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Μεταπτυχιακή Εργασία

VARIATIONAL INEQUALITY FORMULATION OF A MULTICRITERIA SHIPPERS AND CONSUMERS NETWORK EQUILIBRIUM MODEL

υπό

ΚΩΝΣΤΑΝΤΙΝΟΥ ΤΣΕΝΤΖΙΟΥ

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Κώστας Τσέντζιος

VARIATIONAL INEQUALITY FORMULATION OF A MULTICRITERIA SHIPPERS AND CONSUMERS NETWORK EQUILIBRIUM MODEL

ΚΩΝΣΤΑΝΤΙΝΟΣ ΤΣΕΝΤΖΙΟΣ

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Περίληψη

Η μεταφορά φορτίων αποτελεί ουσιώδες στοιχείο μιας οικονομίας. Υποστηρίζει κάθε εμπορική δραστηριότητα εξασφαλίζοντας τη μετακίνηση και την έγκαιρη διαθεσιμότητα πρώτων υλών και ολοκληρωμένων αγαθών. Τα μεταφορικά έξοδα αποτελούν ένα σημαντικό τμήμα του τελικού κόστους ενός προϊόντος και αντιπροσωπεύουν ένα μεγάλο ποσοστό των εθνικών εξόδων μιας χώρας. Η πρόβλεψη της ροής φορτίων σε ένα δίκτυο μεταφορών μπορεί να φανεί πολύ χρήσιμη στη διαδικασία λήψης αποφάσεων ενός συστήματος μεταφοράς φορτίων.

Σε αυτή την μεταπτυχιακή εργασία παρουσιάζεται η θεωρία χωρικής ισορροπίας σε δίκτυα μεταφορών και μελετάται η θεωρία των variational inequalities, η οποία αποτελεί ισχυρό εργαλείο για τη μελέτη καταστάσεων ισορροπίας. Οι θεωρίες αυτές εφαρμόζονται αρχικά σε δίκτυα πολλαπλών βαθμίδων και στη συνέχεια σε δίκτυα πολλαπλών κλάσεων, όπου οι παράγοντες κάθε κλάσης λαμβάνουν υπόψη τους πολλαπλά κριτήρια.

Τέλος, περιγράφουμε ένα δίκτυο για τη μελέτη, ανάλυση και υπολογισμό λύσεων σε προβλήματα, όπου για τη λήψη αποφάσεων σε κάθε μία από τις δύο βαθμίδες του δικτύου, λαμβάνονται υπόψη πολλαπλά κριτήρια. Συγκεκριμένα, κάθε ένας από τους αποστολείς αγαθών και τους καταναλωτές των αγαθών αυτών στις διάφορες αγορές, αντιμετωπίζει πολλαπλά κριτήρια όσον αφορά αποφάσεις για την παραγωγή και κατανάλωση των αγαθών αντίστοιχα. Διατυπώνονται οι συνθήκες ισορροπίας με τη μορφή των variational inequalities, η επίλυση των οποίων δίνει τις ροές των φορτίων και τις αντίστοιχες τιμές σε ισορροπία.

Καθώς ο χρόνος και το κόστος που σχετίζονται με τις μεταφορές προϊόντων έχουν ιδιαίτερη σημασία, τα αποτελέσματα του μοντέλου που κατασκευάζουμε σε αυτή την εργασία, μπορούν να βοηθήσουν μία εταιρία στην αξιολόγηση των πραγματικών ροών και τιμών των προϊόντων.

UNIVERSITY OF THESSALY

SCHOOL OF ENGINEERING

DEPARTMENT OF MECHANICAL & INDUSTRIAL ENGINEERING

Postgraduate Work

VARIATIONAL INEQUALITY FORMULATION OF A MULTICRITERIA SHIPPERS AND CONSUMERS NETWORK EQUILIBRIUM MODEL

by

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The approval of this postgraduate work by the Department of Mechanical and Industrial Engineering of the School of Engineering of the University of Thessaly does not imply acceptance of the writer's opinions (Law 5343/32 article 202 par.2).

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Abstract

Freight transportation is a vital component of the economy. It supports production, trade and consumption activities by ensuring the efficient movement and timely availability of raw materials and finished goods. Transportation accounts for a significant part of the final cost of products and represents an important component of the national expenditures of any country. The prediction of freight flows over a multimodal network can be very helpful in the decision-making process in a freight transportation system.

In this postgraduate work, the theory of spatial equilibration in transport networks is first presented and the study of the variational inequalities (VI) theory, which is a powerful tool for the study of the equilibrium states, follows. These theories are applied first to multitiered networks and then to multiclass networks, in which the decision-makers consider multiple criteria.

Finally, we describe a network framework for the formulation, analysis and computation of solutions to problems in which the decision-makers on each of the two tiers of the network consider multiple criteria. In particular, the shippers, which are spatially separated and the consumers located at the demand markets, each face multiple criteria in making their production / consumption decisions. The variational inequality formulation of the governing equilibrium conditions is derived. Resolving this variational inequality problem, using known algorithms for variational inequalities, we get the product shipment pattern, as well as the demand price pattern in equilibrium.

Since time and cost associated with product deliveries are of particular importance, the outputs of the model we construct in this research, provide a benchmark against which an industry can evaluate both real prices and product flows.

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Chapter 1 Introduction

1.1 Motivation and Background

Freight transportation is a vital component of the economy. It supports production, trade and consumption activities by ensuring the efficient movement and timely availability of raw materials and finished goods. Transportation accounts for a significant part of the final cost of products and represents an important component of the national expenditures of any country.

The freight transportation industry must achieve high performance levels in terms of economic efficiency and quality of service. The former, because a transportation firm must make a profit while evolving in an increasingly open, competitive, and still mainly cost-driven market. The latter, because transportation services must conform to the high standards imposed by the current paradigms of production and management such as small or no inventory associated with just-in-time procurement, production and distribution, on-time personalized services, and customer-driven quality control of the entire logistics chain. For the transportation firm, these standards concern particularly total delivery time and service reliability, which are often translated into objectives such as “be there fast but within the specified limits” or “offer high quality service and consistent performance”.

The political evolution of the world impacts the transportation sector as well. The emergence of free trade zones together with the opening of new markets due to political changes and the resulting globalization of the economy have tremendous consequences for the evolution of transportation systems, not all of which are yet apparent or well understood. For example, open borders generally mean that firms are no longer under obligation to maintain a major distribution center in each country. In consequence, distribution systems are reorganized around fewer but bigger warehouses and transportation services are operated over longer distances. A significant increase in road traffic is a normal consequence of this process, as may be observed in Europe.

Changes to the regulatory environment have an equally powerful impact on the operation and competitive environment of transportation firms. The deregulation drive of the 1980s has seen governments remove numerous rules and restrictions, especially with regard to the entry of new firms in the market and the fixing of tariffs and routes. This resulted in a more competitive industry and in changes to the number and characteristics of transportation firms. A number of new policies and regulations resulting from quality-of-life concerns started to significantly impact the operations of the freight transportation-related firms. Two major examples: (i) more stringent safety regulations; (ii) policies aimed towards increasing the volume of inter (and multi) modal freight movements while decreasing the utilization of trucks. The latter results from environmental and energy efficiency concerns that are particularly important in Europe. The evolution of technology is another major factor that modifies how freight transportation is organized and operated. This is not a new trend. Indeed, transportation has followed the industrial innovations and adapted, for example, to advances in traction technologies and fuels. What is new is that the major technological factor inflecting the evolution of transportation has to do with information and software rather than the traditional hardware. The tremendous expansion of Internet and the electronic-society, eloquently illustrated by the growing importance of electronic market places and business-to-business exchanges, dramatically alters the interactions of carriers and shippers. More complex planning and operating procedures are a direct result of all new policies, requirements, technologies, and challenges.

Freight transportation must adapt to and perform within these rapidly changing political, social, and economic conditions and trends. In addition, freight transportation is itself in a complex domain: many different firms, organizations, and institutions, each with their own set of objectives and means, make up the industry; infrastructure and even service modifications usually require long implementation delays; important decision processes are often strongly interrelated. It is thus a domain where accurate and efficient methods and tools are required to assist and enhance the analysis, planning, operation, and control processes.

Demand for freight transportation derives from the interplay between *producers* and *consumers* and the significant distances that usually separate them. Producers of goods require transportation services to move raw materials and intermediate products, and to distribute final goods in order to meet demands. *Carriers* supply transportation services. Railways, shipping lines, trucking companies, and intermodal container and postal services are examples of carriers. Considering the type of service they provide, ports, intermodal platforms, and other such facilities may be described as carriers as well. *Shippers*, which may be producers of goods or some intermediary firm, attribute demand to supply. Governments contribute the infrastructure: roads and highways, as well as a significant portion of ports, internal navigation, and rail facilities. Governments also regulate (e.g. dangerous and toxic goods transportation) and tax the industry.

From all the above, one can say that transportation systems appear as rather complex organizations that involve a great deal of human and material resources and that exhibit intricate relationships and tradeoffs among the various decisions and management policies affecting their different components. It is convenient to classify these policies according to the following three planning levels:

1. *Strategic* (long-term) planning at the firm level typically involves the highest level of management and requires large capital investments over long-term horizons. Strategic decisions determine general development policies and broadly shape the operating strategies of the system. These include the design of the physical network and its evolution, the location of major facilities (e.g., terminals), the acquisition of major resources such as motive power units, and the definition of broad service and tariff policies.

Strategic planning also takes place at the international, national and regional levels, where the transportation networks or services of several carriers are simultaneously considered. National or regional transportation departments, consultants, international shippers and forwarders, for example, engage in this type of activity.

2. *Tactical* (medium-term) planning aims to determine, over a medium-term horizon, an efficient allocation and utilization of resources to achieve the best possible performance of the whole system. Typical tactical decisions concern the design of the service network and may include issues related to the determination of the routes and types of service to operate, service schedules, vehicle and traffic routing, repositioning of the fleet for use in the next planning period.

3. *Operational* (short-term) planning is performed by local management, yard masters and dispatchers, for example, in a highly dynamic environment where the time factor plays an important role and detailed representations of vehicles, facilities and activities are essential. Important operational decisions concern: the implementation and adjustment of schedules for services, crews, and maintenance activities; the routing and dispatching of vehicles and crews; the dynamic allocation of scarce resources.

This classification highlights how data flows among decision-making levels and how policy guidelines are set. The strategic level sets the general policies and guidelines for decisions taken at the tactical level, which determines goals, rules and limits for operational and real-time decisions. The data flow follows the reverse route, each level of planning supplying information essential for the decision making process at a higher level. This hierarchical relationship emphasizes the differences in scope, data, and complexity among the various planning issues, prevents the formulation of a unique model for the planning of freight transportation systems, and calls for different model formulations that address specific problems at particular levels of decision making.

Strategic planning issues involve the evolution of a given transportation system and its response to various modifications in its environment: changes to existing infrastructure, construction of new facilities, evolution of the "local" or international socio-economic environment resulting in modifications to the patterns and volumes of production and consumption, variations in energy prices, changes to labour conditions, new environment-motivated policies and legislation, carrier mergers, introduction of new technologies, and so on.

The prediction of freight flows over a multimodal network is an important component of transportation science and has attracted significant interest in recent years. One notes, however, that, perhaps due to the inherent difficulties and complexities of such problems, the study of freight flows at the national or regional level has not yet achieved full maturity, in contrast to passenger transportation where the prediction of car and transit flows over multimodal networks has been studied extensively and several of the research results have been transferred to practice.

A strategic planning tool appears as a set of models and procedures. Other than data manipulation (e.g., collection, fusion, updating, validation, etc.) and result analysis (e.g., cost-benefit, environmental impacts, energy consumption policies, etc.) tools, the main components are: (i) *Supply modelling* representing the transportation modes, infrastructure, carriers, services, and lines; vehicles and convoys; terminals and inter-modal facilities; capacities and congestion; economic, service, and performance measures and criteria. (ii) *Demand modelling* that captures the product definitions, identifies shippers and intermediaries and represents production, consumption, and zone-to-zone (region-to-region) distribution volumes, as well as mode choices. (iii) *Assignment* of multi-product flows (from the demand model) to the multi-mode network (the supply representation). This procedure simulates the behaviour of the transportation system and its output forms the basis for the analysis of the strategic plan. Therefore, it has to be both precise in reproducing current situation and general to produce robust analysis of future scenarios based on forecast data.

In this postgraduate work we focus on demand modelling and specifically on a class of models that is well studied for the prediction of interregional commodity flows, the *spatial price equilibrium model* and its variants. This class of models determines simultaneously the flows between producing and consuming regions, as well as the selling and buying prices that satisfy the spatial equilibrium conditions. Simply stated, a spatial equilibrium is reached provided that for all pairs of supply and demand regions with a positive commodity flow, the unit supply price plus the unit transportation cost is equal to the unit demand price; the sum is larger than this price for all pairs of regions with no exchanges. The transportation network used in these models is usually represented in a simplistic way (bipartite networks). These models deal mostly with specific products which have a particular importance, such as crude oil, coal or milk products.

The main contribution of the research reported in this work is that we describe a network framework for the formulation, analysis and computation of solutions to problems in which the decision-makers on each of the two tiers of the network consider multiple criteria. The model we construct, brings together multicriteria decision-makers on the production side and on the consumption side in a network framework. In particular, the shippers, which are spatially separated and the consumers located at the demand markets, each face multiple criteria in making their production / consumption decisions. The *variational inequality* formulation of the governing equilibrium conditions is derived. Resolving this variational inequality problem, using known algorithms for variational inequalities, we get the product shipment pattern, as well as the demand price pattern in equilibrium.

1.2 Literature Review

Freight transportation has always played an important role in both regional and national economics by being a vital link between the supply and demand in the supply chain. However, in contrast to the research on urban transportation, freight transportation assignment modeling has received little attention because of the inherent difficulties and the complexities of the interactions among the components of the system.

The earlier freight transportation models achieved system optimal equilibrium using shortest path calculations. One of the first such multimodal predictive freight models was introduced by Kresge and Roberts [15]. Bronzini [5] developed a non-linear programming formulation based on non-linear cost and delay functions obtained by simulating different railway and waterway operating environments. It was a fixed demand, multicommodity, multimodal freight network model where freight routing results were exclusively from the decisions of shippers seeking the minimum cost paths.

The first approach that took into consideration the roles of both shippers and carriers was by Friesz, Gottfried and Morlok [13]. They proposed a sequential network equilibrium model for predicting freight flows. A survey of optimization models for long-haul freight transportation was presented by Crainic [7].

The foundations of the equilibration of transport networks were laid by Wardrop [31], Beckmann, McGuire and Winsten [1] and Samuelson [26]. Dafermos and Sparrow [12] invented the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which respectively, users act unilaterally, in their own self-interest in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total cost in the system is minimized. In case of network models with asymmetric link costs, variational inequalities (VI) theory is used to formulate the equilibrium conditions. VI theory was introduced by Hartman and Stampacchia [14] and with the contribution of Smith [29] and Dafermos [9], many applications have been studied using this methodology.

The topic of supply chain analysis has been the subject of a growing body of literature (Stadtler and Kilger [30]). Nagurney, Dong and Zhang [21] proposed a supply chain network equilibrium model and gave its finite-dimensional variational inequality formulation. Later, Nagurney and Liu [22] developed a fixed demand version of that model and also gave its variational inequality formulation.

Schneider [27] and Quandt [25] studied multicriteria network equilibrium models by identifying travel cost and travel time as criteria in route selection. These ideas were further developed by Nagurney [19], who proposed a multiclass, multicriteria network equilibrium model in which each class of travellers perceive its travel disutility associated with a route, as a subjective weighting of two criteria given by the travel time and the travel cost. Nagurney and Dong [20] developed also a multiclass, multicriteria network equilibrium model, but with elastic demand.

1.3 Structure of Postgraduate Work

The rest of this postgraduate work is divided into five chapters. Specifically:

In Chapter 2, we study the foundations of spatial equilibration in transport networks. After introducing some basic decision-making concepts, considering system – optimization versus user – optimization, we present network models in which the user cost on a link is no longer dependent solely on the flow on that link. We introduce the variational inequalities (VI) theory and we make the variational inequality formulations of fixed and elastic demand problems.

In Chapter 3, we apply the theory of spatial equilibration in transport networks to the field of supply chain networks. A supply chain network equilibrium model, which consists of distinct tiers of decision-makers, is examined. We consider the manufacturers and the retailers and develop their optimality conditions. Then, we focus on the consumers and form their equilibrium conditions. Finally, the variational inequality formulation of the equilibrium conditions of the entire supply chain network is given.

In Chapter 4, we focus on multiclass, multicriteria network equilibrium models. We examine two multiclass traffic network equilibrium models considering two criteria. In the first model, the demand is considered fixed and the weights associated with the criteria are fixed and only class-dependent. In the second model, the demand is considered elastic and the weights associated with the criteria are fixed, but class- and link-dependent. The variational inequality formulations of the governing equilibrium conditions of both models are also given.

In Chapter 5, we construct a model that brings together multicriteria decision-makers on the production side and on the consumption side in a network framework. We describe the network defining the nodes and links of its structure. We first consider the shippers and develop their optimality conditions and then we focus on the consumers and we derive their equilibrium conditions. Finally, the integrated model is constructed and we derive the variational inequality formulation of the governing equilibrium conditions, giving also two numerical examples.

In Chapter 6, this postgraduate work is summarized and directions for further research are given.

Chapter 2 Spatial Equilibration in Transport Networks

2.1 Introduction

Models of freight networks are closely related to transport network equilibrium models. Transport networks are complex, large-scale spatial systems, and come in a variety of forms, ranging from road networks to air, rail, and waterway networks. They provide the foundation for the functioning of our economies and societies through the movement of people, goods, and services, and allow for the connectivity of residential locations with places of employment, schools, leisure activities, etc. From an economic perspective, the supply in such network systems is represented by the underlying network topology and the cost characteristics, whereas the demand is represented by the users of the transportation system. Before making any policy decisions on transport networks one needs to identify the underlying behavioral mechanisms regarding route selection. For example, in the case of urban transport networks, travelers select their routes from an origin to a destination so as to minimize their own travel cost or travel time, which although optimal from a user's perspective (user-optimization) may not be optimal from a societal one (system-optimization) where a decision-maker or central controller has control of the flows on the network and seeks to allocate the flows so as to minimize the total cost in the network.

This chapter is structured as follows: In Section 2.2 we introduce some basic decision – making concepts, considering system – optimization versus user – optimization. In Section 2.3 we consider network models in which the user cost on a link is no longer dependent solely on the flow on that link. In this section we introduce the *variational inequalities theory* and we make the variational inequality formulations of fixed and elastic demand problems. Finally, in Section 2.4, we conclude this chapter.

2.2 Basic Decision – Making Concepts and Models

Half a century ago, Wardrop [31] explicitly recognized alternative possible behaviors of users of transport networks, notably, urban transport networks and stated two principles, which are commonly named after him:

First Principle: The journey times of all routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route.

Second Principle: The average journey time is minimal.

The first principle corresponds to the behavioral principle in which travelers seek to (unilaterally) determine their minimal costs of travel whereas the second principle corresponds to the behavioral principle in which the total cost in the network is minimal.

Beckmann, McGuire and Winsten [1] were the first to rigorously formulate these conditions mathematically, as had Samuelson [26] in the framework of spatial price equilibrium problems in which there were, however, no congestion effects. Specifically, Beckmann, McGuire, and Winsten [1] established the equivalence between the *traffic network equilibrium* conditions, which state that all used paths connecting an origin/destination (O/D) pair will have equal and minimal travel times (or costs) (corresponding to Wardrop's first principle), and the Kuhn-Tucker [16] conditions of an appropriately constructed optimization problem, under a symmetry assumption on the underlying functions. Hence, in this case, the equilibrium link and path flows could be obtained as the solution of a mathematical programming problem. Their approach made the formulation, analysis, and subsequent computation of solutions to traffic network problems based on actual transportation networks realizable.

Dafermos and Sparrow [12] invented the terms *user-optimized* (U-O) and *system-optimized* (S-O) transportation networks to distinguish between two distinct situations in which, respectively, users act unilaterally, in their own self-interest, in selecting their routes, and in which users select routes according to what is optimal from a societal point of view, in that the total cost in the system is minimized. In the latter problem, marginal total costs rather than average costs are equilibrated. The former problem coincides with Wardrop's first principle, and the latter with Wardrop's second principle. In Table 2.1, one can distinguish the two behavioral principles underlying transportation networks.

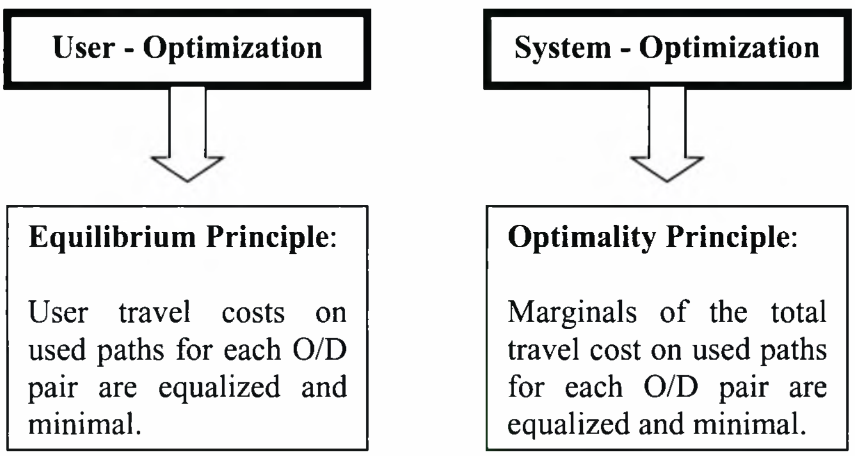

Table 2.1: Distinct Behavior on Transportation Networks

The concept of "system-optimization" is also relevant to other types of "routing models" in transportation, as well as in communications, including those concerned with the routing of freight and computer messages, respectively. Dafermos and Sparrow [12] also provided explicit computational procedures, that is, *algorithms*, to compute the solutions to such network problems in the case where the user travel cost on a link was an increasing (in order to handle congestion) function of the flow on the particular link, and linear.

In Subsections 2.2.1 and 2.2.2 we present the basic transport network models that are due to Beckmann, McGuire and Winsten [1] and Dafermos and Sparrow [12]. First we consider the system – optimized network model and then the user – optimized network model.

2.2.1 The System – Optimized Problem

To present the system – optimized network model, we consider a general network $G=[N, L]$, where N denotes the set of nodes, and L the set of directed links. Let a denote a link of the network connecting a pair of nodes, and let p denote a path consisting of a sequence of links connecting an O/D pair. In transportation networks, nodes correspond to origins and destinations, as well as to intersections. Links, on the other hand, correspond to roads/streets in the case in the case of urban transportation networks and to railroad segments in the case of train networks. A path in its most basic setting thus, is a sequence of "roads" which comprise a route from an origin to a destination.

Let P_w denote the set of paths connecting the origin/destination (O/D) pair of nodes w. Let P denote the set of all paths in the network and assume that there are J origin/destination pairs of nodes in the set Ω. Let x_p represent the flow on path p and let f_a denote the flow on link a. The path flows on the network are grouped into the column vector $x \in \Re_+^{n_p}$, where n_p denotes the number of paths in the network. The link flows, in turn, are grouped into the column vector $f \in \Re_+^n$, where n denotes the number of links in the network.

The following conservation of flow equation must hold:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \quad \forall a \in L, \tag{2.1}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Expression (2.1) states that the flow on a link a is equal to the sum of all the path flows on paths p that traverse link a.

Moreover, if one lets d_w denote the demand associated with O/D pair w, then one must have that

$$d_w = \sum_{p \in P_w} x_p \, , \ \forall w \in \Omega , \tag{2.2}$$

where $x_p \geq 0, \ \forall p \in P$; that is, the sum of all the path flows between an origin/destination pair w must be equal to the given demand d_w.

Let c_a denote the user link cost associated with traversing link a, and let C_p denote the user cost associated with traversing the path p.

Assume that the user link cost function is given by the separable function

$$c_a = c_a(f_a) \, , \ \forall a \in L \, , \tag{2.3}$$

where c_a is assumed to be an increasing function of the link flow f_a in order to model the effect of the link flow on the cost.

The total cost on link a, denoted by $\hat{c}_a(f_a)$, is given by:

$$\hat{c}_a(f_a) = c_a(f_a) \times f_a \, , \ \forall a \in L \, , \tag{2.4}$$

that is, the total cost on a link is equal to the user link cost on the link times the flow on the link. Here the cost is interpreted in a general sense. From a transportation engineering perspective, however, the cost on a link is assumed to coincide with the travel time on a link.

In the system – optimized problem, there exists a central controller who seeks to minimize the total cost in the network system, where the total cost is expressed as

$$\sum_{a \in L} \hat{c}_a(f_a), \tag{2.5}$$

where the total cost on a link is given by expression (2.4).

The system – optimization problem is, thus, given by:

$$\text{Minimize} \quad \sum_{a \in L} \hat{c}_a(f_a) \tag{2.6}$$

subject to:

$$\sum_{p\in P_w} x_p = d_w, \ \forall w \in \Omega, \tag{2.7}$$

$$f_a = \sum_{p\in P} x_p \delta_{ap}, \ \forall a \in L, \tag{2.8}$$

$$x_p \geq 0, \ \forall p \in P. \tag{2.9}$$

The constraints (2.7) and (2.8), along with (2.9), are commonly referred to in network terminology as conservation of flow equations. In particular, they guarantee that the flow in the network, that is, the users don't "get lost".

The total cost on a path, denoted by $\hat{C}_p$, is the user cost on a path times the flow on a path, that is:

$$\hat{C}_p = C_p x_p, \ \forall p \in P, \tag{2.10}$$

Where the user cost on a path, C_p, is given by the sum of the user costs on the links that comprise the path, that is:

$$C_p = \sum_{a\in L} c_a(f_a)\delta_{ap}. \tag{2.11}$$

In view of (2.8), one may express the cost on a path p as a function of the path flow variables and, hence, an alternative version of the above system – optimization problem can be stated in path flow variables only, where one has now the problem:

$$\text{Minimize} \quad \sum_{p\in P} C_p(x) x_p \tag{2.12}$$

subject to constraints (2.7) and (2.9).

System – Optimality Conditions

Under the assumption of increasing user link cost functions, the objective function in the System – Optimized problem is convex, and the feasible set consisting of the linear constraints is also convex. Therefore, the optimality conditions, that is, the Kuhn – Tucker conditions are: For each origin/destination pair $w \in \Omega$, and each path $p \in P_w$, the flow pattern x and (link flow pattern f), satisfying (2.7) – (2.9) must satisfy:

$$\hat{C}'_p \begin{cases} = \mu_w \text{, if } x_p > 0 \\ \\ \geq \mu_w \text{, if } x_p = 0, \end{cases} \tag{2.13}$$

where $\hat{C}'_p$ denotes the marginal of the total cost on path p, given by:

$$\hat{C}'_p = \sum_{a \in L} \frac{\partial \hat{c}_a(f_a)}{\partial f_a} \delta_{ap}, \tag{2.14}$$

and in (2.13) it is evaluated at the solution.

As we can see, in the System – Optimized problem, according to the optimality conditions (2.13), it is the marginal of the total cost on each used path connecting an origin/destination pair which is equalized and minimal. Indeed, conditions (2.13) state that a system – optimized flow pattern is such that for each origin/destination pair the incurred marginals of the total cost on all used paths are equal and minimal (Table 2.1).

2.2.2 The User - Optimized Problem

At this subsection we describe the user – optimized network problem, also commonly referred to in the transportation literature as the traffic assignment problem or the traffic network equilibrium problem. Again, as in the system – optimized problem of the previous subsection, the network $G = [N,L]$, the demands associated with the origin/destination pairs, as well as the user link cost functions are assumed as given. As we have already stated, user – optimization follows Wardrop's first principle.

Network Equilibrium Conditions

In the user – optimized problem, one seeks to determine the path flow pattern x^* (and link flow pattern f^*) which satisfies the conservation of flow equations (2.7), (2.8), and the nonnegativity assumption on the path flows (2.9), and which also satisfies the network equilibrium conditions given by the following statement:

For each origin/destination pair $w \in \Omega$ and each path $p \in P_w$:

$$C_p \begin{cases} = \lambda_w, & \text{if } \; x_p^* > 0 \\ \\ \geq \lambda_w, & \text{if } \; x_p^* = 0. \end{cases} \tag{2.15}$$

Hence, in the user – optimization problem there is no explicit optimization concept, since now users of the network system act independently, in a noncooperative manner, until they can't improve on their situations unilaterally and, thus, an equilibrium is achieved, governed by the above equilibrium conditions. Indeed, conditions (2.15) are simply a restatement of Wardrop's first principle mathematically and mean that only those paths connecting an origin/destination pair will be used which have equal and minimal user costs. In (2.15) the minimal cost for a given origin/destination pair is denoted by λ_w and its value is obtained once the equilibrium flow pattern is determined. Otherwise, a user of the network could improve upon his situation by switching to a path with lower cost. User – optimization represents decentralized decision – making, whereas system – optimization represents centralized decision – making (Table 2.1).

In order to obtain a solution to the above problem, Beckmann, McGuire, and Winsten [1] established that the solution to the equilibrium problem, in the case of user link cost functions in which the cost on a link only depends on the flow on that link could be obtained by solving the following optimization problem:

$$\text{Minimize} \quad \sum_{a \in L} \int_0^{f_a} c_a(y)dy \tag{2.16}$$

subject to:

$$\sum_{p \in P_w} x_p = d_w, \ \forall w \in \Omega, \tag{2.17}$$

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \ \forall a \in L, \tag{2.18}$$

$$x_p \geq 0, \ \forall p \in P. \tag{2.19}$$

We observe that the conservation of flow equations are identical in both the user – optimized network problem (see (2.17) – (2.19)) and the system – optimized problem (see (2.7) – (2.9)). The behavior of the individual decision – makers termed "users", however, is different. Users of the network system, which generate the flow on the network now act independently, and aren't controlled by a centralized controller.

The objective function given by (2.16) is simply a device constructed to obtain a solution using general purpose convex programming algorithms. It doesn't possess the economic meaning of the objective function encountered in the system – optimization problem given by (2.6), equivalently, by (2.12).

2.3 Models with Asymmetric Link Costs

In the previous section we presented network models where we assumed separable user link cost functions. In this section, we consider network models in which the user cost on a link is no longer dependent solely on the flow on that link.

Hence, we assume that user link cost functions are now of a general form, that is, the cost on a link may depend not only on the flow on the link but on other link flows on the network:

$$c_a = c_a(f)\,,\ \forall\, a \in L. \qquad (2.20)$$

In the case where the symmetry assumption exists, that is, $\frac{\partial c_a(f)}{\partial f_b} = \frac{\partial c_b(f)}{\partial f_a}$, for all links $a,b \in L$, one can still reformulate the solution to the network equilibrium problem satisfying equilibrium conditions (2.15) as the solution to an optimization problem, although again, with an objective function that is artificial and simply a mathematical device. However, when the symmetry assumption is no longer satisfied, such an optimization reformulation no longer exists and one must appeal to *variational inequality theory*. Models of traffic networks with asymmetric cost functions are important since they allow for the formulation, qualitative analysis, and, ultimately, solution to problems in which the cost on a link may depend on the flow on another link in a different way than the cost on the other link depends on that link's flow. Such a generalization allows for the more realistic treatment of intersections, two-way links, multiple modes of transport as well as distinct classes of users of the network.

The system – optimization problem in the case of nonseparable user link cost functions becomes:

$$\text{Minimize} \sum_{a \in L} \hat{c}_a(f)\,, \qquad (2.21)$$

subject to (2.7) – (2.9), where $\hat{c}_a(f) = c_a(f) \times f_a,\ \forall a \in L$.

The system – optimality conditions remain as in (2.13), but now the marginal of the total cost on a path becomes, in this more general case:

$$\hat{C}'_p = \sum_{a,b \in L} \frac{\partial \hat{c}_b(f)}{\partial f_a} \delta_{ap}\,, \quad \forall p \in P \qquad (2.22)$$

The user – optimization problem in turn, in the case where the user link cost functions are no longer symmetric, can't be solved using standard optimization algorithms. We emphasize again, that such general cost functions are very important from an application standpoint since they allow for asymmetric interactions on the network. For example, allowing for asymmetric cost functions, permits one to handle the situation when the flow on a particular link affects the cost on another link in a different way than the cost on the particular link is affected by the flow on the other link.

In this section we make an introduction to the theory of finite – dimensional variational inequalities and then we present variational inequality formulations of both fixed demand and elastic demand traffic network equilibrium problems. In these formulations we provide the variational inequality of the network equilibrium conditions in path flows, as well as in link flows.

2.3.1 The Variational Inequality Problem

Variational inequalities were introduced by Hartman and Stampacchia [14], mainly for the study of problems arising in the field of mechanics. The research focused on infinite – dimensional variational inequalities, rather than on finite – dimensional variational inequalities, which are the kind utilized in this postgraduate work. Smith [29] presented a formulation of the equilibrium conditions of the traffic network equilibrium problem that were then identified as a finite – dimensional variational inequality by Dafermos [9]. From this connection, much research has been conducted on such variational inequality problems and many applications, ranging from oligopolistic market equilibrium problems to general economic and financial equilibrium problems, have been studied, both qualitatively and computationally, using this methodology.

Variational inequality theory is a powerful tool for the study of many equilibrium problems, since the variational inequality problem contains, as special cases, important problem classes which are widely utilized in economics and in engineering, such as systems of nonlinear equations, optimization problems and complementarity problems. In this subsection, we first present the formal definition of a variational inequality problem and then the relationship between the variational inequality problem and optimization problems.

The finite – dimensional variational inequality problem, VI (F,K), is to determine a vector $X^* \in K$ *such that*

$$\langle F(X^*), X - X^* \rangle \geq 0, \qquad \forall X \in K, \qquad (2.23)$$

where F is a given continuous function from K to R^N*, K is a given closed convex set and* $\langle .,. \rangle$ *denotes the inner product in* R^N*.*

Variational inequality (2.23) is referred to as being in *standard form*. Hence, for a given equilibrium problem, one must determine the function *F* that enters the variational inequality problem, the vector of variables *X*, as well as the feasible set *K*.

The variational inequality problem has a geometric interpretation. In particular, it states that $F(X^*)$ is "orthogonal" to the feasible set *K* at the point X^*. In Figure 2.1, the geometric interpretation is provided.

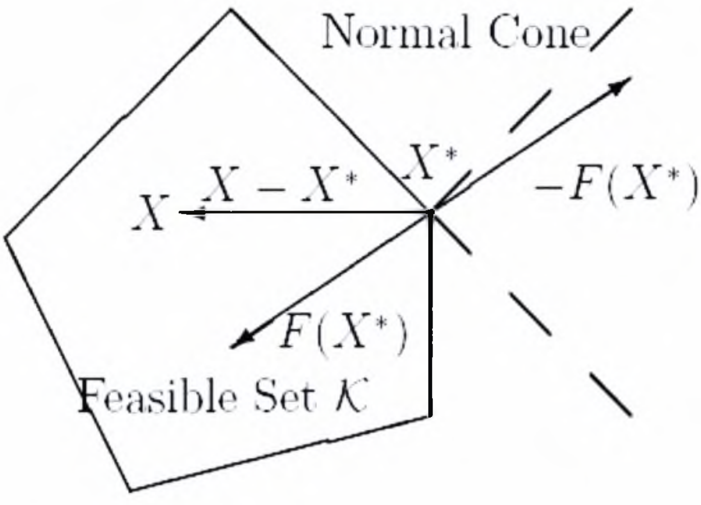

Figure 2.1: Geometric Interpretation of *VI (F,K)*

In order to emphasize the relationship between the variational inequality problem and optimization problems, we now present without proofs, the following results:

Proposition 2.1

Let X^* be the solution to the following optimization problem:

$$\text{Minimize } f(X) \tag{2.24}$$

subject to:

$$X \in K,$$

where f is a continuously differentiable function and K is closed and convex. Then X^* is a solution of the variational inequality problem:

$$\langle \nabla f(X^*), X - X^* \rangle \geq 0, \qquad \forall X \in K, \tag{2.25}$$

where $\nabla f(X^*)$ denotes the gradient of f with respect to X with components:

$$\nabla f(X)^T = \left(\frac{\partial f(X^*)}{\partial X_1}, \frac{\partial f(X^*)}{\partial X_2}, ..., \frac{\partial f(X^*)}{\partial X_N} \right).$$

Proposition 2.2

If f(X) is a convex function and X^* is a solution to VI (∇f, K) given by (2.25), then X^* is a solution to the optimization problem (2.24).

In the special case, when $K=R^N$, then the above optimization problem (2.24) is an *unconstrained* problem.

On the other hand, if a certain symmetry condition holds, the variational inequality problem can also be reformulated as an optimization problem.

Theorem 2.1

Assume that F(X) is continuously differentiable on K and that the Jacobian matrix:

$$\nabla F(X) = \begin{pmatrix} \frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_N} \\ \vdots & & \vdots \\ \frac{\partial F_N}{\partial X_1} & \cdots & \frac{\partial F_N}{\partial X_N} \end{pmatrix} \tag{2.26}$$

is symmetric and positive semidefinite, so that F is convex. Then, there exists a real – valued function f: $K \mapsto R$ satisfying

$$\nabla f(X) = F(X) \tag{2.27}$$

with X^* the solution of VI (F, K) also being the solution of the optimization problem (2.24).

Hence, one can see that the variational inequality problem encompasses the optimization problem and that the variational inequality problem can be reformulated as a convex optimization problem only when the symmetry and the positive semidefiniteness conditions hold.

Therefore, the variational inequality problem is a more general problem in that it can also handle a function *F(X)* with an asymmetric Jacobian, that is, when in (2.26): $\frac{\partial F_i}{\partial X_j} \neq \frac{\partial F_j}{\partial X_i}$.

Hence, the appeal of the use of the variational inequality formulation and associated theory, as it can adequately handle asymmetric cost functions as well as multiple modes and multiple classes on the networks realistically.

Another result that demonstrates how a class of constrained optimization problems over specific types of constraints can be formulated as a variational inequality problem, is presented at Bertsekas and Tsitsiklis [2] as follows:

Consider the convex constrained optimization problem:

$$\text{Minimize } \sum_{i=1}^{m} f_i(X_i) \tag{2.28}$$

subject to:

$$a_j^T X \leq b_j, \qquad j=1,\ldots,r \tag{2.29}$$

$$X_i \in K_i, \qquad i=1,\ldots,m,$$

where f_i: $R^{n_i} \mapsto R$ is a convex differentiable function and a_j^T is a row vector of coefficients corresponding to the *j*-th constraint and *X* is vector consisting of the vectors $\{X_1, \ldots, X_m\}$. Then, this problem is equivalent to the variational inequality problem of finding $X_i^* \in K_i$ and $u_j^* \geq 0$, such that

$$\sum_{i=1}^{m} \left\langle (\nabla f_i(X_i^*) + \sum_{j=1}^{r} u_j^* a_{ji})^T, (X_i - X_i^*) \right\rangle + \sum_{j=1}^{r} (b_j - a_j^T X^*) \times (u_j - u_j^*) \geq 0, \tag{2.30}$$

$$\forall X_i \in K_i, \qquad u_j \geq 0, \qquad \forall j,$$

where u_j^* is the Lagrange multiplier in the solution associated with inequality constraint j in the minimization problem. The coefficient a_{ji} corresponds to the ith component of the vector a_j.

2.3.2 Variational Inequality Formulations of Fixed Demand Problems

In Subsection 2.2.2 we described the user – optimized network problem. As we stated there, in the case of the user – optimization problem, one seeks to determine the path flow pattern x^* (and link flow pattern f^*) which satisfies equations (2.7), (2.8), (2.9) and (2.15). In these problems, d_w, the demand associated with origin/destination pair w, is considered to be given and fixed. In this subsection we present without proof, the variational inequality formulation of network equilibrium with fixed demands, in path flows, as well as in link flows.

Theorem 2.2 Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Path Flow Version

A vector $x^* \in K^1$ is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (2.15) if and only if it satisfies the variational inequality problem:

$$\sum_{w \in \Omega} \sum_{p \in P_w} C_p(x^*) \times (x - x^*) \geq 0, \quad \forall x \in K^1, \tag{2.31}$$

or, in vector form:

$$\left\langle C(x^*), x - x^* \right\rangle \geq 0, \qquad \forall x \in K^1 \tag{2.32}$$

where C is the n_p-dimensional column vector of path user costs and K^1 is defined as: $K^1 \equiv \{x \geq 0$, such that (2.17) holds$\}$.

Theorem 2.3 Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version

A vector $f^* \in K^2$ *is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:*

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) \geq 0, \qquad \forall f \in K^2, \tag{2.33}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle \geq 0, \qquad \forall f \in K^2, \tag{2.34}$$

where c is the n-dimensional column vector of link user costs and K^2 *is defined as:* $K^2 \equiv \{f \mid$ *there exists a* $x \geq 0$ *and satisfying (2.17) and (2.18)*$\}$.

We observe that we may put variational inequality (2.32) into standard form (2.23) by letting $F \equiv C$, $X \equiv x$ and $K \equiv K^1$. Also, we may put variational inequality (2.34) into standard form, where now $F \equiv c$, $X \equiv f$ and $K \equiv K^2$.

The presentation of the variational inequality formulations of the fixed demand models given above was in the context of single mode (or single class) transport networks. We emphasize, however, that in view of the generality of the functions considered (see equation (2.20)), the modeling framework described above can also be adapted to multimodal/multiclass problems in which there are multiple modes of transport available and/or multiple classes of users, each of whom perceives the cost on the links of the network in an individual manner. Dafermos [8] demonstrated how, through a formal model, a multiclass traffic network could be cast into a single-class network through the construction of an expanded network consisting of as many copies of the original network as there were classes.

2.3.3 Variational Inequality Formulations of Elastic Demand Problems

In this subsection we describe a general network equilibrium model with demands, which are no longer fixed, but are now variables and we present without proof, the variational inequality formulations of the network equilibrium conditions in this case of elastic demand.

Specifically, it is assumed that one has associated with each origin/destination pair w in the network a travel disutility function (inverse demand function) λ_w, where here the general case is considered in which the disutility may depend upon the entire vector of demands, which as we stated, are variables, that is,

$$\lambda_w = \lambda_w(d), \qquad \forall w \in \Omega, \qquad (2.35)$$

where d is the J-dimensional column vector of the demands.

The notation, otherwise, is as described earlier in this chapter, except that here we also consider user link cost functions which are general, that is, of the form of equation (2.20). The conservation of flow equations are given by:

$$f_a = \sum_{p \in P} x_p \delta_{ap}, \qquad \forall a \in L, \qquad (2.36)$$

$$d_w = \sum_{p \in P_w} x_p, \qquad \forall w \in \Omega, \qquad (2.37)$$

$$x_p \geq 0, \qquad \forall p \in P. \qquad (2.38)$$

Hence, in the elastic demand case, the demands in equation (2.37) are now variables and no longer given, as was the case for the fixed demand expression in (2.2).

In the elastic demand case, the network equilibrium conditions take on the following form: For every origin/destination pair $w \in \Omega$ and each path $p \in P_w$, a vector of path flows and demands (x^*, d^*) satisfying (2.37) and (2.38) (which induces a link flow pattern f^* through (2.36)) is a network equilibrium pattern if it satisfies:

$$C_p(x^*) \begin{cases} = \lambda_w(d^*), & \text{if } x_p^* > 0 \\ \geq \lambda_w(d^*), & \text{if } x_p^* = 0. \end{cases} \tag{2.39}$$

Equilibrium conditions (2.39) state that the costs on used paths for each origin/destination pair are equal and minimal and equal to the disutility associated with that origin/destination pair. Costs on unutilized paths can exceed the disutility. Hence, this model allows one to ascertain the attractiveness of different origin/destination pairs based on the ultimate equilibrium demand associated with the specific origin/destination pairs.

Also, as described in the case of fixed demands, the elastic demand traffic network model can be adapted to multimodal/multiclass problems. In that case, in equilibrium, the used paths for a given mode and origin/destination pair must have minimal and equal user path costs, which in turn, must be equal to the travel disutility for that mode and origin/destination pair at the equilibrium demand.

In the next two theorems, both the path flow version and the link flow version of the variational inequality formulations of the network equilibrium conditions (2.39) are presented.

Theorem 2.4 Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Path Flow Version

A vector $(x^*, d^*) \in K^3$ *is a network equilibrium path flow pattern, that is, it satisfies equilibrium conditions (2.39) if and only if it satisfies the variational inequality problem:*

$$\sum_{w \in \Omega} \sum_{p \in P_w} C_p(x^*) \times (x - x^*) - \sum_{w \in \Omega} \lambda_w(d^*) \times (d_w - d_w^*) \geq 0, \quad \forall (x,d) \in K^3, \tag{2.40}$$

or, in vector form:

$$\langle C(x^*), x - x^* \rangle - \langle \lambda(d^*), d - d^* \rangle \geq 0, \quad \forall (x,d) \in K^3, \tag{2.41}$$

where λ is the J-dimensional vector of disutilities and K^3 is defined as: $K^3 \equiv \{x \geq 0$, such that (2.39) holds$\}$.

Theorem 2.5 Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version

A vector $(f^*, d^*) \in K^4$ is a network equilibrium link flow pattern if and only if it satisfies the variational inequality problem:

$$\sum_{a \in L} c_a(f^*) \times (f_a - f_a^*) - \sum_{w \in \Omega} \lambda_w(d^*) \times (d_w - d_w^*) \geq 0, \qquad \forall (f, d) \in K^4, \tag{2.42}$$

or, in vector form:

$$\langle c(f^*), f - f^* \rangle - \langle \lambda(d^*), d - d^* \rangle \geq 0, \ \forall (f, d) \in K^4 \tag{2.43}$$

where $K^4 \equiv \{(f, d)$, such that there exists a $x \geq 0$ satisfying (2.36) and (2.38)$\}$.

2.4 Conclusions

In this chapter, we have presented the foundations of the equilibration of transport networks and we have introduced the variational inequality theory. Finite – dimensional variational inequality theory permits formulation and computation of network equilibrium models in which the cost on a link may depend on the flow on another link in a different way than the cost on the other link depends on that link's flow. Variational inequality formulations of both fixed demand and elastic demand traffic network equilibrium problems were also presented.

As demonstrated by Dafermos and Nagurney [11] in the context of a single commodity, and subsequently, by Dafermos [10] in the case of multiple commodities, spatial price equilibrium problems are *isomorphic* to traffic network equilibrium problems over appropriately constructed networks. Hence, the discussed in this chapter theory of traffic networks can be transferred to the study of commodity flows in the case of spatial price equilibrium in which the equilibrium production, consumption, and commodity trade flows are to be determined satisfying the equilibrium conditions that there will be a positive flow (in equilibrium) of the commodity between a pair of supply and demand markets if the supply price at the supply market plus the unit cost of transportation is equal to the demand price at the demand market.

Chapter 3 Multitiered Network Equilibrium Modeling

3.1 Introduction

In this chapter, we will apply the theory of spatial equilibration in transport networks, provided in Chapter 2, to the field of supply chain networks. Supply chain networks are considered to be multitiered, since they consist of distinct tiers of decision-makers, whose behavior affects the variables on the networks in the form of flows, as well as prices. As we have already noted, spatial price equilibrium problems are isomorphic to traffic network equilibrium problems over appropriately constructed networks. Hence, the development of an equilibrium model of a competitive supply chain network has attracted researchers' interest.

The topic of supply chain analysis involves manufacturing, transportation, logistics, as well as retailing and marketing. It has been the subject of a growing body of literature (Stadtler and Kilger [30]), with the associated research being both conceptual in nature (Mentzer [17], Bovet [3]), due to the complexity of the problem and the numerous agents such as manufacturers, retailers and consumers involved in the transactions, as well as analytical (Slats et al. [28], Bramel and Simchi-Levi [4], Miller [18]). Nagurney, Dong and Zhang [21] proposed a supply chain network equilibrium model and gave its finite-dimensional variational inequality formulation. In this chapter, we study a fixed demand version of that model, that was developed by Nagurney and Liu [22].

This chapter is organized as follows: In Section 3.2, we describe the network defining the nodes and links of its structure. In Section 3.3, we consider the manufacturers and develop their optimality conditions and in Section 3.4, we examine the behavior of the retailers and give their optimality conditions. In Section 3.5, we focus on the consumers and form their equilibrium conditions. In Section 3.6, the variational inequality formulation of the equilibrium conditions of the supply chain is given and finally, in Section 3.7, we conclude this chapter.

3.2 The Supply Chain Network Model

The supply chain network structure, proposed by Nagurney and Liu [22], is depicted in Figure 3.1.

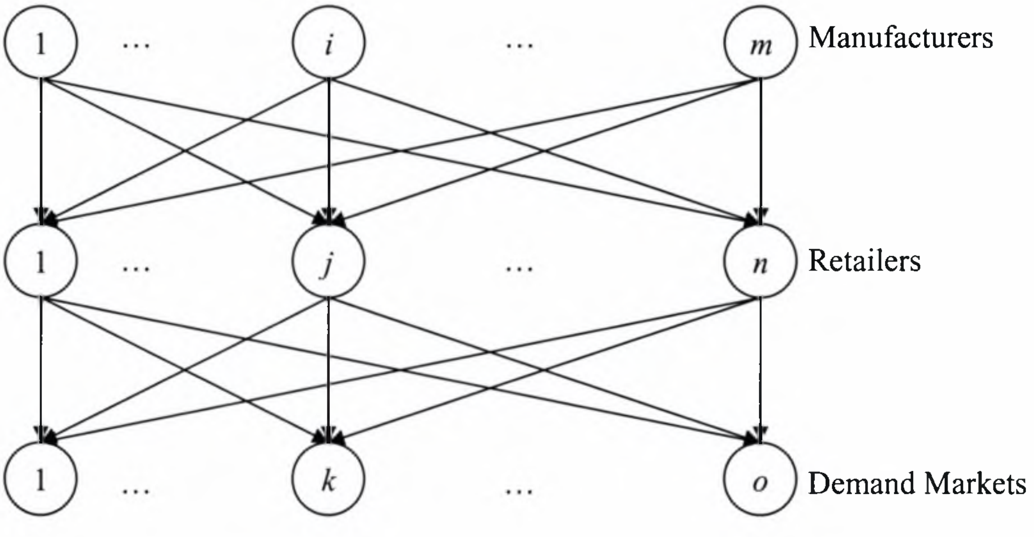

Figure 3.1: The Network Structure of the Supply Chain

Specifically, there are considered *m* manufacturers, who are involved in the production of a product, which can then be purchased by *n* retailers, who in turn, make the product available to consumers located at *o* demand markets. A typical manufacturer is denoted by *i*, a typical retailer by *j* and a typical demand market by *k*. In Figure 3.1, the manufacturing firms are located at the top tier of nodes in the network, the retailers are located at the middle tier, whereas the demand markets are located at the bottom tier. The links in the supply chain network denote transportation / transaction links.

In this model, manufacturers are assumed to be involved in the production of a homogeneous product, which is then shipped to the retailers. Manufacturers obtain a price for the product (which is endogenous) and seek to determine their optimal production and shipment quantities, given the production costs, as well as the transaction costs associated with conducting business with the different retailers. Retailers, in turn, must agree with the manufacturers as to the volume of shipments, since they are faced with the handling cost associated with having the product in their retail outlet. In addition, they seek to maximize their profits, with the price that the consumers are willing to pay for the product being endogenous. Consumers take into account the prices charged by the retailers and the unit transaction costs incurred to obtain the product, in making their consumption decisions. It is assumed that the demand for the product at each demand market, is fixed and known.

In the following four sections, it is first described the behavior of the manufacturers and the retailers. Then, the behavior of the consumers at the demand markets is discussed and the equilibrium conditions for the supply chain network are stated. Finally, the finite-dimensional variational inequality, governing the equilibrium, is given. The equilibrium solution is denoted by "*".

3.3 The Behavior of the Manufacturers and their Optimality Conditions

In this section, the focus is on the behavior of the manufacturers and their optimality conditions.

Let q_i denote the nonnegative production output of the product by manufacturer *i*. The production outputs of all manufacturers are grouped into the column vector $q \in R^m_+$. It is assumed that each manufacturer *i*, is faced with a production cost function f_i, which can depend in general, on the entire vector of production outputs, that is,

$$f_i = f_i(q) \qquad \forall i. \qquad (3.1)$$

A manufacturer may ship the product to the retailers, with the amount of the product shipped (or transacted) between manufacturer *i* and retailer *j* denoted by q_{ij}. A transaction cost denoted by c_{ij} is associated with each manufacturer and retailer pair (*i*, *j*). The transaction cost includes the cost of shipping the product. The product shipments between the manufacturers and the retailers are grouped into the *mn*-dimensional column vector Q^1. The transaction cost between a manufacturer and a retailer is given by:

$$c_{ij} = c_{ij}(q_{ij}), \qquad \forall i, j. \qquad (3.2)$$

Manufacturer *i*'s transactions with retailers are depicted in Figure 3.2.

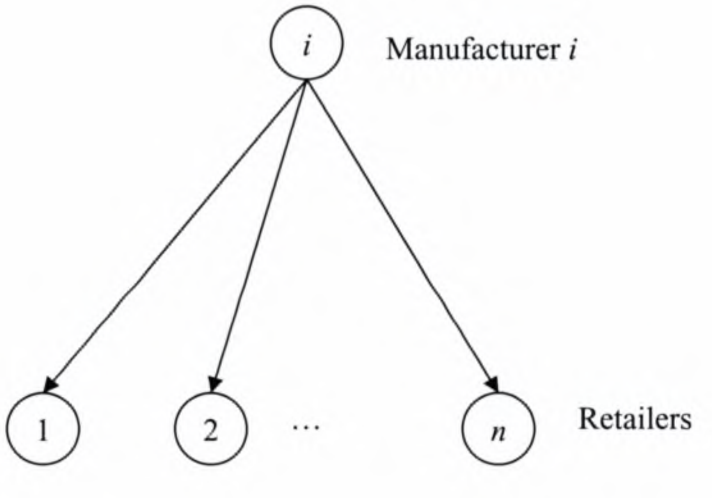

Figure 3.2: The Network Structure of Manufacturer i's Transactions with Retailers

The quantity produced by manufacturer *i* must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n} q_{ij} , \tag{3.3}$$

which states that the quantity produced by manufacturer i is equal to the sum of the quantities shipped from the manufacturer to all retailers.

The total costs incurred by a manufacturer i, thus, are equal to the sum of his production cost plus the total transaction costs. His revenue, in turn, is equal to the price that the manufacturer charges for the product (and the retailers are willing to pay) times the total quantity obtained / purchased of the product from the manufacturer by all the retail outlets. Since equation (3.3) holds, the production cost function f_i can be reexpressed as a function of the flows Q^1, that is,

$$f_i(q) \equiv f_i(Q^1) . \tag{3.4}$$

Let ρ_{1ij}^{*} denote the price charged for the product by manufacturer i in transacting with retailer j. This price is an endogenous variable and will be determined once the entire supply chain network equilibrium model is solved.

Hence, assuming that the manufacturers are profit – maximizers, the optimization problem faced by manufacturer i is:

$$\text{Maximize} \qquad \sum_{j=1}^{n} \rho_{1ij}^{*} q_{ij} - f_i(Q^1) - \sum_{j=1}^{n} c_{ij}(q_{ij}) , \tag{3.5}$$

subject to: $q_{ij} \geq 0$, for all j.

It is assumed that the manufacturers compete in a noncooperative manner in the sense of Cournot [6] and Nash [23], [24], seeking to determine their own optimal production and shipment quantities. According to what we have examined in Chapter 2, if the production cost functions and the transaction cost functions for each manufacturer are continuously differentiable and convex, then the optimality conditions for all manufacturers simultaneously, can be expressed as the following variational inequality:

Determine $Q^{1*} \in R_{+}^{mn}$, satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^{*})}{\partial q_{ij}}-\rho_{1ij}^{*}\right]\times\left[q_{ij}-q_{ij}^{*}\right]\geq 0, \qquad \forall Q^{1} \in R_{+}^{mn}. \tag{3.6}$$

The optimality conditions as expressed by (3.6) have an economic interpretation, which is that a manufacturer will ship a positive amount of the product to a retailer (and the flow on the corresponding link will be positive) if the price that the retailer is willing to pay for the product is precisely equal to the manufacturer's marginal production and transaction costs associated with that retailer. If the manufacturer's marginal production and transaction costs exceed what the retailer is willing to pay for the product, then the flow on the link will be zero.

3.4 The Behavior of the Retailers and their Optimality Conditions

The retailers are involved in transactions both with the manufacturers since they wish to obtain the product for their retail outlets, as well as with the consumers, who are the ultimate purchasers of the product. Hence, the network structure of retailer *j*'s transactions is as depicted in Figure 3.3. Thus, a retailer conducts transactions both with the manufacturers as well as with the consumers at the demand markets.

A retailer *j* is faced with a handling cost, which may include, for example, the display and storage cost associated with the product. This cost is denoted by c_j and, in the simplest case, it is a function of $\sum_{i=1}^{m} q_{ij}$, that is, the handling cost of a retailer is a function of how much of the product he has obtained from the various manufacturers. However, for the sake of generality, it is allowed the function to depend also on the amounts of the product held by other retailers and, therefore, it is

$$c_j = c_j(Q^1) \qquad \forall j. \tag{3.7}$$

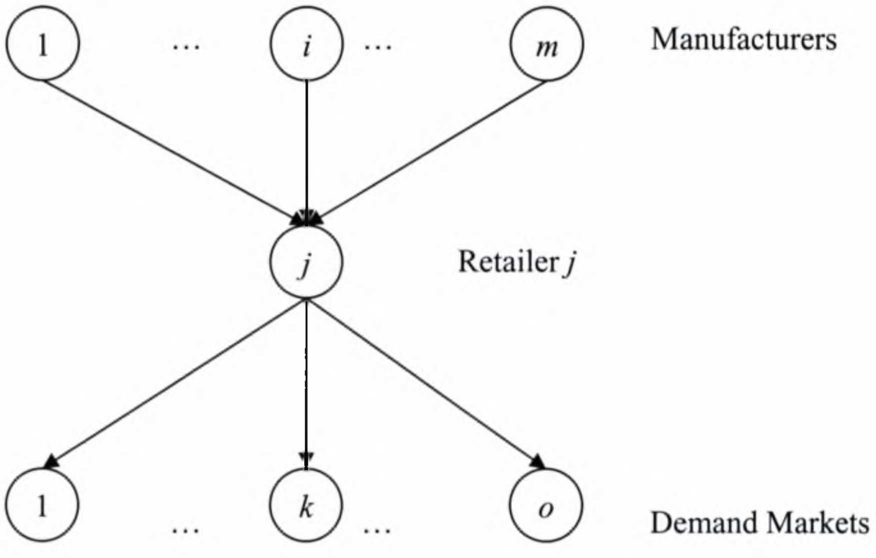

Figure 3.3: The Network Structure of Retailer *j*'s Transactions

The amount of product shipped (or transacted) between retailer *j* and demand market *k* is denoted by q_{jk}. The product shipments between the retailers and the demand markets are grouped into the *no*-dimensional column vector Q^2. The retailers associate a price with the product at their retail outlet, which is denoted by ρ^*_{2j} for retailer *j*. This price will also be determined endogenously after the model is solved.

Hence, assuming that the retailers are also profit – maximizers, the optimization problem faced by retailer *j* is given by:

$$\text{Maximize} \qquad \sum_{k=1}^{o} \rho^*_{2j} q_{jk} - c_j(Q^1) - \sum_{i=1}^{m} \rho^*_{1ij} q_{ij} \tag{3.8}$$

subject to:

$$\sum_{k=1}^{o} q_{jk} \leq \sum_{i=1}^{m} q_{ij} \tag{3.9}$$

and the nonnegativity constraints: $q_{ij} \geq 0$ and $q_{jk} \geq 0$, for all i,k.

Objective function (3.8) expresses that the difference between the revenues minus the handling cost and the payout to the manufacturers should be maximized. Constraint (3.9) simply expresses that consumers cannot purchase more from a retailer than is held in stock.

It is assumed that the retailers also compete in a noncooperative manner, seeking to determine not only the optimal amounts purchased by the consumers from their specific retail outlet but also, the amount that they wish to obtain from the manufacturers. In equilibrium, all the shipments between the tiers of network agents will have to coincide. Assuming that the handling cost for each retailer is continuous and convex, the optimality conditions for all the retailers coincide with the solution of the variational inequality:

Determine $(Q^{1*}, Q^{2*}, \gamma^*) \in R_+^{mn+no+n}$, satisfying

$$\begin{aligned}&\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial c_j(Q^{1*})}{\partial q_{ij}} + \rho_{1ij}^* - \gamma_j^*\right]\times\left[q_{ij} - q_{ij}^*\right] + \sum_{j=1}^{n}\sum_{k=1}^{o}\left[-\rho_{2j}^* + \gamma_j^*\right]\times\left[q_{jk} - q_{jk}^*\right]\\ &+\sum_{j=1}^{n}\left[\sum_{i=1}^{m} q_{ij}^* - \sum_{k=1}^{o} q_{jk}^*\right]\times\left[\gamma_j - \gamma_j^*\right] \geq 0 \quad \forall (Q^1, Q^2, \gamma) \in R_+^{mn+no+n},\end{aligned} \tag{3.10}$$

where the term γ_j is the Lagrange multiplier associated with constraint (3.9) for retailer j and γ the n-dimensional column vector of all the multipliers.

Retailers' optimality conditions, as expressed by (3.10), have also an economic interpretation. From the second term in inequality (3.10), we have that, if consumers at demand market k purchase the product from a particular retailer j, that is, if the q_{jk}^* is positive, then the price charged by retailer j, ρ_{2j}^*, is precisely equal to γ_j^*, which, from the third term in the inequality, serves as the price to clear the market from retailer j. Also, from the second term, we see that if no product is sold by a particular retailer, then the price associated with holding the product can exceed the price charged to the consumers. Furthermore, from the first term in inequality (3.10), we can conclude that, if a manufacturer transacts with a retailer resulting in a positive flow of the product between the two, then the price γ_j^* is precisely equal to the retailer j's payment to the manufacturer, ρ_{1ij}^*, plus its marginal cost of handling the product.

3.5 The Consumers at the Demand Markets and the Equilibrium Conditions

In this section, the behavior of the consumers located at the demand markets is examined and their equilibrium conditions are derived.

The consumers take into account the prices charged by the retailers and the transaction costs incurred to obtain the product, in making their consumption decisions. The transaction cost associated with obtaining the product by consumers at demand market k from retailer j is denoted by c_{jk}. This transaction cost is assumed to be continuous, positive and of the general form:

$$c_{jk} = c_{jk}(Q^2) \qquad \forall j,k \qquad (3.11)$$

where, as it has already been stated, Q^2 is the no-dimensional column vector of product flows between the retailers and the demand markets.

In Figure 3.4, the network of transactions between the retailers and the consumers at demand market k is depicted. The n retailers and demand market k are represented by nodes and the transactions by links, as previously.

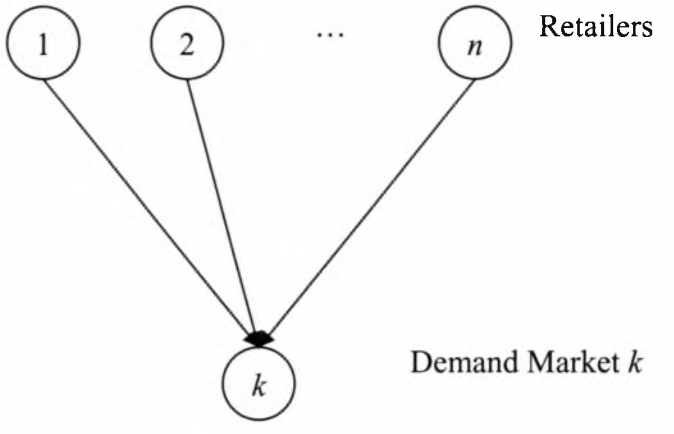

Figure 3.4: The Network Structure of Consumers' Transactions at Demand Market k

In this model, it is assumed that the demand for the product at each demand market is fixed and known. Specifically, the demand for the product at demand market k is denoted by d_k and all the demands are grouped into the o-dimensional column vector d. The following conservation of flow equation must hold:

$$d_k = \sum_{j=1}^{n} q_{jk}, \qquad k = 1,\ldots,o \tag{3.12}$$

Let now ρ_{3k}^{*} denote the price that the consumers pay for the product at demand market k. The consumers take the price charged by the retailers for the product (ρ_{2j}^{*}) plus the transaction cost associated with obtaining the product $(c_{jk}(Q^2))$, in making their consumption decisions. According to what we have examined in Chapter 2, the equilibrium conditions for consumers at demand market k take the form:

For each retailer j:

$$\rho_{2j}^* + c_{jk}(Q^{2*}) \begin{cases} = \rho_{3k}^*, & \text{if} \quad q_{jk}^* > 0 \\ \\ \geq \rho_{3k}^*, & \text{if} \quad q_{jk}^* = 0. \end{cases} \tag{3.13}$$

Conditions (3.13) state that in equilibrium, if the consumers at demand market k purchase the product from retailer j, then the price the consumers pay is exactly equal to the price charged by the retailer plus the transaction cost. However, if the sum of the price charged by the retailer and the transaction cost exceeds the price that the consumers are willing to pay at the demand market, there will be no transaction between this retailer / demand market pair.

Conditions (3.13), that in equilibrium must hold simultaneously for all demand markets, can be expressed as the following variational inequality:

Determine $Q^{2*} \in K^1$, such that

$$\sum_{j=1}^{n}\sum_{k=1}^{o}\left[\rho_{2j}^* + c_{jk}(Q^{2*})\right]\times\left[q_{jk} - q_{jk}^*\right] \geq 0 \qquad \forall Q^2 \in K^1 \tag{3.14}$$

where $K^1 \equiv \{Q^2 \mid Q^2 \in R_+^{no}$ and (3.12) holds$\}$.

3.6 Variational Inequality Formulation of the Equilibrium Conditions of the Supply Chain

In equilibrium, the optimality conditions of all the manufacturers, the optimality conditions of all the retailers and the equilibrium conditions for all the demand markets must be simultaneously satisfied, so that no decision-maker has any incentive to alter his transactions.

The shipments of the product that the manufacturers ship to the retailers must be equal to the shipments that the retailers accept from the manufacturers. In addition, the amounts of the product purchased by the consumers at the demand markets must be equal to the amounts sold by the retailers. Furthermore, the equilibrium shipment and price pattern in the supply chain must satisfy the sum of inequalities (3.6), (3.10) and (3.14).

The summation of inequalities (3.6), (3.10) and (3.14) yields, after algebraic simplification, the following variational inequality:

Determine $(Q^{1*}, Q^{2*}, \gamma^*) \in K^2$, satisfying

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left[\frac{\partial f_i(Q^{1*})}{\partial q_{ij}}+\frac{\partial c_{ij}(q_{ij}^*)}{\partial q_{ij}}+\frac{\partial c_j(Q^{1*})}{\partial q_{ij}}-\gamma_j^*\right]\times\left[q_{ij}-q_{ij}^*\right]$$
$$+\sum_{j=1}^{n}\sum_{k=1}^{o}\left[c_{jk}(Q^{2*})+\gamma_j^*\right]\times\left[q_{jk}-q_{jk}^*\right]+\sum_{j=1}^{n}\left[\sum_{i=1}^{m}q_{ij}^*-\sum_{k=1}^{o}q_{jk}^*\right]\times\left[\gamma_j-\gamma_j^*\right]\geq 0, \qquad (3.15)$$
$$\forall (Q^1, Q^2, \gamma) \in K^2$$

where $K^2 \equiv \{(Q^1, Q^2, \gamma) \mid (Q^1, Q^2, \gamma) \in R_+^{mn+no+n}$ and (3.12) holds$\}$.

The above inequality is the variational inequality formulation of the governing equilibrium conditions of the entire supply chain network.

3.7 Conclusions

In this chapter, we studied multitiered networks which consist of distinct tiers of decision-makers, whose behavior affects the variables on the networks in the form of flows, as well as prices. We applied the theory of spatial equilibration in transport networks to the field of supply chain networks.

The supply chain network equilibrium model that was studied, assumed that the demand for the product at each demand market was fixed and known. It also assumed imperfect competition between manufacturers and between retailers. The production cost associated with a manufacturer could depend not only on the amount that he produced, but also on the amounts produced by the other manufacturers. Respectively, the handling cost associated with a retailer could depend not only on the amount that he handled, but also on the amounts handled by the other retailers.

In this chapter, we saw how the variational inequalities theory is used to formulate the governing equilibrium conditions of an entire supply chain network.

Chapter 4 Multiclass, Multicriteria Network Equilibrium Modeling

4.1 Introduction

In this chapter we focus on multiclass, multicriteria network equilibrium models. The term "multiclass" means that we allow in our models, more than one class of decision-makers in the network. The term "multicriteria" captures the multiplicity of criteria that decision-makers are often faced with in making their choices. Criteria which can be considered as part of the decision-making process include cost minimization, time minimization, profit maximization etc. Each class of decision-maker is allowed to have weights associated with the criteria under consideration. Hence, the models that we present in this chapter are important since they allow for the individual weighting of distinct criteria associated with decision-making on networks and especially transport networks. Moreover, as we have already seen in this work, spatial price equilibrium problems are isomorphic to traffic network equilibrium problems over appropriately constructed networks. For this reason, the theory of multiclass, multicriteria traffic networks can be transferred to the study of the freight network model that we will construct later in this postgraduate work.

This chapter is organized as follows: In Section 4.2 we present a bicriteria fixed demand traffic network equilibrium model in which the weights are fixed and only class-dependent and in Section 4.3 we formulate the governing traffic network equilibrium conditions of this model as a finite – dimensional variational inequality problem. In Section 4.4 we present an elastic demand traffic network problem with two criteria and weights which are fixed but class- and link-dependent. In Section 4.5 we give the finite – dimensional variational inequality formulation of the above problem. Finally, in Section 4.6, we conclude this chapter.

4.2 The Fixed Demand, Multiclass, Multicriteria Network Equilibrium Model

In this section, we present the multiclass, multicriteria traffic network equilibrium model that was developed by Nagurney [19].

We consider a general network $G=[N, L]$, where N denotes the set of nodes in the network and L the set of directed links. Let a denote a link of the network connecting a pair of nodes and let p denote a path, assumed to be acyclic, consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. Let W denote the set of O/D pairs. The set of paths connecting the O/D pair w is denoted by P_w and the entire set of paths in the network by P. There are n links in the network and n_p paths.

We assume that there are k classes of travellers in the network with a typical class denoted by i. Let f_a^i denote the flow of class i on link a and let x_p^i denote the nonnegative flow of class i on path p. The relationship between the link flows by class and the path flows is:

$$f_a^i = \sum_{p \in P} x_p^i \delta_{ap}, \qquad \forall i, a \tag{4.1}$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow of a class of traveller on a link is equal to the sum of the flows of the class on the paths that contain that link.

Let d_w^i denote the demand of class i for O/D pair w, where one must have that

$$d_w^i = \sum_{p \in P_w} x_p^i, \qquad \forall i, w \qquad (4.2)$$

that is, the travel demand of a class of traveller for an O/D pair is equal to the sum of the flows of that class on paths that connect the O/D pair and is assumed to be fixed and given.

In addition, let f_a denote the total flow on link a, where

$$f_a = \sum_{i=1}^{k} f_a^i, \qquad \forall a \in L. \qquad (4.3)$$

We group the class link flows into the kn-dimensional column vector $\tilde{f}$ with components: $\{f_1^1, \ldots, f_n^1, \ldots, f_1^k, \ldots, f_n^k\}$ and the total link flows: $\{f_1, \ldots, f_n\}$ into the n-dimensional column vector f. Also, we group the class path flows into the kn_p-dimensional column vector $\tilde{x}$ with components: $\{x_{p_1}^1, \ldots, x_{p_{n_p}}^k\}$. Further, we define the feasible set $K \equiv \{\tilde{x} \mid \exists \tilde{x} \geq 0, \text{ and satisfying (4.1) and (4.2)}\}$.

We are now ready to describe the disutility functions associated with the links. We assume, as given, a travel time function t_a associated with each link a in the network, where

$$t_a = t_a(f), \qquad \forall a \in L \qquad (4.4)$$

and a travel cost function c_a associated with each link a, that is

$$c_a = c_a(f) \qquad \forall a \in L \qquad (4.5)$$

with both these functions assumed to be continuous. We notice that here we allow for the general situation in which both the travel time and the travel cost can depend on the entire link flow pattern.

We associate with each class of traveller i, the weights w_1^i and w_2^i, which are assumed to be nonnegative, but not both equal to zero, where w_1^i denotes the weight associated with class i's travel time and w_2^i denotes the weight associated with its travel cost. We then construct the disutility of class i associated with link a, and denoted by u_a^i, as:

$$u_a^i = w_1^i t_a + w_2^i c_a, \qquad \forall i, a. \tag{4.6}$$

From all the above, we may write

$$u_a^i = u_a^i(\tilde{f}), \qquad \forall i, a \tag{4.7}$$

and group the link disutilities into the kn-dimensional column vector u with components: $\{u_1^1, \ldots, u_n^1, \ldots, u_1^k, \ldots, u_n^k\}$.

A possible weighting scheme would be $w_1^i = \psi^i$ and $w_2^i = (1-\psi^i)$ with ψ^i lying in the range from zero to one with $\psi^i = 1$ denoting a class of traveller which is only concerned with the travel time, and with $\psi^i = 0$ denoting a class of traveller only concerned about travel cost.

Let u_p^i denote the travel disutility of class i associated with travelling on path p, where

$$u_p^i = \sum_{a \in L} u_a^i(\tilde{f}) \delta_{ap}, \qquad \forall i, p \tag{4.8}$$

Hence, the disutility, as perceived by a class, associated with travelling on a path is its weighting of the travel times and the travel costs on links which comprise the path.

Here, we have finished the development of the bicriteria, fixed demand traffic network equilibrium model. In the next section, we formulate the governing traffic network equilibrium conditions and give their variational inequality formulation.

4.3 Variational Inequality Formulation of the Fixed Demand, Multiclass, Multicriteria Network Equilibrium Model

The traffic network equilibrium conditions, following the theory provided in the previous chapters of this work, in the generalized context of the multiclass, multicriteria traffic network equilibrium problem, take on the form:

For each class i, for all O/D pairs w, and for all paths $p \in P_w$, the flow pattern $\tilde{x}^*$ is said to be in equilibrium if the following condition holds:

$$u_p^i(\tilde{f}^*) \begin{cases} = \lambda_w^i, & \text{if } x_p^{i*} > 0 \\ \\ \geq \lambda_w^i, & \text{if } x_p^{i*} = 0, \end{cases} \qquad (4.9)$$

where λ_w^i is an indicator, whose value is not known a priori. In other words, all utilized paths by a class connecting an O/D pair, have equal and minimal travel disutilities.

We now present without proof, the variational inequality formulation of the equilibrium conditions (4.9) in link flows.

Theorem 4.1 Variational Inequality Formulation of Network Equilibrium with Fixed Demands – Link Flow Version

A multiclass, link flow pattern $\tilde{f}^* \in K$ is a traffic network equilibrium, that is, it satisfies equilibrium conditions (4.9), if and only if it satisfies the variational inequality problem:

$$\sum_{i=1}^{k} \sum_{a \in L} u_a^i(\tilde{f}^*) \times (f_a^i - f_a^{i*}) \geq 0, \qquad \forall \tilde{f} \in K \qquad (4.10)$$

or, in vector form:

$$\left\langle u(\tilde{f}^*), \tilde{f} - \tilde{f}^* \right\rangle \geq 0 \qquad \forall \tilde{f} \in K \qquad (4.11)$$

where $\langle .,. \rangle$ *denotes the inner product in kn-dimensional Euclidean space.*

4.4 The Elastic Demand, Multiclass, Multicriteria Network Equilibrium Model

In this section, we present the multiclass, multicriteria traffic network equilibrium model with elastic demand that was developed by Nagurney and Dong [20].

A significant feature of this model is that it includes weights associated with the two criteria of travel time and travel cost which are not only class-dependent, but also, explicitly, link-dependent. These weights may incorporate such subjective factors as the relative safety or risk associated with particular links, the relative comfort etc. The model also treats demand functions (rather than their inverses) which are very general and not separable functions. Specifically, the demand associated with a class and origin/destination pair can depend not only on the travel disutility of different classes travelling between the particular origin/destination pair, but can also be influenced by the disutilities of the classes travelling between other origin/destination pairs.

We also here consider a general network $G=[N, L]$, where N denotes the set of nodes in the network and L the set of directed links. Let a denote a link of the network connecting a pair of nodes and let p denote a path, assumed to be acyclic, consisting of a sequence of links connecting an origin/destination (O/D) pair of nodes. There are n links in the network and n_p paths. Let Ω denote the set of J O/D pairs. The set of paths connecting the O/D pair w is denoted by P_w and the entire set of paths in the network by P.

We assume that there are k classes of travellers in the network with a typical class denoted by i. Let f_a^i denote the flow of class i on link a and let x_p^i denote the nonnegative flow of class i on path p. The relationship between the link flows by class and the path flows is:

$$f_a^i = \sum_{p \in P} x_p^i \delta_{ap}, \qquad \forall i, a \qquad (4.12)$$

where $\delta_{ap} = 1$, if link a is contained in path p, and 0, otherwise. Hence, the flow of a class of traveller on a link is equal to the sum of the flows of the class on the paths that contain that link.

In addition, let f_a denote the total flow on link a, where

$$f_a = \sum_{i=1}^{k} f_a^i, \qquad \forall a \in L. \qquad (4.13)$$

We group the class link flows into the kn-dimensional column vector $\tilde{f}$ with components: $\{f_1^1, \ldots, f_n^1, \ldots, f_1^k, \ldots, f_n^k\}$ and the total link flows: $\{f_1, \ldots, f_n\}$ into the n-dimensional column vector f. Also, we group the class path flows into the kn_p-dimensional column vector $\tilde{x}$ with components: $\{x_{p_1}^1, \ldots, x_{p_{n_p}}^k\}$.

We are now ready to describe the functions associated with the links. We assume, as given, a travel time function t_a associated with each link a in the network, where

$$t_a = t_a(f), \qquad \forall a \in L \qquad (4.14)$$

and a travel cost function c_a associated with each link a, that is

$$c_a = c_a(f) \qquad \forall a \in L \qquad (4.15)$$

with both these functions assumed to be continuous. We notice, as in the fixed demand case, that here we allow for the general situation in which both the travel time and the travel cost can depend on the entire link flow pattern.

We assume that each class of traveller i has his own perception of the trade-off between travel time and travel cost which are represented by the nonnegative weights w_{1a}^i and w_{2a}^i. Here w_{1a}^i denotes the weight associated with class i's travel time on link a and w_{2a}^i denotes the weight associated with class i's travel cost on link a. The weights w_{1a}^i and w_{2a}^i are link-dependent and, hence, can incorporate such link-dependent factors as safety, comfort, and view. For example, in the case of a pleasant view on a link, travellers may weight the travel cost higher than the travel time on such a link. However, if a link has a rough surface or is noted for unsafe road conditions such as ice in the winter, travellers may then assign a higher weight to the travel time than the travel cost. Link-dependent weights provide a greater level of generality and flexibility in modeling travel decision-making than weights that are identical for the travel time and for the travel cost on all links for a given class.

We then construct the *generalized* cost/disutility of class i associated with link a and denoted by u_a^i, as:

$$u_a^i = w_{1a}^i t_a + w_{2a}^i c_a, \qquad \forall i, a. \tag{4.16}$$

From all the above, we may write

$$u_a^i = u_a^i(\tilde{f}), \qquad \forall i, a \tag{4.17}$$

and group the link generalized costs into the kn-dimensional column vector u with components: $\{u_1^1, \ldots, u_n^1, \ldots, u_1^k, \ldots, u_n^k\}$.

A possible weighting scheme would be $w_{1a}^i = \psi_a^i$ and $w_{2a}^i = (1 - \psi_a^i)$ with ψ_a^i lying in the range from zero to one with ψ_a^i=1 denoting a class of traveller who is only concerned with the travel time on a particular link a, and with ψ_a^i=0 denoting a class of traveller only concerned about travel cost on link *a*; with weights within the range reflecting classes who perceive travel time and travel cost as per the disutility functions accordingly.

Let u_p^i denote the *generalized* cost of class *i* associated with travelling on path *p*, where

$$u_p^i = \sum_{a \in L} u_a^i(\tilde{f})\delta_{ap}, \qquad \forall i, p \tag{4.18}$$

Hence, the generalized cost, as perceived by a class, associated with travelling on a path is the sum of the generalized link costs on links comprising the path.

Let d_w^i denote the travel demand of class *i* traveller between origin/destination pair *w* and let λ_w^i denote the travel disutility associated with class *i* traveller, travelling between the origin/destination pair *w*. We group the travel demands into a *kJ*-dimensional column vector *d* and the origin/destination pair travel disutilities into a *kJ*-dimensional column vector λ.

The path flow vector $\tilde{x}$ induces the demand vector *d* with components

$$d_w^i = \sum_{p \in P_w} x_p^i, \qquad \forall i, w. \tag{4.19}$$

We assume that the travel demands are determined by the origin/destination travel disutilities, that is

$$d_w^i = d_w^i(\lambda), \qquad \forall i, w \tag{4.20}$$

and denote the *kJ*-dimensional row vector of demand functions by $d(\lambda)$.

We must note that the travel demand function (4.20) is quite general as it allows the demand for a class associated with an O/D pair to depend not only on the travel disutilities of different classes associated with that O/D pair, but also on those associated with other O/D pairs.

Here, we have finished the development of the bicriteria, elastic demand traffic network equilibrium model. In the next section, we formulate the governing traffic network equilibrium conditions and give their variational inequality formulation.

4.5 Variational Inequality Formulation of the Elastic Demand, Multiclass, Multicriteria Network Equilibrium Model

The traffic network equilibrium conditions, following the theory provided in the previous chapters of this work, in the generalized context of the multiclass, multicriteria traffic network equilibrium problem, take on the form:

For each class *i*, for all O/D pairs $w \in W$ and for all paths $p \in P_w$, the flow pattern $\tilde{x}^*$ is said to be in equilibrium if the following conditions hold:

$$u_p^i(\tilde{f}^*) \begin{cases} = \lambda_w^{i*}, & \text{if } x_p^{i*} > 0 \\ \geq \lambda_w^{i*}, & \text{if } x_p^{i*} = 0, \end{cases} \tag{4.21}$$

and

$$d_w^i(\lambda^*) \begin{cases} = \sum_{p \in P_w} x_p^{i*}, & \text{if } \lambda_w^{i*} > 0 \\ \leq \sum_{p \in P_w} x_p^{i*}, & \text{if } \lambda_w^{i*} = 0. \end{cases} \tag{4.22}$$

In other words, all utilized paths by a class connecting an O/D pair have equal and minimal generalized path costs. Meanwhile, if the travel disutility associated with travelling between O/D pair *w* of class *i* is positive, then the market clears for this O/D pair and this class. That is, the sum of the path flows of this class of travellers on paths connecting this O/D pair is equal to the demand associated with this O/D pair. If the travel disutility is zero, then the sum of the path flows can exceed the demand of this class of travellers.

Hence, in the elastic demand framework, different classes of travellers can also choose their O/D pairs, in addition to their paths. Thus, this model allows one to capture the relative attractiveness of different O/D pairs as perceived by the distinct classes of travellers through the travel disutilities.

We now present without proof, the variational inequality formulation of the equilibrium conditions (4.21) and (4.22) in link flows.

Theorem 4.2 Variational Inequality Formulation of Network Equilibrium with Elastic Demands – Link Flow Version

A multiclass, multicriteria link flow, travel demand and O/D travel disutility pattern $(\tilde{f}^*, d^*, \lambda^*) \in K$ *is a traffic network equilibrium, that is, satisfies equilibrium conditions (4.21) and (4.22) if and only if it satisfies the variational inequality problem:*

$$\sum_{i=1}^{k}\sum_{a\in L} u_a^i(\tilde{f}^*)\times(f_a^i - f_a^{i*}) - \sum_{i=1}^{k}\sum_{w\in W}\lambda_w^{i*}\times(d_w^i - d_w^{i*}) + \sum_{i=1}^{k}\sum_{w\in W}(d_w^{i*} - d_w^i(\lambda^*))\times(\lambda_w^i - \lambda_w^{i*}) \geq 0$$

$$\forall(\tilde{f}, d, \lambda) \in K \qquad (4.23)$$

where we define the feasible set K as: $K \equiv \{(\tilde{f}, d, \lambda) \mid \lambda \geq 0$ *and* $\exists \tilde{x} \geq 0$, *such that (4.12), (4.13) and (4.19) hold}.*

4.6 Conclusions

In this chapter, we have described two multiclass, multicriteria network equilibrium models. In both these models we have used as criteria associated with the decision-making, travel time and travel cost. About the type of demand, in the first model we considered fixed demand and in the second, elastic. After the development of the two models, the variational inequality formulations of the governing traffic network equilibrium conditions were given.

A remark that can be made on the subject of this chapter, is that the elastic demand model of Section 4.4 reduces to the fixed demand, multiclass, multicriteria model of Section 4.2, in the case that the travel demands are fixed and the weights are not link-dependent but are class-dependent (specifically, if $w_{1a}^i = w_1^i$ and $w_{2a}^i = w_2^i$ for all links $a \in L$ and classes i).

The theory of multiclass, multicriteria traffic networks provided in this chapter, will be used in the construction of the freight network model in the next chapter of this postgraduate work.

Chapter 5 Development of a Multicriteria Shippers and Consumers Network Equilibrium Model using Variational Inequalities Theory

5.1 Introduction

In this chapter, we will use the theory provided in the previous chapters of this research, in order to construct a multicriteria shippers and consumers network equilibrium model and formulate it as a variational inequality problem.

In Chapter 1, we point out the important role of freight transportation in both regional and national economics by being a vital link between the supply and demand in the supply chain. Due to the trend of globalization in the world economic development, logistics management has recently become a major issue for many companies in their effort to secure an edge over their competitors. At the same time, higher labor and energy costs, coupled with increasing congestion on the road networks have increased the proportion of the transportation cost to total cost of a finished product. Transportation time has also become an important factor in logistics as a component of the "lead time", which represents the time between ordering and receiving finished goods. The variance in transportation time has assumed importance as it leads to higher inventory cost. Furthermore, deregulation in some transportation industries and the strategic alliance between different transportation service providers has allowed more competitive prices and more intermodal options becoming available to users.

All the above, create the need to the industries for tools that will help the decision-making process. Here, we consider a network consisting of shippers, who produce a certain product and consumers, who want to buy that product. A model, which in equilibrium, outputs a product shipment and demand price pattern is constructed using the VI theory.

This chapter is organized as follows: In Section 5.2 we describe the network defining the nodes and links of its structure. In Section 5.3 we consider the shippers and develop their optimality conditions and in Section 5.4 we focus on the consumers and we derive their equilibrium conditions. In Section 5.5 the integrated model is constructed and we derive the variational inequality formulation of the governing equilibrium conditions. In Section 5.6, we present two numerical examples. Finally, in Section 5.7 we conclude this chapter.

5.2 The Network Structure

In this chapter, we describe a network framework for the analysis and computation of solutions to problems in which the shippers and the consumers are multicriteria decision-makers. Here, the problem consists of two tiers of decision-makers, the shippers and the consumers, in contrast to the multitiered network studied in supply chains in Chapter 3, which consists of three tiers. Moreover, in this chapter, the decision-makers consider multiple criteria.

In particular, the shippers, which are spatially separated and are assumed to produce a certain product and the consumers, located at the demand markets, each face multiple criteria in making their production / consumption decisions. It is assumed that each shipper seeks to maximize its profit, where profit is the difference between the revenue and costs, which include not only production cost but also the total processing costs associated with selecting different shipment options to each demand market. The shipment alternatives to each demand market are represented by links characterized by specific transportation cost and transportation time functions. Hence, a specific shipment option or link, may have a low associated transportation time but a high transportation cost, whereas, another may have a high associated transportation time and a low cost. Each shipper also seeks to maximize its output (market share). An individual shipper assigns its own weights to the two criteria of profit maximization and output maximization.

The consumers, in turn, correspond to different classes and weight the transportation time and transportation cost associated with obtaining the product from each shipper in an individual manner. Thus, one class of consumers may be more time-sensitive in obtaining the product, whereas, another may be more cost-sensitive. The consumers of each class, therefore, base their consumption decisions not only on the price set by the producers, but also on the *generalized* cost, which includes the transportation time and cost associated with obtaining the product at a particular demand market, from a specific shipper.

The shipment alternatives associated with shipper i and demand market j are depicted as links in Figure 5.1.

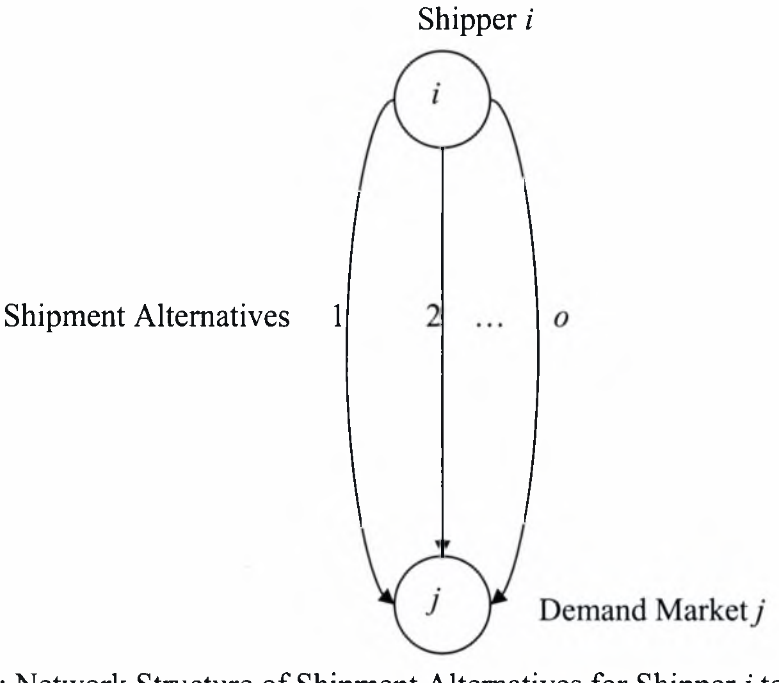

Figure 5.1: Network Structure of Shipment Alternatives for Shipper i to Demand Market j

The whole network is a bipartite one with multiple links connecting the top-tier nodes corresponding to the shippers and the bottom-tier nodes denoting the demand markets, with the links representing different shipment alternatives (Figure 5.2 in Section 5.5).

5.3 The Behavior of the Shippers and Their Optimality Conditions

In this section, we describe the behavior of the shippers and we develop their optimality conditions.

As we have explained in the previous section, we consider *m* shippers that produce a certain homogeneous product, which is consumed by consumers located at *n* demand markets. We denote a typical shipper by *i* and a typical demand market by *j*. Let q_i denote the nonnegative production output of shipper *i* and group the production outputs of all shippers into the column vector $q \in R_+^m$. Assume that each shipper *i* is faced with a production cost function f_i, which depends on the entire vector of production outputs, that is:

$$f_i = f_i(q), \ \forall i. \tag{5.1}$$

We consider this dependence in order to show that the production cost function of a shipper *i* is affected not only by its own output level q_i, but also reflects the impact of the other shippers production patterns on shipper *i*'s cost. This impact is about competition for the resources, consumption of raw materials, etc.

Each shipper can ship the product to each demand market using one or more of *o* possible shipment alternatives, which can represent mode/route alternatives. Denote a typical shipment alternative by *l*. Associated with shipper *i* selecting shipment alternative *l* to demand market *j* is the total shipment processing cost, denoted by tc_{ijl}, and given by:

$$tc_{ijl} = \hat{c}_{ijl}(q_{ijl})q_{ijl}, \tag{5.2}$$

where $\hat{c}_{ijl}$ denotes the unit cost of processing the shipment of the product from shipper i to demand market j using alternative l and q_{ijl} denotes the quantity of the product produced by shipper i and shipped to demand market j using alternative l.

The quantity produced by shipper i must satisfy the following conservation of flow equation:

$$q_i = \sum_{j=1}^{n}\sum_{l=1}^{o} q_{ijl}\,, \tag{5.3}$$

which states that the quantity produced by shipper i is equal to the sum of the quantities shipped from the shipper to all demand markets via all the shipment alternatives.

The total costs incurred by a shipper i, thus, are equal to the sum of the shipper's production cost plus the total cost of processing the shipments of the product along all shipment links to all demand markets. The shipper's revenue, in turn, is equal to the price charged for the product times the total quantity consumed of the product from the shipper at all the demand markets. Let ρ^*_{ijl} denote the price charged for the product by the shipper i for shipment to demand market j via alternative l (the supply price). Then, the criterion of profit maximization for shipper i can be expressed as:

$$\text{Maximize} \qquad \sum_{j=1}^{n}\sum_{l=1}^{o} \rho^*_{ijl} q_{ijl} - f_i(q) - \sum_{j=1}^{n}\sum_{l=1}^{o} \hat{c}_{ijl}(q_{ijl}) q_{ijl} \tag{5.4}$$

subject to:

$$q_i = \sum_{j=1}^{n}\sum_{l=1}^{o} q_{ijl} \tag{5.5}$$

$$q_{ijl} \geq 0, \quad \forall j, l \tag{5.6}$$

In addition, since multicriteria decision – making is being considered, assume that each shipper seeks to maximize its production output, that is, shipper i seeks to also:

$$\text{Maximize} \qquad \sum_{j=1}^{n}\sum_{l=1}^{o} q_{ijl} \tag{5.7}$$

subject to $q_{ijl} \geq 0$ for all j,l.

We now describe how to construct a value function associated with the two criteria facing each shipper, which are profit and output maximization, based on the analysis in the previous chapters. As we have seen in Chapter 4, each shipper can associate a nonnegative weight with every criterion under consideration. In this case, we associate a nonnegative weight w_i with the output maximization criterion and we set the weight associated with the profit maximization criterion equal to 1, for purposes of easier model development. Consequently, we can express the optimization problem faced by shipper i as:

$$\text{Maximize} \qquad \sum_{j=1}^{n}\sum_{l=1}^{o} \rho_{ijl}^{*} q_{ijl} - f_i(q) - \sum_{j=1}^{n}\sum_{l=1}^{o} \hat{c}_{ijl}(q_{ijl}) q_{ijl} + w_i \sum_{j=1}^{n}\sum_{l=1}^{o} q_{ijl} \ , \tag{5.8}$$

subject to $q_{ijl} \geq 0$ for all j,l, and satisfying (5.3).

According to (5.8), each shipper has its own production cost function, its own total shipment processing cost function, as well as its weight associated with the output maximization criterion.

As in the case of the supply chain network model of Chapter 3, the shippers are assumed to behave in a noncooperative manner in the sense of Cournot [6] and Nash ([23], [24]), seeking to determine their own optimal production and shipment quantities. According to what has been examined in Chapter 2, if the production cost function and the total shipment processing cost function for each shipper, are continuously differentiable and convex, then the optimality conditions take the form of a variational inequality problem as follows:

Determine $(q^{*}, Q^{*}) \in K$, such that

$$\sum_{i=1}^{m}\left[\frac{\partial f_i(q^{*})}{\partial q_i} - w_i\right] \times \left[q_i - q_i^{*}\right] + \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{o}\left[\frac{\partial \hat{c}_{ijl}(q_{ijl}^{*})}{\partial q_{ijl}} q_{ijl}^{*} + \hat{c}_{ijl}(q_{ijl}^{*}) - \rho_{ijl}^{*}\right] \times \left[q_{ijl} - q_{ijl}^{*}\right] \geq 0,$$

$$\forall (q,Q) \in K, \tag{5.9}$$

where Q denotes the mno – dimensional column vector of product shipments, and $K \equiv \{(q,Q) \mid Q \geq 0$, and satisfies (5.3)$\}$.

Here, with the variational inequality formulation of the shippers' optimization problem, we have finished the part of the model that has to do with the production side. In the next sections of this chapter, the consumption side is addressed and the variational inequality formulation of the governing equilibrium conditions is derived.

5.4 The Behavior of the Multiclass, Multicriteria Consumers and the Equilibrium Conditions

In this section, we describe the behavior of the consumers located at the demand markets and we derive their equilibrium conditions.

We assume that there are k classes of consumers, with a typical class denoted by r, at each of the n demand markets. Each class of consumer takes into account in making consumption decisions not only the prices charged for the product by the shippers, but also the transportation time and transportation cost to obtain the product. Hence, they, as are the shippers, are multicriteria decision makers.

Let c_{ijl} denote the transportation cost associated with shipping the product from shipper i to demand market j along link l. Assume that the transportation cost is continuous and of the form:

$$c_{ijl} = c_{ijl}(q_{ijl}), \qquad \forall i, j, l. \tag{5.10}$$

In addition, let t_{ijl} denote the transportation time associated with shipping the product from shipper i to j via l, where the function is continuous and of the form:

$$t_{ijl} = t_{ijl}(q_{ijl}), \qquad \forall i, j, l. \tag{5.11}$$

We also let q_{ijlr} denote the quantity of the product shipped from i to j via l and going to class r, where:

$$q_{ijl} = \sum_{r=1}^{k} q_{ijlr}, \qquad \forall i, j, l, \tag{5.12}$$

that is, the total amount of the product shipped between a shipper and a demand market along a link, is equal to the sum of all the class product shipments shipped on that link.

In Chapter 4, we studied two multiclass, multicriteria network equilibrium models in order to understand how the individual weighting of distinct criteria associated with decision-making on networks is implemented. Hence, in this model, we assume that members of a class of consumers at each demand market perceive the transportation cost and the transportation time associated with obtaining the product in an individual manner and weight these two criteria, which they wish to minimize, accordingly. In particular, we let w^1_{jr} denote the nonnegative weight associated with the transportation cost as perceived by class r at demand market j and we let w^2_{jr} denote the nonnegative weight associated with the transportation time as perceived by class r at demand market j. Thus, the generalized cost, as perceived by class r at demand market j, of obtaining the product from shipper i via shipment alternative l is given by the expression:

$$w^1_{jr} c_{ijl}(q_{ijl}) + w^2_{jr} t_{ijl}(q_{ijl}). \tag{5.13}$$

We let now λ_{jr} denote the generalized demand price of the product as perceived by class r at demand market j and group the generalized demand prices into the column vector $\lambda \in R^{nk}_{+}$. Further, we denote the demand of class r at demand market j by d_{jr} and we assume as given, the continuous demand functions:

$$d_{jr} = d_{jr}(\lambda), \qquad \forall j, r. \tag{5.14}$$

The classes of consumers located at the demand markets take the price charged by the producers for the product, which was denoted by ρ^*_{ijl}, plus the generalized cost as perceived by the class associated with shipping the product to the demand market, in making their consumption decisions. In equilibrium, we know that this sum must be equal to the demand price that the consumers of that class are willing to pay to obtain the product. Hence, the equilibrium conditions take the form:

For all *i*, *j*, *l*, *r*:

$$\rho^*_{ijl} + w^1_{jr} c_{ijl}(q^*_{ijl}) + w^2_{jr} t_{ijl}(q^*_{ijl}) \begin{cases} = \lambda^*_{jr}, \text{ if } q^*_{ijlr} > 0 \\ \geq \lambda^*_{jr}, \text{ if } q^*_{ijlr} = 0 \end{cases} \tag{5.15}$$

and

$$d_{jr}(\lambda^*) \begin{cases} = \sum_{i=1}^{m} \sum_{l=1}^{o} q^*_{ijlr}, \text{ if } \lambda^*_{jr} > 0 \\ \leq \sum_{i=1}^{m} \sum_{l=1}^{o} q^*_{ijlr}, \text{ if } \lambda^*_{jr} = 0. \end{cases} \tag{5.16}$$

Conditions (5.15) and (5.16) have the form of conditions (4.21) and (4.22) respectively, since we also here develop an elastic demand, multiclass, multicriteria network equilibrium model. Condition (5.16) state that if the demand price for the product of a class at a demand market is positive, then the sum of the product shipments for that class from the shippers along all shipment alternatives, is precisely equal to the demand for that class and demand market at the equilibrium demand price vector. If the equilibrium demand price for that class and demand market is zero, then there may be an excess of product shipments over the demand for that class and demand market.

According to Theorem 4.2, equilibrium conditions (5.15) and (5.16) can be expressed as the following variational inequality problem:

Determine $(\hat{Q}^*, \lambda^*) \in K$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{o}\sum_{r=1}^{k}\left[\rho_{ijl}^* + w_{jr}^1 c_{ijl}(q_{ijl}^*) + w_{jr}^2 t_{ijl}(q_{ijl}^*) - \lambda_{jr}^*\right]\times\left[q_{ijlr} - q_{ijlr}^*\right]$$
$$+\sum_{j=1}^{n}\sum_{r=1}^{k}\left[\sum_{i=1}^{m}\sum_{l=1}^{o} q_{ijlr}^* - d_{jr}(\lambda^*)\right]\times\left[\lambda_{jr} - \lambda_{jr}^*\right] \geq 0, \forall(\hat{Q}, \lambda) \in K, \quad (5.17)$$

where $\hat{Q}$ is the *mnok*-dimensional vector of class product shipments with components *ijlr* given by q_{ijlr} and $K \equiv \left\{(\hat{Q}, \lambda) \mid (\hat{Q}, \lambda) \in R_+^{mnok+nk}\right\}$.

By giving the variational inequality formulation of the equilibrium conditions (5.15) and (5.16), we have finished the second part of the model, that of the consumption side. In the next section, we combine the two parts to construct the integrated model and we give the variational inequality formulation of the governing equilibrium conditions.

5.5 The Integrated Model and the Variational Inequality Formulation of the Governing Equilibrium Conditions

In this section, we construct the integrated model, which synthesizes the optimality conditions of the multicriteria shippers and the equilibrium conditions of the multiclass, multicriteria consumers on the bipartite network.

As in the case of the supply chain network model that we studied in Chapter 3, in order to obtain an equilibrium of the spatial network system, the sum of the optimality conditions of the shippers as given by (5.9) and the equilibrium conditions of the consumers as in (5.17) must be satisfied.

In Figure 5.2, we depict the network structure, consisting of all the shippers, all the demand markets and classes of consumers and all the shipment alternatives. We note again that the network is bipartite, except that there are multiple links connecting each top-tier node with each bottom-tier node to represent shipment alternatives.

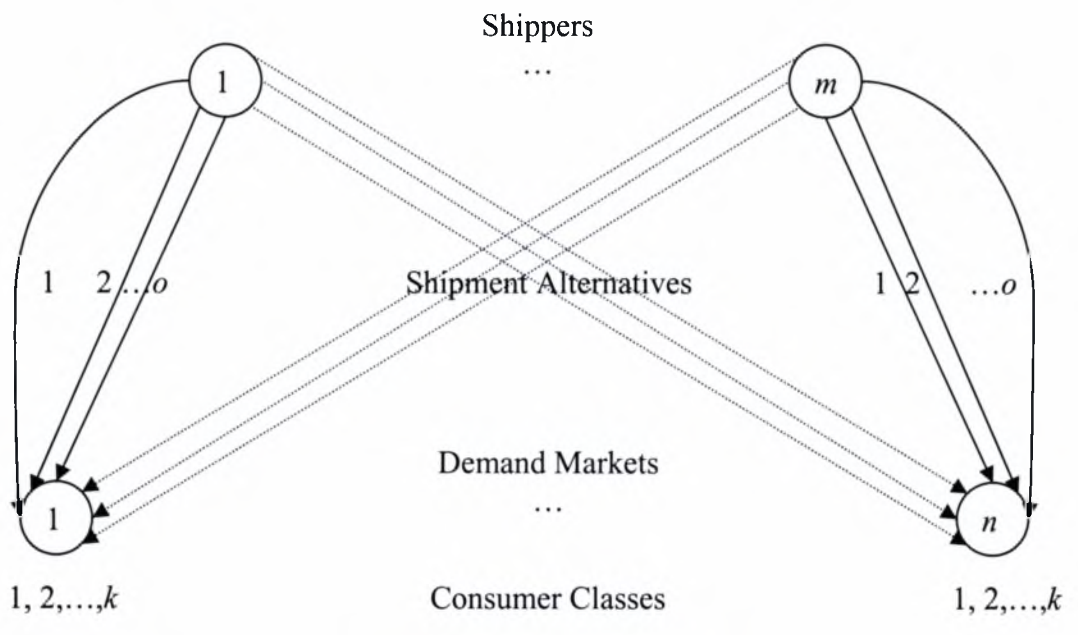

Figure 5.2: The Network Structure

Now, we construct a single variational inequality problem governing both the production and the consumption sides. First, we note that the shipper production output quantities (q_i), their product shipments to the demand markets (q_{ijl}) and the class product shipments (q_{ijlr}) are related by equations (5.3) and (5.12). As we considered in Chapter 3, $f_i(q) \equiv f_i(Q^1)$ (see (3.4)), here, we let for the marginal production costs $\frac{\partial f_i(q)}{\partial q_i}$, $\frac{\partial f_i(\hat{Q})}{\partial q_{ijlr}}$:

$$\frac{\partial f_i(\hat{Q})}{\partial q_{ijlr}} \equiv \frac{\partial f_i(q)}{\partial q_i}, \qquad \forall i,j,l,r. \tag{5.18}$$

We also let:

$$mtc_{ijlr}(\hat{Q}) \equiv \frac{\partial tc_{ijl}}{\partial q_{ijl}}, \qquad \forall i,j,l,r, \tag{5.19}$$

which, using (5.2) becomes:

$$mtc_{ijlr}(\hat{Q}) \equiv \frac{\partial \hat{c}_{ijl}(q_{ijl})}{\partial q_{ijl}} \times q_{ijl} + \hat{c}_{ijl}(q_{ijl}), \quad \forall i,j,l,r. \tag{5.20}$$

The term mtc_{ijlr} denotes the marginal cost of processing the shipment of the product from *i* to *j* via *l* for class *r*. In addition, we define the generalized cost $gc_{ijlr}(\hat{Q})$ for obtaining the product by class *r*, at demand market *j*, from shipper *i*, using shipment alternative *l*, as:

$$gc_{ijlr}(\hat{Q}) \equiv w^1_{jr} c_{ijl}(q_{ijl}) + w^2_{jr} t_{ijl}(q_{ijl}), \quad \forall i,j,l,r, \tag{5.21}$$

using expression (5.13).

As we have already stated, an equilibrium of the spatial network system consisting of multicriteria shippers and consumers is attained when the sum of the optimality conditions for all shippers, as denoted by inequality (5.9), and the spatial equilibrium conditions, as represented by inequality (5.17), is satisfied by the class product shipment variables and the demand price variables. The sum of inequalities (5.9) and (5.17), using identities (5.18), (5.20) and (5.21), gives the following inequality:

$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{o}\sum_{r=1}^{k}\left[\frac{\partial f_i(\hat{Q}^*)}{\partial q_{ijlr}} + mtc_{ijlr}(\hat{Q}^*) - w_i - \rho^*_{ijl}\right]\times\left[q_{ijlr} - q^*_{ijlr}\right] \\
&+\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{o}\sum_{r=1}^{k}\left[\rho^*_{ijl} + gc_{ijlr}(\hat{Q}^*) - \lambda^*_{jr}\right]\times\left[q_{ijlr} - q^*_{ijlr}\right] \\
&+\sum_{j=1}^{n}\sum_{r=1}^{k}\left[\sum_{i=1}^{m}\sum_{l=1}^{o} q^*_{ijlr} - d_{jr}(\lambda^*)\right]\times\left[\lambda_{jr} - \lambda^*_{jr}\right] \geq 0, \forall (\hat{Q},\lambda) \in K.
\end{aligned} \tag{5.22}$$

From the above inequality, according to the variational inequalities theory, we can easily recover the supply prices ρ_{ijl}^{*}, once we have obtained the solution. In equilibrium, from the first term in (5.22) we get that if q_{ijlr}^{*} is positive, then $\rho_{ijl}^{*} = \frac{\partial f_i(\hat{Q}^*)}{\partial q_{ijlr}} + mtc_{ijlr}(\hat{Q}^*) - w_i$.

Also, from the second term in (5.22) we have that if q_{ijlr}^{*} is positive, then $\rho_{ijl}^{*} = \lambda_{jr}^{*} - gc_{ijlr}(\hat{Q}^*)$.

Further simplification of (5.22), gives us the following variational inequality problem:

Determine $(\hat{Q}^*, \lambda^*) \in K$, such that

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{o}\sum_{r=1}^{k}\left[\frac{\partial f_i(\hat{Q}^*)}{\partial q_{ijlr}} + mtc_{ijlr}(\hat{Q}^*) - w_i + gc_{ijlr}(\hat{Q}^*) - \lambda_{jr}^*\right]\times\left[q_{ijlr} - q_{ijlr}^*\right]$$
$$+\sum_{j=1}^{n}\sum_{r=1}^{k}\left[\sum_{i=1}^{m}\sum_{l=1}^{o} q_{ijlr}^* - d_{jr}(\lambda^*)\right]\times\left[\lambda_{jr} - \lambda_{jr}^*\right] \geq 0, \forall(\hat{Q}, \lambda) \in K \tag{5.23}$$

where $K \equiv \left\{(\hat{Q}, \lambda) \mid (\hat{Q}, \lambda) \in R_{+}^{mnok+nk}\right\}$.

The above inequality is the variational inequality formulation of the governing equilibrium conditions of the model we constructed in this chapter of the postgraduate work.

5.6 Numerical Examples

In this section, two numerical examples are presented. We consider a network consisting of two shippers, two demand markets and with two available shipping alternatives from each shipper to each demand market. We also consider two classes of consumers. This network structure is depicted in Figure 5.3.

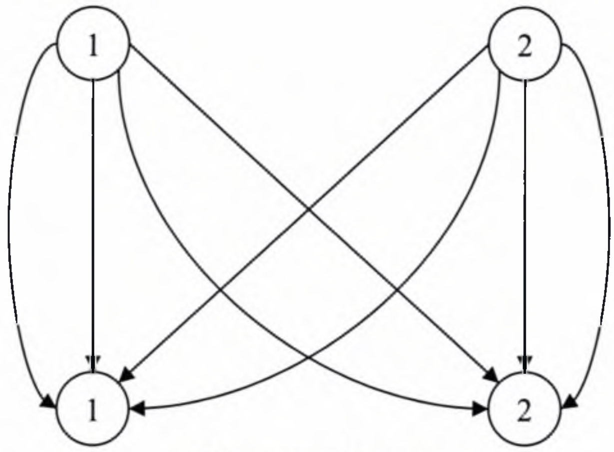

Figure 5.3: The Network Structure for the Examples

5.6.1 Example 1

In the first example, the production cost functions of the two shippers are given by:

$f_1(q) = q_1 + q_1 q_2$,

$f_2(q) = q_2 + q_1 q_2$.

The unit shipment processing cost functions and the transportation cost and time functions are as given in Table 5.1.

The class demand functions are:

$d_{11}(\lambda) = 1000 - 2\lambda_{11} - 1.5\lambda_{21}$,

$d_{12}(\lambda) = 1000 - 2\lambda_{12} - 1.5\lambda_{22}$,

$d_{21}(\lambda) = 1000 - 2\lambda_{21} - 1.5\lambda_{11}$,

$d_{22}(\lambda) = 1000 - 2\lambda_{22} - 1.5\lambda_{12}$.

i, j, l	$\hat{c}_{ijl}$	c_{ijl}	t_{ijl}
1, 1, 1	$q_{111}+1$	$q_{111}+5$	$2q_{111}+10$
1, 1, 2	$0.5q_{112}+0.5$	$3q_{112}+15$	$q_{112}+3.5$
1, 2, 1	$q_{121}+1$	$q_{121}+5$	$2q_{121}+10$
1, 2, 2	$0.5q_{122}+0.5$	$3q_{122}+15$	$q_{122}+3.5$
2, 1, 1	$q_{211}+1$	$q_{211}+5$	$2q_{211}+10$
2, 1, 2	$0.5q_{212}+0.5$	$3q_{212}+15$	$q_{212}+3.5$
2, 2, 1	$q_{221}+1$	$q_{221}+5$	$2q_{221}+10$
2, 2, 2	$0.5q_{222}+0.5$	$3q_{222}+15$	$q_{222}+3.5$

Table 5.1: The Unit Shipment Processing Cost, Transportation Cost and Transportation Time Functions

The weights are:

$w_1 = w_2 = 1$,

$w_{11}^1 = w_{12}^1 = w_{21}^1 = w_{22}^1 = w_{11}^2 = w_{12}^2 = w_{21}^2 = w_{22}^2 = 1$.

From Table 5.1, it is obvious that shipment alternative 2 is faster than shipment alternative 1, but it has a higher transportation cost.

Solving the system of equations that results from inequality (5.23), we have the product shipment pattern reported in Table 5.2.

Shipment	Class r=1	Class r=2
q^*_{111r}	14.1612	14.1612
q^*_{112r}	13.8612	13.8612
q^*_{121r}	14.1612	14.1612
q^*_{122r}	13.8612	13.8612
q^*_{211r}	14.1612	14.1612
q^*_{212r}	13.8612	13.8612
q^*_{221r}	14.1612	14.1612
q^*_{222r}	13.8612	13.8612

Table 5.2: The Equilibrium Product Shipment Pattern for Example 1

The computed equilibrium class demand prices at the two demand markets are:

$\lambda^*_{11} = \lambda^*_{12} = \lambda^*_{21} = \lambda^*_{22} = 269.7015$

and the computed equilibrium production outputs of the shippers, using equations (5.3) and (5.12), are:

$q^*_1 = q^*_2 = 112.0896.$

Using equation $\rho^*_{ijl} = \lambda^*_{jr} - gc_{ijlr}(\hat{Q}^*)$, we have for the prices ρ^*_{ijl} that the shipper i charges to the consumers at demand market j for the product, if delivered through shipment alternative 1 or 2:

$\rho^*_{ij1} = 169.7342 \qquad \forall i, j,$

$\rho^*_{ij2} = 140.3119 \qquad \forall i, j.$

5.6.2 Example 2

In the second example, we kept all the data as in Example 1, except that now we assumed that shipper 1 has altered his market share weight from 1 to 10, that is, $w_1 = 10$.

Solving the system of equations that results from inequality (5.23), we have the product shipment pattern reported in Table 5.3.

Shipment	Class r=1	Class r=2
q^*_{111r}	16.6463	16.6463
q^*_{112r}	16.3463	16.3463
q^*_{121r}	16.6463	16.6463
q^*_{122r}	16.3463	16.3463
q^*_{211r}	12.1463	12.1463
q^*_{212r}	11.8463	11.8463
q^*_{221r}	12.1463	12.1463
q^*_{222r}	11.8463	11.8463

Table 5.3: The Equilibrium Product Shipment Pattern for Example 2

The computed equilibrium class demand prices at the two demand markets are:

$$\lambda^*_{11} = \lambda^*_{12} = \lambda^*_{21} = \lambda^*_{22} = 269.4328$$

and the computed equilibrium production outputs of the shippers, using equations (5.3) and (5.12), are:

$q_1^* = 131.9704,$

$q_2^* = 95.9704.$

Using equation $\rho_{ijl}^* = \lambda_{jr}^* - gc_{ijlr}(\hat{Q}^*)$, we have for the prices ρ_{ijl}^* that the shipper i charges to the consumers at demand market j for the product, if delivered through shipment alternative l:

for shipper 1, $\rho_{111}^* = \rho_{121}^* = 154.555$ and $\rho_{112}^* = \rho_{122}^* = 120.1624$,

for shipper 2, $\rho_{211}^* = \rho_{221}^* = 181.555$ and $\rho_{212}^* = \rho_{222}^* = 156.1624$.

Comparing the two examples, we observe that in Example 2, each shipper charges a different price, according to the shipment alternative, to both demand markets. We also observe that in Example 2, the total output of shipper 1 increased in comparison to its output in Example 1, whereas that of shipper 2 decreased. Finally, the change in the class demand prices is negligible.

5.7 Conclusions

In this chapter, we examine a bipartite network with shippers and consumers, that both are assumed to be multicriteria decision-makers. Using variational inequalities theory, we developed a model that outputs the equilibrium production, consumption and product flows.

Inequality (5.22), where the variables are the product shipments and the class demand prices, can also be seen from an economic perspective. From the first term in (5.22), we can conclude that the price ρ_{ijl}^* charged by a shipper i for the product to consumers at demand market j using shipment along l, must be precisely equal to the shipper's marginal cost of production plus the marginal of the total cost of shipment processing along l minus the weight associated with the output maximization criterion (if the product shipment is positive between the shipper/demand market pair along the link and for that class). Otherwise, if the marginal costs associated with that combination of production / shipment alternative / class / demand market exceed the supply price that the consumers are willing to pay for the product, then there will be zero of that product shipped on that link from that shipper to the class at that demand market.

We emphasize again that time and cost associated with product deliveries are of a particular importance in today's economy regarding decision-making, not only for shippers, but also for consumers. The model developed in this chapter can be very helpful in that decision-making process.

Chapter 6 Concluding Remarks

In this postgraduate work, we studied network equilibrium models. Networks provide the infrastructure for business, science, technology and education. Transportation networks give us the means to cross physical distance in order to see clients and conduct business, as well as to visit colleagues and friends. They enable manufacturing processes through the supply of the necessary input components and the ultimate distribution of the finished products to the consumers. Freight transportation is a vital component of the economy. It supports production, trade and consumption activities by ensuring the efficient movement and timely availability of raw materials and finished goods.

The decision-making process in a freight transportation system is a very complicated procedure, since it involves many agents with conflict of interests. The network framework allows us to formalize the alternatives available to decision-makers, to model their individual behavior (characterized by particular criteria, which they wish to optimize) and ultimately to compute the flows on the network, as well as the associated prices. The theory of spatial equilibration in transport networks is first presented and the study of the variational inequalities (VI) theory, which is a powerful tool for the study of the equilibrium states, follows.

These theories are then applied to multitiered networks that consist of distinct tiers of decision-makers. A supply chain network model is studied, with the emphasis given to the equilibrium conditions and their VI formulation. An equilibrium approach is necessary and valuable since it provides a benchmark against which one can evaluate both prices and product flows. Moreover, it captures the independent behavior of the various decision-makers, as well as the effect of their interactions.

After multitiered networks, we study two multiclass, multicriteria network equilibrium models. In these models, we allow more than one class of decision-makers in the network that consider multiple criteria in making their choices. A fixed demand network equilibrium model is first presented and it is formulated as a variational inequality problem. Then, en elastic demand network problem is addressed and its variational inequality formulation is also given.

The main contribution of the research reported in this work is that we described a network framework for the formulation, analysis and computation of solutions to problems in which the decision-makers on each of the two tiers of the network consider multiple criteria. The model we constructed, brings together multicriteria decision-makers on the production side and on the consumption side in a network framework. In particular, the shippers, which are spatially separated and the consumers located at the demand markets, each face multiple criteria in making their production / consumption decisions. The variational inequality formulation of the governing equilibrium conditions is derived. Resolving this variational inequality problem, using known algorithms for variational inequalities, we get the product shipment pattern, as well as the demand price pattern in equilibrium.

The network equilibrium model constructed in this research can be very helpful to the industries involved in freight transportation, in their decision making process. Since time and cost associated with product deliveries are of particular importance, the outputs of this model provide a benchmark against which an industry can evaluate both real prices and product flows. For example, an industry can compute the impact to product shipments and prices of changes to its market share weight or changes to consumers' time-sensitivity, etc.

Such a sensitivity analysis can be done as a further research on the constructed network equilibrium model. The adjustment of this model, in order to allow time-varying demands, can also be a topic of further research.

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