3.3, but as we shall demonstrate in section 3.4, Figure 3.3 can be met even if Minh's condition is violated. Also, without specifying the type of lead-time demand distribution, Das (1988) offered a general condition for distinguishing the case of Figure 3.3 from the cases of Figures 3.1 and 3.2. However, with Normal or Log-Normal lead-time demand, it can be verified numerically that again Figure 3.3 can be met even if Das' general condition does not hold. The reason is that, although in his analysis the author allowed R to take any value on $(-\infty,+\infty)$, unfortunately he used (3.1) without having replaced $S(R)$ by $S(R)-S(Q+R)$.

Our last remark refers to the degeneracy problem, as this was named by Lau & Lau (2002). This problem appears when at some iteration in the implementation of H-W iterative procedure the cycle service level expressed by $F(R)$ becomes negative. If this happens the iterative procedure breaks down. This problem is perfectly illustrated through Figure 3.3. Following the staircase line with the arrows, in the third iteration the resulting value $Q^{(3)}$ is greater than the quantity $(s/h)D$. This means that in (3.4) the expression within parentheses is negative and this in turn leads to a negative $F(R)$. This anomaly places particular stress on the need to have the required condition for distinguishing the case of Figure 3.3 from the cases of Figures 3.1 and 3.2. If such a condition was available and the model parameters values satisfied this condition then the application of H-W iterative procedure would not be allowed. But at the same time, for the case of Figure 3.3 we need a formal justification why $R^* = 0$ is generally an optimal solution as Gross & Ince (1975) suggested for the case of Poisson demand.

Taking into account the remarks above, in the current work we follow a completely different approach for determining the minimum of (3.1). Instead of starting with Figures 3.1, 3.2, 3.3, which describe the two equations obtained from the first-order conditions minimizing (3.1), our analysis is based on graphs in the two-dimensional space of a univariate cost function of the reorder point, R. Such a univariate function is obtained when the order quantity, Q, in (3.1) is replaced by the expression on the right-hand side of (3.3). The new graphs form the basis of an algebraic approach which mathematically establishes conditions for the cases of Figures 3.1, 3.2, 3.3 that should be examined at the process of finding the minimum of the H-W cost function. In the analysis which follows in the next section we show that the monotony of this univariate cost function depends on the sign of the first derivative $\mathrm{d}f(\mathrm{R})/\mathrm{dR}$. This implies that general procedures for minimizing (3.1) can be developed for those families of distributions for the lead-time demand for which the shape of $f(\mathrm{R})$ displays the same monotony.

Between different types of lead-time demand distribution, in the current work we have chosen the class of unimodal distributions for which the probability density function vanishes when $R = 0$ and $R \to \infty$. The Normal distribution as well as distributions heavily skewed to the right (like the Log-Normal, Gamma, Weibull etc) are included in this class, and the justification for this choice is given in Section 3.4. For this class of distributions, in the next section we show that when condition (3.5) holds, the H-W cost function defined in (3.1), although non-convex, has a single minimum. Additionally, we give the mathematical conditions for distinguishing the cases of Figures 3.2 and 3.3. Particularly, for the case of Figure 3.2, we prove that the local maximum of the univariate cost function is attained at a positive value of R which is located to the left of the corresponding R value for which the local minimum appears. This implies that the cost at the lowest permissible value of the reorder point, namely at $R = 0$, appears as a second local minimum, and so the minimum of (3.1) is obtained by comparing the two local minima. On the other hand, for the case of Figure 3.3 we show that the univariate cost function is either strictly increasing or non-decreasing for any $R \geq 0$ justifying algebraically that the minimum cost is obtained directly setting $R = 0$.

3.4 Optimal solutions for unimodal lead-time demand distribution

The second-order derivatives of the H-W cost function (HWCF) defined in (3.1) with respect to Q and R are

$$\frac{\partial^2 C_{HW}}{\partial Q^2} = \frac{2 \cdot s \cdot D}{Q^3}\left\{\frac{A}{s} + S(R)\right\} > 0, \quad \frac{\partial^2 C_{HW}}{\partial R^2} = \frac{s \cdot D}{Q} \cdot f(R) > 0, \quad \frac{\partial^2 C_{HW}}{\partial Q \partial R} = \frac{s \cdot D}{Q^2}\left[1 - F(R)\right].$$

Thus, the Hessian determinant is written in the form $|H| = \left(s^2 D^2 / Q^4\right) g_1(R)$, where

$$g_1(R) = 2\left[\frac{A}{s} + S(R)\right] f(R) - \left[1 - F(R)\right]^2. \tag{3.6}$$

It is concluded, therefore, that the sign of $g_1(R)$, and by extension the sign of the Hessian determinant, is formed independently of the size of the order quantity, Q. This implies that the monotony of (3.1) could be examined in a clear and comprehensible manner (compared to Figures 3.1, 3.2, 3.3) if (3.1) was transformed to a univariate function of the reorder point R. This is succeeded by replacing Q in (3.1) with the right-hand side of (3.3), taking in that way the transformed univariate cost function

$$C_1(R) = h\left(Q + R - \mu_L\right), \tag{3.7}$$

where Q is given in (3.3).

Proof 3.2: See in the Appendix at the end of the chapter.

To examine the convexity of (3.7), we consider its first and second derivatives which are given in the next two Lemmas:

Lemma 3.1: $C_1'(R) = \frac{dC_1}{dR} = -h \cdot V_1(R)$, where

$$V_1(R) = \frac{s}{h} D Q^{-1}\left[1 - F(R)\right] - 1. \tag{3.8}$$

Proof 3.3: See in the Appendix at the end of the chapter.

Lemma 3.2: $C_1''(R) = \frac{d^2 C_1}{dR} = -h \cdot V_1'(R)$, where

$$V_1'(R) = -\left(\frac{s}{h} D\right)^2 Q^{-3} \cdot g_1(R), \tag{3.9}$$

and $g_1(R)$ is defined in (3.6).

Proof 3.4: See in the Appendix at the end of the chapter.

Additionally, the first derivative of $g_1(\mathrm{R})$ is

$$g_1'(\mathrm{R}) = 2\left\{\frac{\mathrm{A}}{\mathrm{s}} + \mathrm{S}(\mathrm{R})\right\}\frac{\mathrm{d}f(\mathrm{R})}{\mathrm{dR}}, \tag{3.10}$$

as $2f(\mathrm{R})\dfrac{\mathrm{dS}(\mathrm{R})}{\mathrm{dR}} = \dfrac{\mathrm{d}}{\mathrm{dR}}[1-F(\mathrm{R})]^2 = -2f(\mathrm{R})[1-F(\mathrm{R})]$.

From Lemmas 3.1 and 3.2, it is evident that the mathematical properties of $g_1(\mathrm{R})$ determine whether (3.7) is convex or not. Particularly, the range of $g_1(\mathrm{R})$ determines not only the sign of $\mathrm{V}_1'(\mathrm{R})$ and by extension the sign of $\mathrm{C}_1''(\mathrm{R})$ but also through $\mathrm{V}_1'(\mathrm{R})$ the monotony of $\mathrm{V}_1(\mathrm{R})$ which shows how many times $\mathrm{C}_1'(\mathrm{R})$ changes sign. But from (3.10) it is also realized that the monotony of $g_1(\mathrm{R})$, as well as its range, depend upon the sign of $g_1'(\mathrm{R})$ which, in turn, is determined by the mathematical behavior of the first derivative of the lead-time demand probability density function $\mathrm{d}f(\mathrm{R})/\mathrm{dR}$. So, to obtain general results for the convexity of (3.7), between different types of lead-time demand distribution, we have chosen the class of unimodal distributions for which the probability density function vanishes at $\mathrm{R}=0$ and when $\mathrm{R}\to\infty$, and thus $\mathrm{d}f(\mathrm{R})/\mathrm{dR}$ changes sign only once. For this class of unimodal distributions, Lemmas 3.3 and 3.4 give the range of $g_1(\mathrm{R})$ and $\mathrm{V}_1(\mathrm{R})$ respectively.

Lemma 3.3: $g_1(\mathrm{R})$ has range the interval $[-1,0)$ and a unique maximum attained at $\mathrm{R}=\mathrm{R}_\mathrm{m}$, where R_m is the mode of the lead-time demand distribution.

Proof 3.5: See in the Appendix at the end of the chapter.

From Lemma 3.3 it is deduced that there is only one value, say R_o, for which $g_1(\mathrm{R}_\mathrm{o})=0$ and $g_1(\mathrm{R})<0$ for any R less than R_o. This, in turn, implies that for some interval of R values both the Hessian determinant, $|\mathrm{H}| = (\mathrm{s}^2\mathrm{D}^2/\mathrm{Q}^4)\,g_1(\mathrm{R})$, and the second derivative, $\mathrm{C}_1''(\mathrm{R})$, take on negative values. This proves that when the lead-time demand has the specific class of unimodal distributions the cost functions $\mathrm{C}_{\mathrm{HW}}(\mathrm{Q},\mathrm{R})$ defined in (3.1) or $\mathrm{C}_1(\mathrm{R})$ defined in (3.7) are not convex.

Lemma 3.4: $V_1(R)$ has range the interval $\left[V_1(0)\ ,\ -1\right)$ and a unique maximum attained at $R = R_o$, where

$$V_1(0) = \frac{\frac{s}{h}D}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu_L}} - 1. \tag{3.11}$$

and R_o satisfies the equation $g_1(R_o) = 0$.

Proof 3.6: See in the Appendix at the end of the chapter.

From the results of the four lemmas stated above, we conclude that to determine the minimum of the transformed univariate function, $C_1(R)$, and by extension the optimal values Q^* and R^* minimizing the bivariate function, $C_{HW}(Q,R)$, three cases should be examined. The condition in order each case to hold, as well as, the values of R where the minimum of $C_1(R)$ is attained are given in Proposition 3.1.

Proposition 3.1: Three cases should be examined to determine the minimum of the cost function $C_1(R)$ defined in (3.7):

(a) If $V_1(0) \geq 0$ then $C_1(R)$ has a unique minimum at a value R_1 satisfying the equation $V_1(R_1) = 0$ with $0 < R_o < R_1 < +\infty$,

(b) If $V_1(0) < 0$ and $V_1(R_o) > 0$ then the minimum of $C_1(R)$ is the smallest between $C_1(0)$ and $C_1(R_3)$, where R_3 is the largest of the two roots R_2, R_3 of the equation $V_1(R) = 0$ with $0 < R_2 < R_o < R_3 < +\infty$,

(c) If $V_1(0) < 0$ and $V_1(R_o) \leq 0$ then $C_1(R)$ has its unique minimum at $R = 0$.

Proof 3.7: See in the Appendix at the end of the chapter.

The three cases of Proposition 3.1 are illustrated respectively in Figures 3.4, 3.5, and 3.6. The correspondence between these three figures and the set of Figures 3.1, 3.2, 3.3 of section 3.2 is easily verified. Specifically, R_1 in Figure 3.4 is R^* of Figure 3.1, R_{min} and R_{max} in Figure 3.2 are respectively R_2 and R_3 of Figure 3.5, and Figure 3.6 illustrates clearly, compared to Figure 3.3, why the minimum cost is attained at the smallest permissible value of the reorder point which is $R = 0$. Besides, some algebraic manipulations transform condition $V_1(0) \geq 0$ into (3.5).

Now, using the conditions stated in Proposition 3.1, the following general algorithm for the class of unimodal lead-time demand distributions under consideration has been developed to obtain the minimum of the H-W cost function $C_{HW}(Q,R)$:

Step 1: Specify values for parameters A, h, s, D, μ_L, and σ_L^2.

Step 2: Find $V_1(0)$ from (3.11). If $V_1(0) > 0$ then go to Step 3, otherwise go to Step 4.

Step 3: Find R_1 from (3.8) solving the equation $V_1(R_1) = 0$, set $R^* = R_1$ and go to Step 7.

Step 4: Find R_o from (3.6) solving the equation $g_1(R_o) = 0$ and compute $V_1(R_o)$.

Step 5: If $V_1(R_o) > 0$ then find the two roots R_2, R_3 of equation $V_1(R) = 0$ with $R_2 < R_o < R_3$ and go to Step 6, otherwise go to Step 8.

Step 6: If $C_1(R_3) < C_1(0)$ then set $R^* = R_3$ and go to Step 7, otherwise, go to Step 8.

Step 7: Compute $S(R^*)$ from (3.2), the optimal order quantity $Q^* = \sqrt{2\frac{A}{h}D + 2\frac{s}{h} \cdot D \cdot S(R^*)}$ and the minimum total cost $C_{HW}(Q^*, R^*) = C_1(R^*) = h(Q^* + R^* - \mu_L)$, and go to Step 9.

Step 8: Set the optimal reorder point $R^* = 0$, and compute the optimal order quantity $Q^* = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot \mu_L}$ and the minimum total cost $C_{HW}(Q^*, 0) = h(Q^* - \mu_L)$.

Step 9: End of algorithm.

Figure 3.4 Graphs of $V_1(R)$ and $C_1(R)$ when $V_1(0) \geq 0$.

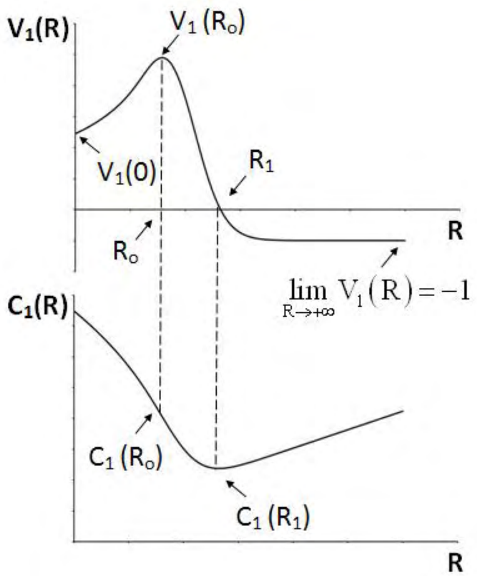


Figure 3.5A Graphs of $V_1(R)$, $C_1(R)$ when $V_1(0) < 0$ and $V_1(R_o) > 0$ with $C_1(0) > C_1(R_3)$.

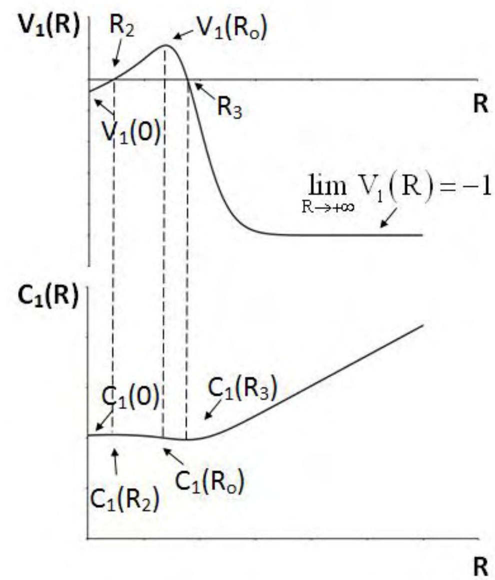


Figure 3.5B Graphs of $V_1(R)$, $C_1(R)$ when $V_1(0) < 0$ and $V_1(R_o) > 0$ with $C_1(0) < C_1(R_3)$.

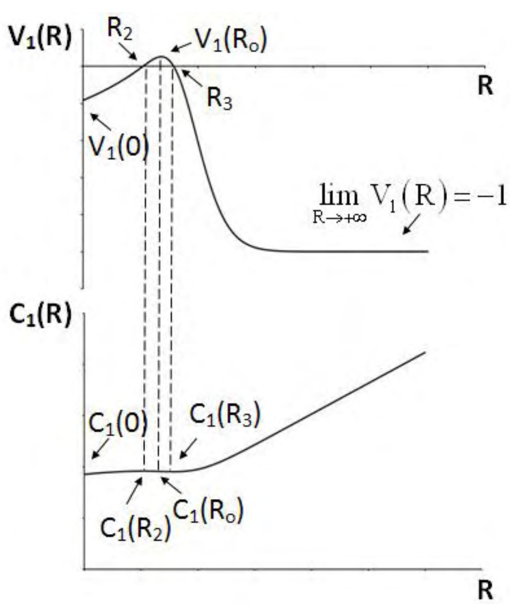
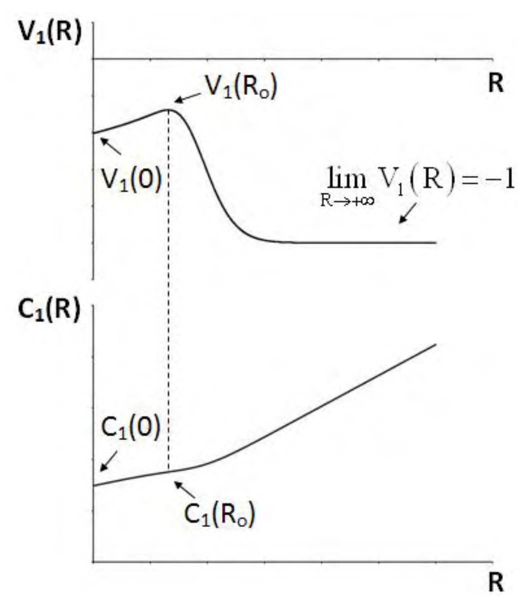


Figure 3.6 Graphs of $V_1(R)$ and $C_1(R)$ when $V_1(0) < 0$ and $V_1(R_o) \leq 0$.



3.5 Applications of the algorithm

In the current section, we illustrate the application of the general algorithm through some numerical examples when the lead-time demand is Normal or Log-Normal. Firstly, the choice of the two distributions has been made as the area under their probability density functions can be calculated through the standard Normal distribution and thus the key-functions of the general algorithm, $S(R)$, $g_1(R)$, $V_1(R)$, and $C_1(R)$ can be expressed in terms of the standardized Normal random variable. Further, in the field of inventory control of finished goods, the Normal distribution adequately models the demand for fast-moving items (e.g. Burgin, 1975), provided that its coefficient of variation (CV) is relatively low, preferably equal or below 0.3 (e.g. Lau, 1997; Janssen et al., 2009; Kevork, 2010; Su & Pearn, 2011). Further, Log-Normal lead-time demand is positive for any size of CV and especially when CV is getting larger the demand distribution becomes heavily skewed to the right representing in that way moderate-moving items.

3.5.1 Normal lead-time demand

For Normal lead-time demand X with mean μ_L and variance σ_L^2, define the standardized Normal variable $z = (R - \mu_L)/\sigma_L$. Then, for the key-functions of the general algorithm, the following specifications hold:

$$S(R) = S_{NM}(z) = \sigma_L\left[\varphi(z) - z \cdot \Phi(-z)\right] \qquad \text{(e.g. Lau et al., 2002b),} \qquad (3.12)$$

$$g_1(R) = g_{NM}(z) = 2\left[\frac{A}{s} + S_{NM}(z)\right]\frac{\varphi(z)}{\sigma} - \left[\Phi(-z)\right]^2,$$

and

$$V_1(R) = V_{NM}(z) = \frac{(s/h)D \cdot \Phi(-z)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{NM}(z)}} - 1,$$

where $\phi(z)$ and $\Phi(z)$ are respectively the probability density and the cumulative distribution function of the standard Normal evaluated at z.

What follows are two examples of how the general algorithm is applied when Case 1 or Case 3 are met.

Case 1

Step 1: $A = 70$, $h = 0.6$, $s = 3$, $D = 300$, $\mu_L = 100$, and $\sigma_L = 20$.

Step 2: From (3.11) we compute $V_1(0) = 1.465985 > 0$.

Step 3: Solving the equation $V_{NM}(z_1) = 0$ we take $z_1 = 0.9010$,

and $R^* = R_1 = \mu_L + z_1\sigma_L = 118.0191$.

Step 7: $S(R^*) = 2.0051$, $Q^* = 275.7088$, and $C_{HW}(Q^*, R^*) = C_1(R^*) = 176.2367$.

Case 3

Keeping the same values for h, s, D, μ_L, σ_L^2 as in Step 1 of case 1 and increasing the fixed ordering cost from $A = 70$ to $A = 2200$, we take:

Step 2: $V_1(0) = -0.05132 < 0$.

Step 4: Solving the equation $g_{NM}(z_o) = 0$ we take $z_o = -2.6283$,

and $V_{NM}(z_o) = -0.027315 < 0$.

Step 8: $R^* = 0$, $Q^* = 1581.1388$, and $C_{HW}(Q^*, R^*) = C_1(R^*) = 888.6833$.

Minh (1975) stated that if it holds $\frac{h \cdot \sigma}{s \cdot D} < \phi(0)$ then Case 3 is not met. This is not verified in our example as Case 3 appears while, unfortunately, this condition is true.

3.5.2 Log-Normal lead-time demand

When X has the Log-Normal distribution it holds (e.g. Tadikamalla, 1979; Gallego et al., 2007) $\ln X \sim N(\lambda, \theta^2)$ with $\theta > 0$, $\mu_L = e^{\lambda + \theta^2/2}$ and $\sigma_L^2 = e^{2\lambda + \theta^2}\left(e^{\theta^2} - 1\right)$. Thus the parameters λ and θ are determined respectively from $\theta = \sqrt{\ln(1 + CV^2)}$ and $\lambda = \ln \mu_L - \theta^2/2$, where $CV = \sigma_L/\mu_L$ is the coefficient of variation. The algorithm key functions specifications are

$$S(R) = S_{LN}(r) = \mu_L \Phi(\theta - r) - e^{\lambda + r\cdot\theta}\Phi(-r) \qquad \text{(e.g. Silver, 1980),} \qquad (3.13)$$

$$g_1(R) = g_{LN}(r) = 2\left[\frac{A}{s} + S_{LN}(r)\right]\frac{\varphi(r)}{\theta \cdot e^{\lambda + r\cdot\theta}} - \left[\Phi(-r)\right]^2,$$

and

$$V_1(R) = V_{LN}(r) = \frac{(s/h)D \cdot \Phi(-r)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}(r)}} - 1,$$

where $r = (\ln(R) - \lambda)/\theta$ stands for the standardized Normal variable.

The following example illustrates the application of the general algorithm when the lead-time demand is Log-Normal and Case 2 of the Proposition appears.

Step 1: $A = 1951$, $h = 0.6$, $s = 3$, $D = 300$, $\mu_L = 100$, and $\sigma_L = 20$.

Step 2: $V_1(0) = -0.0002221 < 0$.

Step 4: Solving the equation $g_{LN}(r_o) = 0$ we take $r_o = -2.7962$, and

$V_{LN}(r_o) = 0.036891 > 0$.

Step 5: The two roots of the equation $V_{LN}(r) = 0$ are $r_2 = -28.7014$ and $r_3 = -1.6672$, thus using $\theta = \sqrt{\ln 1.04} = 0.19804$ and $\lambda = 4.58556$ we take $R_2 = \lambda + r_2\theta = 0.3334$ and $R_3 = \lambda + r_3\theta = 70.4835$.

Step 6: Since $C_1(R_3) = 839.3290 < C_1(0) = 840.2000$ we set $R^* = R_3$.

Step 7: $S(R^*) = 29.7738$, $Q^* = 1428.3982$, and $C_{HW}(Q^*, R^*) = C_1(R^*) = 839.3290$.

On the contrary, if instead of $A = 1951$ we set $A = 2107$ and we followed the same steps as above, in steps 4 and 5 we would find $r_o = -2.8238$, $V_{LN}(r_o) = 0.000106 > 0$, $R_2 = 54.8498$, $R_3 = 57.1749$ and finally $C_1(0) = 870.8706$ which would be smaller than $C_1(R_3) = 871.4016$. In this case, instead of proceeding to step 7, we would go to step 8 and the optimal target inventory measures would be

$$R^* = 0,\ Q^* = 1551.4509,\ \text{and}\ C_{HW}(Q^*, R^*) = C_1(0) = 870.8706.$$

For Case 3 Das (1988) derived the condition $f(R_m) < \frac{s \cdot D}{4h}$. Using the parameter values of the above example, we find that the mode of Log-Normal is $R_m = e^{\lambda - \theta^2} = 94.2866$. We see, therefore, that Das' condition is true but unfortunately with the specific parameter values we are in Case 2 and not in Case 3. The reason why Das' condition does not hold has been explained in Sections 3.1 and 3.2.

3.6 Relating the optimal solutions to cost parameter values

For Log-Normal lead-time demand, we give in Tables 3.1 and 3.2 the optimal values Q^* and R^* of the order quantity and the reorder point respectively, together with the cycle service level associated with R^* and the minimum of the H-W cost function. For case 1, Q^* and R^* are obtained from solving the first-order conditions. For case 2, Q^* ensures the smallest between the cost at $R = 0$ and the cost associated with the sizes of Q and R which are obtained after solving the first-order conditions. Finally, for case 3, the optimal order quantity and the minimum cost are computed for $R^* = 0$. The same information as above is given in Tables 3.3 and 3.4, with the exception that the lead-time demand is Normal for which, cases 1, 2 and 3 have the same meaning as in the Log-Normal. In each Table, the computation of the target inventory measures was performed at different values of the cost parameters A, h and s, when each time we changed one of them and kept the other two fixed. The choice of the values for A, h, and s was made following the numerical examples given in the work of Zhao et al. (2012).

For both forms of the lead-time demand distribution, we observe in Tables 3.1-3.4 that when the value of s reduces or the value of A increases, we move gradually from case 1 to case 3. Further, case 1 is met even when the cycle service levels are below 0.50. On the other hand, if the values of the cost parameter lead to case 2, then cost attained at $R^* = 0$ is eventually the global minimum when s takes on the relative lower values and A the relatively higher values. Besides, if case 2 occurs when we change A, we end up in small up to negligible differences between the cost at $R = 0$ and the cost associated with the values of Q and R obtained from the first order conditions.

Tables 3.1-3.4 also demonstrate the implications of changes of the cost parameters on the optimal values Q^* and R^*. When s declines, to attain the minimum cost from the solution of the first order conditions, the optimal inventory policy aims to larger order quantities and smaller reorder points. In this way, the size of backorders increases and this is justified from the reduction of the unit shortage cost. Increasing Q^* and reducing R^* is also the optimal inventory policy when the fixed ordering cost rises. In this way the firm manages to reduce the number of orders in the year. Finally, as the holding cost increases, it is less costly for the firm to keep small amounts of inventories. In this case the optimal policy imposes the reduction of Q^* and the increase of R^*.

Table 3.1 Optimal solutions for different values of ordering cost when the demand distribution is Log-Normal; Q^* is the optimal order quantity; R^* is the optimal reorder point; $C_{HW}\left(Q^*,R^*\right)$ is the minimum cost associated with solving the first order conditions; $C(0)$ is the minimum cost at $R=0$; h=0.6, D=300 and μ_L=100.

		s=5h=3					s=15h=9				
CV	**A**	**Case**	**Cycle Service Level**	$\left(Q^*,R^*\right)$	$C_{HW}\left(Q^*,R^*\right)$	$C(0)$	**Case**	**Cycle Service Level**	$\left(Q^*,R^*\right)$	$C_{HW}\left(Q^*,R^*\right)$	$C(0)$
	30		0.88	(186.93,123.19)	126.08	-		0.96	(185.58,138.30)	134.33	-
	70		0.81	(278.90,117.03)	177.56	-		0.94	(277.20,133.07)	186.16	-
	300		0.62	(564.06,104.39)	341.07	-		0.88	(561.11,123.18)	350.57	-
	800	**Case 1**	0.39	(913.88,92.82)	544.02	-		0.80	(908.68,115.69)	554.62	-
	1100		0.29	(1070.20,87.70)	634.74	-		0.76	(1063.44,113.05)	645.89	-
	1400		0.20	(1206.93,82.73)	713.79	-		0.73	(1198.17,110.96)	725.48	-
0.2	1700		0.11	(1330.75,77.14)	784.73	-	**Case 1**	0.71	(1319.09,109.22)	796.98	-
	1951		0.05	(1428.40,70.48)	839.33	840.20		0.69	(1412.26,107.94)	852.12	-
	1955		0.05	(1429.94,70.34)	840.17	841.00		0.69	(1413.69,107.92)	852.97	-
	1960	**Case 2**	0.05	(1431.88,70.15)	841.22	842.00		0.69	(1415.48,107.90)	854.03	-
	1970		0.04	(1435.75,69.77)	843.31	843.99		0.68	(1419.06,107.85)	856.15	-
	2107		0	(1551.45,0)	870.87	870.87		0.67	(1467.16,107.22)	884.63	-
	2200	**Case 3**	0	(1581.14,0)	-	888.68		0.67	(1498.93,106.80)	903.44	-
	30		0.85	(223.92,146.15)	162.04	-		0.95	(225.80,194.37)	192.10	-
	70		0.79	(313.24,131.17)	206.64	-		0.93	(314.40,179.68)	236.45	-
	300		0.60	(595.47,101.19)	357.99	-		0.87	(594.63,151.55)	387.71	-
	800	**Case 1**	0.37	(944.59,76.49)	552.65	-		0.79	(940.44,131.13)	582.94	-
	1100		0.27	(1101.15,66.57)	640.63	-		0.76	(1094.71,124.25)	671.38	-
	1400		0.17	(1238.51,57.45)	717.57	-		0.73	(1229.12,118.94)	748.83	-
0.5	1700		0.09	(1363.66,47.60)	786.76	-	**Case 1**	0.70	(1349.79,114.59)	818.63	-
	1951		0.02	(1465.13,34.92)	840.03	840.20		0.68	(1442.80,111.46)	872.56	-
	1955	**Case 2**	0.02	(1466.84,34.58)	840.85	841.00		0.68	(1444.23,111.42)	873.39	-
	1960		0.02	(1469.00,34.12)	841.87	842.00		0.68	(1446.02,111.36)	874.43	-
	1970		0.02	(1473.39,33.12)	843.91	843.99		0.68	(1449.59,111.24)	876.50	-
	2107	**Case 3**	0	(1551.45,0)	-	870.87		0.67	(1497.61,109.70)	904.39	-
	2200		0	(1581.14,0)	-	888.68		0.66	(1529.34,108.70)	922.82	-
	30		0.79	(307.79,140.32)	208.87	-		0.92	(342.99,232.75)	285.44	-
	70		0.74	(383.76,122.11)	243.52	-		0.91	(414.95,213.50)	317.07	-
	300		0.57	(646.38,81.74)	376.87	-		0.85	(671.09,168.11)	443.52	-
	800	**Case 1**	0.34	(983.76,50.63)	560.63	-		0.78	(1003.83,133.34)	622.30	-
	1100		0.24	(1136.35,39.53)	645.53	-		0.74	(1154.25,121.90)	705.69	-
	1400		0.15	(1270.37,30.16)	720.32	-		0.71	(1285.83,113.26)	779.45	-
1	1700		0.07	(1392.29,20.93)	787.93	-	**Case 1**	0.69	(1404.27,106.33)	846.36	-
	1951		0.01	(1491.83,8.49)	840.19	840.20		0.67	(1495.73,101.43)	898.29	-
	1955	**Case 2**	0.00	(1493.82,7.84)	841.00	841.00		0.67	(1497.13,101.36)	899.10	-
	1960		0	(1503.33,0)	842.00	842.00		0.67	(1498.89,101.27)	900.10	-
	1970		0	(1506.65,0)	-	843.99		0.67	(1502.41,101.09)	902.10	-
	2107	**Case 3**	0	(1551.45,0)	-	870.87		0.66	(1549.68,98.70)	929.03	-
	2200		0	(1581.14,0)	-	888.68		0.65	(1580.92,97.17)	946.85	-

Table 3.2 Optimal solutions for different values of ordering cost when the demand distribution is Log-Normal; Q^* is the optimal order quantity; R^* is the optimal reorder point; $C_{HW}\left(Q^*,R^*\right)$ is the minimum cost associated with solving the first order conditions; $C\left(0\right)$ is the minimum cost at $R=0$; h=0.1, D=300 and μ_L=100.

		s=5h=0.5					s=15h=1.5				
CV	**A**	**Case**	**Cycle Service Level**	$\left(Q^*,R^*\right)$	$C_{HW}\left(Q^*,R^*\right)$	$C\left(0\right)$	**Case**	**Cycle Service Level**	$\left(Q^*,R^*\right)$	$C_{HW}\left(Q^*,R^*\right)$	$C\left(0\right)$
0.2	30	**Case 1**	0.71	(439.69,109.22)	44.89	-	**Case 1**	0.90	(437.32,126.80)	46.41	-
	70		0.56	(665.21,100.86)	66.61	-		0.85	(661.72,120.70)	68.24	-
	300		0.09	(1370.04,74.88)	134.49	-		0.70	(1356.98,108.69)	136.57	-
	800	**Case 3**	0	(2258.32,0)	-	215.83		0.51	(2208.48,98.51)	220.70	-
	1100		0	(2626.79,0)	-	252.68		0.42	(2587.82,94.45)	258.23	-
	1400		0	(2949.58,0)	-	284.96		0.35	(2918.26,90.93)	290.92	-
	1700		0	(3240.37,0)	-	314.04		0.29	(3215.02,87.65)	320.27	-
	1951		0	(3464.97,0)	-	336.50		0.23	(3443.86,84.97)	342.88	-
	1955		0	(3468.43,0)	-	336.84		0.23	(3447.39,84.93)	343.23	-
	1960		0	(3472.75,0)	-	337.28		0.23	(3451.79,84.87)	343.67	-
	1970		0	(3481.38,0)	-	338.14		0.23	(3460.58,84.76)	344.53	-
	2107		0	(3597.50,0)	-	349.75		0.20	(3578.84,83.28)	356.21	-
	2200		0	(3674.23,0)	-	357.42		0.19	(3656.98,82.25)	363.92	-
0.5	30	**Case 1**	0.69	(471.92,112.36)	48.43	-	**Case 1**	0.90	(471.97,161.75)	53.37	-
	70		0.54	(696.22,93.33)	68.95	-		0.85	(694.58,144.67)	73.93	-
	300		0.06	(1403.80,43.60)	134.74	-		0.69	(1387.61,113.30)	140.09	-
	800	**Case 3**	0	(2258.32,0)	-	215.83		0.50	(2238.25,89.72)	222.80	-
	1100		0	(2626.79,0)	-	252.68		0.42	(2617.51,81.14)	259.87	-
	1400		0	(2949.58,0)	-	284.96		0.34	(2948.00,74.07)	292.21	-
	1700		0	(3240.37,0)	-	314.04		0.28	(3244.92,67.81)	321.27	-
	1951		0	(3464.97,0)	-	336.50		0.23	(3473.98,62.90)	343.69	-
	1955		0	(3468.43,0)	-	336.84		0.23	(3477.51,62.82)	344.03	-
	1960		0	(3472.75,0)	-	337.28		0.23	(3481.91,62.72)	344.46	-
	1970		0	(3481.38,0)	-	338.14		0.22	(3490.71,62.53)	345.32	-
	2107		0	(3597.50,0)	-	349.75		0.20	(3609.12,59.90)	356.90	-
	2200		0	(3674.23,0)	-	357.42		0.18	(3687.39,58.10)	364.55	-
1	30	**Case 1**	0.65	(529.26,96.84)	52.61	-	**Case 1**	0.88	(556.16,185.31)	64.15	-
	70		0.50	(743.07,71.40)	71.45	-		0.83	(766.32,156.34)	82.27	-
	300		0.05	(1431.27,17.36)	134.86	-		0.68	(1441.45,104.30)	144.57	-
	800	**Case 3**	0	(2258.32,0)	-	215.83		0.49	(2281.25,69.69)	225.09	-
	1100		0	(2626.79,0)	-	252.68		0.41	(2656.98,58.45)	261.54	-
	1400		0	(2949.58,0)	-	284.96		0.34	(2984.69,49.79)	293.45	-
	1700		0	(3240.37,0)	-	314.04		0.27	(3279.24,42.59)	322.18	-
	1951		0	(3464.97,0)	-	336.50		0.22	(3506.51,37.26)	344.38	-
	1955		0	(3468.43,0)	-	336.84		0.22	(3510.01,37.18)	344.72	-
	1960		0	(3472.75,0)	-	337.28		0.22	(3514.38,37.08)	345.15	-
	1970		0	(3481.38,0)	-	338.14		0.22	(3523.11,36.87)	346.00	-
	2107		0	(3597.50,0)	-	349.75		0.19	(3640.58,34.15)	357.47	-
	2200		0	(3674.23,0)	-	357.42		0.17	(3718.21,32.34)	365.06	-

Table 3.3 Optimal solutions for different values of ordering cost when the demand distribution is Normal; Q^* is the optimal order quantity; R^* is the optimal reorder point; $C_{HW}\left(Q^*,R^*\right)$ is the minimum cost associated with solving the first order conditions; $C(0)$ is the minimum cost at $R=0$; h=0.6, D=300 and μ_L=100.

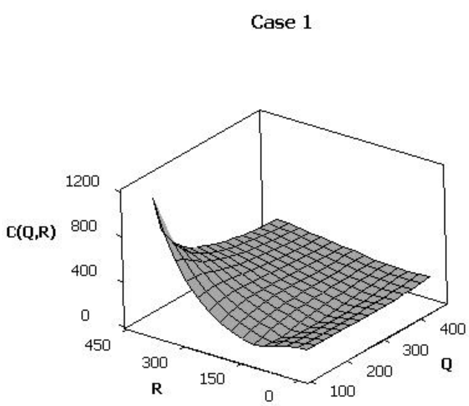
		s=5h=3					s=15h=9				
CV	**A**	**Case**	**Cycle Service Level**	$\left(Q^*,R^*\right)$	$C_{HW}\left(Q^*,R^*\right)$	$C(0)$	**Case**	**Cycle Service Level**	$\left(Q^*,R^*\right)$	$C_{HW}\left(Q^*,R^*\right)$	$C(0)$
0.1	30	**Case 1**	0.88	(178.18,111.81)	114.00	-	**Case 1**	0.96	(177.28,117.58)	116.91	-
	70		0.82	(270.06,109.15)	167.52	-		0.94	(268.91,115.57)	170.69	-
	200		0.70	(453.62,105.17)	275.28	-		0.90	(451.98,112.79)	278.86	-
	500		0.52	(714.92,100.59)	429.30	-		0.84	(712.37,110.01)	433.43	-
	1000		0.33	(1009.90,95.51)	603.24	-		0.78	(1005.78,107.60)	608.03	-
	1500		0.18	(1237.28,90.66)	736.76	-		0.73	(1230.91,106.02)	742.16	-
	1800		0.10	(1356.67,86.93)	806.16	-		0.70	(1348.00,105.26)	811.95	-
	1960	**Case 2**	0.06	(1417.25,84.03)	840.77	842.00		0.69	(1406.46,104.89)	846.81	-
	1980		0.05	(1424.76,83.57)	844.99	845.98		0.69	(1413.59,104.84)	851.06	-
	1990		0.05	(1428.51,83.32)	847.10	847.96		0.69	(1417.15,104.82)	853.18	-
	2000		0.05	(1432.26,83.06)	849.19	849.95		0.68	(1420.69,104.80)	855.29	-
	2100		0	(1549.19,0)	869.52	869.52		0.68	(1455.68,104.58)	876.15	-
	2200	**Case 3**	0	(1581.14,0)	-	888.68		0.67	(1489.84,104.37)	896.52	-
0.2	30	**Case 1**	0.88	(183.37,123.28)	123.99	-	**Case 1**	0.96	(181.48,134.94)	129.85	-
	70		0.82	(275.71,118.02)	176.24	-		0.94	(273.35,130.97)	182.59	-
	200		0.69	(460.19,110.10)	282.17	-		0.90	(456.81,125.46)	289.36	-
	500		0.52	(722.91,100.91)	434.29	-		0.84	(717.70,119.93)	442.58	-
	1000		0.32	(1020.07,90.64)	606.43	-		0.78	(1011.61,115.12)	616.04	-
	1500		0.17	(1250.36,80.63)	738.60	-		0.73	(1237.12,111.96)	749.45	-
	1800		0.08	(1372.86,72.52)	807.23	-		0.70	(1354.41,110.43)	818.90	-
	1960	**Case 2**	0.04	(1436.89,65.46)	841.41	842.00		0.69	(1412.96,109.69)	853.59	-
	1980		0.04	(1445.12,64.16)	845.57	845.98		0.68	(1420.11,109.60)	857.83	-
	1990		0.03	(1449.29,63.45)	847.65	847.96		0.68	(1423.67,109.56)	859.94	-
	2000		0.03	(1453.52,62.67)	849.71	849.95		0.68	(1427.23,109.51)	862.04	-
	2100	**Case 3**	0	(1549.19,0)	-	869.52		0.68	(1426.27,109.08)	882.81	-
	2200		0	(1581.14,0)	-	888.68		0.67	(1496.49,108.66)	903.09	-
0.3	30	**Case 1**	0.87	(188.76,134.39)	133.89	-	**Case 1**	0.96	(185.80,152.08)	142.73	-
	70		0.81	(281.54,126.59)	184.88	-		0.94	(277.88,146.21)	194.45	-
	200		0.69	(466.92,114.77)	289.01	-		0.90	(461.72,138.01)	299.83	-
	500		0.51	(731.10,100.95)	439.23	-		0.84	(723.09,129.75)	451.70	-
	1000		0.31	(1030.55,85.38)	609.55	-		0.77	(1017.50,122.55)	624.03	-
	1500		0.16	(1264.09,69.83)	740.35	-		0.72	(1243.38,117.82)	756.72	-
	1800		0.07	(1390.62,56.37)	808.19	-		0.70	(1360.87,115.52)	825.84	-
	1960	**Case 2**	0.03	(1461.64,41.49)	841.88	842.00		0.68	(1419.52,114.41)	860.36	-
	1980		0.02	(1472.74,37.21)	845.97	845.98		0.68	(1426.69,114.28)	864.58	-
	1990		0	(1513.28,0)	847.97	847.96		0.68	(1430.25,114.21)	866.68	-
	2000	**Case 3**	0	(1516.58,0)	-	849.95		0.68	(1433.81,114.15)	868.77	-
	2100		0	(1549.20,0)	-	869.52		0.67	(1468.91,113.49)	889.44	-
	2200		0	(1581.14,0)	-	888.68		0.67	(1503.19,112.86)	909.63	-

Table 3.4 Optimal solutions for different values of ordering cost when the demand distribution is Normal; Q^* is the optimal order quantity; R^* is the optimal reorder point; $C_{HW}(Q^*, R^*)$ is the minimum cost associated with solving the first order conditions; $C(0)$ is the minimum cost at $R = 0$; h=0.1, D=300 and μ_L=100.

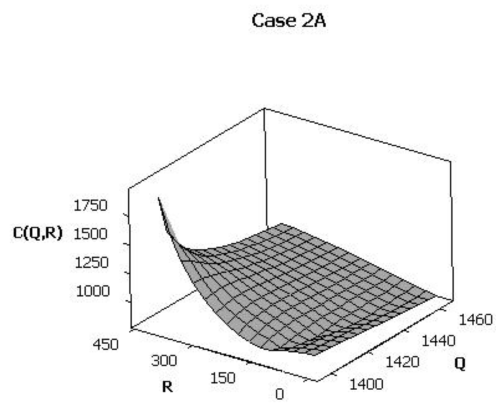
		s=5h=0.5					s=15h=1.5				
CV	**A**	**Case**	**Cycle Service Level**	(Q^*, R^*)	$C_{HW}(Q^*, R^*)$	$C(0)$	**Case**	**Cycle Service Level**	(Q^*, R^*)	$C_{HW}(Q^*, R^*)$	$C(0)$
0.1	30	**Case 1**	0.71	(430.56,105.62)	43.62	-	**Case 1**	0.90	(428.98,113.09)	44.21	-
	70		0.56	(655.55,101.59)	65.71	-		0.85	(653.23,110.57)	66.38	-
	200		0.26	(1106.27,93.64)	109.99	-		0.76	(1101.39,106.91)	110.83	-
	500	**Case 3**	0	(1816.59,0)	-	171.66		0.61	(1739.08,102.89)	174.20	-
	1000		0	(2509.98,0)	-	241.00		0.45	(2457.92,98.84)	245.68	-
	1500		0	(3049.59,0)	-	294.96		0.33	(3009.81,95.63)	300.54	-
	1800		0	(3331.67,0)	-	323.17		0.27	(3297.05,93.79)	329.08	-
	1960		0	(3472.75,0)	-	337.28		0.24	(3440.54,92.79)	343.33	-
	1980		0	(3489.99,0)	-	339.00		0.23	(3458.06,92.66)	345.07	-
	1990		0	(3498.57,0)	-	339.86		0.23	(3466.79,92.60)	345.94	-
	2000		0	(3507.14,0)	-	340.71		0.23	(3475.49,92.53)	346.80	-
	2100		0	(3591.66,0)	-	349.17		0.21	(3561.41,91.89)	355.33	-
	2200		0	(3674.23,0)	-	357.42		0.19	(3645.33,91.22)	363.65	-
0.2	30	**Case 1**	0.71	(437.01,110.99)	44.80	-	**Case 1**	0.90	(433.77,126.05)	45.98	-
	70		0.56	(663.19,102.91)	66.61	-		0.85	(658.45,121.05)	67.95	-
	200		0.26	(1117.46,86.82)	110.43	-		0.75	(1107.38,113.74)	112.11	-
	500	**Case 3**	0	(1816.59,0)	-	171.66		0.61	(1746.17,105.69)	175.19	-
	1000		0	(2509.98,0)	-	241.00		0.45	(2466.42,97.58)	246.40	-
	1500		0	(3049.59,0)	-	294.96		0.33	(3019.70,91.14)	301.08	-
	1800		0	(3331.67,0)	-	323.17		0.26	(3307.88,87.43)	329.53	-
	1960		0	(3472.75,0)	-	337.28		0.23	(3451.92,85.41)	343.73	-
	1980		0	(3489.99,0)	-	339.00		0.23	(3469.51,85.16)	345.47	-
	1990		0	(3498.57,0)	-	339.86		0.23	(3478.28,85.03)	346.33	-
	2000		0	(3507.14,0)	-	340.71		0.23	(3487.02,84.90)	347.19	-
	2100		0	(3591.66,0)	-	349.17		0.21	(3573.31,83.59)	355.69	-
	2200		0	(3674.23,0)	-	357.42		0.19	(3657.64,82.23)	363.99	-
0.3	30	**Case 1**	0.70	(443.63,116.10)	45.97	-	**Case 1**	0.90	(438.63,138.88)	47.75	-
	70		0.55	(671.02,103.97)	67.50	-		0.85	(663.73,131.42)	69.52	-
	200		0.25	(1129.03,79.51)	110.85	-		0.75	(1113.43,120.48)	113.39	-
	500	**Case 3**	0	(1816.59,0)	-	171.66		0.61	(1753.30,108.41)	176.17	-
	1000		0	(2509.98,0)	-	241.00		0.45	(2474.98,96.23)	247.12	-
	1500		0	(3049.59,0)	-	294.96		0.33	(3029.69,86.53)	301.62	-
	1800		0	(3331.67,0)	-	323.17		0.26	(3318.82,80.93)	329.98	-
	1960		0	(3472.75,0)	-	337.28		0.23	(3463.43,77.87)	344.13	-
	1980		0	(3489.99,0)	-	339.00		0.23	(3481.10,77.48)	345.86	-
	1990		0	(3498.57,0)	-	339.86		0.22	(3489.90,77.28)	346.72	-
	2000		0	(3507.14,0)	-	340.71		0.22	(3498.68,77.09)	347.58	-
	2100		0	(3591.66,0)	-	349.17		0.20	(3585.37,75.10)	356.05	-
	2200		0	(3674.23,0)	-	357.42		0.18	(3670.13,73.04)	364.32	-

Before closing the section 3.6, using Log-Normal and Normal distributed lead-time demands we present the 3D graphs for the three cases where it is observed that when the ordering cost increases then the unique minimum is attained for lower values of the reorder point and greater values of the order quantity.

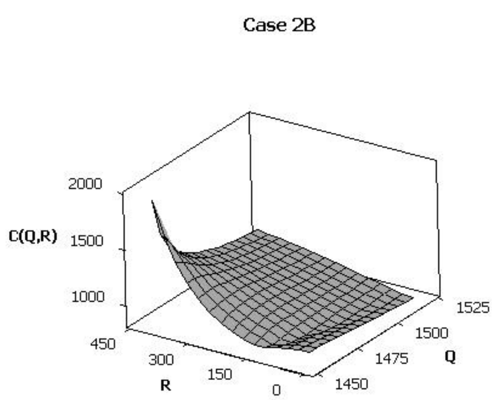
(a) $A = 14$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 20$ and $s = 3$



(b) $A = 1951$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 20$ and $s = 3$



(c) $A = 2107$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 20$ and $s = 3$



(d) $A = 2200$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 20$ and $s = 3$

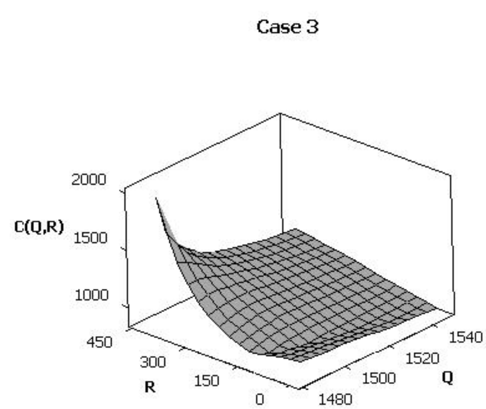
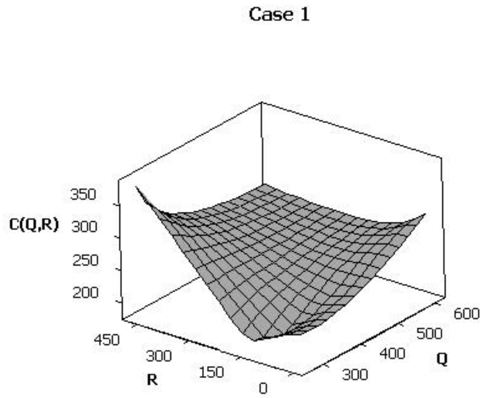
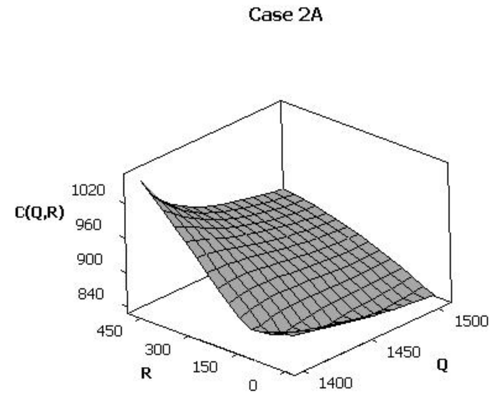


Figure 3.7 Graph of the cost function $C_{HW}(Q,R)$ under Log-Normal lead-time demand.

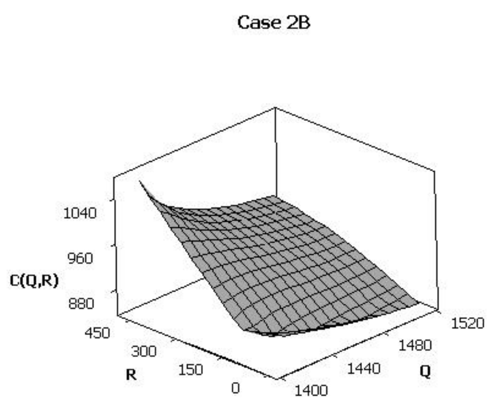
(a) $A = 70$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 30$ and $s = 3$



(b) $A = 1960$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 30$ and $s = 3$



(c) $A = 1990$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 30$ and $s = 3$



(d) $A = 2200$, $h = 0.6$, $D = 300$, $\mu_L = 100$, $\sigma_L = 30$ and $s = 3$

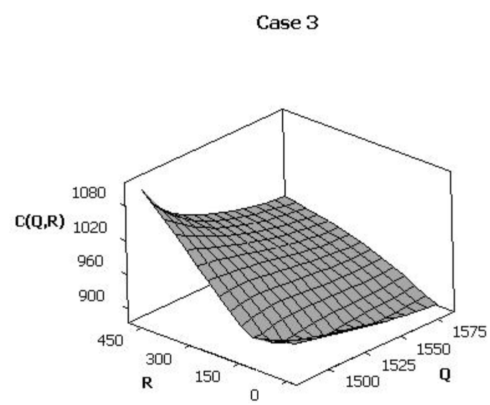


Figure 3.8 Graph of the cost function $C_{HW}(Q,R)$ under Normal lead-time demand.

3.7 Degeneracy problem and cost parameter values

According to Lau & Lau (2002), the degeneracy problem happens when the minimization procedure of the cost function breaks-down and the unique minimum occurs at zero reorder point. Thus, in this section, for each distribution, we examine the range values of the three cost elements in order the unique minimum to be attained for $R^* > 0$ or $R^* = 0$. In particular, in Table 3.5, we give the range values of s, A and h and we observe that these threshold values are not independent of the form of the lead-time demand distribution as in order to compute S(R) and Φ(R) the r_o and z_o values need to be determined.

Table 3.5 Interval values of the cost parameters for a minimum cost at a positive reorder point.

Log-Normal

$$0 \le A \le s\left\{\frac{sD\left[\Phi\left(-r_o\right)\right]^2}{2h} - S_{LN}\left(r_o\right)\right\}$$

$$\frac{h}{D}\left\{\frac{S_{LN}\left(r_o\right)+\sqrt{\left[S_{LN}\left(r_o\right)\right]^2+2\frac{A}{h}D\left[\Phi\left(-r_o\right)\right]^2}}{\left[\Phi\left(-r_o\right)\right]^2}\right\} \le s < +\infty$$

$$0 \le h \le \frac{s^2D^2\left[\Phi\left(-r_o\right)\right]^2}{2\cdot A\cdot D+2\cdot s\cdot D\cdot S_{LN}\left(r_o\right)}$$

Normal

$$0 \le A \le s\left\{\frac{sD\left[\Phi\left(-z_o\right)\right]^2}{2h} - S_{NM}\left(z_o\right)\right\}$$

$$\frac{h}{D}\left\{\frac{S_{NM}\left(z_o\right)+\sqrt{\left[S_{NM}\left(z_o\right)\right]^2+2\frac{A}{h}D\left[\Phi\left(-z_o\right)\right]^2}}{\left[\Phi\left(-z_o\right)\right]^2}\right\} \le s < +\infty$$

$$0 \le h \le \frac{s^2D^2\left[\Phi\left(-z_o\right)\right]^2}{2\cdot A\cdot D+2\cdot s\cdot D\cdot S_{NM}\left(z_o\right)}$$

Proof 3.10: See in the Appendix at the end of the chapter.

3.8 Managerial Implications

The Hadley-Whitin (H-W) expression $I_{HW} = \left[(Q/2) + R - \mu_L \right]$ constitutes an important factor at the process of constructing the cost function defined in (3.1). This expression is used as an approximation of the exact expected on-hand inventory at any point in time which is given in (2.3) as

$$I_{ex} = I_{HW} + \frac{\Theta(R)}{2Q}.$$

It is further well known that this approximation is accurate only when the minimization of (3.1) leads to optimal R values which give sufficiently large probabilities (cycle service level, CSL) for not observing stock-out during the lead-time. In an effort of investigating how large these probabilities should be, Lau & Lau (2002) suggested for Normal lead-time demand the use of I_{HW} only when the minimization of (3.1) results in optimal R values leading to CSLs greater than 0.6. Additionally to that, from the numerical examples discussed by the authors it seems that Cases 2 and 3 of the general algorithm are met at very low CSLs. If that is in fact the situation then the value of the algorithm is reduced significantly as Cases 2 and 3 are met at CSLs where finally the H-W approximation is not valid and should not be used.

Contrary to what is generally accepted about the validity of the H-W approximation, in the current section we are promoting the value of the general algorithm by illustrating for Normal and Log-Normal lead-time demands that (a) the determinants of the validity of the H-W approximation are the lead-time demand coefficient of variation (CV) and the fixed ordering cost, A, and (b) there are combinations of CV and A sizes for which acceptable approximations occur even when CSLs are zero. The last two remarks are verified by comparing the results for the optimal target inventory measures reported in Table 3.6 for the case of Normal and Table 3.7 for the Log-Normal. These results were obtained for different combinations of CV and A sizes, given the values for the rest of the (Q,R) inventory model parameters, h, s, D, μ_L, by minimizing first the H-W approximate cost function given in (3.1) and then the corresponding exact cost function given in (2.2b)

$$C_{ex}(Q,R) = \frac{A \cdot D}{Q} + h \cdot I_{ex} + \frac{s \cdot D}{Q} S(R). \qquad (3.14)$$

For the minimization of $C_{ex}(Q,R)$, specifications of S(R) and Θ(R) are required according to the shape of the lead-time demand distribution. For Normal and Log-Normal lead-time demands, $S(R)$ is obtained from (3.12) and (3.13) respectively. Regarding the specifications of $\Theta(\mathrm{R})$, for the Normal distribution, we use the formula (26) of Lau et al. (2002b) while for the Log-Normal case, the computational expression of $\Theta(\mathrm{R})$ is obtained using Table 2.1 of chapter 2. Finally, regarding the minimization process of $C_{ex}(Q,R)$, for both distributions we used the algorithm suggested in chapter 2. Particularly, when the condition

$$(s/h)^2 D^2 - (2A/h)D - \sigma_L^2 > 0 \tag{3.15}$$

holds, the application of any iterative procedure for solving the equations obtained from the first-order conditions minimizing (3.14) leads to a unique optimal pair (Q_{ex}^*, R_{ex}^*) and a unique minimum exact cost $C_{ex}(Q_{ex}^*, R_{ex}^*)$. On the other hand, if (3.15) is not true then the minimum exact cost occurs at $R_{ex}^* = 0$ and $Q_{ex}^* = \sqrt{2(A/h)D + 2(s/h)D\mu_L + \mu_L^2 + \sigma_L^2}$.

With respect to the choice of values for the parameters h and s, this was made according to the suggestions of Zhao et al. (2012). Particularly, the authors recommended that values for h should be chosen from the interval $[0.1, 3.0]$, and then given h, the parameter s will take on values on the interval $[5h, 15h]$. On the other hand, no restrictions were imposed for the values of A. Further, for the Normal distribution, CV was restricted to sizes less than or equal to 0.3 for reasons which were explained in the previous section. Contrary to that, for the Log-Normal distribution, the CV sizes were set at levels both less and greater than 0.3.

Having determined for each combination of parameters values the minima of (3.1) and (3.14) next we computed the percentage cost penalty (Gross & Ince, 1975; Lau & Lau, 2002) defined as

$$\mathrm{PCP} = \frac{C_{ex}(Q^*, R^*) - C_{ex}(Q_{ex}^*, R_{ex}^*)}{C_{ex}(Q_{ex}^*, R_{ex}^*)} \times 100,$$

which is presented in the last column of Tables 3.6 and 3.7. As it was expected, PCP takes on lower values for sufficiently small stock-out probabilities. Something, however, which is not expected, is the trend of the PCP values for $\mathrm{CV} \le 0.5$ when Case 3 is met. Particularly, as A increases and at the same time CSL decreases (as a result of raising A), we find out that PCP initially is getting larger until Case 3 is met, but, being at Case 3, PCP values will start to follow a decreasing trend if A continues to rise. This means that very small PCP indicating accurate approximations can be observed not only when CSL is very high but also when CSL is very low, or even when CSL is zero. According to the size of the coefficient of variation, we find out that PCP takes on increasingly higher sizes when CV gets larger providing that the values of the remaining parameters are kept fixed.

Table 3.6 Comparison of the exact $C_{ex}\left(Q_{ex}^{*}, R_{ex}^{*}\right)$ and Hadley-Whitin $C_{HW}\left(Q^{*}, R^{*}\right)$ cost functions under Normal lead-time demand with s=3, h=0.6, D=300 and μ_L=100.

			Hadley & Whitin			Exact				
CV		**A**	**Cycle Service Level**	$\left(Q^{*}, R^{*}\right)$	$C_{HW}\left(Q^{*}, R^{*}\right)$	**Cycle Service Level**	$\left(Q_{ex}^{*}, R_{ex}^{*}\right)$	$C_{ex}\left(Q_{ex}^{*}, R_{ex}^{*}\right)$	$C_{ex}\left(Q^{*}, R^{*}\right)$	**PCP**
0.1	Case 1	20	0.90	(146.21,112.96)	95.50	0.90	(146.20,112.98)	95.51	95.51	0.00%
		70	0.82	(270.06,109.15)	167.52	0.82	(270.05,109.18)	167.53	167.53	0.00%
		200	0.70	(453.62,105.17)	275.28	0.70	(453.61,105.21)	275.29	275.29	0.00%
		500	0.52	(714.92,100.59)	429.30	0.53	(714.89,100.65)	429.32	429.32	0.00%
		1000	0.33	(1009.90,95.51)	603.24	0.33	(1099.82,95.63)	603.27	603.27	0.00%
		1500	0.18	(1237.28,90.66)	736.76	0.18	(1237.08,90.92)	736.80	736.80	0.00%
		1800	0.10	(1356.67,86.93)	806.16	0.10	(1356.26,87.44)	806.22	806.22	0.00%
	Case 2	1960	0.06	(1417.25,84.03)	840.77	0.07	(1416.47,84.93)	840.84	840.85	0.00%
		1980	0.05	(1424.76,83.57)	844.99	0.06	(1423.89,84.55)	845.07	845.07	0.00%
		1990	0.05	(1428.51,83.32)	847.10	0.06	(1427.60,84.36)	847.17	847.18	0.00%
		2000	0.05	(1432.26,83.06)	849.19	0.06	(1431.30,84.15)	849.27	849.27	0.00%
		2100	0	(1549.19,0)	869.52	0.03	(1468.26,81.68)	869.96	869.97	0.00%
	Case 3	2200	0	(1581.14,0)	888.68	0.01	(1506.48,77.09)	890.14	890.60	0.05%
0.2	Case 1	20	0.90	(151.22,125.54)	106.06	0.90	(151.20,125.61)	106.09	106.09	0.00%
		70	0.82	(275.71,118.02)	176.24	0.82	(275.67,118.12)	176.28	176.28	0.00%
		200	0.69	(460.19,110.10)	282.17	0.70	(460.13,110.25)	282.23	282.23	0.00%
		500	0.52	(722.91,100.91)	434.29	0.52	(722.79,101.16)	434.37	434.37	0.00%
		1000	0.32	(1020.07,90.64)	606.43	0.33	(1019.76,91.14)	606.54	606.55	0.00%
		1500	0.17	(1250.36,80.63)	738.60	0.18	(1249.56,81.73)	738.77	738.78	0.00%
		1800	0.08	(1372.86,72.52)	807.23	0.10	(1371.02,74.76)	807.47	807.48	0.00%
	Case 2	1960	0.04	(1436.89,65.46)	841.41	0.07	(1433.10,69.74)	841.71	841.74	0.00%
		1980	0.04	(1445.12,64.16)	845.57	0.06	(1440.82,68.98)	845.88	845.92	0.00%
		1990	0.03	(1449.29,63.45)	847.65	0.06	(1444.68,68.59)	847.96	848.00	0.00%
		2000	0.03	(1453.52,62.67)	849.71	0.06	(1448.54,68.18)	850.04	850.08	0.01%
	Case 3	2100	0	(1549.19,0)	869.52	0.03	(1487.56,63.23)	870.47	871.53	0.12%
		2200	0	(1581.14,0)	888.68	0.01	(1529.93,54.02)	890.37	890.66	0.03%
0.3	Case 1	20	0.90	(156.47,137.72)	116.52	0.90	(156.42,137.89)	116.59	116.59	0.00%
		70	0.81	(281.54,126.59)	184.88	0.81	(281.46,126.83)	184.97	184.97	0.00%
		200	0.69	(466.92,114.77)	289.01	0.69	(466.78,115.11)	289.14	289.14	0.00%
		500	0.51	(731.10,100.95)	439.23	0.52	(730.81,101.53)	439.40	439.40	0.00%
		1000	0.31	(1030.55,85.38)	609.55	0.33	(1029.83,86.53)	609.81	609.82	0.00%
		1500	0.16	(1264.09,69.83)	740.35	0.18	(1262.17,72.41)	740.75	740.77	0.00%
		1800	0.07	(1390.62,56.37)	808.19	0.10	(1385.94,61.96)	808.74	808.79	0.01%
	Case 2	1960	0.03	(1461.64,41.49)	841.88	0.06	(1449.90,54.42)	842.59	842.77	0.02%
		1980	0.02	(1472.74,37.21)	845.97	0.06	(1457.92,53.28)	846.72	846.96	0.03%
		1990	0	(1513.28,0)	847.97	0.06	(1461.93,52.69)	848.78	849.07	0.04%
	Case 3	2000	0	(1516.58,0)	849.95	0.06	(1465.96,52.08)	850.83	852.10	0.15%
		2100	0	(1549.20,0)	869.52	0.03	(1507.06,44.63)	871.01	871.63	0.07%
		2200	0	(1581.14,0)	888.69	0.01	(1553.64,30.74)	890.63	890.75	0.01%

Table 3.7 Comparison of the exact $C_{ex}\left(Q_{ex}^{*}, R_{ex}^{*}\right)$ and Hadley-Whitin $C_{HW}\left(Q^{*}, R^{*}\right)$ cost functions under Log-Normal lead-time demand with s=3, h=0.6, D=300 and μ_L=100.

			Hadley & Whitin			Exact				
CV		**A**	**Cycle Service Level**	$\left(Q^{*}, R^{*}\right)$	$C_{HW}\left(Q^{*}, R^{*}\right)$	**Cycle Service Level**	$\left(Q_{ex}^{*}, R_{ex}^{*}\right)$	$C_{ex}\left(Q_{ex}^{*}, R_{ex}^{*}\right)$	$C_{ex}\left(Q^{*}, R^{*}\right)$	**PCP**
0.2	**Case 1**	14	0.91	(131.74,128.22)	95.98	0.91	(131.72,128.34)	96.04	96.04	0.00%
		70	0.81	(278.90,117.03)	177.56	0.82	(278.87,117.18)	177.63	177.63	0.00%
		300	0.62	(564.06,104.39)	341.07	0.63	(563.99,104.61)	341.16	341.16	0.00%
		800	0.39	(913.88,92.82)	544.02	0.40	(913.71,93.19)	544.14	544.14	0.00%
		1100	0.29	(1070.20,87.70)	634.74	0.30	(1069.93,88.21)	634.88	634.89	0.00%
		1400	0.20	(1206.93,82.73)	713.79	0.21	(1206.47,83.46)	713.96	713.96	0.00%
		1700	0.11	(1330.75,77.14)	784.73	0.13	(1329.88,78.34)	784.93	784.94	0.00%
	Case 2	1951	0.05	(1428.40,70.48)	839.33	0.07	(1426.38,72.92)	839.58	839.60	0.00%
		1955	0.05	(1429.94,70.34)	840.17	0.07	(1427.88,72.82)	840.42	840.44	0.00%
		1960	0.05	(1431.88,70.15)	841.22	0.07	(1429.76,72.69)	841.47	841.49	0.00%
		1970	0.04	(1435.75,69.77)	843.31	0.06	(1433.53,72.42)	843.57	843.58	0.00%
		2107	0	(1551.45,0)	870.87	0.03	(1485.03,67.86)	871.73	871.85	0.01%
	Case 3	2200	0	(1581.14,0)	888.68	0.01	(1521.61,62.22)	890.30	890.66	0.04%
0.5	**Case 1**	14	0.89	(171.86,157.85)	137.83	0.89	(171.98,159.13)	138.67	138.68	0.01%
		70	0.79	(313.24,131.17)	206.64	0.80	(313.28,132.49)	207.46	207.47	0.01%
		300	0.60	(595.47,101.19)	357.99	0.61	(595.36,102.67)	358.82	358.83	0.00%
		800	0.37	(944.59,76.49)	552.65	0.39	(944.14,78.41)	553.53	553.55	0.00%
		1100	0.27	(1101.15,66.57)	640.63	0.29	(1100.39,68.90)	641.57	641.59	0.00%
		1400	0.17	(1238.51,57.45)	717.57	0.20	(1237.19,60.44)	718.58	718.61	0.00%
		1700	0.09	(1363.66,47.60)	786.76	0.13	(1361.10,52.00)	787.86	787.91	0.01%
	Case 2	1951	0.02	(1465.13,34.92)	840.03	0.07	(1458.30,43.84)	841.28	841.41	0.02%
		1955	0.02	(1466.84,34.58)	840.85	0.06	(1459.82,43.69)	842.10	842.24	0.02%
		1960	0.02	(1469.00,34.12)	841.87	0.06	(1461.72,43.50)	843.13	843.27	0.02%
		1970	0.02	(1473.39,33.12)	843.91	0.06	(1465.52,43.12)	845.18	845.33	0.02%
	Case 3	2107	0	(1551.45,0)	870.87	0.03	(1517.68,36.89)	872.74	873.29	0.06%
		2200	0	(1581.14,0)	888.68	0.01	(1554.97,29.88)	890.90	891.06	0.02%
1	**Case 1**	14	0.82	(268.31,152.06)	192.22	0.83	(273.82,156.41)	198.13	198.29	0.08%
		70	0.74	(383.76,122.11)	243.52	0.76	(387.92,126.57)	248.69	248.81	0.05%
		300	0.57	(646.38,81.74)	376.87	0.59	(649.35,85.85)	381.12	381.20	0.02%
		800	0.34	(983.76,50.63)	560.63	0.38	(985.87,54.63)	564.30	564.38	0.01%
		1100	0.24	(1136.35,39.53)	645.53	0.28	(1138.03,43.72)	649.05	649.12	0.01%
		1400	0.15	(1270.37,30.16)	720.32	0.20	(1271.44,34.81)	723.75	723.83	0.01%
		1700	0.07	(1392.29,20.93)	787.93	0.12	(1392.14,26.74)	791.33	791.43	0.01%
	Case 2	1951	0.01	(1491.83,8.49)	840.19	0.06	(1486.33,19.76)	843.65	843.89	0.03%
		1955	0.00	(1493.82,7.84)	841.00	0.06	(1487.79,19.64)	844.46	844.71	0.03%
		1960	0	(1503.33,0)	842.00	0.06	(1489.62,19.49)	845.47	845.75	0.03%
	Case 3	1970	0	(1506.65,0)	843.99	0.06	(1493.28,19.18)	847.48	847.97	0.06%
		2107	0	(1551.45,0)	870.87	0.03	(1543.12,14.47)	874.55	874.74	0.02%
		2200	0	(1581.14,0)	888.68	0.01	(1577.73,9.67)	892.43	892.48	0.00%

Particularly, examining Tables 3.6 and 3.7, we see that for all the combinations of parameter values, the absolute value of PCP is considerably lower than 1%, even in cases where CSLs are zero, indicating in that way very accurate approximations when using the H-W expression in the cost function. The same holds for the Log-Normal distribution provided that CV is less than or equal to 1. At this point it is important to mention that absolute values of PCP lower than 1% are observed for all the combinations of A, h and s which are suggested by Zhao et al. (2012). This demonstrates the usefulness, but also the added value of the proposed general algorithm in the relevant inventory literature when the lead-time demand is described by the unimodal distributions under consideration. It is proper to say here that the threshold value of 1% ensuring acceptable approximations when using the H-W approximate expression was established by Gross & Ince (1975).

Except the percentage cost penalty (PCP) an alternative way to test the validity of the H-W expression is to compute the percentage approximation error (PAE) defined as

$$\mathrm{PAE} = \frac{C_{HW}\left(Q^{*}, R^{*}\right) - C_{ex}\left(Q_{ex}^{*}, R_{ex}^{*}\right)}{C_{ex}\left(Q_{ex}^{*}, R_{ex}^{*}\right)} \times 100,$$

which is presented in Table 3.8 using the parameter values of Tables 3.6 and 3.7. We observe that PAE takes on increasingly higher sizes when CV gets larger and the values of the remaining parameters are kept fixed. However, the most important finding is that PAE initially is getting smaller as A increases and at the same time, as a result of raising A, CSL decreases. Of course, PAE starts to increase again from some CSL which can be either very low or zero. This means that very small PAE indicating accurate approximations can be observed even when large stock-out probabilities exist. Therefore, from the results of PCP and PAE we summarize that the Hadley & Whitin’s approximate expression is accurate even for sufficiently small cycle service levels.

Table 3.8 The values of the percentage approximation error (PAE) under Log-Normal and Normal lead-time demands with s=3, h=0.6, D=300 and μ_L=100.

Normal				Log-Normal			
CV	**Case**	**A**	**PAE**	**CV**	**Case**	**A**	**PAE**
0.1	Case 1	20	-0.01%	0.2	Case 1	14	-0.06%
		70	-0.01%			70	-0.04%
		200	0.00%			300	-0.03%
		500	0.00%			800	-0.02%
		1000	0.00%			1100	-0.02%
		1500	-0.01%			1400	-0.02%
		1800	-0.01%			1700	-0.03%
	Case 2	1960	-0.01%		Case 2	1951	-0.03%
		1980	-0.01%			1955	-0.03%
		1990	-0.01%			1960	-0.03%
		2000	-0.01%			1970	-0.03%
		2100	-0.05%			2107	-0.10%
	Case 3	2200	-0.16%		Case 3	2200	-0.18%
0.2	Case 1	20	-0.03%	0.5	Case 1	14	-0.61%
		70	-0.02%			70	-0.39%
		200	-0.02%			300	-0.23%
		500	-0.02%			800	-0.16%
		1000	-0.02%			1100	-0.15%
		1500	-0.02%			1400	-0.14%
		1800	-0.03%			1700	-0.14%
	Case 2	1960	-0.04%		Case 2	1951	-0.15%
		1980	-0.04%			1955	-0.15%
		1990	-0.04%			1960	-0.15%
		2000	-0.04%			1970	-0.15%
	Case 3	2100	-0.11%		Case 3	2107	-0.21%
		2200	-0.19%			2200	-0.25%
0.3	Case 1	20	-0.06%	1	Case 1	14	-2.98%
		70	-0.05%			70	-2.08%
		200	-0.04%			300	-1.12%
		500	-0.04%			800	-0.65%
		1000	-0.04%			1100	-0.54%
		1500	-0.05%			1400	-0.47%
		1800	-0.07%			1700	-0.43%
	Case 2	1960	-0.08%		Case 2	1951	-0.41%
		1980	-0.09%			1955	-0.41%
		1990	-0.10%			1960	-0.41%
	Case 3	2000	-0.10%		Case 3	1970	-0.41%
		2100	-0.17%			2107	-0.42%
		2200	-0.22%			2200	-0.42%

Closing this section, we note that if we set A at lower values than those of Tables 3.6, 3.7 and 3.8 then we would take optimal Q smaller than the optimal R, something which would violate the assumption that at each inventory cycle the order quantity should exceed the lead-time demand. Finally, by increasing CV, we observe that values of A being very close to the limits of Cases 1 and 2 or 2 and 3, are moving to the next Case with the lower CSLs.

3.9 Summary

In the current chapter we considered the continuous review (Q, R) inventory model with backorders and fixed lead-time, when (a) the Hadley-Whitin (H-W) expression $(Q/2 + \text{safety stock})$ is used to evaluate the expected annual inventory carrying cost, and (b) the cost per unit backordered is used to calculate the annual expected shortage cost. For this model we showed that, given a non-negative reorder point the convexity of the H-W cost function depends on the monotony of the first derivative of the lead-time probability density function. Next, selecting the class of unimodal lead-time demand distributions for which the probability density function vanishes at $R = 0$ and when $R \to \infty$, we derive general conditions for determining the optimal solution in terms of Q and R values ensuring the minimum of H-W cost function. These general conditions distinguish the following three mutually exclusive events:

Case 1: There is a unique optimal solution which is obtained after solving the equations resulted from the first-order conditions minimizing the H-W cost function.

Case 2: The minimum of the H-W cost function is attained after comparing the cost at $R = 0$ with the “local” minimum cost at the optimal solution obtained in case 1.

Case 3: The minimum of the H-W cost function occurs at $R = 0$.

The three cases with the corresponding conditions were integrated to a general algorithm for which its added value in the relevant literature is illustrated through some new comparative results when the lead-time demand is described by the Normal or Log-Normal distribution. In particular, these comparative results refer to the target inventory measures which are taken minimizing first the H-W cost function and then the corresponding exact cost function. The latter one is obtained by replacing in the cost function the H-W expression $(Q/2 + \text{safety stock})$ with the exact expression of the expected on-hand inventory at any point in time. Contrary to what is believed about the validity of the H-W expression, we show that valid approximations using this expression occur even when the cycle service level (CSL) is zero, provided that the coefficient of variation is low, preferably below 1.

Appendix

Proof 3.1:

Taking the first derivative of (3.3) we obtain

$$\frac{dQ}{dR}=\frac{1}{2}\left[2\frac{A}{h}D+2\frac{s}{h}D\cdot S(R)\right]^{-1/2}\cdot 2\frac{s}{h}D\left\{-\left[1-F(R)\right]\right\}=$$

$$=-\frac{1}{Q}\frac{s}{h}D\left[1-F(R)\right].$$

Then,

$$\frac{d^2Q}{dR^2}=\frac{d}{dR}\left\{-Q^{-1}\frac{s}{h}D\left[1-F(R)\right]\right\}=\frac{s}{h}D\frac{d}{dR}\left\{-Q^{-1}\left[1-F(R)\right]\right\}=$$

$$=-\frac{s}{h}D\left\{\left[1-F(R)\right]\frac{dQ^{-1}}{dQ}\frac{dQ}{dR}+Q^{-1}\frac{d}{dR}\left[1-F(R)\right]\right\}=$$

$$=-\frac{s}{h}D\left\{\left[1-F(R)\right](-1)Q^{-2}\cdot\left(-Q^{-1}\frac{s}{h}D\left[1-F(R)\right]\right)+Q^{-1}\left(-f(R)\right)\right\}=$$

$$=-\frac{s}{h}D\left\{\frac{1}{Q^3}\frac{s}{h}D\left[1-F(R)\right]^2-\frac{1}{Q}f(R)\right\}=$$

$$=-\frac{s\cdot D}{h\cdot Q}\left\{\frac{s\cdot D}{h}\left(\frac{1-F(R)}{Q}\right)^2-f(R)\right\}.$$

Proof 3.2:

$$C_1(R)=\frac{A\cdot D}{Q(R)}+\frac{s\cdot D}{Q(R)}S(R)+h\frac{Q(R)}{2}+h(R-\mu_L)=$$

$$=\frac{h}{2\cdot Q(R)}\left\{2\frac{A}{h}D+2\frac{s}{h}\cdot D\cdot S(R)\right\}+h\frac{Q(R)}{2}+h(R-\mu_L)=$$

$$=\frac{h\left[Q(R)\right]^2}{2\cdot Q(R)}+h\frac{Q(R)}{2}+h(R-\mu_L)=$$

$$=h\frac{Q(R)}{2}+h\frac{Q(R)}{2}+h(R-\mu_L),$$

and finally

$$C_1(R)=h\left\{Q(R)+R-\mu_L\right\}.$$

Proof 3.3:

Proof of Lemma 3.1

Taking the first derivative of (3.7), we obtain $C_1'(R) = h\left\{\frac{dQ}{dR} + 1\right\}$. But

$\frac{dS(R)}{dR} = -[1 - F(R)]$ (e.g. Hadley & Whitin, 1963, pp. 167),

and $\frac{dQ}{dR} = -\frac{s}{h} DQ^{-1}\left[1 - F(R)\right]$.

Thus

$$C_1'(R) = h\frac{dQ(R)}{R} + h = h\left\{\frac{dQ(R)}{R} + 1\right\} = h\left\{-\frac{s}{h} D\left[Q(R)\right]^{-1}\left[1 - F(R)\right] + 1\right\} = -h \cdot V_1(R).$$

Proof 3.4:

Proof of Lemma 3.2

From Lemma 3.1, we take

$$C_1''(R) = -h \cdot V_1'(R),$$

where $V_1'(R) = \frac{s}{h} D\left\{\left[1 - F(R)\right]\frac{dQ^{-1}}{dQ}\frac{dQ}{dR} + Q^{-1}\frac{d}{dR}\left[1 - F(R)\right]\right\}$.

By using, also, from Proof 3.1, $\frac{dQ}{dR} = -\frac{s}{h} DQ^{-1}\left[1 - F(R)\right]$,

$$V_1'(R) = \frac{s}{h} D\left\{\left[1 - F(R)\right](-1)\left[Q(R)\right]^{-2}\left[-\frac{s}{h} D\left[Q(R)\right]^{-1}\left[1 - F(R)\right]\right] + \left[Q(R)\right]^{-1}\left(-f(R)\right)\right\} =$$

$$= \frac{s}{h} D\left\{\frac{s}{h} D[Q(R)]^{-3}[1 - F(R)]^2 - f(R)[Q(R)]^{-1}\right\} =$$

$$= \frac{s}{h} D[Q(R)]^{-3}\left\{\frac{s}{h} D[1 - F(R)]^2 - f(R)[Q(R)]^2\right\} =$$

$$= \frac{s}{h} D[Q(R)]^{-3}\left\{\frac{s}{h} D[1 - F(R)]^2 - f(R)\left[2\frac{A}{h} D + 2\frac{s}{h} D \cdot S(R)\right]\right\} =$$

$$= \frac{s}{h} D[Q(R)]^{-3}\left\{\frac{s}{h} D[1 - F(R)]^2 - \frac{s}{h} D\left[2\left(\frac{A}{h} + S(R)\right) f(R)\right]\right\} =$$

$$= \left(\frac{s}{h} D\right)^2 [Q(R)]^{-3}\left\{[1 - F(R)]^2 - \left[2\left(\frac{A}{h} + S(R)\right) f(R)\right]\right\}.$$

But $g_1(\mathrm{R})=2\left[\frac{\mathrm{A}}{\mathrm{s}}+\mathrm{S}(\mathrm{R})\right]f(\mathrm{R})-\left[1-F(\mathrm{R})\right]^2$ and hence

$$\mathrm{V}_1'(\mathrm{R})=-\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\right)^2\left[\mathrm{Q}(\mathrm{R})\right]^{-3}\left\{\left[2\left(\frac{\mathrm{A}}{\mathrm{h}}+\mathrm{S}(\mathrm{R})\right)f(\mathrm{R})\right]-\left[1-F(\mathrm{R})\right]^2\right\}=$$

$$=-\left(\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\right)^2\left[\mathrm{Q}(\mathrm{R})\right]^{-3}\cdot g_1(\mathrm{R}).$$

Proof 3.5:

Proof of Lemma 3.3

Consider the case of unimodal distributions with the mode at $\mathrm{R}=\mathrm{R}_\mathrm{m}$ and $f(\mathrm{R})$ vanishing at the extreme values of R, namely, $f(0)=0$ and $\lim\limits_{\mathrm{R}\to\infty}f(\mathrm{R})=0$. For $\mathrm{R}<\mathrm{R}_\mathrm{m}$ it holds $\mathrm{d}f(\mathrm{R})/\mathrm{dR}>0$ and thus $g_1'(\mathrm{R})>0$, while for $\mathrm{R}>\mathrm{R}_\mathrm{m}$ we take $\mathrm{d}f(\mathrm{R})/\mathrm{dR}<0$ and $g_1'(\mathrm{R})<0$. When $\mathrm{R}=0$, the backorders size equals to the lead-time demand and thus it holds $\mathrm{S}(0)=\mu_\mathrm{L}$. On the contrary, when $\mathrm{R}\to\infty$ then the backorders size tends to zero and hence $\lim\limits_{\mathrm{R}\to\infty}\mathrm{S}(\mathrm{R})=0$. So, at the extreme values of R we take

$$g_1(0)=2\left[\frac{\mathrm{A}}{\mathrm{s}}+\mathrm{S}(0)\right]f(0)-\left[1-F(0)\right]^2=2\left[\frac{\mathrm{A}}{\mathrm{s}}+\mu_\mathrm{L}\right]\times 0-(1-0)^2=-1,$$

$$\lim_{\mathrm{R}\to\infty}g_1(\mathrm{R})=2\left[\frac{\mathrm{A}}{\mathrm{s}}+\lim_{\mathrm{R}\to\infty}\mathrm{S}(\mathrm{R})\right]\lim_{\mathrm{R}\to\infty}f(\mathrm{R})-\left[1-F(\infty)\right]^2=2\frac{\mathrm{A}}{\mathrm{s}}\times 0-(1-1)^2=0.$$

Summarizing, therefore, for $0\le\mathrm{R}<\mathrm{R}_\mathrm{m}$ the function $g_1(\mathrm{R})$ is strictly increasing taking values on the interval $\left[-1\,,\,g_1(\mathrm{R}_\mathrm{m})\right)$, and for $\mathrm{R}_\mathrm{m}<\mathrm{R}<\infty$ the function $g_1(\mathrm{R})$ is strictly decreasing with values on $\left(g_1(\mathrm{R}_\mathrm{m})\,,\,0\right)$. Hence the continuous function $g_1(\mathrm{R})$ has its unique maximum at the mode of the lead-time demand distribution R_m.

Proof 3.6:

Proof of Lemma 3.4

At the extreme values of R we have

$$\mathrm{Q}_\mathrm{o}=\sqrt{2\frac{\mathrm{A}}{\mathrm{h}}\mathrm{D}+2\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\mathrm{S}(0)}=\sqrt{2\frac{\mathrm{A}}{\mathrm{h}}\mathrm{D}+2\frac{\mathrm{s}}{\mathrm{h}}\mathrm{D}\cdot\mu_\mathrm{L}}\ ,$$

and

$$\lim_{R\to\infty} Q = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D\cdot\lim_{R\to\infty} S(R)} = \sqrt{2\frac{A}{h}D} = Q_W,$$

where Q_W is the known Wilson economic order quantity. Further

$$V_1(0) = \frac{s}{h}DQ_o^{-1}\left[1-F(0)\right]-1 = \frac{\frac{s}{h}D}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L}}-1,$$

and

$$\lim_{R\to\infty} V_1(R) = \frac{s}{h}D\left[\lim_{R\to\infty} Q\right]^{-1}\left[1-F(\infty)\right]-1 = \frac{\frac{s}{h}D}{Q_W}(1-1)-1=-1.$$

Using also (3.10) and the result of Lemma 3.3 we conclude the following:

(i) For $0 \le R < R_o$ it holds $g_1(R)<0$ which in turn gives $V_1'(R)>0$ leading to a strictly increasing $V_1(R)$ with range $\left[V_1(0)\ ,R_o\right)$,

(ii) For $R_o < R < \infty$, we have $g_1(R)>0$ and hence $V_1'(R)<0$ leading to a strictly decreasing $V_1(R)$ with range $(R_o\ ,-1)$,

(iii) $g_1(R_o)=0$ and $V_1'(R_o)=0$.

Hence, the continuous function $V_1(R)$ has its unique maximum at $R = R_o$.

Proof 3.7:

Proof of Proposition 3.1

CASE 1: $V_1(0)\ge 0$

If $V_1(0)\ge 0$ then $V_1(R_o)$ is always positive and the graph of $V_1(R)$ will intersect the horizontal R-axis, apart from the extreme value zero, at a single point R_1 with $0<R_o<R_1<+\infty$. For $R<R_1$ it holds $V_1(R)>0$ while for $R>R_1$ we shall have $V_1(R)<0$. As it holds $C_1'(R)=-h\cdot V_1(R)$, we deduce that when R increases up to R_1 the cost function $C_1(R)$ is strictly decreasing taking values on the interval $[C_1(0), C_1(R_1))$. If R continues to increase taking values greater than R_1 then $C_1(R)$ becomes strictly increasing with values on the interval $(C_1(R_1),+\infty)$. Thus, $C_1(R)$ has a unique local minimum attained at $R=R_1$.

CASE 2: $V_1(0)<0$ and $V_1(R_o)>0$

In this case the graph of $V_1(R)$ crosses the R-axis at two points, say R_2 and R_3. As the maximum of $V_1(R)$ is attained at R_o, it also holds that $0<R_2<R_o<R_3<+\infty$. For any R smaller than R_2 or greater than R_3, $V_1(R)$ is negative, while for $R_2<R<R_3$, $V_1(R)$ is positive. Again from $C_1'(R)=-h\cdot V_1(R)$ it follows that $C_1(R)$ (i) is strictly increasing on the interval $[0,R_2)$, (ii) becomes strictly decreasing on (R_2,R_3) and (iii) is again strictly increasing on $(R_3,+\infty)$. Therefore, $C_1(R)$ has a local maximum at $R=R_2$ and a local minimum at $R=R_3$. But as the extreme value $R=0$ is located left of R_3, the cost at $R=0$, which is given by

$$C_1(0)=h\left\{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\mu_L}-\mu_L\right\},$$

appears to be a second local minimum. Hence, a comparison between the two local minima $C_1(0)$ and $C_1(R_3)$ should be carried out and the actual minimum cost will be smallest of the two.

CASE 3: $V_1(0)<0$ and $V_1(R_o)\le 0$

If $V_1(R_o)<0$ then $C_1(R)$ is strictly increasing function of R on the interval $[0,+\infty)$. Additionally when $V_1(R_o)=0$, the function $C_1(R)$ becomes non-decreasing. But in both occasions, as unique local minimum we should consider the value of the cost function $C_1(R)$ at the smallest permissible R, namely, at $R=0$.

Proof 3.8:

(1) Log-Normal

Based on the three cases the degeneracy problem happens when $V_{LN}(r_o)\le 0$. So, solving the inequality with respect to one of the cost parameters keeping the other two fixed we obtain:

(A) threshold value for the shortage cost

$$V_{LN}\left(r_o\right) \le 0 \Leftrightarrow \frac{\frac{s}{h} D\Phi\left(-r_o\right)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right)}} - 1 \le 0 \Leftrightarrow$$

$$\frac{s^2}{h^2} D^2 \left[\Phi\left(-r_o\right)\right]^2 \le 2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right) \Leftrightarrow$$

$$\frac{s^2}{h^2} D^2 \left[\Phi\left(-r_o\right)\right]^2 - 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right) - 2\frac{A}{h}D \le 0 \Leftrightarrow$$

$$s_{1,2} = \frac{h}{D} \frac{S_{LN}\left(r_o\right) \pm S_{LN}\left(r_o\right)\sqrt{1 + 2\frac{A}{h}D\frac{\left[\Phi\left(-r_o\right)\right]^2}{\left[S_{LN}\left(r_o\right)\right]^2}}}{\left[\Phi\left(-r_o\right)\right]^2} .$$

Since $S_{LN}\left(r_o\right)\sqrt{1 + 2\frac{A}{h}D\frac{\left[\Phi\left(-r_o\right)\right]^2}{\left[S_{LN}\left(r_o\right)\right]^2}} > S_{LN}\left(r_o\right)$ in order to have positive value for the shortage cost we take

$$s = \frac{h}{D} \frac{S_{LN}\left(r_o\right) + S_{LN}\left(r_o\right)\sqrt{1 + 2\frac{A}{h}D\frac{\left[\Phi\left(-r_o\right)\right]^2}{\left[S_{LN}\left(r_o\right)\right]^2}}}{\left[\Phi\left(-r_o\right)\right]^2} .$$

(B) threshold value for the ordering cost

$$V_{LN}\left(r_o\right) \le 0 \Leftrightarrow \frac{\frac{s}{h} D\Phi\left(-r_o\right)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right)}} - 1 \le 0 \Leftrightarrow$$

$$\frac{s^2}{h^2} D^2 \left[\Phi\left(-r_o\right)\right]^2 \le 2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right) \Leftrightarrow$$

$$A \geq \frac{h}{2D}\left[\frac{s^2}{h^2}D^2\left[\Phi\left(-r_o\right)\right]^2 - 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right)\right] \Leftrightarrow$$

$$A \geq s\left[\frac{s}{2h}D\left[\Phi\left(-r_o\right)\right]^2 - S_{LN}\left(r_o\right)\right].$$

(C) threshold value for the holding cost

$$V_{LN}\left(r_o\right) \leq 0 \Leftrightarrow \frac{\frac{s}{h}D\Phi\left(-r_o\right)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right)}} - 1 \leq 0 \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2\left[\Phi\left(-r_o\right)\right]^2 \leq 2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{LN}\left(r_o\right) \Leftrightarrow$$

$$\frac{s^2}{h}D^2\left[\Phi\left(-r_o\right)\right]^2 - 2 \cdot A \cdot D - 2 \cdot s \cdot D \cdot S_{LN}\left(r_o\right) \leq 0 \Leftrightarrow$$

$$h \geq \frac{s^2 D^2\left[\Phi\left(-r_o\right)\right]^2}{2 \cdot A \cdot D + 2 \cdot s \cdot D \cdot S_{LN}\left(r_o\right)}.$$

(2) Normal

Based on the three cases the degeneracy problem happens when $V_{NM}\left(z_o\right) \leq 0$. So, solving the inequality with respect to one of the cost parameters keeping the other two fixed we obtain:

(A) threshold value for the shortage cost

$$V_{NM}\left(z_o\right) \leq 0 \Leftrightarrow \frac{\frac{s}{h}D\Phi\left(-z_o\right)}{\sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{NM}\left(z_o\right)}} - 1 \leq 0 \Leftrightarrow$$

$$\frac{s}{h}D\Phi\left(-z_o\right) \leq \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D \cdot S_{NM}\left(z_o\right)} \Leftrightarrow$$

$$\frac{s^2}{h^2}D^2\left[\Phi\left(-z_o\right)\right]^2-2\frac{s}{h}D\cdot S_{NM}\left(z_o\right)-2\frac{A}{h}D\le 0\Leftrightarrow$$

$$s_{1,2}=\frac{h}{D}\frac{S_{NM}\left(z_o\right)\pm S_{NM}\left(z_o\right)\sqrt{1+2\frac{A}{h}D\frac{\left[\Phi\left(-z_o\right)\right]^2}{\left[S_{NM}\left(z_o\right)\right]^2}}}{\left[\Phi\left(-z_o\right)\right]^2}.$$

Since $S_{NM}\left(z_o\right)\sqrt{1+2\frac{A}{h}D\frac{\left[\Phi\left(-z_o\right)\right]^2}{\left[S_{NM}\left(z_o\right)\right]^2}}>S_{NM}\left(z_o\right)$ in order to have positive value for the shortage cost we take

$$s=\frac{h}{D}\frac{S_{NM}\left(z_o\right)+S_{NM}\left(z_o\right)\sqrt{1+2\frac{A}{h}D\frac{\left[\Phi\left(-z_o\right)\right]^2}{\left[S_{NM}\left(z_o\right)\right]^2}}}{\left[\Phi\left(-z_o\right)\right]^2}.$$

(B) threshold value for the ordering cost

$$V_{NM}\left(z_o\right)\le 0\Leftrightarrow\frac{\frac{s}{h}D\Phi\left(-z_o\right)}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot S_{NM}\left(z_o\right)}}-1\le 0\Leftrightarrow$$

$$\frac{s^2}{h^2}D^2\left[\Phi\left(-z_o\right)\right]^2\le 2\frac{A}{h}D+2\frac{s}{h}D\cdot S_{NM}\left(z_o\right)\Leftrightarrow$$

$$A\ge\frac{h}{2D}\left[\frac{s^2}{h^2}D^2\left[\Phi\left(-z_o\right)\right]^2-2\frac{s}{h}D\cdot S_{NM}\left(z_o\right)\right]\Leftrightarrow$$

$$A\ge s\left[\frac{s}{2h}D\left[\Phi\left(-z_o\right)\right]^2-S_{NM}\left(z_o\right)\right].$$

(C) threshold value for the holding cost

$$V_{NM}\left(z_o\right)\le 0\Leftrightarrow\frac{\frac{s}{h}D\Phi\left(-z_o\right)}{\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot S_{NM}\left(z_o\right)}}-1\le 0\Leftrightarrow$$

$$\frac{s^2}{h^2}D^2\left[\Phi\left(-z_o\right)\right]^2 \le 2\frac{A}{h}D + 2\frac{s}{h}D\cdot S_{NM}\left(z_o\right) \Leftrightarrow$$

$$\frac{s^2}{h}D^2\left[\Phi\left(-z_o\right)\right]^2 - 2\cdot A\cdot D - 2\cdot s\cdot D\cdot S_{NM}\left(z_o\right) \le 0 \Leftrightarrow$$

$$h \ge \frac{s^2D^2\left[\Phi\left(-z_o\right)\right]^2}{2\cdot A\cdot D + 2\cdot s\cdot D\cdot S_{NM}\left(z_o\right)}.$$

Chapter 4

Estimation in (Q,R) inventory systems with uncorrelated demand

4.1 Introduction

In inventory management the knowledge of demand distribution is necessary in order to formulate optimal inventory policies. However, in real life conditions neither the process of generating demand data nor the values of demand parameters are known. To resolve this problem, a number of studies have been conducted, which are available in the relevant literature offering alternative estimation processes. However, a choice between them depends upon the type and length of historical data concerning the demand per period.

The first classification of these estimation processes reflects the situation in which either demand is fully observed or demand occurring when the stock level drops to zero is lost and thus it cannot be observed. Specifically, for the latter case, it is extremely difficult to measure that part of demand which is not met, especially when sales are conducted in an impersonal environment. In such cases, when the available sample constitutes of sales data, underestimation of the real demand exists at periods in the sample where stockouts are occurred. Under such circumstances, demand should be modelled through censored or truncated distributions in order the estimation process to take into account the unobserved lost part of demand. References on this area include the works of Nahmias (1994), Lau & Lau (1996), Ernst & Kamrad (2006), and Halkos & Kevork (2011).

On the other hand, when historical demand data are available, it would not make any difference in the analysis if stockouts do or do not occur in the periods included in the sample. Under such circumstances, Rossi et al. (2014) classify the estimation procedures in accordance with the knowledge of the form of demand distribution. When the form of demand distribution is known but its parameters must be estimated from historical data, parametric estimation processes are appropriate. For this case, Berk et al. (2007) recognize two general stochastic approaches for which random variables follow known distributions with unknown parameters: the Bayesian and the Frequentist. By using either collateral data or subjective judgment, the Bayesian approach, selects a prior distribution for the demand distribution parameters and then using this prior distribution, the posterior distribution is derived. After that, the posterior distribution is continuously updated as the sample is enriched with new data for demand. Finally, the optimum value of the objective function is estimated by using the posterior distribution. Early works in this area constitute the papers of Scarf (1959b), Iglehart (1964) and Azoury (1985). When following this approach, since the unknown parameters must be expressed as a prior distribution of the demand, if available supporting information does not exist, Hill (1999) suggested the use of uninformative priors, which, unfortunately, introduce a strong bias, especially under limited available data, at the stage of performing Bayesian updating. On the contrary, for the Frequentist approach, point estimate for the unknown parameters of the known parametric demand distribution are obtained using historical data (see e.g. Kevork, 2010; Halkos & Kevork, 2013; Rossi et al., 2014). Specially, Halkos & Kevork (2013) distinguish for the newsvendor problem the following three estimation policies: the direct estimation policy (DEP), the unbiased percentile estimation policy (UPEP) and the Hayes (1969) estimation policy (HEP).

For the case in which the class of demand distribution cannot be identified, there are two alternatives estimation processes for determining optimal inventory policies. The first alternative is to follow a nonparametric approach that includes the sampling-based policy or the use of order statistics and bootstrapping techniques. In the sampling-based policy, demand is modeled by the empirical distribution function of historical demand data. The second alternative is followed when partial information about the demand distribution (mean, variance, symmetry, unimodality, etc.) is available. In this case, using the so-called distribution free procedure, the optimal inventory policy is determined by maximizing (minimizing) the worst case expected profit (cost) considering all distributions with the same values of the available moments. For the two alternatives of this category of estimation processes, the relevant literature review can be found in Liyanage & Shanthikumar (2005), Janssen et al. (2009), and Akcay et al. (2011).

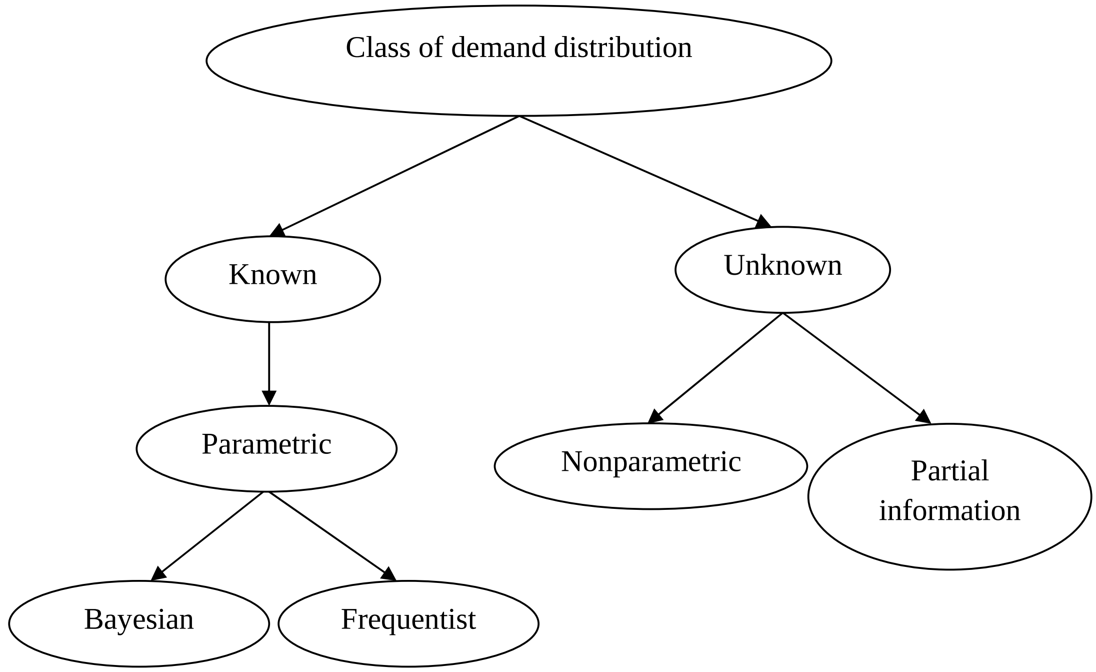


Figure 4.1 Estimation processes.

Under the above consideration, the research of this thesis is classified in the area of the Frequentist approach. More specific, between the alternative estimation processes we consider the direct estimation policy which relates to the replacement of the parameters of demand distribution with their estimates in the theoretical formulas which determine the three target inventory measures. By considering fixed lead-time and normally distributed lead-time demand, as well as, assuming that demand distribution parameters are unknown, in the current chapter we address for the first time the issue of estimating the Hadley & Whitin's (1963) cost function. In particular, in Section 4.2 we give the necessary theoretical background and present the model assumptions required for the analysis which follows. In Section 4.3, by considering Maximum Likelihood estimators (ML) for the demand parameters we explain how the estimation policy constitutes the basis for constructing estimation formulae for the optimal reorder point and the optimal order quantity, which lead to developing asymptotic confidence intervals for the minimum of the Hadley & Whitin's cost function. In Section 4.4, we test the validity of the asymptotic confidence intervals through Monte-Carlo simulations. Finally, the last section concludes chapter 4 summarizing the most important findings.

4.2 Model assumptions and terminology

For the continuous review (Q,R) inventory model with stochastic demand, fixed lead-time, and backorders, the majority of studies start with a certain form of the lead-time demand without making any specification about the length of the lead-time. Specifically, they are based on the assumption that the lead-time demand follows a probability distribution (discrete or continuous) with known demand parameters and consider that the procedure for determining the expression for the total cost of the inventory model is usually carried out for one year. Also, due to the use of discrete time when in fact this inventory model should be operated under continuous time, for ensuring the validity of results, it is also assumed that any undershoot of the reorder point is negligible compared to the magnitude of the total lead-time demand (e.g. Silver et al., 1998).

Contrary to the above procedure, in the current chapter the analysis starts by defining a standard discrete time unit, t, (e.g. day, month, etc), and then both the lead-time and the reference period where the total cost is defined will be considered as multiples of this standard time unit. In particular, we assume that the lead-time consists of L standard time units while the reference period of β time units. For the size of undershoot of the reorder point we follow the Silver et al.'s (1998) assumption.

The second assumption of our analysis is related to the process of determining the demand. We assume that demand is formed independently between the standard time units while for each time unit it follows the Normal distribution with the same mean, μ_t, and the same variance, σ_t^2. As a result, the expected value and the variance of the demand in both the lead-time and the reference period will be multiples of μ_t and σ_t^2 respectively.

The last assumption deals with the cycle service level, namely the probability the lead-time demand not to exceed the reorder point. In most of the studies in continuous review models, this probability is considered as a decision variable and is determined by the optimal values of Q and R which are obtained by solving the first order conditions of the minimization process of the total cost function in the reference period. Opposite to this practice, in the current chapter we consider that this probability is constant and is initially defined by the Management. The reason for this assumption is two-fold. The first is coming from practice as inventory managers do not like the cycle service level to be determined by a mathematical algorithm but, on the contrary, they wish to control the value of this parameter in accordance with their individual preferences. The second reason is technical. The simultaneous consideration of both estimators for Q and R, where each one is a function of the other, makes very difficult the task to study the asymptotic distribution of the estimator for the minimum total cost in the reference period, when both Q and R are treated as random variables. Assuming, therefore, the cycle service level as constant, the expression for reorder point is given by (e.g. Urban, 2000)

reorder point	=	expected value of the lead-time demand	+	safety stock

which allows, asymptotically at least, the study of the statistical properties of the minimum cost estimator.

Let X be a continuous non-negative random variable representing the demand in the lead-time with mean $\mu_L = L \cdot \mu_t$ and variance $\sigma_L^2 = L \cdot \sigma_t^2$, where μ_t and σ_t^2 are the mean and variance respectively of the demand size D_t occurred during lead-time t. Given now the aforementioned assumptions, the total cost function in the reference period with Normal distributed lead-time demand, $X = \sum_{t=1}^{L} D_t \sim N\left(\mu_L\ , \sigma_L^2\right)$, (see chapter 3 for more details) is given by

$$C_{HW}\left(Q,R\right) = A\frac{D}{Q} + h\left(\frac{Q}{2} + z_P \cdot \sigma_L\right) + s \cdot \frac{D}{Q} \cdot S\left(R\right), \tag{4.1}$$

where $D = \beta \cdot \mu_t$ and $S\left(R\right) = \sigma_L \cdot \psi\left(z_P\right)$, with $\psi\left(z_P\right) = \varphi\left(z_P\right) - z_P \cdot \Phi\left(-z_P\right)$ and P to be the fixed cycle service level defined as $P = \Pr\left(X \leq R\right) = \Pr\left(Z \leq \frac{R - \mu_L}{\sigma_L}\right) = \Pr\left(Z \leq z_P\right)$.

Differentiating with respect to Q and equating the first derivative to zero we take the optimal order quantity minimizing the total cost

$$Q^* = \sqrt{2 \cdot \frac{A}{h} \cdot D + 2 \cdot \frac{s}{h} \cdot D \cdot S\left(R\right)}. \tag{4.2}$$

Then, substituting Q^* for Q in (4.1), the minimum total cost in the reference period is equal to

$$C_{HW}^*\left(z_P\right) = h \cdot \sqrt{2 \cdot \frac{A}{h} \cdot D + 2 \cdot \frac{s}{h} \cdot D \cdot \sigma_L \cdot \psi\left(z_P\right)} + h \cdot B\left(P\right), \tag{4.3}$$

where $B\left(P\right) = R - \mu_L = z_P \cdot \sigma_L$.

4.3 Estimation policy for the minimum cost

Suppose that the demand size, D_t, is available for a sample of n consecutive time units $(t = 1,2,...,n)$. Considering Maximum Likelihood estimators (ML) for the expected demand per unit time, $\hat{\mu}_t = \sum_{t=1}^{n} D_t \big/ n$ and for the variance, $\hat{\sigma}_t^2 = \sum_{t=1}^{n} (D_t - \hat{\mu}_t)^2 \big/ n$ and taking the cycle service level we explain below how the estimation policy for the optimal reorder point and the optimal order quantity leads us to develop asymptotic confidence intervals for the minimum cost of the reference period.

Direct replacement of ML estimators (DEP)

The direct estimation policy (e.g. Janssen et al., 2009; Kevork, 2010) relates to the use of the estimators $\hat{\mu}_t$ and $\hat{\sigma}_t$ instead of the corresponding population parameters on the sizes that determine the three target inventory measures, namely the optimal reorder point, the optimal order quantity and the minimum cost. Thus, with the direct replacement of ML estimators, $\hat{\mu}_t$ and $\hat{\sigma}_t^2$, in the theoretical formulas which give the expected lead-time demand, μ_L, the standard deviation of the lead-time demand, σ_L, the expected demand in the reference period, D, and the expected size of backorders in each inventory cycle, $S(R)$, we take respectively the estimators for the optimal reorder point

$$\hat{R} = \hat{\mu}_L + z_P \cdot \hat{\sigma}_L , \tag{4.4}$$

the optimal order quantity

$$\hat{Q}^* = \sqrt{2 \cdot \frac{A}{h} \cdot \hat{D} + 2 \cdot \frac{s}{h} \cdot \hat{D} \cdot \hat{S}(R)} , \tag{4.5}$$

and the minimum cost of the reference period

$$\hat{C}_{HW}^* = f(\hat{\mu}_t, \hat{\sigma}_t) = h\left[\hat{Q}^* + z_P \cdot \hat{\sigma}_L\right], \tag{4.6}$$

where $\hat{\mu}_L = L \cdot \hat{\mu}_t$, $\hat{\sigma}_L = \sqrt{L} \cdot \hat{\sigma}_t$, $\hat{D} = \beta \cdot \hat{\mu}_t$, $\hat{S}(R) = \hat{\sigma}_L \cdot \psi(z_P)$, as for the last equation it holds

$z_P = \left(\hat{R} - \hat{\mu}_L\right)/\hat{\sigma}_L$. Then, in Proposition 4.1 which follows we develop through the application of the bivariate Delta method the asymptotic distribution of the estimator $\hat{C}^*_{HW}$.

Proposition 4.1: If $D_1, D_2, ..., D_n$ are i.i.d. random variables with $D_t \sim N\left(\mu_t, \sigma_t^2\right)$, $t = 1,2,...,n$, then the statistic $\sqrt{n}\left(\hat{C}^*_{HW} - C^*_{HW}\right)$ is asymptotically Normal with mean zero and variance

$$\mathbf{M}' \cdot \mathbf{\Sigma} \cdot \mathbf{M} = \left\{\beta \cdot \sigma_t \left[\frac{A + s \cdot S(R)}{Q^*}\right]\right\}^2 + \frac{1}{2}\left\{\sigma_L \left[\frac{s \cdot D \cdot \psi(z_P)}{Q^*} + h \cdot z_P\right]\right\}^2$$

where $\mathbf{M}' = \left[\left.\frac{\partial f}{\partial \hat{\mu}_t}\right|_{\substack{\hat{\mu}_t = \mu_t \\ \hat{\sigma}_t = \sigma_t}} \quad \left.\frac{\partial f}{\partial \hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t = \mu_t \\ \hat{\sigma}_t = \sigma_t}}\right]$ and $\mathbf{\Sigma} = \begin{bmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_t^2/2 \end{bmatrix}$.

Proof 4.1: See in the Appendix at the end of chapter 4.

It is easily deduced from Proposition 4.1 that, for n sufficiently large, the $(1-\alpha)\cdot 100\%$ confidence interval for the minimum cost when the inventory system operates for a reference period consisting of β time units will be given from

$$\hat{C}^*_{HW} - z_{1-\alpha/2}\sqrt{\frac{\mathbf{M}' \cdot \mathbf{\Sigma} \cdot \mathbf{M}}{n}} \le C^*_{HW} \le \hat{C}^*_{HW} + z_{1-\alpha/2}\sqrt{\frac{\mathbf{M}' \cdot \mathbf{\Sigma} \cdot \mathbf{M}}{n}}. \tag{4.7}$$

4.4 Validity of asymptotic confidence intervals in finite samples

To study in finite samples the performance of confidence intervals for C^*_{HW} given in (4.7), 10.000 replications of maximum size 500 observations were generated from the Normal distribution $N\left(300, 60^2\right)$. The required sequences of random numbers were generated using the method of Box and Muller (Law, 2007). More details about the random number generator and its validity can be found in Kevork (1990).

Firstly, using each replication, estimates for the demand parameters μ_t and σ_t^2 were taken at different sample sizes. Then, for each sample size n, corresponding estimates for the optimal order quantity, optimal reorder point and minimum cost were computed for different values of the parameters which are given in Table 4.1, using (4.4)-(4.6) for DEP. Finally, for different combinations of sample size n, and cost parameters a set of 10.000 different confidence intervals for the minimum cost, C^*_{HW}, were computed using the asymptotic form (4.7).

Recall from Chapter 3 that by using the cost parameters A, h and s which are suggested by Zhao et al. (2012) the Hadley & Whitin's cost function approximates accurately the exact cost function even for sufficiently large stockout probabilities. Given this valuable finding, in the current Chapter we extent our analysis by examining the validity of the Hadley & Whitin's cost function under the same cost parameters values "*when unknown demand parameters exist*".

Table 4.1 Parameter combinations.

The fixed cycle service level P %	99.9, 99, 95, 90, 80, 60, 40, 20, 10 and 5
The reference period β	200, 1000 and 9000
The holding cost per unit per time h	0.6
The shortage cost per unit backordered s	3, 6 and 9
The ordering cost A	70, 100 and 500
The lead-time L	2, 5 and 10
The demand D_t	Normal distribution with μ_t=300 and σ_t=60

Having available 10.000 different confidence intervals for each combination of sample size, s, A, h and L, we compute two types of coverage (COVs) which are the percentages of the 10.000 confidence intervals containing either the true minimum value of the Hadley & Whitin's cost function, namely C^*_{HW} or the true minimum value of the exact cost function, namely

$$C^*_{ex} = h \cdot \sqrt{2 \cdot \frac{A}{h} \cdot D + 2 \cdot \frac{s}{h} \cdot D \cdot S(R) + \Theta(R)} + h \cdot B(P), \qquad (4.8)$$

where $\Theta(R) = \sigma_L^2 \cdot \Phi(-z_P) - (R - \mu_L) \cdot S(R)$ (see Lau et al., 2002b).

(**Proof 4.2:** See in the Appendix at the end of the chapter)

In Tables 4.2, 4.3 and 4.4, taking the cost parameters $h = 0.6$ and $s = 9$, we give for the direct estimation policy the values for COVs at 95% nominal confidence level under different values of β, P, L, A and n. We observe that for all the combinations of parameter values acceptable COVs greater than 90% always exist. According to the cycle service level we find out that when stockout probabilities increase then COVs are getting marginally smaller. For example, with $L = 2$, $\beta = 9000$ and sample size equal to 25 observations, COVs for H-W are 93% for P=0.999 and 92% for P=0.80. However, the most important finding is that similar results exist for the two coverages. In particular, this means that the estimated confidence interval of the Hadley-Whitin's cost function includes the true minimum value of the exact cost function, confirming the previous results of chapter 3 where we found out that small deviations exist between the two cost functions.

Table 4.2 Coverage of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $A = 70$, $h = 0.6$, $s = 9$, $L = 2$ and $D_t \sim N(300, 60^2)$.

		Hadley & Whitin					Exact				
	CSL	n=25	n=50	n=100	n=300	n=500	n=25	n=50	n=100	n=300	n=500
β=200	0.999	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.9	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
β=1000	0.999	93%	94%	95%	95%	95%	93%	94%	95%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.9	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
β=9000	0.999	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.99	93%	94%	95%	95%	95%	93%	94%	95%	95%	95%
	0.95	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.9	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%

Table 4.3 Coverage of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $A = 70$, $h = 0.6$, $s = 9$, $L = 5$ and $D_t \sim N(300, 60^2)$.

		Hadley & Whitin					Exact				
	CSL	n=25	n=50	n=100	n=300	n=500	n=25	n=50	n=100	n=300	n=500
β=200	0.999	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.9	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
β=1000	0.999	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	92%	94%	94%	95%	95%	92%	94%	94%	95%	95%
	0.9	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
β=9000	0.999	93%	94%	95%	95%	95%	93%	94%	95%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.9	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%

Table 4.4 Coverage of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $A = 70$, $h = 0.6$, $s = 9$, $L = 10$ and $D_t \sim N(300, 60^2)$.

		Hadley & Whitin					Exact				
	CSL	n=25	n=50	n=100	n=300	n=500	n=25	n=50	n=100	n=300	n=500
β=200	0.999	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.9	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
β=1000	0.999	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.9	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
β=9000	0.999	93%	94%	95%	95%	95%	93%	94%	95%	95%	95%
	0.99	93%	94%	94%	95%	95%	93%	94%	94%	95%	95%
	0.95	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.9	92%	93%	94%	94%	95%	92%	93%	94%	94%	95%
	0.8	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.6	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.4	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.2	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.1	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%
	0.05	92%	93%	94%	95%	95%	92%	93%	94%	95%	95%

At this point it is important to note that for the sake of brevity we display the results only for $s = 9$ since from the experimentation we find out that acceptable COVs greater than 90% always exist for all the combinations of parameter values. In particular, regarding the cost parameters A and s, we observe that when the ordering cost increase then the coverages are getting marginally larger, while, on the contrary, COVs marginally decrease for larger values of the shortage cost. Concerning the value of L we observe that when lead-time increases then coverages are getting marginally smaller. While, regarding the reference period, β, it is observed that for small sample sizes COVs marginally decrease with larger values of β.

Since from the results of Tables 4.2, 4.3 and 4.4 we observe that for all the combinations of parameter values acceptable coverages always exist, in the remaining of this section we examine the expected half length (EHL)

$$z_{1-\alpha/2}\sqrt{\frac{\mathbf{M}'\cdot\mathbf{\Sigma}\cdot\mathbf{M}}{n}}$$

of the estimated confidence intervals in order to give some crucial managerial recommendations. Specifically, using the parameter values of Table 4.1 we find from Tables 4.5-4.8 that EHLs increase for larger stockout probabilities. For example, with $L = 2$, $\beta = 1000$, $A = 500$, $s = 9$ and sample size equal to 100 observations, EHL is 261.91 for P=0.999 and 662.05 for P=0.40. According to the sample size n we observe that EHLs are getting smaller when n increases. Further, concerning the value of L we observe that when lead-time decreases then the expected half lengths are getting smaller. For instance, having the parameter values $n = 50$, $\beta = 200$, $A = 70$, $s = 9$, $P = 0.999$ and lead-time equal to 2 the expected half length is 68.79, while on the contrary EHL is 92.72 for $L = 10$. Regarding the value of β it is observed that EHLs decrease when the reference period is getting smaller. For example, with $A = 70$, $s = 9$, $n = 500$, $P = 0.9$ and $L = 2$ the expected half length is 257.16 for $\beta = 9000$, while reducing the value of β from $\beta = 9000$ to $\beta = 200$ EHL is 41.06. Finally, regarding the cost parameters A and s, we observe that when the ordering cost is getting larger then EHLs either increase or decrease according to the parameter values. For example, with $n = 25$, $s = 9$, $\beta = 1000$, $L = 5$ and $P = 0.95$, EHL is 330.38 for $A = 70$ and 537.04 for $A = 500$. If P is decreasing further and reaching the size of 0.8 then EHL is 847.84 for $A = 70$ and 729.46 for $A = 500$. On the other hand, EHLs are clearly getting larger when the shortage cost increases. For instance, with $n = 50$, $\beta = 200$, $L = 10$, $A = 70$ and $P = 0.9$, EHL is 122.58 for $s = 3$ and 218.45 for $s = 9$.

Under the above consideration, we suggest to practice that in order to take a decision regarding the choice of parameter values of the continuous review model, as a first priority, it is required to examine the expected half lengths of the estimated confidence intervals only when acceptable coverages are attained for all the combinations of parameter values. Therefore, we recommend that confidence intervals for the minimum cost with higher precision, namely smaller EHLs, are achieved when:

a) the sample size, n, increases,

b) large cycle service levels, P, are set,

c) the lead-time, L, is getting smaller,

d) the reference period, β, decreases,

e) the shortage cost, s, is getting smaller,

f) the ordering cost, A, decreases but in accordance with the values of the other parameters as there are cases for which EHLs increase, mainly when the stockout probabilities are getting larger.

Table 4.5 Precision of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $h = 0.6$, $n = 25$ and $D_t \sim N(300, 60^2)$.

		Expected half length											
		s=3						s=9					
		A=70			A=500			A=70			A=500		
β	CSL	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10
200	0.999	95.39	108.73	127.89	231.94	237.68	246.96	95.73	109.43	129.03	232.01	237.83	247.22
	0.99	92.89	102.81	117.37	230.61	234.28	240.22	96.83	110.68	130.09	231.41	235.90	243.02
	0.95	99.08	114.12	134.38	231.30	235.08	240.86	126.83	162.83	204.59	236.48	245.47	258.62
	0.9	113.21	139.17	170.65	233.63	239.38	247.82	178.48	240.39	304.55	247.32	266.70	293.26
	0.8	153.10	201.86	253.75	240.97	253.57	271.11	288.83	388.47	483.65	282.86	332.53	393.06
	0.4	372.31	491.06	600.51	321.27	395.89	479.54	714.99	919.23	1105.76	578.03	771.01	958.03
	0.2	519.80	673.66	813.27	416.36	541.42	669.37	971.73	1235.99	1477.01	823.19	1092.12	1342.35
	0.05	709.48	906.15	1083.52	573.23	759.51	937.89	1299.28	1640.95	1952.48	1157.74	1513.46	1837.94
1000	0.999	195.64	202.64	213.77	511.66	514.33	518.73	196.13	203.65	215.52	511.78	514.54	519.09
	0.99	196.07	202.75	213.06	511.49	513.52	516.77	202.23	215.31	234.40	512.81	516.03	520.94
	0.95	210.49	230.90	258.80	514.22	518.26	524.09	265.86	330.38	405.32	523.86	537.04	555.73
	0.9	241.25	287.41	343.58	519.44	528.03	540.14	381.80	507.70	637.15	546.95	582.38	630.58
	0.8	331.78	432.64	539.44	535.68	559.63	592.54	633.10	847.84	1051.63	625.65	729.46	855.88
	0.4	836.45	1104.39	1351.81	720.95	890.16	1079.93	1602.83	2061.92	2481.71	1296.14	1730.09	2150.98
	0.2	1175.65	1527.64	1848.78	941.66	1229.36	1524.53	2186.43	2785.28	3333.18	1853.53	2462.87	3031.37
	0.05	1612.80	2068.07	2482.15	1305.25	1737.46	2153.90	2931.91	3711.42	4425.50	2614.62	3425.59	4168.64
9000	0.999	574.31	577.03	581.45	1530.28	1531.25	1532.83	575.21	578.74	584.32	1530.54	1531.70	1533.53
	0.99	578.57	584.68	593.63	1531.43	1533.01	1535.25	592.13	611.42	638.90	1534.65	1538.65	1544.08
	0.95	617.55	661.06	720.07	1539.67	1547.38	1557.62	771.09	939.73	1135.35	1565.42	1596.14	1638.35
	0.9	706.24	826.48	972.72	1554.62	1575.05	1602.79	1117.96	1475.59	1841.38	1632.00	1726.67	1854.48
	0.8	978.62	1268.04	1573.60	1602.42	1667.96	1757.26	1878.77	2509.98	3106.74	1866.17	2166.10	2531.12
	0.4	2515.73	3323.42	4070.06	2167.05	2678.47	3252.21	4815.06	6196.21	7459.96	3894.30	5200.08	6467.15
	0.2	3548.56	4617.43	5595.34	2842.33	3718.50	4618.72	6581.27	8390.70	10048.91	5581.38	7422.33	9142.38
	0.05	4881.09	6272.06	7542.62	3953.94	5276.03	6553.87	8838.82	11202.51	13373.09	7885.71	10343.80	12601.33

Table 4.6 Precision of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $h = 0.6$, $n = 50$ and $D_t \sim N(300, 60^2)$.

		Expected half length											
		s=3						s=9					
		A=70			A=500			A=70			A=500		
β	CSL	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10
200	0.999	68.54	78.13	91.90	166.66	170.79	177.46	68.79	78.63	92.72	166.71	170.90	177.64
	0.99	66.75	73.88	84.34	165.71	168.35	172.61	69.58	79.54	93.48	166.29	169.51	174.63
	0.95	71.20	82.01	96.56	166.21	168.92	173.08	91.14	116.97	146.92	169.94	176.40	185.86
	0.9	81.36	99.99	122.58	167.88	172.02	178.09	128.17	172.52	218.45	177.74	191.66	210.75
	0.8	109.97	144.92	182.08	173.17	182.23	194.84	207.13	278.35	346.34	203.28	238.93	282.35
	0.4	266.72	351.50	429.59	230.84	284.35	344.27	511.30	657.02	790.11	414.79	552.80	686.42
	0.2	371.98	481.74	581.32	299.01	388.55	480.07	694.45	883.00	1054.98	590.07	782.14	960.72
	0.05	507.28	647.54	774.04	411.31	544.48	671.87	928.15	1171.94	1394.24	828.97	1082.80	1314.26
1000	0.999	140.58	145.61	153.61	367.66	369.57	372.74	140.93	146.34	154.87	367.74	369.73	373.00
	0.99	140.89	145.69	153.11	367.53	369.00	371.33	145.33	154.73	168.44	368.49	370.81	374.34
	0.95	151.27	165.94	185.97	369.51	372.42	376.61	191.04	237.33	291.07	376.46	385.94	399.38
	0.9	173.37	206.50	246.80	373.27	379.46	388.17	274.17	364.34	456.97	393.07	418.54	453.17
	0.8	238.32	310.59	387.04	384.97	402.19	425.84	453.99	607.46	753.00	449.63	524.14	614.82
	0.4	599.24	790.52	967.09	518.03	639.36	775.31	1146.22	1473.78	1773.32	930.09	1240.44	1541.17
	0.2	841.36	1092.50	1321.60	676.26	882.27	1093.42	1562.59	1989.91	2380.91	1328.65	1763.86	2169.65
	0.05	1153.24	1478.03	1773.44	936.57	1245.62	1543.10	2094.53	2650.83	3160.47	1872.19	2450.94	2981.08
9000	0.999	412.68	414.63	417.81	1099.60	1100.29	1101.43	413.32	415.86	419.87	1099.78	1100.62	1101.93
	0.99	415.75	420.15	426.59	1100.43	1101.57	1103.18	425.52	439.39	459.14	1102.75	1105.64	1109.55
	0.95	443.81	475.08	517.47	1106.37	1111.93	1119.31	554.10	675.08	815.31	1124.94	1147.05	1177.41
	0.9	507.54	593.85	698.73	1117.15	1131.87	1151.83	802.80	1058.89	1320.58	1172.86	1240.92	1332.74
	0.8	702.94	910.30	1129.02	1151.58	1198.72	1262.90	1347.23	1798.29	2224.42	1341.14	1556.42	1818.22
	0.4	1802.30	2378.94	2911.77	1557.10	1923.82	2334.85	3443.38	4428.85	5330.61	2794.50	3728.37	4633.72
	0.2	2539.60	3302.29	4000.01	2041.23	2668.65	3312.66	4703.58	5994.77	7178.19	4000.87	5315.82	6543.64
	0.05	3490.40	4482.83	5389.40	2837.14	3782.56	4695.50	6314.56	8001.53	9550.82	5646.63	7400.99	9011.78

Table 4.7 Precision of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $h = 0.6$, $n = 100$ and $D_t \sim N(300, 60^2)$.

		Expected half length											
		s=3						s=9					
		A=70			A=500			A=70			A=500		
β	CSL	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10
	0.999	48.82	55.64	65.45	118.70	121.64	126.39	48.99	56.00	66.04	118.73	121.71	126.52
	0.99	47.54	52.62	60.07	118.02	119.90	122.93	49.56	56.65	66.58	118.43	120.73	124.37
	0.95	50.71	58.41	68.77	118.38	120.31	123.27	64.91	83.30	104.62	121.03	125.64	132.37
200	0.9	57.94	71.21	87.29	119.57	122.52	126.84	91.26	122.81	155.47	126.59	136.51	150.10
	0.8	78.31	103.17	129.60	123.34	129.79	138.77	147.39	198.01	246.31	144.78	170.16	201.06
	0.4	189.72	249.93	305.38	164.41	202.48	245.10	363.40	466.86	561.36	295.25	393.35	488.28
	0.2	264.46	342.39	413.08	212.91	276.59	341.64	493.43	627.30	749.41	419.83	556.26	683.07
	0.05	360.51	460.09	549.88	292.76	387.40	477.89	659.35	832.46	990.30	589.51	769.74	934.06
	0.999	100.12	103.70	109.40	261.85	263.21	265.46	100.37	104.22	110.30	261.91	263.32	265.65
	0.99	100.35	103.76	109.05	261.76	262.80	264.46	103.50	110.20	119.97	262.44	264.09	266.61
	0.95	107.74	118.19	132.45	263.16	265.24	268.23	136.06	169.01	207.25	268.12	274.87	284.45
1000	0.9	123.48	147.07	175.75	265.85	270.26	276.46	195.22	259.35	325.21	279.96	298.10	322.76
	0.8	169.71	221.12	275.49	274.18	286.45	303.30	323.07	432.11	535.49	320.24	373.29	437.82
	0.4	426.25	562.10	687.47	368.94	455.28	551.98	814.66	1047.24	1259.92	662.05	882.65	1096.31
	0.2	598.19	776.50	939.16	481.53	628.04	778.14	1110.29	1413.70	1691.34	945.32	1254.48	1542.65
	0.05	819.62	1050.21	1259.94	666.64	886.29	1097.63	1487.98	1883.00	2244.91	1331.41	1742.37	2118.77
	0.999	293.91	295.30	297.57	783.13	783.63	784.44	294.37	296.18	299.04	783.27	783.86	784.80
	0.99	296.10	299.23	303.82	783.72	784.54	785.68	303.06	312.95	327.02	785.38	787.44	790.23
	0.95	316.10	338.37	368.56	787.97	791.93	797.19	394.64	480.75	580.52	801.20	816.96	838.60
9000	0.9	361.49	422.93	497.58	795.65	806.14	820.37	571.61	753.74	939.78	835.35	883.84	949.23
	0.8	500.57	648.08	803.60	820.19	853.78	899.49	958.69	1279.17	1581.85	955.21	1108.48	1294.79
	0.4	1282.00	1691.54	2069.89	1108.95	1369.92	1662.30	2447.35	3147.06	3787.36	1989.17	2652.96	3296.20
	0.2	1805.62	2347.16	2842.54	1453.45	1899.68	2357.50	3342.12	4258.94	5099.28	2846.58	3780.69	4652.64
	0.05	2480.72	3185.34	3829.03	2019.47	2691.43	3340.03	4486.01	5683.94	6784.15	4015.64	5261.42	6405.13

Table 4.8 Precision of 95% asymptotic confidence intervals in finite samples for the direct estimation policy, given that $h = 0.6$, $n = 500$ and $D_t \sim N(300, 60^2)$.

		Expected half length											
		s=3						s=9					
		A=70			A=500			A=70			A=500		
β	CSL	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10	L=2	L=5	L=10
	0.999	21.96	25.04	29.45	53.41	54.73	56.87	22.05	25.20	29.71	53.42	54.77	56.93
	0.99	21.39	23.68	27.03	53.10	53.95	55.31	22.30	25.49	29.96	53.29	54.32	55.96
	0.95	22.82	26.28	30.95	53.26	54.13	55.47	29.21	37.48	47.07	54.46	56.53	59.56
200	0.9	26.07	32.04	39.28	53.80	55.13	57.07	41.06	55.24	69.91	56.96	61.43	67.54
	0.8	35.23	46.41	58.29	55.50	58.40	62.44	66.28	89.01	110.69	65.15	76.57	90.47
	0.4	85.28	112.30	137.19	73.98	91.10	110.26	163.23	209.66	252.07	132.79	176.86	219.48
	0.2	118.82	153.79	185.51	95.79	124.40	153.63	221.58	281.66	336.47	188.75	250.00	306.92
	0.05	161.92	206.61	246.90	131.67	174.18	214.80	296.05	373.74	444.58	264.92	345.81	419.55
	0.999	45.05	46.66	49.23	117.82	118.43	199.44	45.16	46.89	49.63	117.84	118.48	119.53
	0.99	45.15	46.69	49.07	117.78	118.28	118.99	46.57	49.59	53.98	118.09	118.83	119.96
	0.95	48.48	53.18	59.60	118.41	119.35	120.69	61.22	76.05	93.24	120.64	123.68	128.00
1000	0.9	55.57	66.18	79.08	119.62	121.61	124.40	87.82	116.65	146.24	125.98	134.14	145.24
	0.8	76.36	99.47	123.90	123.38	128.90	136.48	145.27	194.23	240.65	144.10	167.97	196.99
	0.4	191.59	252.57	308.84	166.01	204.84	248.31	365.93	470.31	565.76	297.77	396.86	492.80
	0.2	268.77	348.79	421.79	216.64	282.48	349.91	498.60	634.78	759.39	425.00	563.80	693.15
	0.05	368.14	471.62	565.75	299.82	398.48	493.37	668.12	845.41	1007.85	598.33	782.78	951.70
	0.999	132.24	132.87	133.89	352.37	352.59	352.96	132.45	133.27	134.55	352.43	352.70	353.12
	0.99	133.23	134.64	136.71	352.64	353.00	353.52	136.37	140.82	147.15	353.38	354.31	355.56
	0.95	142.24	152.26	165.85	354.55	356.33	358.70	177.58	216.31	261.17	360.51	367.61	377.35
9000	0.9	162.67	190.31	223.88	358.01	362.74	369.14	257.16	339.01	422.59	375.89	397.72	427.14
	0.8	225.22	291.53	361.41	369.06	384.18	404.76	431.07	574.98	710.86	429.83	498.78	582.57
	0.4	576.24	760.08	929.89	499.00	616.34	747.77	1099.30	1413.33	1700.70	894.66	1192.83	1481.67
	0.2	811.28	1054.33	1276.65	653.90	854.44	1060.12	1500.87	1912.36	2289.53	1279.78	1699.17	2090.56
	0.05	1114.26	1430.50	1719.39	908.26	1210.10	1501.33	2014.27	2551.96	3045.80	1804.63	2363.79	2877.09

4.5 Summary

In this chapter, considering fixed lead-time, normally distributed lead-time demand and by assuming that the values of demand parameters are unknown we address for the first time the issue of the Hadley & Whitin's (1963) cost function estimation. To study asymptotically the statistical properties of the estimator for the minimum cost of the reference period we make an estimation policy with the assumption that the cycle service level is determined a-priori and is treated as a fixed quantity. Using maximum likelihood estimators for the parameters of the Normal distribution we develop, for the first time, estimators for the optimal order quantity, optimal reorder point and minimum cost of the reference period. Based on the asymptotic distribution of the estimators, confidence intervals for the minimum cost are derived whose validity is tested through Monte-Carlo simulations for different sample sizes. To evaluate the performance of the confidence intervals, we consider the coverage, that is, the actual probability the interval to include the true minimum value of the Hadley & Whitin's cost function. Experimenting with different values of the cycle service level, we find out that acceptable COVs greater than 90% are always attained for all the combinations of parameter values.

Subsequently, we extend the analysis to examine also the expected half length of the estimated confidence intervals in order to give some crucial managerial recommendations regarding to the parameter values of the continuous review model. We suggest to practice that better precision, namely smaller EHLs, are achieved when either the sample size and the cycle service level increase or the lead-time, the reference period and the shortage cost are getting smaller.

Finally, we investigate how accurately the Hadley-Whitin's cost function approximates the exact cost function. Using again simulation analysis we develop, this time, the percentage of the 10.000 confidence intervals containing the true minimum value of the exact cost function. We find out that there are similar values for the two types of coverage which are greater than 90% for all the combinations of parameters values and this result strengthens the approximation accuracy of the Hadley-Whitin's cost function. Further, these results confirm the previous finding of chapter 3 where we observe that accurate approximations (lower than 1%) exist when the demand parameters are known.

Appendix

Proof 4.1:

For the ML estimators it holds $\operatorname{plim}\hat{\mu}_t = \mu_t$ and $\operatorname{plim}\hat{\sigma}_t = \sigma_t$. It follows therefore that

$$\operatorname{plim}\hat{S}(R) = \psi(z_P)\cdot\sqrt{L}\cdot\operatorname{plim}\hat{\sigma}_t = S(R),$$

$$\operatorname{plim}\hat{Q}^* = \left(2\cdot\frac{A}{h}\cdot\beta\cdot\operatorname{plim}\hat{\mu}_t + 2\cdot\frac{s}{h}\cdot\beta\cdot\operatorname{plim}\hat{\mu}_t\cdot\operatorname{plim}\hat{S}(R)\right)^{1/2} = Q^*,$$

and finally $\operatorname{plim}\hat{C}^*_{HW} = h\left[\operatorname{plim}\hat{Q}^* + z_P\cdot\sqrt{L}\cdot\operatorname{plim}\hat{\sigma}_t\right] = C^*_{HW}$,

so $\hat{C}^*_{HW}$ is a consistent estimator for C^*_{HW}.

It is also known (e.g. Knight 2000, page 258) that the vector $\sqrt{n}\left[\hat{\mu}_t - \mu_t \quad \hat{\sigma}_t - \sigma_t\right]'$ converges in distribution to the bivariate Normal $N_2(\mathbf{0},\mathbf{\Sigma})$. Hence from the application of the bivariate Delta method we conclude that the statistic

$$\sqrt{n}\left\{f(\hat{\mu}_t,\hat{\sigma}_t) - f(\mu_t,\sigma_t)\right\} = \sqrt{n}\left(\hat{C}^*_{HW} - C^*_{HW}\right)$$

converges in distribution to the Normal $N(0,\mathbf{M}'\cdot\mathbf{\Sigma}\cdot\mathbf{M})$. Further, given that

$$\left.\frac{\partial\hat{D}}{\partial\hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = \beta \quad \text{and} \quad \left.\frac{\partial\hat{D}}{\partial\hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = 0,$$

$$\left.\frac{\partial\hat{\sigma}_L}{\partial\hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = 0 \quad \text{and} \quad \left.\frac{\partial\hat{\sigma}_L}{\partial\hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = \sqrt{L},$$

$$\left.\frac{\partial\hat{B}(P)}{\partial\hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = 0 \quad \text{and} \quad \left.\frac{\partial\hat{B}(P)}{\partial\hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = z_P\cdot\sqrt{L},$$

$$\left.\frac{\partial\hat{S}(R)}{\partial\hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = 0 \quad \text{and} \quad \left.\frac{\partial\hat{S}(R)}{\partial\hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = \psi(z_P)\cdot\sqrt{L},$$

$$\left.\frac{\partial\hat{Q}^*}{\partial\hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = \frac{\beta\left\{A + s\cdot S(R)\right\}}{h\cdot Q^*} \quad \text{and} \quad \left.\frac{\partial\hat{Q}^*}{\partial\hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = \frac{s\cdot D\cdot\psi(z_P)\cdot\sqrt{L}}{h\cdot Q^*},$$

we obtain

$$\left.\frac{\partial f}{\partial \hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = h\left\{\left.\frac{\partial \hat{Q}^*}{\partial \hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} + \left.\frac{\partial \hat{B}(P)}{\partial \hat{\mu}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}}\right\} = \beta\left\{\frac{A+s\cdot S(R)}{Q^*}\right\},$$

and

$$\left.\frac{\partial f}{\partial \hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} = h\left\{\left.\frac{\partial \hat{Q}^*}{\partial \hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}} + \left.\frac{\partial \hat{B}(P)}{\partial \hat{\sigma}_t}\right|_{\substack{\hat{\mu}_t=\mu_t\\ \hat{\sigma}_t=\sigma_t}}\right\} = \sqrt{L}\left\{\frac{s\cdot D\cdot \psi(z_P)}{Q^*} + h\cdot z_P\right\}.$$

Substituting into the vector $\mathbf{M}$ and performing the operation $\mathbf{M}'\cdot\mathbf{\Sigma}\cdot\mathbf{M}$ completes the proof.

Proof 4.2:

For the exact cost function we know from chapter 2 that

$$Q(R) = \sqrt{2\frac{A}{h}D + 2\frac{s}{h}D\cdot S(R) + \Theta(R)}.$$

Thus,

$$C_{ex} = \frac{A\cdot D}{Q(R)} + \frac{s\cdot D}{Q(R)}S(R) + h\frac{Q(R)}{2} + h(R-\mu_L) + h\frac{\Theta(R)}{2Q(R)} =$$

$$= \frac{h}{2\cdot Q(R)}\left\{2\frac{A}{h}D + 2\frac{s}{h}\cdot D\cdot S(R) + \Theta(R)\right\} + h\frac{Q(R)}{2} + h(R-\mu_L) =$$

$$= \frac{h\left[Q(R)\right]^2}{2\cdot Q(R)} + h\frac{Q(R)}{2} + h(R-\mu_L) =$$

$$= h\frac{Q(R)}{2} + h\frac{Q(R)}{2} + h(R-\mu_L),$$

and finally

$$C_{ex} = h\left\{Q(R) + R - \mu_L\right\}.$$

Chapter 5

Estimation in (Q,R) inventory systems with correlated demand

5.1 Introduction

In the previous chapters we dealt with the determination of optimal inventory policies for the continuous review (Q,R) inventory model under the assumption that the demand sizes are independent. The purpose of this chapter is to investigate the effects of using correlated demand on the target inventory measures for the Hadley & Whitin's cost function, by assuming that the demand process follows the stationary autoregressive, AR(1), model. Specially, considering that the parameters of the autocorrelated demand are unknown, we develop for the first time estimators for the optimal order quantity, optimal reorder point and minimum cost. Further, extending the analysis of Chapter 4 regarding to the estimated confidence intervals for the minimum of the Hadley & Whitin's cost function, in this Chapter, through Monte-Carlo simulations for different sample sizes, we examine under which conditions the asymptotic confidence interval of the independent demand model (see chapter 4) includes the true minimum cost when in fact the demand follows the AR(1) model.

Based on the aforementioned discussion and remarks the rest of the chapter is organized as follows. Section 5.2 contains a literature review for autocorrelated demand models. In Section 5.3 assuming that the demand process follows the stationary autoregressive, AR(1), model we present the lead-time demand parameters. In Section 5.4, using maximum likelihood estimators for the parameters of the Normal distribution we develop estimators for the optimal order quantity, optimal reorder point and minimum cost of the reference period. In Section 5.5, we study under which conditions the independent demand model can be a good approximation to handle the autocorrelated demand. Finally, the last section concludes chapter 5 summarizing the most important findings.

5.2 Relevant literature review

Most operations research textbooks in inventory theory note that in order to determine the optimal pair (Q,R) and the minimum value of the cost function it is necessary to investigate the demand during the lead-time. Further, most inventory models assume that the demand sizes are independent and normally distributed with some mean μ_t and some variance σ_t^2. In many practical problems in real world, however, these assumptions can sometimes be violated.

In the situations in which the demand rate on an item is independent of the inventory level but normality for the lead-time demand does not exist many studies have been developed in inventory literature, as it has been noted earlier in section 1.2.5. On the contrary, when the assumption of independence is violated then in such situations the demand sizes follow a serially correlated process. This means that the demand size in period t can be expressed as a function of the demand sizes in previous periods. Many studies have been conducted in inventory theory to investigate the effect of serially correlated demand processes on the optimal inventory policies. Ray (1980) investigated the effects of the first order autoregressive demand process on the first two moments of the lead-time demand for three different distributions of lead-time. He found that, under normally distributed lead-time demand, the stock is underestimated for negative serial correlation but will be overestimated when positive correlation occurs. The same author in 1981 extended his previous work (Ray, 1980) and by maintaining the first four moments of the normally distributed lead-time demand, investigated the effects of the first order autoregressive and moving average demand processes on the reorder point for Geometric, Uniform and truncated Poisson lead-times. Ray (1982) considering ARIMA forecasting models, investigated the effect of the autocorrelation parameter on the reorder point for either fixed or random lead-time. Especially, he compared the variance of the lead-time demand in the two cases of unconditional (method of moments) and forecast demands for both AR(1) and MA(1) models. He found that for either fixed or random lead-time the effect of positive correlation is to increase the reorder point while on the other hand negative correlation reduces it. Between the method of moments and method of forecasts the differences are comparatively small except when the autocorrelation parameter approaches unity.

If, on the other hand, the demand rate of an item is dependent on the inventory level but non-normally distributed lead-time demand exists then a number of studies have been conducted. Lau & Wang (1987) developed a formula for estimating the first four moments of lead-time demand when the inventory item's demand follows a probability distribution with any combination of skewness and kurtosis. Considering that the demand sizes can be modeled either by a first order autoregressive or a first order moving average process, they found that if autocorrelation or non-normality of demand is not properly considered then significant errors in inventory decisions may result. Further, Fotopoulos et al. (1988), assuming that the early works have computational difficulties, developed a simple approach for computing the approximate safety stock and reorder points by using an upper bound rather than an exact solution when the demand sizes follow AR(1) and MA(1) processes. Considering that the mean and the variance of demand sizes changed whenever the value of the autocorrelation parameter changed, they found through numerical investigation that the effect of the autocorrelation is very important for determining the safety stock. Marmorstein & Zinn (1993) based on the work of Fotopoulos et al. (1988) but assuming that the mean and the variance of demand sizes held constant, investigated the effect of autocorrelation demand on the safety stock determination. Finally, An et al. (1989) derived the exact first four moments of lead-time demand for AR(1) and MA(1) demand processes. They estimated reorder points of an inventory system based on the Pearson system and a Normal approximation.

Other studies have also investigated the effect of correlated demand on specific inventory models. Among others, Veinott (1965) firstly have shown the optimality of myopic policies for independent stochastically demand processes. Then, under the condition for which it holds that in each period the orders lead to specific inventory levels, he extended the analysis to the case of dependent demand over time. Johnson & Thompson (1975) based on the work of Veinott (1965) investigated the optimality of myopic replenishment rules for periodic inventory systems for both stationary and nonstationary processes with nonnegative demand. Urban (2000) examined and compared for a continuous review (Q,R) inventory model with fixed lead-time three different approaches (traditional, method of moments, method of forecasts) of determining accurate reorder points for first order autoregressive and moving average demand processes. Through simulation analysis with normally distributed error terms he found that existing approaches of managing serially correlated demand can result in excessive inventories and shortages for high levels of autocorrelation. Further, the same author in 2005 investigated, for a periodic review model with some common functional forms for the demand rate, the effect of AR(1) and MA(1) demand processes on the safety stock required in the system. Urban found that the effect of lead-time on safety stock is greatly amplified as the level of autocorrelation increases and to a lesser extent as the effect of the demand dependency on inventory increases. Strijbosch et al. (2000), assuming the (Q,R) continuous review inventory system with intermittent demand, compared the simple system with Normal demand distribution to an advanced system with compound Bernoulli distribution. Through Monte-Carlo simulations they found that the advanced system, using the Croston's (1972) method, gives desired service level under circumstances while the simple system is not consistent. But, at this point it is important to note that Syntetos & Boylan (2001, 2005) have shown that Croston's method is biased. Erkip et al. (1990) based on the work of Eppen & Schrage (1981) considered a multilevel inventory/warehousing system and developed optimal stock policies which are computed as explicit functions of the correlation coefficient. In particular, they found correlations between successive monthly demands of about 0.7. Syntetos & Boylan (2008), in a more recent paper, studied the interactions between forecasting and stock control for a periodic review system and through empirical data from the RAF they found that there is a scope to improve the performance of parametric stock control systems. Finally, the last years, many studies have been conducted by several researchers for the impacts of forecasting methods on the bullwhip effect in supply chain models. Among others, are: Graves (1999), Lee et al. (2000), Chen et al. (2000a,b), Luong (2007), Duc et al. (2008), Ali et al. (2012) and Babai et al. (2013).

Under the above consideration, we find out that for the continuous review (Q,R) inventory model most of the research has focused on specifying optimal inventory policies with the assumption that parameters of the demand distribution are known. However, the extent of applicability of such models to managerial aspects of inventories depends on the estimation of demand parameters. To the extent of our knowledge, research is very limited concerning topics on developing estimators for the target inventory measures of the Hadley & Whitin's cost function and determining optimal inventory policies when the demand in successive periods is autocorrelated and at the same time the parameters of demand distribution are unknown. For this reason we address for the first time in this thesis the issues for estimating target inventory measures when both correlated demand occurs and parameters of demand distribution are unknown.

Except the continuous review (Q,R) inventory model, we find out that for the classical newsvendor model there is also a limited research on determining the order quantity for autocorrelated demand and unknown parameters of demand distribution. Many studies have been developed to solve the problem of not knowing the parameters of demand distribution. Among them are: Ritchken & Sankar (1984), Liyanage & Shanthikumar (2005), Kevork (2010), Akcay et al. (2011) and Halkos & Kevork (2012). But, these works assume that demand in successive periods is formed independently. However, Akcay et al. (2012)

addressed in the classical newsvendor model the issues of both the correlated demand and the demand parameters estimation. Specially, by using a simulation-based sampling algorithm, they quantified the expected cost due to parameter uncertainty when the demand process is an autoregressive-to-anything time series, and the marginal demand distribution is represented by the Johnson translation system with unknown parameters.

5.3 Modeling demand as correlated

To estimate the demand distribution parameters, according to Ray (1982) and Urban (2000) there are two methods: (a) the method of moments and (b) the method of forecasts. The method of moments, by using the relevant moments of the lead-time demand, gives a fixed reorder point in each inventory cycle and provides an expected demand during lead-time which is constant and it is independent of the most recently observed demand. Further, this approach obtains an effective estimate of the distribution parameters during a long run lead-time. On the contrary, in the method of forecasts a variable reorder point is provided using appropriate forecasts and the demand parameters are updated every period, conditional on the most recent observed demand. Finally, this forecasting approach gives the chance to study the short term effect of the autocorrelation on the distribution parameters. In this chapter, in order to have a fixed reorder point in each inventory cycle we use the relevant moments of the lead-time demand (method of moments).

Assume that the demand process follows the first order autoregressive model which is given by:

$$D_t = \delta + \varphi D_{t-1} + \varepsilon_t \tag{5.1}$$

where the error terms, $\varepsilon_t' s$, are uncorrelated and Normal variables with mean zero, $E(\varepsilon_t) = 0$, and constant variance, $Var(\varepsilon_t) = \sigma_\varepsilon^2$. This process is stationary when $|\varphi| < 1$ with mean $\mu_t = \delta/(1-\varphi)$, variance $\sigma_t^2 = \sigma_\varepsilon^2/(1-\varphi^2)$ and covariance $Cov(X_t, X_{t+s}) = \varphi^s \sigma_t^2$. Considering, also, that the lead-time demand $X \sim N(\mu_L \ , \sigma_{L(AR)}^2)$ is given by equation (5.2)

$$X = \sum_{t=1}^{L} D_t = D_1 + D_2 + \ldots + D_L \tag{5.2}$$

then, the mean and the variance of the lead-time demand, based on Ray (1982), are given by equations (5.3) and (5.4) respectively,:

$$E(X)=\mu_L=E\left(\sum_{t=1}^{L}D_t\right)=L\cdot E(D_t)=L\cdot\mu_t \tag{5.3}$$

$$Var(X)=\sigma^2_{L(AR)}=\frac{\sigma^2_{\varepsilon}}{(1-\varphi)^2}\left\{L-\frac{2\varphi}{1-\varphi^2}+\frac{2\varphi^{L+1}}{1-\varphi^2}\right\}. \tag{5.4}$$

5.4 Hadley & Whitin's cost function with correlated demand

We start our analysis using the Hadley & Whitin's cost function which was given in previous chapters but considering, now, that demand follows an AR(1) model. Replacing in (4.1) the formulas of μ_L and $\sigma^2_{L(AR)}$ which are given in (5.3) and (5.4) we take

$$C^{AR}_{HW}(Q_{AR},R_{AR})=A\frac{D}{Q_{AR}}+h\left(\frac{Q_{AR}}{2}+\left(\frac{R_{AR}-\mu_L}{\sigma_{L(AR)}}\right)\sigma_{L(AR)}\right)+s\frac{D}{Q_{AR}}\sigma_{L(AR)}\psi\left(\frac{R_{AR}-\mu_L}{\sigma_{L(AR)}}\right). \tag{5.5}$$

Assuming also a fixed cycle service level (see chapter 4 for more details), defined as $P=\Pr(X\le R_{AR})=\Pr\left(Z\le\frac{R_{AR}-\mu_L}{\sigma_{L(AR)}}\right)=\Pr(Z\le z_P)$, with z_P to be the p^{th} percentile of the standard Normal, we take the reorder point as $R^*_{AR}=\mu_L+z_P\sigma_{L(AR)}$.

Taking, now, the partial derivatives of $C^{AR}_{HW}(Q_{AR},R_{AR})$ with respect to Q_{AR} and R_{AR}, we obtain the first order conditions

$$\frac{\partial C^{AR}_{HW}(Q_{AR},R_{AR})}{\partial Q_{AR}}=-\frac{A\cdot D}{Q^2_{AR}}+\frac{h}{2}-\frac{s\cdot D}{Q^2_{AR}}\sigma_{L(AR)}\psi(z_P), \tag{5.6}$$

$$\frac{\partial C^{AR}_{HW}(Q_{AR},R_{AR})}{\partial R_{AR}}=h-\frac{s\cdot D}{Q_{AR}}\left[1-F(z_P)\right], \tag{5.7}$$

from which we take

$$Q^*_{AR}=\sqrt{2\frac{A}{h}D+2\frac{s}{h}D\cdot\left[\frac{\sigma^2_{\varepsilon}}{(1-\varphi)^2}\left\{L-\frac{2\varphi}{1-\varphi^2}+\frac{2\varphi^{L+1}}{1-\varphi^2}\right\}\right]^{1/2}\psi(z_P)}, \tag{5.8}$$

$$F^*_{AR}(z_P)=1-\frac{h\cdot Q_{AR}}{s\cdot D}. \tag{5.9}$$

Further, substituting Q^*_{AR} for Q_{AR}, the minimum total cost is equal to

$$C^{AR^*}_{HW}(z_P)=h\cdot Q^*_{AR}+h\cdot z_P\cdot\sigma_{L(AR)}. \tag{5.10}$$

In Figures 5.1-5.5, assuming the parameter values of Table 4.1 (see chapter 4), we present the minimum value of the Hadley & Whitin's cost function for a range of autoregressive parameter values, $(\varphi = -0.8,\ -0.5,\ -0.2,\ 0.2,\ 0.5,\ 0.8)$. Specially, we find out that for bigger values of φ the minimum cost $C_{HW}^{AR^*}$ is getting larger. Concerning the value of L we observe from Figure 5.1 that when lead-times decrease then the optimal pairs $\left(Q_{AR}^*, R_{AR}^*\right)$ lead to smaller minimum costs. Further, regarding the value of the cycle service level it is observed that $C_{HW}^{AR^*}$ decreases when the cycle service level is getting larger (see Figure 5.2). Finally, according to the reference period β we find from Figure 5.3 that $C_{HW}^{AR^*}$ increases for larger values of β. Besides, regarding the cost parameters A and s, we observe from Figures 5.4 and 5.5 that when the ordering and shortage costs increase then $C_{HW}^{AR^*}$ is getting larger. At this point it is important to mention that the minimum value of the Hadley & Whitin's cost function has been computed for all the combinations of parameter values. But, for the sake of brevity, indicative results are presented in Figures 5.1-5.5 in order to observe the trend of $C_{HW}^{AR^*}$. Similar results exist using different parameter values.

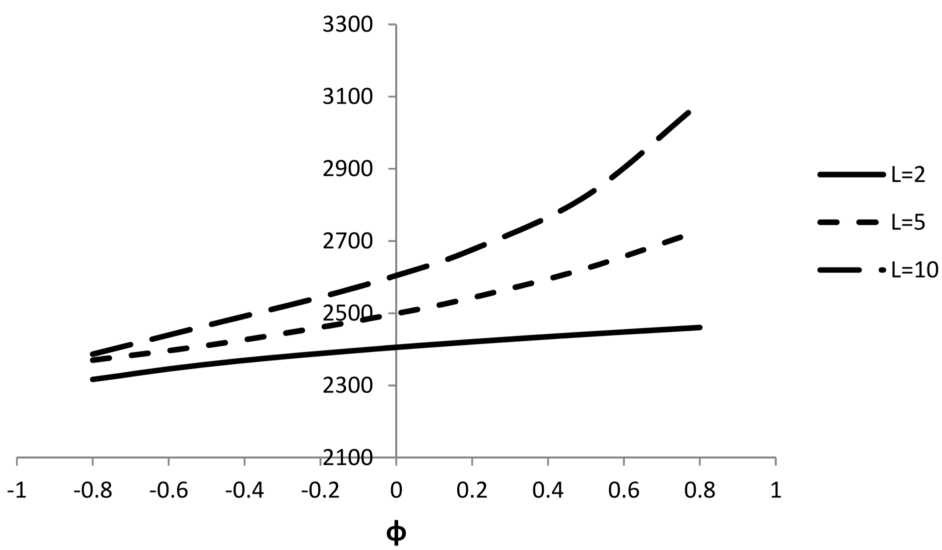


Figure 5.1 Minimum cost of C_{HW}^{AR} with $A = 70$, $h = 0.6$, $s = 9$, $\beta = 200$ and $P = 0.999$.

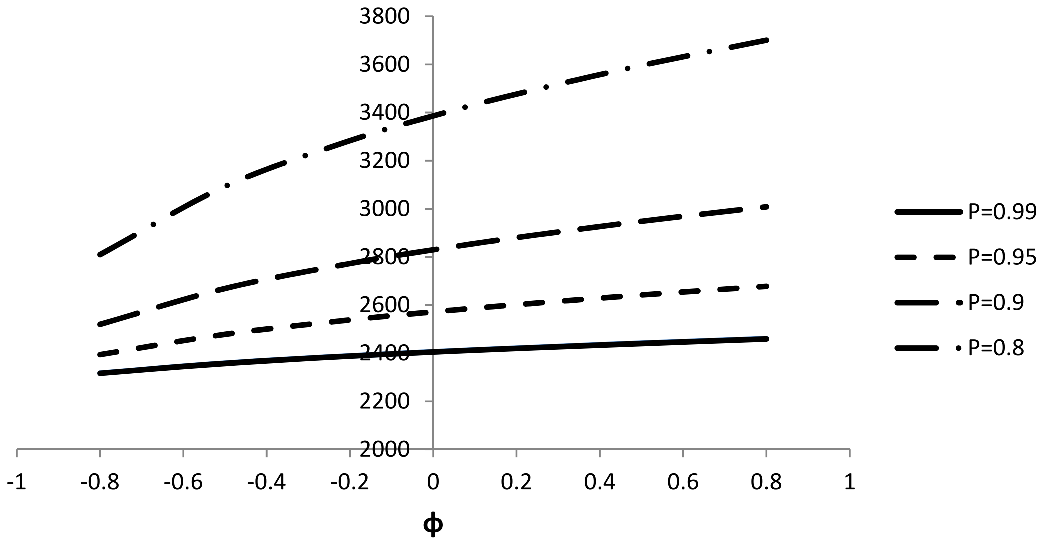


Figure 5.2 Minimum cost of C_{HW}^{AR} with $A = 70$, $h = 0.6$, $s = 9$, $\beta = 200$ and $L = 2$.

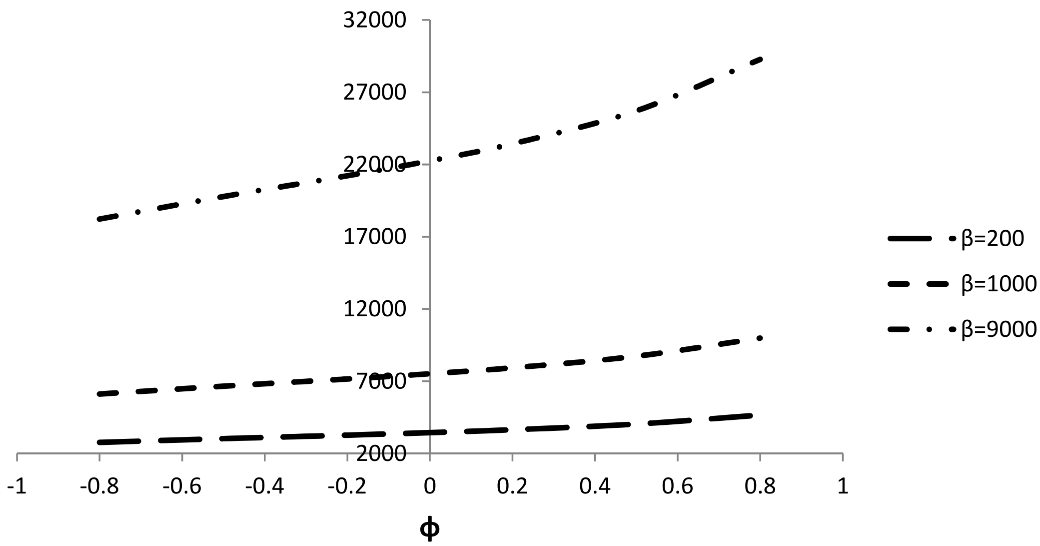


Figure 5.3 Minimum cost of C_{HW}^{AR} with $A = 70$, $h = 0.6$, $s = 9$, $L = 10$ and $P = 0.9$.

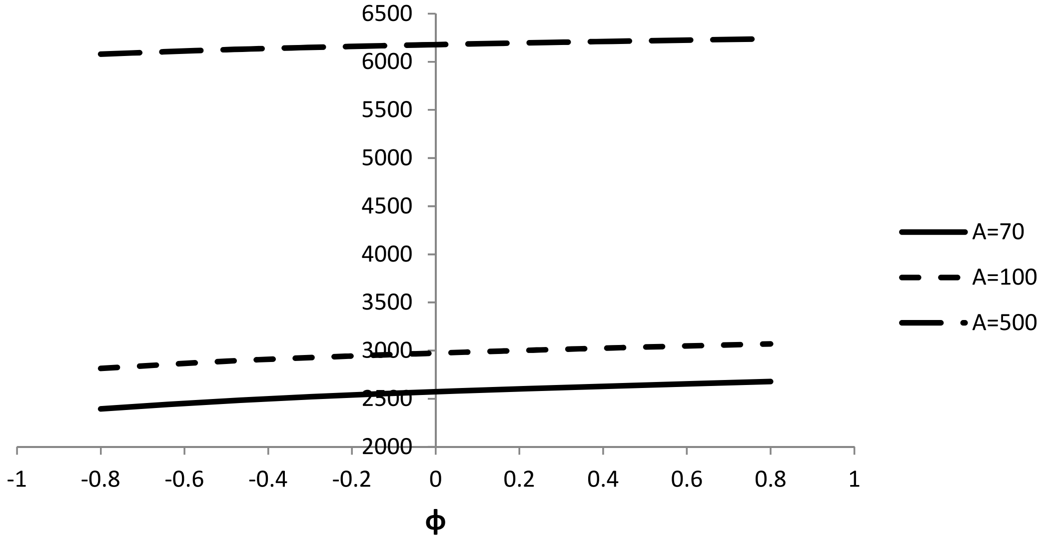


Figure 5.4 Minimum cost of C_{HW}^{AR} with $L = 2$, $h = 0.6$, $s = 9$, $\beta = 200$ and $P = 0.95$.

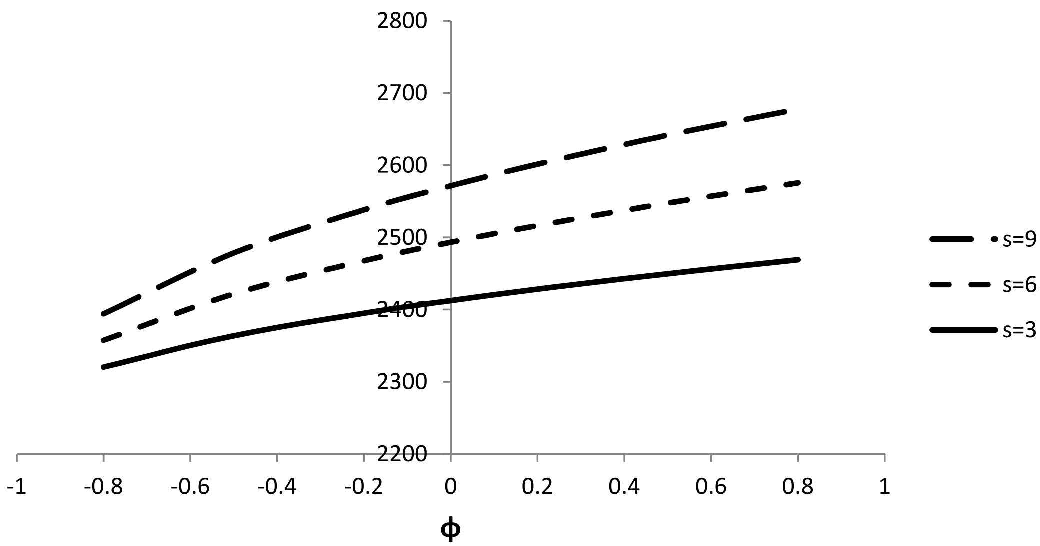


Figure 5.5 Minimum cost of C_{HW}^{AR} with $A = 70$, $h = 0.6$, $L = 2$, $\beta = 200$ and $P = 0.95$.

Estimating the Hadley & Whitin's cost function

Suppose, now, that $D_1, D_2, ..., D_n$ is a sequence of random variables representing demand on time t $(t = 1, 2, ..., n)$.

Further, let $\hat{\boldsymbol{\psi}} = \begin{bmatrix} \hat{\mu}_t \\ \hat{\varphi} \\ \hat{\sigma}_\varepsilon^2 \end{bmatrix}$ be the maximum likelihood estimators of $\boldsymbol{\psi} = \begin{bmatrix} \mu_t \\ \varphi \\ \sigma_\varepsilon^2 \end{bmatrix}$ and assume, also, that the conditional Log Likelihood function is given by

$$\ln L = -\frac{n-1}{2}\ln 2\pi - \frac{n-1}{2}\ln\hat{\sigma}_\varepsilon^2 - \frac{1}{2\hat{\sigma}_\varepsilon^2}\sum_{t=2}^{n}\left[D_t - \hat{\mu}_t - \hat{\varphi}\left(D_{t-1} - \hat{\mu}_t\right)\right]^2 . \tag{5.11}$$

Then, based on Harvey (1981) the maximization of the likelihood function of an AR(1) model becomes equivalent to minimizing the sum of squares function

$$\sum_{t=2}^{n}\varepsilon_t^2 = \sum_{t=2}^{n}\left(D_t - \delta - \varphi D_{t-1}\right)^2 . \tag{5.12}$$

Writing the classical linear regression model as

$$\mathbf{D} = \mathbf{X}\cdot\boldsymbol{\beta} + \boldsymbol{\varepsilon} , \tag{5.13}$$

where $\mathbf{D} = \begin{bmatrix} D_2 \\ D_3 \\ \cdots \\ D_n \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 & D_1 \\ 1 & D_2 \\ \cdots & \cdots \\ 1 & D_{n-1} \end{bmatrix}$ and $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_2 \\ \varepsilon_3 \\ \cdots \\ \varepsilon_n \end{bmatrix}$, the OLS estimator is then

$$\hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\delta} \\ \hat{\varphi} \end{bmatrix} = \left(\mathbf{X'X}\right)^{-1}\left(\mathbf{X'Y}\right), \tag{5.14}$$

with $\left(\mathbf{X}'\mathbf{X}\right)^{-1} = \begin{bmatrix} n-1 & \sum_{t=2}^{n} D_{t-1} \\ \sum_{t=2}^{n} D_{t-1} & \sum_{t=2}^{n} D_{t-1}^2 \end{bmatrix}^{-1}$ and $\left(\mathbf{X}'\mathbf{Y}\right) = \begin{bmatrix} \sum_{t=2}^{n} D_t \\ \sum_{t=2}^{n} D_t D_{t-1} \end{bmatrix}$.

Therefore the estimators of the AR(1) model are computed as

$$\hat{\delta} = \frac{1}{\det\left(\mathbf{X}'\mathbf{X}\right)}\left[\sum_{t=2}^{n} D_{t-1}^2 \sum_{t=2}^{n} D_t - \sum_{t=2}^{n} D_{t-1} \sum_{t=2}^{n} D_t D_{t-1}\right], \tag{5.15}$$

$$\hat{\varphi} = \frac{1}{\det\left(\mathbf{X}'\mathbf{X}\right)}\left[-\sum_{t=2}^{n} D_{t-1} \sum_{t=2}^{n} D_t + \left(n-1\right)\sum_{t=2}^{n} D_t D_{t-1}\right], \tag{5.16}$$

$$\hat{\sigma}_\varepsilon^2 = \frac{\sum_{t=2}^{n}\left(D_t - \hat{\delta} - \hat{\varphi} D_{t-1}\right)^2}{\left(n-1\right)-2}, \tag{5.17}$$

$$\hat{\mu}_t = \frac{\hat{\delta}}{1-\hat{\varphi}} \text{ and} \tag{5.18}$$

$$\hat{\sigma}_t^2 = \frac{\hat{\sigma}_\varepsilon^2}{1-\hat{\varphi}^2}. \tag{5.19}$$

Since in practice μ_t, σ_ε^2 and φ are unknown quantities, replacing the ML estimators into (5.8), in the place of μ_t, σ_ε^2 and φ, the resulting estimator for the optimal order quantity takes the form

$$\hat{Q}_{AR}^* = \sqrt{2\frac{A}{h}\hat{D} + 2\frac{s}{h}\hat{D}\cdot\left[\frac{\hat{\sigma}_\varepsilon^2}{\left(1-\hat{\varphi}\right)^2}\left\{L - \frac{2\hat{\varphi}}{1-\hat{\varphi}^2} + \frac{2\hat{\varphi}^{L+1}}{1-\hat{\varphi}^2}\right\}\right]^{1/2}\psi\left(z_P\right)}, \tag{5.20}$$

where $\hat{D} = \beta \cdot \hat{\mu}_t$ and $\hat{R}_{AR}^* = \hat{\mu}_L + z_P \hat{\sigma}_{L(AR)}$.

Therefore, the estimator for the minimum total cost is given by

$$\hat{C}_{HW}^{AR^*} = f\left(\hat{\mu}_t, \hat{\varphi}, \hat{\sigma}_\varepsilon^2\right) = h \cdot \hat{Q}_{AR}^* + h \cdot z_P \cdot \hat{\sigma}_{L(AR)} . \tag{5.21}$$

5.5 The independence approximation

In this section, we investigate under which conditions an independent demand model $(\varphi = 0)$ can be used when lead-time demand is generated by an AR(1) process. In particular, the analysis of the previous sections showed that the optimal target inventory measures should be determined through the use of the AR(1) process, namely Q_{AR}^*, R_{AR}^* and $C_{HW}^{AR^*}\left(Q_{AR}^*, R_{AR}^*\right)$, when the actual demand data generated by autocorrelated structure. On the contrary, using an independent demand model and assuming as optimal the following target inventory measures Q_{IND}^*, R_{IND}^* and $C_{HW}^{IND^*}\left(Q_{IND}^*, R_{IND}^*\right)$ which are developed after substituting $\varphi = 0$ into the formulas of Q_{AR}^*, R_{AR}^* and $C_{HW}^{AR^*}\left(Q_{AR}^*, R_{AR}^*\right)$, then significant errors exist in computing the size of the order quantity, reorder point and minimum cost. To perform the study, we introduce a statistical measure which is related to the approximation error in computing the minimum total cost, $RAE = \left(C_{HW}^{AR^*} - C_{HW}^{IND^*}\right) \Big/ C_{HW}^{IND^*}$.

The values of RAE are presented in Tables 5.1-5.4 for different combinations of A, s , L and β. Considering again the same parameters values with those which are used in chapter 4, we find out that when φ is approaching -1 or 1 the absolute values of RAE increase indicating reduction in the accuracy of approximation. For example, with $L = 2$, $s = 9$, $\beta = 200$, $A = 70$ and $P = 0.999$, RAE is 0.71% for $\varphi = -0.2$ while for $\varphi = -0.8$ is 3.69%. On the contrary, the absolute values of RAE are getting smaller when the stockout probabilities decrease. For example, with $L = 2$, $s = 9$, $\beta = 9000$, $A = 500$ and $\varphi = 0.8$, RAE is 9.64% for P=0.2 while is 0.14% for P=0.999. Concerning the value of L we observe that when lead-time increases then the absolute values of the relative approximation error are getting larger. For instance, having the parameter values $s = 3$, $A = 70$, $\beta = 200$, $\varphi = 0.5$, $P = 0.95$ and lead-time equal to 2 the relative approximation error is 1.54%, while on the contrary RAE is 8.43% for $L = 10$. Regarding the value of β it is observed that when the reference period is getting larger, |RAEs| either increase or decrease according to the parameter values. For example, with $\varphi = -0.2$, $A = 70$, $s = 9$, $L = 2$ and $P = 0.9$, RAE is 2.02% for $\beta = 200$ and 1.84% for $\beta = 9000$. If P is decreasing further and reaching the size of 0.4 then RAE is 4.60% for $\beta = 200$ and 4.62 for $\beta = 9000$. Finally, regarding the cost parameters A and s, we observe that when the ordering cost is getting larger then |RAEs| decrease. For example, with $\varphi = 0.8$, $s = 9$, $\beta = 200$, $L = 10$ and $P = 0.9$, RAE is 35.21% for $A = 70$ while is 11.60% for $A = 500$. On the other hand, the absolute values of RAE are getting larger when the shortage cost increases. For instance, with $\varphi = -0.5$, $A = 70$, $\beta = 200$, $L = 2$ and $P = 0.6$, RAE is 7.84% for $s = 3$ while for $s = 9$ is 11.82%.

The above findings imply that the choice of the independent demand model when autocorrelated demand exists is incorrect. And this happens because the decision cannot be taken independently of the value of the autocorrelation coefficient and without taking into account the size of A, s, L and β. Consequently, we recommend to practice that better approximation, namely smaller |RAE|, is achieved when:

a) the autocorrelation, φ, approaches zero,

b) large cycle service levels, P, exist,

c) the lead-time, L, is getting smaller,

d) the shortage cost, s, is getting smaller,

e) the ordering cost, A, increases.

For the reference period, β, |RAE| either decreases or increases according to the parameter values.

Table 5.1 Relative Approximation Error in computing the minimum total cost, $\mathrm{RAE} \times 100\%$, with $h = 0.6$, $s = 9$ and $A = 70$.

			L=2				
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-3.69%	-1.96%	-0.71%	0.64%	1.50%	2.28%
	0.99	-3.66%	-1.94%	-0.70%	0.63%	1.49%	2.26%
	0.95	-6.90%	-3.62%	-1.30%	1.16%	2.73%	4.13%
	0.9	-10.95%	-5.68%	-2.02%	1.80%	4.19%	6.32%
	0.8	-17.04%	-8.66%	-3.04%	2.68%	6.20%	9.30%
	0.6	-23.82%	-11.82%	-4.10%	3.56%	8.21%	12.25%
	0.4	-27.30%	-13.38%	-4.60%	3.99%	9.15%	13.62%
	0.2	-29.45%	-14.31%	-4.91%	4.23%	9.70%	14.42%
	0.1	-30.33%	-14.69%	-5.03%	4.33%	9.92%	14.73%
	0.05	-30.80%	-14.88%	-5.09%	4.38%	10.03%	14.89%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-0.65%	-0.35%	-0.12%	0.11%	0.27%	0.40%
	0.99	-1.41%	-0.74%	-0.27%	0.24%	0.57%	0.86%
	0.95	-5.52%	-2.89%	-1.03%	0.93%	2.17%	3.28%
	0.9	-10.06%	-5.20%	-1.84%	1.64%	3.82%	5.76%
	0.8	-16.62%	-8.43%	-2.96%	2.60%	6.02%	9.03%
	0.6	-23.74%	-11.78%	-4.08%	3.55%	8.18%	12.19%
	0.4	-27.35%	-13.41%	-4.62%	4.00%	9.18%	13.66%
	0.2	-29.57%	-14.38%	-4.93%	4.26%	9.76%	14.51%
	0.1	-30.48%	-14.78%	-5.06%	4.36%	10.00%	14.85%
	0.05	-30.97%	-14.99%	-5.13%	4.42%	10.12%	15.03%
			L=10				
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-8.36%	-5.32%	-2.30%	2.75%	8.45%	18.35%
	0.99	-8.27%	-5.26%	-2.27%	2.71%	8.31%	18.00%
	0.95	-13.98%	-8.75%	-3.73%	4.36%	13.12%	27.54%
	0.9	-19.64%	-12.10%	-5.08%	5.82%	17.20%	35.21%
	0.8	-26.15%	-15.76%	-6.52%	7.31%	21.22%	42.46%
	0.6	-31.56%	-18.66%	-7.61%	8.40%	24.09%	47.46%
	0.4	-33.82%	-19.82%	-8.04%	8.81%	25.15%	49.25%
	0.2	-35.05%	-20.44%	-8.26%	9.03%	25.68%	50.12%
	0.1	-35.52%	-20.67%	-8.35%	9.10%	25.87%	50.41%
	0.05	-35.76%	-20.79%	-8.39%	9.14%	25.95%	50.53%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-1.58%	-1.00%	-0.43%	0.52%	1.59%	3.46%
	0.99	-3.31%	-2.10%	-0.90%	1.07%	3.29%	7.08%
	0.95	-11.31%	-7.05%	-2.99%	3.47%	10.37%	21.51%
	0.9	-18.11%	-11.11%	-4.65%	5.29%	15.55%	31.55%
	0.8	-25.50%	-15.34%	-6.33%	7.07%	20.47%	40.76%
	0.6	-31.45%	-18.58%	-7.58%	8.36%	23.94%	47.13%
	0.4	-33.89%	-19.87%	-8.06%	8.85%	25.25%	49.49%
	0.2	-35.24%	-20.57%	-8.33%	9.11%	25.94%	50.73%
	0.1	-35.76%	-20.84%	-8.43%	9.21%	26.20%	51.20%
	0.05	-36.04%	-20.98%	-8.48%	9.26%	26.34%	51.44%

Table 5.2 Relative Approximation Error in computing the minimum total cost, $\text{RAE} \times 100\%$, with $h = 0.6$, $s = 9$ and $A = 500$.

			L=2				
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-1.42%	-0.75%	-0.27%	0.25%	0.58%	0.88%
	0.99	-1.21%	-0.64%	-0.23%	0.21%	0.49%	0.75%
	0.95	-1.60%	-0.84%	-0.30%	0.27%	0.65%	0.98%
	0.9	-2.44%	-1.29%	-0.46%	0.42%	0.98%	1.49%
	0.8	-4.45%	-2.33%	-0.84%	0.75%	1.76%	2.67%
	0.6	-8.85%	-4.59%	-1.63%	1.45%	3.39%	5.11%
	0.4	-13.30%	-6.81%	-2.40%	2.12%	4.92%	7.39%
	0.2	-17.84%	-9.00%	-3.15%	2.76%	6.38%	9.55%
	0.1	-20.43%	-10.23%	-3.56%	3.11%	7.17%	10.71%
	0.05	-22.09%	-11.00%	-3.81%	3.32%	7.66%	11.42%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-0.23%	-0.12%	-0.04%	0.04%	0.09%	0.14%
	0.99	-0.30%	-0.16%	-0.06%	0.05%	0.12%	0.19%
	0.95	-0.97%	-0.51%	-0.18%	0.17%	0.39%	0.60%
	0.9	-1.96%	-1.04%	-0.37%	0.34%	0.79%	1.20%
	0.8	-4.16%	-2.18%	-0.78%	0.70%	1.64%	2.49%
	0.6	-8.78%	-4.55%	-1.62%	1.44%	3.36%	5.06%
	0.4	-13.35%	-6.84%	-2.41%	2.13%	4.94%	7.43%
	0.2	-17.98%	-9.08%	-3.17%	2.79%	6.44%	9.64%
	0.1	-20.62%	-10.33%	-3.59%	3.14%	7.25%	10.83%
	0.05	-22.30%	-11.11%	-3.86%	3.36%	7.75%	11.57%
			L=10				
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-3.38%	-2.15%	-0.93%	1.11%	3.42%	7.42%
	0.99	-2.88%	-1.83%	-0.79%	0.95%	2.91%	6.32%
	0.95	-3.75%	-2.38%	-1.03%	1.22%	3.75%	8.10%
	0.9	-5.55%	-3.51%	-1.51%	1.78%	5.44%	11.60%
	0.8	-9.44%	-5.91%	-2.52%	2.94%	8.84%	18.48%
	0.6	-16.42%	-10.10%	-4.24%	4.84%	14.27%	29.04%
	0.4	-21.95%	-13.30%	-5.52%	6.21%	18.06%	36.16%
	0.2	-26.50%	-15.85%	-6.51%	7.24%	20.87%	41.33%
	0.1	-28.72%	-17.06%	-6.98%	7.72%	22.15%	43.64%
	0.05	-30.02%	-17.76%	-7.24%	7.99%	22.87%	44.93%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-0.55%	-0.35%	-0.15%	0.18%	0.56%	1.21%
	0.99	-0.74%	-0.47%	-0.20%	0.24%	0.75%	1.62%
	0.95	-2.30%	-1.46%	-0.63%	0.75%	2.28%	4.91%
	0.9	-4.50%	-2.84%	-1.22%	1.44%	4.37%	9.28%
	0.8	-8.85%	-5.54%	-2.36%	2.74%	8.23%	17.16%
	0.6	-16.29%	-10.02%	-4.20%	4.80%	14.14%	28.74%
	0.4	-22.04%	-13.37%	-5.55%	6.24%	18.17%	36.40%
	0.2	-26.72%	-16.00%	-6.58%	7.33%	21.14%	41.94%
	0.1	-29.00%	-17.25%	-7.06%	7.83%	22.50%	44.43%
	0.05	-30.34%	-17.97%	-7.34%	8.11%	23.26%	45.83%

Table 5.3 Relative Approximation Error in computing the minimum total cost, $\mathrm{RAE} \times 100\%$, with $h = 0.6$, $A = 70$ and $s = 3$.

			L=2				
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-3.64%	-1.93%	-0.70%	0.63%	1.48%	2.25%
	0.99	-3.07%	-1.63%	-0.59%	0.53%	1.25%	1.90%
	0.95	-3.82%	-2.02%	-0.73%	0.66%	1.54%	2.34%
	0.9	-5.48%	-2.88%	-1.03%	0.93%	2.18%	3.31%
	0.8	-9.07%	-4.72%	-1.68%	1.50%	3.50%	5.28%
	0.6	-15.39%	-7.84%	-2.76%	2.43%	5.63%	8.45%
	0.4	-20.25%	-10.15%	-3.53%	3.09%	7.13%	10.65%
	0.2	-24.16%	-11.94%	-4.13%	3.58%	8.24%	12.28%
	0.1	-26.03%	-12.78%	-4.40%	3.81%	8.75%	13.02%
	0.05	-27.11%	-13.25%	-4.55%	3.94%	9.03%	13.43%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-0.60%	-0.32%	-0.11%	0.10%	0.24%	0.37%
	0.99	-0.76%	-0.40%	-0.15%	0.13%	0.31%	0.47%
	0.95	-2.26%	-1.19%	-0.43%	0.39%	0.91%	1.37%
	0.9	-4.35%	-2.28%	-0.82%	0.73%	1.72%	2.60%
	0.8	-8.44%	-4.38%	-1.56%	1.39%	3.24%	4.89%
	0.6	-15.25%	-7.77%	-2.73%	2.40%	5.57%	8.36%
	0.4	-20.35%	-10.21%	-3.55%	3.11%	7.17%	10.72%
	0.2	-24.40%	-12.07%	-4.18%	3.63%	8.35%	12.45%
	0.1	-26.33%	-12.95%	-4.46%	3.87%	8.89%	13.23%
	0.05	-27.45%	-13.44%	-4.63%	4.00%	9.19%	13.68%
			L=10				
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-8.26%	-5.25%	-2.27%	2.71%	8.35%	18.13%
	0.99	-7.05%	-4.48%	-1.94%	2.31%	7.12%	15.45%
	0.95	-8.52%	-5.39%	-2.32%	2.76%	8.43%	18.09%
	0.9	-11.52%	-7.24%	-3.09%	3.63%	10.96%	23.10%
	0.8	-16.99%	-10.50%	-4.43%	5.08%	15.07%	30.94%
	0.6	-24.32%	-14.68%	-6.08%	6.82%	19.79%	39.55%
	0.4	-28.64%	-17.04%	-6.98%	7.73%	22.20%	43.82%
	0.2	-31.52%	-18.56%	-7.54%	8.29%	23.66%	46.33%
	0.1	-32.74%	-19.18%	-7.77%	8.51%	24.23%	47.29%
	0.05	-33.40%	-19.52%	-7.90%	8.62%	24.51%	47.76%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-1.45%	-0.92%	-0.40%	0.48%	1.46%	3.18%
	0.99	-1.84%	-1.17%	-0.50%	0.60%	1.85%	4.01%
	0.95	-5.14%	-3.24%	-1.39%	1.64%	5.00%	10.64%
	0.9	-9.23%	-5.78%	-2.46%	2.87%	8.61%	17.98%
	0.8	-15.85%	-9.77%	-4.11%	4.70%	13.86%	28.27%
	0.6	-24.10%	-14.54%	-6.01%	6.74%	19.54%	38.99%
	0.4	-28.79%	-17.14%	-7.02%	7.78%	22.39%	44.25%
	0.2	-31.88%	-18.80%	-7.66%	8.43%	24.12%	47.40%
	0.1	-33.19%	-19.49%	-7.92%	8.69%	24.82%	48.67%
	0.05	-33.91%	-19.87%	-8.06%	8.83%	25.20%	49.35%

Table 5.4 Relative Approximation Error in computing the minimum total cost, $\mathrm{RAE} \times 100\%$, with $h = 0.6$, $A = 500$ and $s = 3$.

L=2							
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-1.42%	-0.75%	-0.27%	0.24%	0.58%	0.88%
	0.99	-1.12%	-0.59%	-0.21%	0.19%	0.45%	0.69%
	0.95	-1.04%	-0.55%	-0.20%	0.18%	0.42%	0.64%
	0.9	-1.23%	-0.65%	-0.24%	0.21%	0.50%	0.76%
	0.8	-1.87%	-0.99%	-0.36%	0.32%	0.75%	1.14%
	0.6	-3.67%	-1.93%	-0.69%	0.62%	1.46%	2.21%
	0.4	-6.04%	-3.15%	-1.12%	1.01%	2.35%	3.55%
	0.2	-9.21%	-4.76%	-1.69%	1.50%	3.50%	5.26%
	0.1	-11.49%	-5.90%	-2.08%	1.84%	4.28%	6.44%
	0.05	-13.18%	-6.73%	-2.37%	2.09%	4.85%	7.28%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-0.22%	-0.12%	-0.04%	0.04%	0.09%	0.14%
	0.99	-0.21%	-0.11%	-0.04%	0.04%	0.09%	0.13%
	0.95	-0.40%	-0.21%	-0.08%	0.07%	0.16%	0.25%
	0.9	-0.74%	-0.39%	-0.14%	0.13%	0.30%	0.46%
	0.8	-1.55%	-0.82%	-0.29%	0.27%	0.63%	0.95%
	0.6	-3.58%	-1.88%	-0.67%	0.61%	1.42%	2.15%
	0.4	-6.12%	-3.19%	-1.14%	1.02%	2.38%	3.60%
	0.2	-9.44%	-4.88%	-1.73%	1.54%	3.59%	5.41%
	0.1	-11.80%	-6.07%	-2.14%	1.90%	4.41%	6.64%
	0.05	-13.55%	-6.93%	-2.44%	2.16%	5.00%	7.51%
L=10							
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=200	0.999	-3.37%	-2.14%	-0.93%	1.10%	3.40%	7.39%
	0.99	-2.67%	-1.70%	-0.73%	0.88%	2.70%	5.86%
	0.95	-2.50%	-1.59%	-0.69%	0.82%	2.52%	5.47%
	0.9	-2.93%	-1.86%	-0.80%	0.96%	2.94%	6.35%
	0.8	-4.32%	-2.74%	-1.18%	1.39%	4.25%	9.10%
	0.6	-7.94%	-4.98%	-2.12%	2.48%	7.47%	15.64%
	0.4	-12.08%	-7.49%	-3.17%	3.65%	10.84%	22.28%
	0.2	-16.78%	-10.28%	-4.30%	4.88%	14.32%	28.94%
	0.1	-19.68%	-11.96%	-4.97%	5.60%	16.31%	32.65%
	0.05	-21.62%	-13.07%	-5.41%	6.06%	17.57%	34.98%
	P	$\varphi = -0.8$	$\varphi = -0.5$	$\varphi = -0.2$	$\varphi = 0.2$	$\varphi = 0.5$	$\varphi = 0.8$
β=9000	0.999	-0.53%	-0.34%	-0.15%	0.18%	0.54%	1.17%
	0.99	-0.51%	-0.33%	-0.14%	0.17%	0.52%	1.12%
	0.95	-0.98%	-0.62%	-0.27%	0.32%	0.99%	2.14%
	0.9	-1.77%	-1.12%	-0.48%	0.58%	1.76%	3.80%
	0.8	-3.61%	-2.28%	-0.98%	1.16%	3.53%	7.52%
	0.6	-7.75%	-4.86%	-2.07%	2.42%	7.27%	15.22%
	0.4	-12.24%	-7.59%	-3.21%	3.70%	11.00%	22.63%
	0.2	-17.19%	-10.54%	-4.42%	5.03%	14.76%	29.92%
	0.1	-20.21%	-12.31%	-5.12%	5.79%	16.89%	33.97%
	0.05	-22.23%	-13.46%	-5.58%	6.28%	18.25%	36.52%

The above results justify why it is hard for us to recommend the independent demand model in order to handle the autocorrelated demand. For this reason, we have introduced the Relative Approximation Error in computing the minimum total cost and we believe that the values of RAE give to any researcher the option to evaluate the approximation errors in computing the target inventory measures. So, we leave him/her to decide regarding to the parameter values for which he/she thinks that the independent demand model constitutes a good approximation for the autocorrelated demand.

But, at this point, it is important to note that the above analysis is conducted for the case in which the demand parameters are known. In the remaining of this section, we investigate under which conditions the independent demand model can be a good approximation to handle the autocorrelated demand assuming, however, that unknown demand parameters exist. In particular, through simulation analysis which is conducted for the same parameter values of chapter 4 (see Table 4.1) we compute the coverage which is the percentage of the 10.000 confidence intervals of the independent demand model which are given in (4.7) containing the true quantity $C_{HW}^{AR^*}$ of an AR(1) demand process. This true quantity is given in (5.10).

Specially, using the random number generator (see chapter 4 for more details) we generated 10.000 replications of maximum size 1000 observations. To achieve stationarity in each replication of the AR(1) model, D_t was generated from the stationary distribution $N\left(300, 60^2\right)$. Then, for each replication and for different sample sizes we estimated the parameters δ, φ, σ_ε^2, μ_t and σ_t^2 using (5.15)-(5.19) respectively. Replacing the ML estimators of φ, σ_ε^2 and μ_t in (5.10), corresponding estimates for the minimum cost of the autocorrelated demand model were computed for different combinations of parameter values. Finally, replacing $\hat{\mu}_t$ and $\hat{\sigma}_t^2$ in the asymptotic form (4.7) a set of 10.000 different confidence intervals for the minimum cost of the independent demand model were computed. To evaluate the performance of the estimated confidence intervals we developed the coverage (COV) at 95% nominal confidence level which is the actual probability the interval of the independent demand model to include the true minimum cost $C_{HW}^{AR^*}$ when in fact AR(1) autocorrelated demand occurs.

The values of COVs are presented in Tables 5.5-5.8 for different combinations of A, s , L, n, φ and β. We find out that when φ is approaching -1 or 1 the coverages are getting smaller indicating reduction in the accuracy of the independent model. For example, with $L=2$, $\beta=1000$, $A=70$, $s=3$, $n=500$ and $P=0.999$, COV is 84% for $\varphi=0.2$ while for $\varphi=0.8$ is 36% (see Table 5.7). On the contrary, COVs increase when the stockout probabilities are getting larger. For example, with $L=10$, $\beta=200$, $A=500$, $s=9$, $n=100$ and $\varphi=0.5$, COV is 20% for P=0.999 while is 5% for P=0.9 (see Table 5.8). However, even when small P exists there are cases for which COVs increase. For example, with $L=10$, $\beta=200$, $A=500$, $s=3$, $n=25$ and $\varphi=-0.8$, COV is 78% for $P=0.999$ while for $P=0.9$ is 94% (see Table 5.5). Concerning the value of L we observe that when lead-time increases then the coverages are getting smaller. For instance, having the parameter values $s=3$, $A=70$, $\beta=200$, $\varphi=-0.2$, $P=0.999$, $n=50$ and lead-time equal to 2 the coverage is 96%, while on the contrary COV is 79% for $L=10$ (see Table 5.5). Regarding the cost parameters A and s, we observe that when the ordering cost is getting larger then COVs increase. For example, with $\varphi=-0.8$, $\beta=200$, $L=2$, $n=100$, $s=9$ and $P=0.999$, COV is 1% for $A=70$ while is 93% for $A=500$ (see Table 5.6). On the other hand, the coverages are getting smaller when the shortage cost increases. For instance, with $\varphi=-0.2$, $\beta=1000$, $L=10$, $A=70$, $n=50$ and $P=0.9$, COV is 78% for $s=3$ while for $s=9$ is 67% (see Tables 5.5 and 5.6). Finally, regarding the value of β it is observed that when the reference period is getting larger, COVs either increase or decrease according to the parameter values. For example, with $\varphi=0.5$, $s=3$, $A=70$, $L=2$, $n=100$ and $P=0.999$, COV is 59% for $\beta=200$ and 69% for $\beta=1000$ (see Table 5.7). If P is decreasing further and reaching large stockout probabilities then coverages are getting marginally lower for larger values of β. For example, with $P=0.2$, COV is 22.58% for $\beta=200$ while is 22.54% for $\beta=1000$. For the sake of brevity, we don't display these results since the differences are quite small.

Hence, from the results of Tables 5.5-5.8, we summarize that according to the sample size n for negative autocorrelation acceptable COVs, namely greater than 90%, are achieved when:

i) for A=70, s=3 and L=2, the value for φ is either -0.2 or -0.5 (for $\varphi=-0.5$ except in the case of β=200 and P=0.9)

ii) for A=70, s=3 and L=2, the value for φ is -0.8, P=0.999 and β=1000

iii) for A=70, s=3 and L=10, the value for φ is -0.2 except in the case of β=200 and P=0.9

iv) for A=500 and s=3, there is at least one acceptable COV for each combination of P, β, φ and L except in the case of $\varphi=-0.8$, L=10, β=200, and P=0.999

v) for A=70, s=9 and L=2, the value for φ is either -0.2 or -0.5 (for $\varphi = -0.5$ except in the case of P=0.9)

vi) for A=70, s=9 and L=2, the value for φ is -0.8, P=0.999 and β=1000

vii) for A=70, s=9 and L=10, the value for φ is -0.2 and P=0.999

viii) for A=500 and s=9, there is at least one acceptable COV for each combination of P, β, φ and L except in the cases of L=10 and a) $\varphi = -0.8$, P=0.999 and β=200, b) P=0.9 and φ is either -0.8 or -0.5.

On the other hand, for positive autocorrelation acceptable COVs, namely greater than 85%, are achieved when:

i) for A=70, s=3 and L=2, the value for φ is 0.2

ii) for A=500, s=3 and L=2, the value for φ is 0.2

iii) for A=500, s=3 and L=10, the value for φ is 0.2 except in the case of P=0.999 and β=200

iv) for A=70, s=9 and L=2, the value for φ is 0.2 and P=0.999

v) for A=500, s=9 and L=2, the value for φ is 0.2

vi) for A=500, s=9 and L=10, the value for φ is 0.2, P=0.999 and β=1000.

Under the above analysis, we recommend to practice that better coverages are generally achieved when:

a) for positive autocorrelation, φ approaches low autocorrelation levels,

b) for negative autocorrelation, φ approaches low or moderate autocorrelation levels,

c) large cycle service levels, P, exist,

d) the lead-time, L, is getting smaller,

e) the shortage cost, s, is getting smaller,

f) the ordering cost, A, increases.

For the reference period, β, COV either decreases or increases according to the parameter values. These results strengthen the previous findings of Tables 5.1-5.4, confirming the claim which mentions that the choice of the independent demand model when in fact AR(1) autocorrelated demand occurs cannot be taken independently of the parameter values A, s, L, n, φ and β.

Table 5.5 Coverages of the 95% estimated confidence intervals when $\varphi < 0$ and s=3.

			L=2			L=10		
A=70		**n**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**
P=0.999	β=200	25	68%	95%	97%	8%	44%	90%
		50	20%	85%	96%	0%	12%	79%
		100	1%	56%	94%	0%	1%	59%
		500	0%	0%	71%	0%	0%	1%
		1000	0%	0%	43%	0%	0%	0%
	β=1000	25	100%	99%	97%	54%	88%	95%
		50	98%	99%	97%	10%	63%	93%
		100	75%	96%	97%	0%	22%	86%
		500	0%	45%	93%	0%	0%	31%
		1000	0%	7%	87%	0%	0%	5%
P=0.9	β=200	25	33%	87%	96%	4%	34%	88%
		50	5%	63%	95%	0%	7%	75%
		100	0%	26%	91%	0%	0%	51%
		500	0%	0%	50%	0%	0%	0%
		1000	0%	0%	19%	0%	0%	0%
	β=1000	25	46%	92%	96%	6%	40%	90%
		50	8%	74%	95%	0%	10%	78%
		100	0%	38%	92%	0%	0%	56%
		500	0%	0%	60%	0%	0%	1%
		1000	0%	0%	28%	0%	0%	0%
A=500		**n**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**
P=0.999	β=200	25	100%	99%	97%	78%	94%	96%
		50	100%	99%	98%	24%	78%	94%
		100	94%	98%	98%	1%	41%	90%
		500	0%	67%	95%	0%	0%	47%
		1000	0%	23%	91%	0%	0%	14%
	β=1000	25	100%	100%	97%	100%	99%	97%
		50	100%	100%	98%	99%	98%	97%
		100	100%	100%	98%	87%	95%	97%
		500	93%	98%	97%	0%	32%	89%
		1000	40%	93%	97%	0%	2%	77%
P=0.9	β=200	25	100%	99%	97%	94%	96%	96%
		50	100%	99%	98%	47%	88%	95%
		100	98%	99%	98%	4%	60%	92%
		500	2%	80%	96%	0%	0%	60%
		1000	0%	43%	93%	0%	0%	28%
	β=1000	25	100%	100%	97%	99%	98%	97%
		50	100%	100%	98%	90%	96%	96%
		100	100%	99%	98%	33%	84%	95%
		500	41%	93%	97%	0%	4%	78%
		1000	1%	77%	96%	0%	0%	56%

Table 5.6 Coverages of the 95% estimated confidence intervals when $\varphi < 0$ and s=9.

			L=2			L=10		
A=70		**n**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**
P=0.999	**β=200**	**25**	67%	95%	97%	8%	43%	90%
		50	20%	84%	96%	0%	11%	79%
		100	1%	55%	94%	0%	0%	58%
		500	0%	0%	70%	0%	0%	1%
		1000	0%	0%	42%	0%	0%	0%
	β=1000	**25**	100%	99%	97%	50%	87%	95%
		50	98%	99%	97%	8%	60%	92%
		100	70%	96%	97%	0%	20%	85%
		500	0%	41%	93%	0%	0%	29%
		1000	0%	5%	86%	0%	0%	4%
P=0.9	**β=200**	**25**	10%	61%	94%	2%	22%	83%
		50	1%	29%	90%	0%	3%	66%
		100	0%	5%	80%	0%	0%	39%
		500	0%	0%	20%	0%	0%	0%
		1000	0%	0%	2%	0%	0%	0%
	β=1000	**25**	11%	63%	94%	2%	23%	84%
		50	1%	30%	90%	0%	3%	67%
		100	0%	5%	81%	0%	0%	40%
		500	0%	0%	22%	0%	0%	0%
		1000	0%	0%	3%	0%	0%	0%
A=500		**n**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**	**φ=-0.8**	**φ=-0.5**	**φ=-0.2**
P=0.999	**β=200**	**25**	100%	99%	97%	78%	94%	96%
		50	100%	99%	98%	23%	77%	94%
		100	93%	98%	98%	1%	40%	90%
		500	0%	66%	95%	0%	0%	46%
		1000	0%	23%	91%	0%	0%	14%
	β=1000	**25**	100%	100%	97%	100%	99%	97%
		50	100%	100%	98%	99%	98%	97%
		100	100%	100%	98%	85%	95%	97%
		500	93%	98%	97%	0%	30%	88%
		1000	37%	93%	97%	0%	2%	76%
P=0.9	**β=200**	**25**	99%	99%	97%	26%	76%	95%
		50	79%	97%	97%	2%	39%	90%
		100	20%	88%	96%	0%	6%	79%
		500	0%	10%	87%	0%	0%	13%
		1000	0%	0%	75%	0%	0%	1%
	β=1000	**25**	100%	99%	97%	38%	84%	95%
		50	92%	98%	97%	4%	52%	92%
		100	40%	93%	97%	0%	13%	83%
		500	0%	22%	90%	0%	0%	22%
		1000	0%	1%	81%	0%	0%	2%

Table 5.7 Coverages of the 95% estimated confidence intervals when $\varphi > 0$ and s=3.

			L=2			**L=10**		
A=70		**n**	**φ=0.2**	**φ=0.5**	**φ=0.8**	**φ=0.2**	**φ=0.5**	**φ=0.8**
P=0.999	**β=200**	**25**	87%	68%	38%	74%	19%	2%
		50	86%	65%	39%	64%	5%	0%
		100	85%	59%	36%	45%	0%	0%
		500	69%	23%	15%	1%	0%	0%
		1000	51%	6%	4%	0%	0%	0%
	β=1000	**25**	87%	70%	40%	83%	46%	12%
		50	87%	70%	43%	79%	30%	6%
		100	88%	69%	43%	71%	12%	1%
		500	84%	57%	36%	24%	0%	0%
		1000	79%	43%	28%	5%	0%	0%
P=0.9	**β=200**	**25**	86%	65%	36%	72%	16%	2%
		50	85%	60%	34%	61%	4%	0%
		100	82%	50%	30%	40%	0%	0%
		500	57%	9%	6%	0%	0%	0%
		1000	33%	1%	1%	0%	0%	0%
	β=1000	**25**	86%	66%	37%	73%	19%	2%
		50	85%	63%	37%	64%	6%	0%
		100	84%	54%	33%	45%	0%	0%
		500	63%	15%	9%	1%	0%	0%
		1000	42%	3%	2%	0%	0%	0%
A=500		**n**	**φ=0.2**	**φ=0.5**	**φ=0.8**	**φ=0.2**	**φ=0.5**	**φ=0.8**
P=0.999	**β=200**	**25**	87%	70%	40%	84%	52%	18%
		50	87%	71%	43%	81%	38%	10%
		100	88%	70%	44%	75%	21%	3%
		500	86%	62%	40%	35%	0%	0%
		1000	82%	51%	33%	11%	0%	0%
	β=1000	**25**	87%	70%	40%	86%	66%	35%
		50	88%	72%	44%	87%	63%	32%
		100	88%	72%	46%	86%	55%	26%
		500	88%	71%	47%	74%	18%	2%
		1000	88%	69%	45%	60%	4%	0%
P=0.9	**β=200**	**25**	87%	70%	40%	85%	56%	23%
		50	88%	71%	43%	83%	46%	15%
		100	88%	71%	44%	79%	30%	6%
		500	87%	65%	42%	46%	1%	0%
		1000	84%	56%	36%	21%	0%	0%
	β=1000	**25**	87%	70%	40%	86%	62%	30%
		50	88%	72%	43%	85%	56%	24%
		100	88%	72%	45%	83%	44%	15%
		500	88%	69%	45%	63%	5%	0%
		1000	86%	64%	42%	42%	0%	0%

Table 5.8 Coverages of the 95% estimated confidence intervals when $\varphi > 0$ and s=9.

			L=2			L=10		
A=70		**n**	**φ=0.2**	**φ=0.5**	**φ=0.8**	**φ=0.2**	**φ=0.5**	**φ=0.8**
P=0.999	**β=200**	**25**	87%	68%	38%	74%	18%	2%
		50	86%	65%	39%	64%	5%	0%
		100	85%	58%	36%	45%	0%	0%
		500	69%	22%	14%	1%	0%	0%
		1000	51%	6%	4%	0%	0%	0%
	β=1000	**25**	87%	70%	40%	82%	45%	12%
		50	87%	70%	43%	79%	28%	5%
		100	88%	69%	43%	70%	11%	1%
		500	84%	56%	36%	22%	0%	0%
		1000	79%	42%	27%	4%	0%	0%
P=0.9	**β=200**	**25**	84%	58%	27%	69%	11%	1%
		50	81%	48%	24%	55%	2%	0%
		100	75%	33%	17%	32%	0%	0%
		500	34%	1%	1%	0%	0%	0%
		1000	10%	0%	0%	0%	0%	0%
	β=1000	**25**	84%	58%	28%	69%	12%	1%
		50	82%	49%	25%	56%	2%	0%
		100	76%	34%	18%	33%	0%	0%
		500	36%	1%	1%	0%	0%	0%
		1000	12%	0%	0%	0%	0%	0%
A=500		**n**	**φ=0.2**	**φ=0.5**	**φ=0.8**	**φ=0.2**	**φ=0.5**	**φ=0.8**
P=0.999	**β=200**	**25**	87%	70%	40%	84%	52%	18%
		50	87%	71%	43%	81%	38%	10%
		100	88%	70%	44%	75%	20%	3%
		500	86%	62%	40%	35%	0%	0%
		1000	82%	51%	33%	11%	0%	0%
	β=1000	**25**	87%	70%	40%	86%	66%	34%
		50	88%	72%	44%	87%	63%	32%
		100	88%	72%	46%	86%	55%	25%
		500	88%	71%	47%	74%	17%	2%
		1000	87%	69%	45%	59%	3%	0%
P=0.9	**β=200**	**25**	87%	69%	39%	80%	37%	8%
		50	87%	69%	42%	75%	20%	2%
		100	87%	66%	42%	63%	5%	0%
		500	80%	45%	29%	12%	0%	0%
		1000	71%	27%	17%	1%	0%	0%
	β=1000	**25**	87%	70%	40%	82%	42%	11%
		50	87%	70%	42%	78%	26%	4%
		100	87%	68%	42%	67%	9%	1%
		500	82%	51%	32%	18%	0%	0%
		1000	75%	34%	22%	2%	0%	0%

5.6 Summary

In this chapter, considering the continuous review (Q,R) inventory model with the cost function proposed by Hadley & Whitin (1963) we developed a procedure to determine the target inventory measures when the issue of both correlated demand and unknown demand parameters was addressed. Considering that the demand process follows the first order autoregressive model, AR(1), we developed for the first time estimators for the optimal order quantity, optimal reorder point and minimum total cost, using the maximum likelihood estimators for the stationary mean, the stationary variance and the theoretical autocorrelation coefficient at lag one.

At first, we studied under which conditions the independent demand model can be a good approximation to handle the autocorrelated demand. The investigation was based on the size of approximation error in computing the minimum total cost (RAE) for the case in which the demand parameters are known. While, on the other hand, when unknown demand parameters exist the investigation was based on the actual probability (COV) the prediction interval of the independent demand model to include the true minimum cost of an AR(1) demand process.

From the analysis performed, we concluded that the use of the independent model, apart from the size of the first order autocorrelation coefficient, depends on the sizes of A, s, L, n and β. In particular, we find out that better approximation is achieved when:

a) for positive autocorrelation, φ approaches low autocorrelation levels,

b) for negative autocorrelation, φ approaches low or moderate autocorrelation levels,

c) large cycle service levels, P, exist,

d) the lead-time, L, is getting smaller,

e) the shortage cost, s, is getting smaller,

f) the reference period, β, increases but in accordance with the values of the other parameters.

Based on these remarks, we would not recommend to practice any flat permissible sizes for the parameter values in order to make the choice of an independent demand model when the demand data generated by autocorrelated structure. We believe that this choice should be left to the subjective criteria of the researchers. Using the analysis and the results of this chapter we help them to set their own rules for deciding when the independent demand model can give acceptable approximations to target inventory measures in the case where autocorrelated demand is not negligible.

Chapter 6

Contributions, Conclusions and Extensions of the Thesis

6.1 Introduction

In this chapter, we discuss the main issues addressed in our research and summarize our contributions. Further, we identify the limitations of our theoretical work and suggest avenues for further research.

This research aspires to take forward the current state of knowledge on the continuous review (Q,R) inventory model.

The objectives of this research as stated in chapter 1 of the thesis are as follows:

1. To examine the convexity problem and to identify the unique minimum for a cost function with exact or approximate expressions for the expected on-hand inventory level at any point in time
2. To develop algorithms in order the optimal solution to be attained
3. To derive optimal policies for unimodal and J-shaped lead-time demand distributions
4. To investigate how the values of the cost parameters affect the optimal solution
5. To develop estimators for target inventory measures
6. To derive asymptotic confidence interval for a cost function
7. To test the validity of the theoretical results on a set of generated data through Monte-Carlo simulation
8. To identify optimal solutions for correlated demand.

All the objectives have been achieved and the contributions of the thesis are summarized in the following section.

6.2 Contributions

Our contributions can be summarized as follows:

- A theoretically optimization procedure for the Hadley & Whitin's (1963) cost function which results from the sum of the expected annual ordering, inventory carrying and shortage costs is proposed. Using exact or approximate expression for the expected on-hand inventory level at any point in time, new efficient optimization procedures are presented. (Objective 1)

- For the Hadley & Whitin's (1963) cost function the conditions under which convexity exists have derived and displayed. New algorithms are proposed which ensure the unique minimum of this cost function. (Objectives 1 and 2)

- The effects of the cost parameters on the target inventory measures are studied. Threshold values for the cost parameters are obtained in order the unique minimum to be attained either through mathematical optimization or when the optimal reorder point takes on the value zero. (Objectives 1, 2, 3 and 4)

- Optimal solutions are derived for different lead-time demand distributions, namely either J-shaped or unimodal. The managerial implications of increasing lead-time demand variability on the optimal target inventory measures are investigated. (Objectives 1, 2 and 3)

- For the first time estimators for target inventory measures are developed for either independent or correlated demand. (Objectives 5, 7 and 8)

- Based on the asymptotic properties of the estimators, confidence intervals are derived for the true minimum value of the cost function (Objective 6)

- Coverage is used as a statistical measure in order to evaluate the performance of the confidence interval whose validity is tested by means of Monte-Carlo simulation on theoretically generated data. (Objectives 6 and 7)

- The effects of the autocorrelated demand on the optimal solutions are investigated. (Objective 8)

6.3 Conclusions

In this section the main issues explored in this Ph.D. research are drawn together and our conclusions are discussed.

To the extent of our knowledge, for the continuous review (Q,R) inventory system with backorders and fixed lead-time there are many unsolved practical problems. Considering a single echelon structure for the inventory system, we want to determine the optimal order quantity Q and the optimal reorder point R for a given item after minimizing the total cost function. If the system stocks more than a single item, then we assume that there are no interactions between them. Thus, for the average total cost function we study the convexity problem and we explore the existence of a unique minimum either for exact or approximate expressions for the expected on-hand inventory level at any point in time. In particular, we explore the effects of the cost parameters on the optimal solution and we develop algorithms for finding the minimum cost.

Subsequently, since there are not studies which are addressed with the effects of demand estimation on target inventory measures neither for autocorrelated (ARMA) nor independent demand processes, for the first time we estimate the unknown parameters of demand distribution and we develop estimators for the target inventory measures which are explored through Monte-Carlo simulation. Further, we investigate the effect of a serially correlated demand process on the optimal inventory policies.

Our conclusions can be summarized as follows:

A) Exact cost function:

A.1) Independent demand with known demand parameters

- Under J-shaped and unimodal lead-time demand distributions we derive a general condition to identify when the minimum of the exact cost function

  (a) is obtained through mathematical optimization and

  (b) occurs when the reorder point takes on the value zero.

  Interval values of the cost parameters are obtained from the general condition in order the minimum cost to occur at zero reorder point. Further, the limits of these intervals are independent of the form of the lead-time demand distribution and to compute them we need, apart from the cost parameter values, the annual expected demand and the variance of the lead-time demand.

- We offer an algorithm for finding the minimum of the exact cost function. After some numerical experimentation, we observed that as the ordering cost increases we move from a situation where the unique minimum cost is attained at a positive reorder point to a situation where the minimum cost occurs at zero reorder point.

- We investigate the effects of the lead-time demand variability on the target inventory measures. As CV raises with fixed cost parameter values we result in larger optimal order quantities and larger minimum costs while the reorder points and cycle service levels decline. From the managerial aspects of inventory this means that as lead-time demand variability grows the optimal policies lead to excessively large orders, zero reorder points and higher minimum costs.

B) Hadley &Whitin cost function:

B.1) Independent demand with known demand parameters

- For the continuous review (Q, R) inventory model with backorders and fixed lead-time, when (a) the Hadley-Whitin (H-W) expression $(Q/2 + \text{safety stock})$ is used to evaluate the expected annual inventory carrying cost, and (b) the cost per unit backordered is used to calculate the annual expected shortage cost, we showed that, given a non-negative reorder point the convexity of the H-W cost function depends on the monotony of the first derivative of the lead-time probability density function.

- Selecting the class of unimodal lead-time demand distributions for which the probability density function vanishes at $R = 0$ and when $R \to \infty$, we derive general conditions for determining the optimal solution in terms of Q and R values ensuring the minimum of H-W cost function.

- We showed that a unique minimum cost can be found and the process of finding out the associated optimal order quantities and reorder points depends upon the values of the fixed ordering cost, the unit shortage cost, and the unit holding cost.

- The analysis demonstrated that as the unit shortage cost declines, or the fixed ordering cost increases, we move between three cases of optimal solutions which lead to a unique minimum cost. The three cases are:

  1. There is a unique optimal solution which is obtained after solving the equations resulted from the first-order conditions minimizing the H-W cost function.

  2. The minimum of the H-W cost function is attained after comparing the cost at $R = 0$ with the "local" minimum cost at the optimal solution obtained in case 1.

  3. The minimum of the H-W cost function occurs at $R = 0$.

- These three cases are integrated to a general algorithm, and its application is illustrated when the lead-time demand has the Normal and the Log-Normal distribution.

- The added value of the general algorithm in the relevant inventory literature is illustrated by comparing the minimum of HWCF taken after following the algorithm with the corresponding minimum of the exact cost function. The latter one is obtained by replacing in the cost function the H-W expression $\left(Q/2+\text{safety stock}\right)$ with the exact expression of the expected on-hand inventory at any point in time.

- Contrary to what is believed about the validity of the H-W expression, we show that valid approximations using this expression occur even when the cycle service level (CSL) is zero, provided that the coefficient of variation is low, preferably below 1.

B.2) Independent demand with unknown demand parameters

- Considering fixed lead-time, normally distributed lead-time demand and by assuming that the values of demand parameters are unknown we develop an estimation process for the minimum value of the Hadley & Whitin's (1963) cost function.

- We make an estimation policy with the assumption that the cycle service level is constant in order to study asymptotically the statistical properties of the estimator for the minimum cost of the reference period.

- Using ML estimators for the parameters of the Normal distribution we develop, for the first time, estimators for the optimal order quantity, optimal reorder point and minimum cost of the reference period.

- Confidence interval for the minimum cost is derived whose validity is tested through Monte-Carlo simulations in different sample sizes. To evaluate the performance of the confidence interval, we consider the coverage (COV) as a statistical measure.

- Experimenting with different values of the cycle service level (CSL), we find out that when CSL decreases then the coverage is getting marginally lower.

- Acceptable COVs greater than 90% always exist for all the combinations of parameter values. For this reason, in order to give some crucial managerial recommendations regarding to the choice of parameter values of the continuous review model, we extend the analysis and examine also the expected half length of the estimated confidence intervals. We suggest to practice that better precision, namely smaller EHLs, are achieved when either the sample size and the cycle service level increase or the lead-time, the reference period and the shortage cost are getting smaller.

- Investigation about the use of the Hadley-Whitin's cost function instead of the exact cost function has been conducted. Using simulation analysis we find out that the actual probability the confidence interval for the minimum value of the H-W cost function to include the true minimum value of the exact cost function approaches sizes larger than 90% for all the combinations of parameter values. This means that with unknown demand parameters the H-W cost function approximates well the exact cost function even when large stockout probabilities exist.

B.3) Correlated demand with unknown demand parameters

- We develop a procedure to determine the target inventory measures when the issue of both correlated demand and unknown demand parameters is addressed for the Hadley & Whitin's cost function.

- Through numerical experimentation we find out that when the first order autocorrelation coefficient, the lead-time, the stockout probability, the reference period, the ordering cost and the shortage cost increase then the minimum cost is getting larger.

- For the AR(1) model, we develop for the first time estimators for the optimal order quantity, optimal reorder point and minimum total cost, using the maximum likelihood estimators for the stationary mean, the stationary variance and the theoretical autocorrelation coefficient at lag one.

- We study under which conditions the independent demand model can be a good approximation to handle the autocorrelated demand. The investigation was based on a statistical measure which is related to the size of approximation error in computing the minimum total cost for the case in which the demand parameters are known. While, on the other hand, when unknown demand parameters exist the investigation was based on the actual probability the prediction interval of the independent demand model to include the true minimum cost when in fact the demand follows the AR(1) model.

- We find out that better approximation can be achieved when

  a) for positive autocorrelation, φ approaches low autocorrelation levels,

  b) for negative autocorrelation, φ approaches low or moderate autocorrelation levels,

  c) large cycle service levels, P, exist,

  d) the lead-time, L, is getting smaller,

  e) the shortage cost, s, is getting smaller,

  f) the ordering cost, A, increases**,**

  g) the reference period, β, increases but in accordance with the values of the other parameters.

- Concluding, we would not recommend to practice any flat permissible sizes for the parameter values in order to make the choice of an independent demand model when autocorrelated demand occurs. We believe that this choice should be left to the subjective criteria of the researchers.

6.4 Further research

In this section we summarize the limitations of the thesis and suggest avenues for further research.

- Our thesis is built around the continuous review (Q,R) inventory model. An interest direction for future research is the determination of corresponding inventory decision values for periodic review models after estimating the target inventory measures.

- This thesis focuses on a continuous review policy for which when the inventory position (on-hand plus on order minus backorders) drops to or falls below the reorder point R then an order of size Q is placed and is delivered after a fixed period of time (lead-time) has elapsed. For this review policy, we offer a theoretical work assuming either known or unknown demand parameters. An interesting avenue for further research may be the performance of this inventory policy for variable lead-time and *unknown demand parameters*.

- Based on the fact that in this study we examine the backorders model with either known or unknown demand parameters, an interesting aspect to investigate in the future is to consider the lost-sales model with *unknown demand parameters*.

- In this thesis, we examine the exact annual total cost function which results from the sum of the annual expected ordering, holding and shortage costs. Given, however, that different ways to compute the holding and shortage costs have been suggested in the relevant literature, this type of continuous review policy can be differentiated according to the form of the annual total cost objective function.

Therefore, interesting avenues to investigate in the future are to determine the target inventory measures when *unknown demand parameters* exist and we use:

a) Different approximate expressions for the expected on-hand inventory level at any point in time. Specially, except the Hadley & Whitin (1963) case which is examined in this work, in the literature many studies have been conducted which can be analyzed in the future (e.g., Holt et al., 1960; Wagner, 1975; Love, 1979; Yano, 1985).

b) Different shortage cost models. In this thesis, we consider a shortage cost per unit backordered. Other models suggested in the literature and can be used are either a fixed cost per stockout occasion or a shortage cost per unit backordered per year.

- In this work, we make the assumption that the unique minimum cost is obtained for positive reorder points. In order to find the unique minimum when the degeneracy problem exists, namely there is not optimal solution after minimizing the first order conditions of the cost function and thus the reorder point is negative, we consider that the unique minimum occurs at zero reorder point. Therefore, considering that the reorder point can take either positive or negative values, a further research which could be proved useful is to extend our theoretical work for the case in which we use the Lau et al.'s (2002b) cost function.

- In chapter 5, we investigate the effects of the correlated demand on the target inventory measures by using the relevant moments of the lead-time demand which give a fixed reorder point in each inventory cycle. An interesting direction for future research is to consider the method of forecasts which provides a variable reorder point and gives the chance to study the short term effect of the autocorrelation on the distribution parameters.

- In this thesis, the validity of our theoretical work was checked by means of simulation on theoretically generated data. A next stage of research may be intended to assess the performance of the theoretical work in empirical demand data series.

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