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ΣΕΙΡΑ ΕΡΕΥΝΗΤΙΚΩΝ ΕΡΓΑΣΙΩΝ

Convergence in Fuzzy Topological Spaces

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CONVERGENCES IN FUZZY TOPOLOGICAL SPACES*

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Abstract: In this paper we introduce the notions of fuzzy upper limit, fuzzy lower limit and the fuzzy continuous convergence on the set of fuzzy continuous functions. In examining these aforementioned notions in the present paper we find on the one hand many properties of them whilst on the other, the following applications take place: (α) the characterization of fuzzy compact spaces through the contribution of fuzzy upper limit and (β) the characterization of the fuzzy continuous convergence through the assistance of fuzzy upper limit.

Keywords: Fuzzy sets, fuzzy topology, fuzzy upper and lower limit, fuzzy compact spaces and fuzzy continuous convergences.

I. Introduction.

1.1. Fuzzy sets. Throughout this paper, the symbol I will denote the unit interval $[0,1]$. Let X be a nonempty set.

A *fuzzy set* in X is a function with domain X and values in I, that is, an element of I^X. Let $A \in I^X$. The subset of X in which A assumes nonzero values, is known as the *support* of A. (See [Z]).

A member A of I^X is *contained* in a member B of I^Y denoted $A \leq B$ if and only if $A(x) \leq B(x)$, for every $x \in X$. (See [Z]).

Let $A, B \in I^X$. We define the following fuzzy sets (see [Z]):

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(1) $A \wedge B \in I^X$ by $(A \wedge B)(x) = \min\{A(x), B(x)\}$ for every $x \in X$ (intersection).

(2) $A \vee B \in I^X$ by $(A \vee B)(x) = \max\{A(x), B(x)\}$ for every $x \in X$ (union).

(3) $A^c \in I^X$ by $A^c(x) = 1 - A(x)$ for every $x \in X$.

(4) Let $f : X \to Y$, $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup\{A(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x))$, $x \in X$.

Let $A, B \in I^X$, then $(A \wedge B)^c = A^c \vee B^c$ and $(A \vee B)^c = A^c \wedge B^c$.

Let $A \in I^X$ and $B \in I^Y$. Then by $A \times B$ we denote the fuzzy set in $X \times Y$ for which $(A \times B)(x, y) = \min\{A(x), B(y)\}$, for every $(x, y) \in X \times Y$.

1.2. Fuzzy topology. The first definition of a *fuzzy topological space* is due to Chang. (See [C]). According to Chang, a fuzzy topological space is a pair (X, τ), where X is a set and τ is a *fuzzy topology* on it, that is, a family of fuzzy sets ($\tau \subseteq I^X$) satisfying the following three axioms:

(1) $\bar{0}, \bar{1} \in \tau$. By $\bar{0}$ and $\bar{1}$ we denote the characteristic functions $\mathcal{X}_\emptyset$ and $\mathcal{X}_X$, respectively.

(2) If $A, B \in \tau$, then $A \wedge B \in \tau$.

(3) If $\{A_j : j \in J\} \subseteq \tau$, then $\vee\{A_j : j \in J\} \in \tau$.

The elements of τ are called *fuzzy open sets*. A fuzzy set K is called *fuzzy closed set* if $K^c \in \tau$. We denote by τ^c the collection of all fuzzy closed sets in this fuzzy topological space. Obviously, we have: (α) $\bar{0}, \bar{1} \in \tau^c$, (β) if $K, M \in \tau^c$, then $K \vee M \in \tau^c$ and (γ) if $\{K_j : j \in J\} \subseteq \tau^c$, then $\wedge\{K_j : j \in J\} \in \tau^c$.

The *closure* $Cl(A)$ and the *interior* $Int(A)$ of a fuzzy set A of X are defined as

$$Cl(A) = inf\{K : A \leq K, K^c \in \tau\}$$

$$Int(A) = sup\{O : O \leq A, O \in \tau\},$$

respectively.

A fuzzy set in X is called a *fuzzy point* if and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is λ ($0 < \lambda \leq 1$) we denote the fuzzy point by p_x^λ, where the point x is called its *support*. The class of all fuzzy points in X is denoted by $\mathcal{X}$. (See for example [W$_1$] and [M-M$_1$]).

The fuzzy point p_x^λ is said to be *contained* in a fuzzy set A or to belong to A, denoted by $p_x^\lambda \in A$, if and only if $\lambda \leq A(x)$. Evidently, every fuzzy set A can be expressed as the union of all the fuzzy points which belongs to A. (See [M-M$_1$]).

A fuzzy set A in a fuzzy topological space (X, τ) is called a *neighbourhood* of a fuzzy point p_x^λ if and only if there exists a $V \in \tau$ such that $p_x^\lambda \in V \leq A$. (See [M-M$_1$]). A neighbourhood A is said to be open if and only if A is open.

A fuzzy point p_x^λ is said to be *quasi-coincident* with A denoted by $p_x^\lambda q A$ if and only if $\lambda > A^c(x)$ or $\lambda + A(x) > 1$. (See [M-M$_1$]).

A fuzzy set A is said to be *quasi-coinident* with B, denoted AqB, if and only if there exists $x \in X$ such that $A(x) > B^c(x)$ or $A(x) + B(x) > 1$. (See [M-M$_1$]). If A does not quasi-coincident with B, then we write $A \not{q} B$.

A fuzzy set A in a fuzzy topological space (X, τ) is called a *$Q-$neighbourhood* of p_x^λ if and only if there exists $B \in \tau$ such that $p_x^\lambda q B \leq A$. The family of all $Q-$neighbourhoods of p_x^λ is called the system of $Q-$neighbourhoods of p_x^λ. (See [M-M$_1$]). A $Q-$neighbourhood of a fuzzy point generally does not contain the point itself. In what follows by $\mathcal{N}(p_x^\lambda)$ we denote the family of all fuzzy open $Q-$neighbourhoods of the fuzzy point p_x^λ in X. The set $\mathcal{N}(p_x^\lambda)$ with the relation $\leq^*$ (that is, $U_1 \leq^* U_2$ if and only if $U_2 \leq U_1$) form a directed set.

A fuzzy point $p_x^\lambda \in Cl(A)$ if and only if each $Q-$neighbourhood of p_x^λ is quasi-coincident with A. (See Theorem 4.1$'$ of [M-M$_1$]).

A fuzzy point $p_x^\lambda \in Int(A)$ if and only if has a neighbourhood B contained in A. (See [M-M$_1$]).

1.3. Fuzzy functions. A function f from a fuzzy topological space X into a fuzzy topological space Y is *fuzzy continuous* if and only if for every fuzzy point p_x^λ in X and every $Q-$neighbourhood V of $f(p_x^\lambda)$, there exists a $Q-$neighbourhood U of p_x^λ such that $f(U) \leq V$. (See [M-M$_2$]).

Let f be a function from X to Y. Then (see for example [W_2], [M-M_2], [Y], [A-T] and [C]):

(1) $f^{-1}(B^c) = (f^{-1}(B))^c$, for any fuzzy set B in Y.

(2) $f(f^{-1}(B)) \leq B$, for any fuzzy set B in Y.

(3) $A \leq f^{-1}(f(A))$, for any fuzzy set A in X.

(4) Let p be a fuzzy point of X, A be a fuzzy set in X and B be a fuzzy set in Y. Then, we have:

(i) If $f(p)\ q\ B$, then $p\ q\ f^{-1}(B)$.

(ii) If $p\ q\ A$, then $f(p)\ q\ f(A)$.

(5) Let A and B be fuzzy sets in X and Y, respectively. Let p be a fuzzy point in X. Then we have:

(i) $p \in f^{-1}(B)$ if $f(p) \in B$.

(ii) $f(p) \in f(A)$ if $p \in A$.

1.4. Fuzzy nets. Let Λ be a directed set. Let X be an ordinary set. Let $\mathcal{X}$ be the collection of all fuzzy points in X. The function $S : \Lambda \to \mathcal{X}$ is called a *fuzzy net* in X. For every $\lambda \in \Lambda$, $S(\lambda)$ is often denoted by s_λ and hence a net S is often denoted by $\{s_\lambda, \lambda \in \Lambda\}$. (See [M-$M_1$]).

Let $S = \{s_\lambda, \lambda \in \Lambda\}$ be a fuzzy net in X. S is said to be *quasi-coincident* with A if and only if for each $\lambda \in \Lambda$, s_λ is quasi-coincident with A. S is said to be eventually quasi-coincident with A if and only if there is an element m of Λ such that if $\lambda \in \Lambda$ and $\lambda \geq m$, then s_λ is quasi-coincident with A. S is said to be *frequently quasi-coincident* with A if and only if for each $m \in \Lambda$ there is an $\lambda \in \Lambda$ such that $\lambda \geq m$ and s_λ is quasi-coincident with A. S is said to be in A if and only if for each $\lambda \in \Lambda$, $s_\lambda \in A$. (See [M-M_1]).

A fuzzy net $S = \{s_\lambda, \lambda \in \Lambda\}$ in a fuzzy topological space (X, τ) is said to be *convergent* to a fuzzy point e in X relative to τ and write $\lim s_\lambda = e$ if and only if S is eventually quasi-cincident with each $Q-$neighbourhood of e. (See [M-M_1]).

A fuzzy net $\{g_\mu, \mu \in M\}$ in X is called a fuzzy subnet of a fuzzy net $\{s_\lambda, \lambda \in \Lambda\}$ in X if and only if there is a map $N : M \to \Lambda$ such that:

(i) $g_\mu = s_{N(\mu)}$ and

(ii) for the element $\lambda_0 \in \Lambda$ there is $\mu_0 \in M$ such that if $\mu \geq \mu_0$, $\mu \in M$, then $N(\mu) \geq \lambda_0$.

It is known that (see [M-M$_1$] and [M-M$_2$]):

(1) In a fuzzy topological space (X, τ) a fuzzy point $p \in A$ if and only if there is a fuzzy net S in A such that S converges to p.

(2) A fuzzy subset A in a fuzzy topological space (X, τ) is closed if and only if every fuzzy net S cannot converge to a fuzzy point not belonging to A.

(3) A function f from a fuzzy topological space X into a fuzzy topological space Y is fuzzy continuous if and only if for every fuzzy net $S = \{s_\lambda, \lambda \in \Lambda\}$, if S converges to p, then $f \circ S = \{f(s_\lambda), \lambda \in \Lambda\}$ is a fuzzy net in Y and converges to $f(p)$.

II. Fuzzy upper and lower limit.

2.1. Definition. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in a fuzzy topological space Y. Then by $F - \overline{\lim\limits_{N}}(A_n)$, we denote the *fuzzy upper limit* of the net $\{A_n, n \in N\}$ in I^Y, that is, the fuzzy set which is the union of all fuzzy points p_x^a in Y such that for every $n_0 \in N$ and for every fuzzy open $Q-$neighbourhood U of p_x^a in Y there exists an element $n \in N$ for which $n \geq n_0$ and $A_n q U$. In other cases we set $F - \overline{\lim\limits_{N}}(A_n) = \overline{0}$

2.2. Theorem. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in Y. Then the following propositions are true:

(1) The fuzzy upper limit is closed.

(2) $F - \overline{\lim\limits_{N}}(A_n) = F - \overline{\lim\limits_{N}}(Cl(A_n))$.

(3) If $A_n = A$ for every $n \in N$, then $F - \overline{\lim\limits_{N}}(A_n) = Cl(A)$

(4) The fuzzy upper limit is not affected by changing a finite number of the A_n.

(5) $F - \overline{\lim\limits_{N}}(A_n) \leq Cl(\vee\{A_n : n \in N\})$.

Proof. (1) It is sufficient to prove that

$$Cl(F - \overline{\lim\limits_{N}}(A_n)) \leq F - \overline{\lim\limits_{N}}(A_n).$$

Let $p_y^r \in Cl(F-\overline{\lim_N}(A_n))$ and let U be an arbitrary fuzzy open $Q-$neighbourhood U of p_y^r. Then, we have:

$$UqF - \overline{\lim_N}(A_n).$$

Hence, there exists an element $y' \in Y$ such that

$$U(y') + F - \overline{\lim_N}(A_n)(y') > 1.$$

Let $F - \overline{\lim_N}(A_n)(y') = k$. Then, for the fuzzy point $p_{y'}^k$ in Y we have

$$p_{y'}^k \ q \ U \text{ and } p_{y'}^k \in F - \overline{\lim_N}(A_n).$$

Thus, for every element $n_0 \in N$ there exists $n \geq n_0$, $n \in N$ such that $A_n qU$. This means that $p_y^r \in F - \overline{\lim_N}(A_n)$.

(2) Clearly, it is sufficient to prove that for every fuzzy open set U the condition UqA_n is equivalent to $UqCl(A_n)$.

Let UqA_n. Then there exists an element $y \in Y$ such that $U(y) + A_n(y) > 1$. Since $A_n \leq Cl(A_n)$ we have $U(y) + Cl(A_n)(y) > 1$ and therefore $UqCl(A_n)$.

Conversely, let $UqCl(A_n)$. Then there exists an element $y \in Y$ such that $U(y) + Cl(A_n)(y) > 1$.

Let $Cl(A_n)(y) = r$. Then $p_y^r \in Cl(A_n)$ and the fuzzy open set U is a fuzzy open $Q-$neighbourhood of p_y^r. Thus UqA_n.

(3) Follows by Theorem 4.1 of [M-M$_1$] and the definition of the fuzzy upper limit.

(4) Follows by the definition of the fuzzy upper limit.

(5) Let $p_y^r \in F - \overline{\lim_N}(A_n)$ and let U be a fuzzy open $Q-$neighbourhood of p_y^r in Y. Then for every $n_0 \in N$ there exists $n \in N$, $n \geq n_0$ such that $A_n qU$ and therefore $\vee\{A_n : n \in N\}qU$. Thus, $p_y^r \in Cl(\vee\{A_n : n \in N\})$.

2.3. Theorem. Let $\{A_n, n \in N\}$ be a net of fuzzy closed sets in Y such that $A_{n_1} \leq A_{n_2}$ if and only if $n_2 \leq n_1$. Then

$$F - \overline{\lim_N}(A_n) = \wedge\{A_n : n \in N\}.$$

Proof. Let $p_y^r \in \wedge\{A_n : n \in N\}$. Then $p_y^r \in A_n$ or $r \leq A_n(y)$ for every $n \in N$. Let $n_0 \in N$ and U be a fuzzy open $Q-$neighbourhood of p_y^r, that is $r + U(y) > 1$. Then there exists $n \in N$, $n \geq n_0$ such that $A_n(y) + U(y) \geq r + U(y) > 1$. Hence $A_n q U$ and therefore

$$p_y^r \in F - \overline{\lim_N}(A_n).$$

Conversely, let

$$p_y^r \in F - \overline{\lim_N}(A_n)$$

and let $p_y^r \notin \wedge\{A_n : n \in N\}$. Then there exists $n_0 \in N$ such that $p_y^r \notin A_{n_0}$, that is $r > A_{n_0}(y)$. Let $U = (A_{n_0})^c$. Then U is a fuzzy open $Q-$neighbourhood of p_y^r and for every $n \geq n_0$, $U \not{q} A_n$, which is a contradiction.

2.4. Theorem. Let $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ be two nets of fuzzy sets in Y. Then the following propositions are true:

(1) If $A_n \leq B_n$ for every $n \in N$, then $F - \overline{\lim_N}(A_n) \leq F - \overline{\lim_N}(B_n)$.

(2) $F - \overline{\lim_N}(A_n \vee B_n) = F - \overline{\lim_N}(A_n) \vee F - \overline{\lim_N}(B_n)$.

(3) $F - \overline{\lim_N}(A_n \wedge B_n) \leq F - \overline{\lim_N}(A_n) \wedge F - \overline{\lim_N}(B_n)$.

Proof. We prove only the proposition (2).

Clearly, $A_n \leq A_n \vee B_n$ and $B_n \leq A_n \vee B_n$, for every $n \in N$. Hence by proposition (1) $F - \overline{\lim_N}(A_n) \leq F - \overline{\lim_N}(A_n \vee B_n)$ and $F - \overline{\lim_N}(B_n) \leq F - \overline{\lim_N}(A_n \vee B_n)$. Thus $F - \overline{\lim_N}(A_n) \vee F - \overline{\lim_N}(B_n) \leq F - \overline{\lim_N}(A_n \vee B_n)$.

Conversely, let $p_y^r \in F - \overline{\lim_N}(A_n \vee B_n)$. We prove that $p_y^r \in F - \overline{\lim_N}(A_n) \vee F - \overline{\lim_N}(B_n)$. Let us suppose that $p_y^r \notin F - \overline{\lim_N}(A_n) \vee F - \overline{\lim_N}(B_n)$. Then $p_y^r \notin F - \overline{\lim_N}(A_n)$ and $p_y^r \notin F - \overline{\lim_N}(B_n)$. Hence there exists a fuzzy open $Q-$neighbourhood U_1 of p_y^r and an element $n_1 \in N$ such that $A_n \not{q} U_1$, for every $n \in N$, $n \geq n_1$. Also, there exists a fuzzy open $Q-$neighbourhood U_2 of p_y^r and an element $n_2 \in N$ such that $B_n \not{q} U_2$, for every $n \in N$, $n \geq n_2$.

Let $U = U_1 \wedge U_2$ and let $n_0 \in N$ such that $n_0 \geq n_1$ and $n_0 \geq n_2$. Then the fuzzy set U is a fuzzy open $Q-$neighbourhood of p_y^r and $A_n \vee B_n \not{q} U$, for every $n \in N$, $n \geq n_0$. Since $p_y^r \in F - \overline{\lim_N}(A_n \vee B_n)$ this is a contradiction. Thus $p_y^r \in F - \overline{\lim_N}(A_n) \vee F - \overline{\lim_N}(B_n)$.

2.5. Theorem. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in Y. Then we have:

$$F - \overline{\lim_N}(A_n) = \wedge\{Cl(\vee\{A_n : n \geq n_0\}) : n_0 \in N\}.$$

Proof. Let $p_y^r \in F - \overline{\lim_N}(A_n)$ and let $n_0 \in N$. We prove that $p_y^r \in Cl(\vee\{A_n : n \geq n_0\})$. Let U be an arbitrary fuzzy open $Q-$neighbourhood of p_y^r in Y. Then, there exists $n \geq n_0$, $n \in N$ such that UqA_n. Thus $Uq \vee \{A_n : n \geq n_0\}$ and therefore $p_y^r \in Cl(\vee\{A_n : n \geq n_0\})$.

Conversely, let $p_y^r \in \wedge\{Cl(\vee\{A_n : n \geq n_0\}) : n_0 \in N\}$. Then, we have $p_y^r \in Cl(\vee\{A_n : n \geq n_0\})$, for every $n_0 \in N$. We prove that $p_y^r \in F - \overline{\lim_N}(A_n)$. Let U be an arbitrary fuzzy open $Q-$neighbourhood of p_y^r in Y and let $n_0 \in N$. Then $Uq \vee \{A_n : n \geq n_0\}$. We prove that there exists $n \in N$, $n \geq n_0$ such that A_nqU. Let us suppose that $U \not{q} A_n$, for every $n \in N$, $n \geq n_0$. Then, for every $n \in N$, $n \geq n_0$ and for every $y \in Y$ we have $U(y) + A_n(y) \leq 1$ and therefore $U(y) + (\vee\{A_n : n \geq n_0\})(y) \leq 1$, which is a contradiction. Thus $p_y^r \in F - \overline{\lim_N}(A_n)$.

2.6. Definition. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in a fuzzy topological space Y. Then by $F - \underline{\lim_N}(A_n)$, we denote the *fuzzy lower limit* of the net $\{A_n, n \in N\}$ in I^Y, that is, the fuzzy set which is the union of all fuzzy points p_x^a in Y such that for every fuzzy open $Q-$neighbourhood U of p_x^a in Y there exists an element $n_0 \in N$ such that A_nqU, for every $n \in N$, $n \geq n_0$. In other cases we set $F - \underline{\lim_N}(A_n) = \overline{0}$

2.7. Theorem. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in Y. Then the following propositions are true:

(1) The fuzzy lower limit is closed.

(2) $F - \underline{\lim_N}(A_n) = F - \underline{\lim_N}(Cl(A_n))$.

(3) If $A_n = A$ for every $n \in N$, then $F - \underline{\lim_N}(A_n) = Cl(A)$

(4) The fuzzy upper limit is not affected by changing a finite number of the A_n.

(5) $\wedge\{A_n : n \in N\} \leq F - \underline{\lim_N}(A_n)$.

(6) $F - \underline{\lim_N}(A_n) \leq Cl(\vee\{A_n : n \in N\})$.

(7) $\vee\{\wedge\{A_n : n \geq n_0\} : n_0 \in N\} \leq F - \underline{\lim_N}(A_n)$.

Proof. The propositions (1), (2), (3), (4) and (6) can be proved in the same way, as in Theorem 2.2. We prove the propositions (5) and (7).

(5) Let $p_y^r \in \wedge\{A_n : n \in N\}$. We prove that $p_y^r \in F - \underline{\lim}_N(A_n)$. Let us suppose that $p_y^r \notin F - \underline{\lim}_N(A_n)$. Then there exists a fuzzy open $Q-$neighbourhood U of p_y^r such that for every $n \in N$ there exists $n' \geq n$ for which $A_{n'} \not{q} U$. This means that $A_{n'}(x) + U(x) \leq 1$, for every $x \in Y$.

Now, since $p_y^r \in \wedge\{A_n : n \in N\}$ and U is a fuzzy open $Q-$neighbourhood of p_y^r we have that $r \leq A_n(y)$, for every $n \in N$ and $r + U(y) > 1$. Thus $A_n(y) + U(y) > 1$, for every $n \in N$. By the above this is a contradiction. Hence $p_y^r \in F - \underline{\lim}_N(A_n)$.

(7) Let $p_y^r \in \vee\{\wedge\{A_n : n \geq n_0\} : n_0 \in N\}$. Then, there exists $n_0 \in N$ such that $p_y^r \in \wedge\{A_n : n \geq n_0\}$. Hence $p_y^r \in A_n$, for every $n \in N$, $n \geq n_0$. and therefore $r \leq A_n(y)$, for every $n \in N$, $n \geq n_0$.

We prove that $p_y^r \in F - \underline{\lim}_N(A_n)$. Let U be an arbitrary fuzzy open $Q-$neighbourhood of p_y^r in Y. Then we have $p_y^r q U$ or equivalently $r + U(y) > 1$. Since $r \leq A_n(y)$, for every $n \in N$, $n \geq n_0$ we have that $A_n(y) + U(y) > 1$, for every $n \in N$, $n \geq n_0$. Thus $A_n q U$, for every $n \in N$, $n \geq n_0$ and therefore $p_y^r \in F - \underline{\lim}_N(A_n)$.

2.8. Theorem. Let $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ be two nets of fuzzy sets in Y. Then the following propositions are true:

(1) If $A_n \leq B_n$ for every $n \in N$, then $F - \underline{\lim}_N(A_n) \leq F - \underline{\lim}_N(B_n)$.

(2) $F - \underline{\lim}_N(A_n \vee B_n) \geq F - \underline{\lim}_N(A_n) \vee F - \underline{\lim}_N(B_n)$.

(3) $F - \underline{\lim}_N(A_n \wedge B_n) \leq F - \underline{\lim}_N(A_n) \wedge F - \underline{\lim}_N(B_n)$.

Proof. We prove only the proposition (2). Let $p_y^r \in F - \underline{\lim}_N(A_n) \vee F - \underline{\lim}_N(B_n)$. Then either $p_y^r \in F - \underline{\lim}_N(A_n)$ or $p_y^r \in F - \underline{\lim}_N(B_n)$. Let $p_y^r \in F - \underline{\lim}_N(A_n)$. Then for every fuzzy open $Q-$neighbourhood U of p_y^r in Y there exists an element $n_0 \in N$ such that $A_n q U$, for every $n \geq n_0$, $n \in N$. Also $A_n \leq A_n \vee B_n$. Thus $(A_n \vee B_n) q U$, for every $n \in N$, $n \geq n_0$ and therefore $p_y^r \in F - \underline{\lim}_N(A_n \vee B_n)$.

2.9. Theorem. For the fuzzy upper and lower limit we have the relation $F - \underline{\lim}_N(A_n) \leq F - \overline{\lim}_N(A_n)$.

Proof. The proof of this theorem follows by the Definitions 2.1 and 2.6.

2.10. Theorem. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in Y. Then the formula $p \in F - \underline{\lim}_N(A_n)$ is equivalent to the existence of a fuzzy net $\{p_n, n \in N\}$ such that $\lim p_n = p$ and $p_n \in A_n$.

Proof. The proof of this theorem follows by Definition 2.6.

2.11. Theorem. Let $\{A_n, n \in N\}$ be a net of fuzzy sets in Y such that $A_{n_1} \leq A_{n_2}$ if and only if $n_1 \leq n_2$. Then $Cl(\vee\{A_n : n \in N\}) = F - \underline{\lim}_N(A_n)$.

Proof. Let $p_y^r \in Cl(\vee\{A_n : n \in N\})$ and let U be a fuzzy open $Q-$neighbourhood of p_y^r in Y. Then $Uq \vee \{A_n : n \in N\}$. Hence, there exists $n_0 \in N$ such that UqA_{n_0}. By assumption we have UqA_n, for every $n \in N$, $n \geq n_0$. Thus $p_y^r \in F - \underline{\lim}_N(A_n)$.

Conversely, let $p_y^r \in F - \underline{\lim}_N(A_n)$ and let U be an arbitrary fuzzy open $Q-$neighbourhood of p_y^r in Y. Then there exists an element $n_0 \in N$ such that $U\ q\ A_n$, for every $n \in N$, $n \geq n_0$. Hence $U\ q\ \vee\{A_n : n \in N\}$ and therefore $p_y^r \in Cl(\vee\{A_n : n \in N\})$.

2.12. Definition. A net $\{A_n, n \in N\}$ of fuzzy sets in a fuzzy topological space Y is said to be *fuzzy convergent* to the fuzzy set A if $F - \underline{\lim}_N(A_n) = F - \overline{\lim}_N(A_n) = A$.

We then write $F - \lim_N(A_n) = A$.

2.13. Theorem. Let $\{A_n, n \in N\}$ be a convergent net of fuzzy sets in Y. Then the following propositions are true:

(1) $Cl(F - \lim_N(A_n)) = F - \lim_N(A_n) = F - \lim_N(Cl(A_n))$.

(2) If $A_n = A$ for every $n \in N$, then $F - \lim_N(A_n) = Cl(A)$

Proof. The proof of this theorem follows by Theorems 2.2 and 2.7.

2.14. Theorem. Let $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ be two nets of fuzzy sets in Y. Then the following propositions are true (in (1) and (2) the nets $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ are supposed to be convergent):

(1) If $A_n \leq B_n$ for every $n \in N$, then $F - \lim_N(A_n) \leq F - \lim_N(B_n)$.

(2) $F - \lim_N(A_n \vee B_n) = F - \lim_N(A_n) \vee F - \lim_N(B_n)$.

Proof. The proof of Proposition (1) follows by Theorems 2.2 and 2.7. We prove the Proposition (2).

By Theorems 2.4 and 2.8 we have

$$\begin{aligned} F-\overline{\lim_{N}}(A_n \vee B_n) &= F-\overline{\lim_{N}}(A_n) \vee F-\overline{\lim_{N}}(B_n) \\ &\leq F-\underline{\lim}_N(A_n) \vee F-\underline{\lim}_N(B_n) \\ &\leq F-\underline{\lim}_N(A_n \vee B_n) \\ &\leq F-\overline{\lim_{N}}(A_n \vee B_n). \end{aligned}$$

Thus $F-\lim_{N}(A_n \vee B_n) = F-\lim_{N}(A_n) \vee F-\lim_{N}(B_n)$.

2.15. Theorem. Let $\{A_n, n \in N\}$ be a convergent net of fuzzy sets in Y such that $A_{n_1} \geq A_{n_2}$ for $n_1 \leq n_2$, then $F-\lim_{N}(A_n) = \wedge\{Cl(A_n) : n \in N\}$.

Proof. By Theorems 2.2 and 2.7. we have:

$$\begin{aligned} \wedge\{Cl(A_n) : n \in N\} &\leq F-\underline{\lim}_N(Cl(A_n)) \\ &= F-\underline{\lim}_N(A_n) \\ &\leq F-\overline{\lim_{N}}(A_n) \\ &= F-\overline{\lim_{N}}(Cl(A_n)) \\ &= \wedge\{Cl(A_n) : n \in N\}. \end{aligned}$$

Thus $F-\lim_{N}(A_n) = \wedge\{Cl(A_n) : n \in N\}$.

2.16. Theorem. Let $\{A_n, n \in N\}$ be a convergent net of fuzzy sets in Y such that $A_{n_1} \leq A_{n_2}$ for $n_1 \leq n_2$, then $F-\lim_{N}(A_n) = Cl(\vee\{A_n : n \in N\})$.

Proof. By Theorems 2.11 and 2.2 we have:

$$\begin{aligned} Cl(\vee\{A_n : n \in N\}) &\leq F-\underline{\lim}_N(A_n) \\ &\leq F-\overline{\lim_{N}}(A_n) \\ &\leq Cl(\vee\{A_n : n \in N\}). \end{aligned}$$

Thus $F-\lim_{N}(A_n) = Cl(\vee\{A_n : n \in N\})$.

It is not dificult to prove the following theorem.

2.17. Theorem. The following propositions are true:

(1) Let $U_1, A \in I^X$ and $U_2, B \in I^Y$. If $U_1 \times U_2 q A \times B$, then $U_1 q A$ and $U_2 q B$.

(2) Let U_1 and U_2 be fuzzy open $Q-$neighbourhoods of p_x^r and p_y^r in X and Y, respectively. Then the fuzzy set $U_1 \times U_2$ is a fuzzy open $Q-$neighbourhood of $p_{(x,y)}^r$ in $X \times Y$.

2.18. Theorem. Let $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ be two nets of fuzzy sets in Y. Then the following propositions are true (in (3) the nets $\{A_n, n \in N\}$ and $\{B_n, n \in N\}$ are supposed to be convergent):

(1) $F - \overline{\lim_N}(A_n \times B_n) \leq F - \overline{\lim_N}(A_n) \times F - \overline{\lim_N}(B_n)$.

(2) $F - \underline{\lim_N}(A_n \times B_n) \leq F - \underline{\lim_N}(A_n) \times F - \underline{\lim_N}(B_n)$.

(3) $F - \lim_N(A_n \times B_n) \leq F - \lim_N(A_n) \times F - \lim_N(B_n)$.

Proof. (1) Let $p_{(x,y)}^r \in F - \overline{\lim_N}(A_n \times B_n)$. We must prove that

$$p_{(x,y)}^r \in F - \overline{\lim_N}(A_n) \times F - \overline{\lim_N}(B_n)$$

or equivalently

$$r \leq (F - \overline{\lim_N}(A_n) \times F - \overline{\lim_N}(B_n))(x,y).$$

Let $n_0 \in N$, U_1 be an arbitrary fuzzy open $Q-$neighbourhood of p_x^r in X and U_2 be a constant fuzzy open $Q-$neighbourhood of p_y^r in Y. Then, the fuzzy set $U_1 \times U_2$ is a fuzzy open $Q-$neighbourhood of $p_{(x,y)}^r$ in $X \times Y$ (see Theorem 2.17). Hence, there exists $n \in N$, $n \geq n_0$ such that $U_1 \times U_2 q A_n \times B_n$.

By Theorem 2.17 we have $U_1 q A_n$ and $U_2 q B_n$. Thus $p_x^r \in F - \overline{\lim_N}(A_n)$. Similarly, we can prove that $p_y^r \in F - \overline{\lim_N}(B_n)$.

Hence $p_{(x,y)}^r \in F - \overline{\lim_N}(A_n) \times F - \overline{\lim_N}(B_n)$.

Similarly, we can prove the propositions (2) and (3).

III. Compact fuzzy spaces

3.1. Definitions. (See [C]) A family $\mathcal{A}$ of fuzzy sets is a *cover* of a fuzzy set B if and only if $B \leq \vee\{A : A \in \mathcal{A}\}$. It is an *open cover* if and only if each member of $\mathcal{A}$ is an open fuzzy set. A *subcover* of $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover.

A fuzzy topological space X is *compact* if and only if each open cover has a finite subcover.

3.2. Theorem. A fuzzy space X is compact if and only if for every net $\{K_\lambda, \lambda \in \Lambda\}$ of fuzzy closed sets in X such that $F-\overline{\lim_{\Lambda}}(K_\lambda) = \overline{0}$, there exists $\lambda_0 \in \Lambda$ for which $K_\lambda = \overline{0}$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

Proof. Let X be a fuzzy compact space and let $\{K_\lambda, \lambda \in \Lambda\}$ be a net of fuzzy closed sets in X such that $F-\overline{\lim_{\Lambda}}(K_\lambda) = \overline{0}$. Then for every fuzzy point p_x^r in X there exists a fuzzy open $Q-$neighbourhood $U_{p_x^r}$ in X and an element $\lambda_{p_x^r} \in \Lambda$ such that $K_\lambda \; q\!\!\!/ \; U_{p_x^r}$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_{p_x^r}$.

Clearly, the family $\{U_{p_x^r} : p_x^r \in \mathcal{X}\}$ is an open cover of fuzzy sets of X, that is

$$\overline{1} = \vee\{U_{p_x^r} : p_x^r \in \mathcal{X}\}.$$

Since the fuzzy space X is compact, there exist fuzzy points $p_1, p_2, ...p_n \in \mathcal{X}$ such that $\overline{1} = \vee\{U_{p_i} : i = 1, 2..., n\}$.

Let $\lambda_0 \in \Lambda$ such that $\lambda_0 \geq \lambda_{p_i}$ for every $i = 1, 2, ..., n$. Then for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$ we have $K_\lambda \; q\!\!\!/ \; \vee\{U_{p_i} : i = 1, 2, ..., n\}$ or $K_\lambda \; q\!\!\!/ \; \overline{1}$. Thus $K_\lambda = \overline{0}$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

Conversely, suppose that the fuzzy space X satisfies the condition of the theorem. We prove that the fuzzy space X is compact.

Let $\mathcal{A}$ be an open cover of fuzzy set of the space X. Let Λ be the set of all finite subsets of $\mathcal{A}$ directed by inclusion and let $\{K_\lambda, \lambda \in \Lambda\}$ be a net of fuzzy closed sets in X such that $K_\lambda^c = \vee\{A : A \in \lambda\}$. Obviously $K_{\lambda_1} \leq K_{\lambda_2}$ if $\lambda_2 \subseteq \lambda_1$. Hence by Theorem 2.3 it follows that $F-\overline{\lim_{\Lambda}}(K_\lambda) = \wedge\{K_\lambda : \lambda \in \Lambda\}$.

Also, we have:

$$\begin{aligned}\wedge\{K_\lambda : \lambda \in \Lambda\} &= (\vee\{K_\lambda^c : \lambda \in \Lambda\})^c \\ &= (\vee\{A : A \in \mathcal{A}\})^c \\ &= \overline{1}^c = \overline{0}.\end{aligned}$$

Thus $F-\overline{\lim_{\Lambda}}(K_\lambda) = \overline{0}$. By assumption there exists an element $\lambda_0 \in \Lambda$ for which $K_\lambda = \overline{0}$ for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$.

By the above we have

$$\overline{1} = K_\lambda^c = \vee\{A : A \in \lambda_0\}$$

and therefore the fuzzy space X is compact.

IV. Fuzzy continuous convergence.

4.1 Notations. Let Y and Z be fuzzy topological spaces. Then by $FC(Y,Z)$ we denote the set of all fuzzy continuous maps of Y into Z.

4.2. Theorem. Let $f : Y \to Z$ be a fuzzy continuous map, p be a fuzzy point in Y and U, V be fuzzy open $Q-$neighbourhoods of p and $f(p)$, respectively such that $f(U) \not\leq V$. Then there exists a fuzzy point p_1 in Y such that $p_1 q U$ and $f(p_1) \not{q} V$.

Proof. Since $f(U) \not\leq V$. We have $U \not\leq f^{-1}(V)$. Thus there exists $x \in Y$ such that $U(x) > f^{-1}(V)(x)$ or $U(x) - f^{-1}(V)(x) > 0$ and therefore $U(x) + 1 - f^{-1}(V)(x) > 1$ or $U(x) + (f^{-1}(V))^c(x) > 1$. Let $(f^{-1}(V))^c(x) = r$. Clearly, for the fuzzy point p_x^r we have $p_x^r qU$ and $p_x^r \in (f^{-1}(V))^c$. Hence for the fuzzy point $p_1 \equiv p_x^r$ we have $p_1 qU$ and $f(p_1) \not{q} V$.

4.3. Definition. A net $\{f_\mu, \mu \in M\}$ in $FC(Y,Z)$ *fuzzy continuously converges to* $f \in FC(Y,Z)$ if and only if for every fuzzy net $\{p_\lambda, \lambda \in \Lambda\}$ in Y which converges to a fuzzy point p in Y we have that the fuzzy net $\{f_\mu(p_\lambda), (\lambda, \mu) \in \Lambda \times M\}$ converges to the fuzzy point $f(p)$ in Z.

4.4. Theorem. A net $\{f_\mu, \mu \in M\}$ in $FC(Y,Z)$ fuzzy continuously converges to $f \in FC(Y,Z)$ if and only if for every fuzzy point p in Y and for every fuzzy open $Q-$neighbourhood V of $f(p)$ in Z there exist an element $\mu_0 \in M$ and a fuzzy open $Q-$neighbourhood U of p in Y such that

$$f_\mu(U) \leq V,$$

for every $\mu \geq \mu_0$, $\mu \in M$.

Proof. Let p be a fuzzy point in Y and let V be a fuzzy open $Q-$neighbourhood of $f(p)$ in Z such that for every $\mu \in M$ and for every fuzzy open $Q-$neighbourhood U of p in Y there exists $\mu' \geq \mu$ such that

$$f_{\mu'}(U) \not\leq V.$$

Then for every fuzzy open $Q-$neighbourhood U of p in Y we can choose a fuzzy point p_U in Y (see Theorem 4.2) such that

$$p_U \; q \; U \text{ and } f_{\mu'}(p_U) \, \not{q} \, V.$$

Clearly, the fuzzy net $\{p_U, U \in \mathcal{N}(p)\}$ converges to p, but the fuzzy net $\{f_\mu(p_U), (U, \mu) \in \mathcal{N}(p) \times M\}$ does not converge to $f(p)$ in Z.

Conversely, let $\{p_\lambda, \lambda \in \Lambda\}$ be a fuzzy net in $FC(Y, Z)$ which converges to the fuzzy point p in Y and let V be an arbitrary fuzzy open $Q-$neighbourhood of $f(p)$ in Z. By assumption there exists a fuzzy open $Q-$neighbourhood U of p in Y and an element $\mu_0 \in M$ such that $f_\mu(U) \leq V$, for every $\mu \geq \mu_0$, $\mu \in M$. Since the fuzzy net $\{p_\lambda, \lambda \in \Lambda\}$ converges to p in Y. There exists $\lambda_0 \in \Lambda$ such that $p_\lambda q U$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Let $(\lambda_0, \mu_0) \in \Lambda \times M$. Then for every $(\lambda, \mu) \in \Lambda \times M$, $(\lambda, \mu) \geq (\lambda_0, \mu_0)$ we have $f_\mu(p_\lambda) \; q \; f_\mu(U) \leq V$, that is $f_\mu(p_\lambda) \; q \; V$. Thus the net $\{f_\mu(p_\lambda), (\lambda, \mu) \in \Lambda \times M\}$ converges to $f(p)$ and the net $\{f_\mu, \mu \in M\}$ fuzzy continuously converges to f.

4.5. Theorem. Let A and B two fuzzy sets in Y and let $f : Y \to Z$ be a map. If AqB, then $f(A)qf(B)$.

Proof. Let AqB. Then there exists $y \in Y$ such that $A(y) + B(y) > 1$. Let $A(y) = r$. Then for the fuzzy point $p_y^r \in A$ we have $p_y^r qB$. Thus $f(p_y^r)qf(B)$. Since $p_y^r \in A$ we have $f(p_y^r) \in f(A)$ and therefore $f(A)qf(B)$.

4.6. Theorem. A net $\{f_\lambda, \lambda \in \Lambda\}$ in $FC(Y, Z)$ fuzzy continuously converges to $f \in FC(Y, Z)$ if and only if

$$F - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \leq f^{-1}(K), \tag{1}$$

for every fuzzy closed subset K of Z.

Proof. Let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a net in $FC(Y, Z)$, which fuzzy continuously converges to f and let K be an arbitrary fuzzy closed subset of Z. Let $p \in F - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))$ and let W be an arbitrary fuzzy open $Q-$neighbourhood of $f(p)$ in Z.

Since the net $\{f_\lambda,\ \lambda \in \Lambda\}$ fuzzy continuously converges to f, there exist an open open $Q-$neighbourhood V of p in Y and an element $\lambda_0 \in \Lambda$ such that $f_\lambda(V) \leq W$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. (See Theorem 4.4). On the other hand, there exists an element $\lambda \geq \lambda_0$ such that $V q f_\lambda^{-1}(K)$. Hence, $f_\lambda(V) q K$ and therefore $W q K$. This means that $f(p) \in Cl(K) = K$. Thus, $p \in f^{-1}(K)$.

Conversely, let $\{f_\lambda,\ \lambda \in \Lambda\}$ be a net in $FC(Y, Z)$ and $f \in FC(Y, Z)$ such that the relation (1) holds for every fuzzy closed subset K of Z. We prove that the net $\{f_\lambda,\ \lambda \in \Lambda\}$ fuzzy continuously converges to f. Let p be a fuzzy point of Y and W be a fuzzy open $Q-$neighbourhood of $f(p)$ in Z. Since $p \notin f^{-1}(K)$, where $K = W^c$ we have $p \notin F - \overline{\lim\limits_{\Lambda}}(f_\lambda^{-1}(K))$. This means that there exists an element $\lambda_0 \in \Lambda$ and a fuzzy open $Q-$neighbourhood V of p in Y such that $f_\lambda^{-1}(K)\ \not q V$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$. Then we have $V \leq (f_\lambda^{-1}(K))^c = f_\lambda^{-1}(K^c) = f_\lambda^{-1}(W)$ (see Proposition 2.1 of [M-M$_1$]) and, therefore, $f_\lambda(V) \leq W$, for every $\lambda \in \Lambda$, $\lambda \geq \lambda_0$, that is the net $\{f_\lambda,\ \lambda \in \Lambda\}$ fuzzy continuously converges to f.

4.7. Theorem. The following propositions are true:

(1) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $FC(Y, Z)$ such that $f_\lambda = f$, for every $\lambda \in \Lambda$, then the $\{f_\lambda, \lambda \in \Lambda\}$ fuzzy continuously converges to $f \in FC(Y, Z)$.

(2) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $FC(Y, Z)$, which fuzzy continuously converges to $f \in FC(Y, Z)$ and $\{g_\mu, \mu \in M\}$ be a subnet of $\{f_\lambda, \lambda \in \Lambda\}$, then the net $\{g_\mu, \mu \in M\}$ fuzzy continuously converges to f.

(3) If $\{f_\lambda, \lambda \in \Lambda\}$ is a net in $FC(Y, Z)$ which does not fuzzy continuously converges to $f \in FC(Y, Z)$, then there exists a of $\{f_\lambda, \lambda \in \Lambda\}$, no subnet of which fuzzy continuously converges to f.

Proof. We prove only the proposition (3). Since the net $\{f_\lambda, \lambda \in \Lambda\}$ does not fuzzy continuously converges to f, there exists a fuzzy closed set K in Z such that

$$F - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)) \not\leq f^{-1}(K).$$

Hence, there exists $y \in Y$ such that

$$f^{-1}(K)(y) \leq F - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K))(y).$$

Let $f^{-1}(K)(y) = r$. Then for the fuzzy point p_y^r we have that $p_y^r \in f^{-1}(K)$ and therefore

$$p_y^r \in F - \overline{\lim_{\Lambda}}(f_\lambda^{-1}(K)).$$

Let $N = \Lambda \times \mathcal{N}(p_y^r)$ and let φ be a map of N into Λ which defined as follows: If $n = (\lambda, U) \in N$, then by $\varphi(n)$ we denote an element λ' of Λ such that $\lambda' \geq \lambda$ and $f_{\lambda'}^{-1}(K)qU$.

Clearly, the net $\{g_n = f_{\varphi(n)}, n \in N\}$ is a subnet of $\{f_\lambda, \lambda \in \Lambda\}$. Let $\{h_\tau, \tau \in T\}$ be an arbitrary subnet of $\{g_n, n \in N\}$. We prove that the net $\{h_\tau, \tau \in T\}$ does not fuzzy continuously converge to f. Obviously, for this it is sufficient to prove that

$$p_y^r \in F - \overline{\lim_{T}}(h_\tau^{-1}(K)).$$

Since the net $\{h_\tau, \tau \in T\}$ is a subnet of $\{g_n, n \in N\}$, there is a map ψ of T into N such that:

α) $h_\tau = g_{\psi(\tau)}$, for every $\tau \in T$ and

β) for every element $n_1 \in N$, there exists $\tau_1 \in T$ such that if $\tau \in T$, $\tau \geq \tau_1$, then $\psi(\tau) \geq n_1$.

Now, let $\tau_0 \in T$ and U be an arbitrary fuzzy open Q–neighbourhood of p_y^r in Y. We prove that there exists $\tau \geq \tau_0$, $\tau \in T$ such that $h_\tau^{-1}(K)\ q\ U$.

Indeed, let $\psi(\tau_0) = n_0 = (\lambda_0, U_0)$, $W_0 = U \wedge U_0$ and $n_1 = (\lambda_0, W_0)$. Then there exists an element $\tau_1 \in T$, $\tau_1 \geq \tau_0$ such that if $\tau \in T$, $\tau \geq \tau_1$, then $\psi(\tau) \geq n_1 \geq n_0$.

Let $\tau \in T$, $\tau \geq \tau_1$ and $\psi(\tau) = n = (\lambda, V)$. Then we have:

γ) $h_\tau^{-1}(K) = g_{\psi(\tau)}^{-1}(K) = f_{\varphi(\psi(\tau))}^{-1}(K)$ and

δ) $f_{\varphi(\psi(\tau))}^{-1}(K)\ q\ V$.

Since

$$\psi(\tau) = n = (\lambda, V) \geq n_1 = (\lambda_0, W_0)$$

we have that

$$V \leq W_0 \leq U.$$

By the above relation and by relations γ) and δ) we have that $h_\tau^{-1}(K)\ q\ V$ and therefore $h_\tau^{-1}(K)\ q\ U$, where $\tau \in T$, $\tau \geq \tau_0$.

Thus $p_y^r \in F - \overline{\lim_{T}}(h_\tau^{-1}(K))$.

4.8. Remark. For the notion of continuous convergence see [K].

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