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Rational n-Dimensional Spaces and the Property of Universality

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RATIONAL n-DIMENSIONAL SPACES AND THE PROPERTY OF UNIVERSALITY

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In this paper we prove that in the family of all metrizable separable spaces having rational dimension $\leq n$, $n = 1, 2, \ldots$, there exists a universal element.

Introduction. All spaces considered in this paper are separable metrizable. Let Sp be a family of spaces. We define a family $\mathbb{R}(\mathrm{Sp})$ of spaces as follows: a space X belongs to $\mathbb{R}(\mathrm{Sp})$ iff X has a basis $\mathbb{B}$ for open sets such that the boundary of every element of $\mathbb{B}$ belongs to Sp. We set $\mathbb{R}^{-1}(\mathrm{Sp}) = \{\emptyset\}$, $\mathbb{R}^0(\mathrm{Sp}) = \mathrm{Sp}$ and $\mathbb{R}^n(\mathrm{Sp}) = \mathbb{R}(\mathbb{R}^{n-1}(\mathrm{Sp}))$, for $n = 1, 2, \ldots$. In the sequel we denote by $\mathbb{M}$ the family of all countable spaces. (The empty set and finite sets are considered to be countable). Since $\mathbb{M}$ is a *normal family of spaces* (see [H]), for every $n = 1, 2, \ldots$, the family $\mathbb{R}^n(\mathbb{M})$ is also a normal family, that is, every subspace of any element of $\mathbb{R}^n(\mathbb{M})$ is an element of $\mathbb{R}^n(\mathbb{M})$ and a space which is a countable union of closed subsets belonging to $\mathbb{R}^n(\mathbb{M})$, belongs also to $\mathbb{R}^n(\mathbb{M})$. The elements of $\mathbb{R}^n(\mathbb{M})$ are called spaces having *rational dimension* $\leq n$ (see, for example, [N]) or *n-dimensional rational spaces* (see [Me]). Obviously, a space X is *rational* (see [Ku]) iff X is an 1-dimensional rational space, that is, iff $X \in \mathbb{R}(\mathbb{M})$.

A space T is said to be *universal for a family* Sp *of spaces* iff $T \in \mathrm{Sp}$ and for every $X \in \mathrm{Sp}$ there exists an embedding of X into T. In [I_3] (see also [M-T_1]) it has been proved that in the family $\mathbb{R}(\mathbb{M})$ of all rational spaces there exists a universal element. The property of universality for some subfamilies of rational spaces has been studied, for example, in the papers: [I_1], [I_2], [I_4], [I_5], [I-Z], [M-T_2], [Nö].

The main result of the present paper is the following: in the family of all

n-dimensional rational spaces there exists a universal element. The method used for the proof of this result is a modification of the methods of papers [I$_1$], [I$_3$], [I$_4$], [I$_5$].

Throughout this paper we will use the following notations and definitions.

Let F be a subset of a space X. By $\mathrm{Bd}(F)$ (or $\mathrm{Bd}_X(F)$), $\mathrm{Cl}(F)$ (or $\mathrm{Cl}_X(F)$), $\mathrm{Int}(F)$ (or $\mathrm{Int}_X(F)$) and $|F|$ we denote the boundary, the closure, the interior and the cardinality of F respectively. If X is a metric space, then the diameter of F is denoted by $\mathrm{diam}(F)$. Let Q and K be disjoint closed subsets of a space X. We say that an open subset U of X *separates* Q and K iff either $Q \subseteq U$ and $K \subseteq X \setminus \mathrm{Cl}(U)$ or $K \subseteq U$ and $Q \subseteq X \setminus \mathrm{Cl}(U)$. We denote by N the set $\{0, 1, ...\}$.

We use the symbol $''\equiv''$ in a relation $A \equiv B$ in two cases: (α) in order to introduce two distinct notations, A and B, for the same object (set, ordered set, space, map, etc.), and (β) in order to introduce a notation, A or B (if B or A, respectively is a known notation), without mentioning this fact.

We denote by L_n, $n = 1, 2, ...$, the set of all ordered n-tuples $i_1...i_n$, where $i_t = 0$ or 1, $t = 1, ..., n$. Also we set $L_0 = \{\emptyset\}$ and $L = \bigcup\{L_n : n = 0, 1, ...\}$. For $n = 0$, by $i_1...i_n$ we denote the element $\emptyset$ of L. We say that the element $i_1...i_n$ of L is a *part* of the element $j_1...j_m$ and we write $i_1...i_n \leq j_1...j_m$ iff either $n = 0$, or $0 < n \leq m$ and $i_t = j_t$ for every $t \leq n$. The elements of L are denoted by $\bar{i}$, $\bar{j}$, $\bar{i_1}$, etc. If $\bar{i} = i_1...i_n$, then by $\bar{i}0$ (respectively, $\bar{i}1$) we denote the element $i_1...i_n0$ (respectively, $i_1...i_n1$) of L.

We denote by Λ_n, $n = 1, 2, ...$, the set of all ordered n-tuples $i_1...i_n$, where i_t, $t = 1, ..., n$, is a positive integer. We set $\Lambda = \bigcup\{\Lambda_n : n = 1, 2, ...\}$. The elements of Λ are denoted by $\bar{\alpha}$, $\bar{\beta}$, etc. Let $\bar{\alpha} = i_1...i_n$ and $\bar{\beta} = j_1...j_m$. We say that $\bar{\alpha}$ is a *part* of $\bar{\beta}$ and we write $\bar{\alpha} \leq \bar{\beta}$ iff $1 \leq n \leq m$ and $i_t = j_t$ for every $t \leq n$. Obviously, if $\bar{\alpha}, \bar{\beta} \in \Lambda_n$ and $\bar{\alpha} \leq \bar{\beta}$, then $\bar{\alpha} = \bar{\beta}$. Also, for every $\bar{\alpha} \in \Lambda_n$ the set of all elements $\bar{\beta} \in \Lambda_{n+1}$ such that $\bar{\alpha} \leq \bar{\beta}$ is a countable non-finite set.

We denote by C the Cantor ternary set. By $C_{\bar{i}}$, where $\bar{i} = i_1...i_n \in L$, $n \geq 1$, we denote the set of all points of C for which the t^{th} digit in the ternary expansion, $t = 1, ..., n$, coincides with 0 if $i_t = 0$ and with 2 if $i_t = 1$. Also we set $C_\emptyset = C$. For every point a of C and for every integer $n \in N$, by $\bar{i}(a, n)$ we denote the uniquely determined element $\bar{i} \in L_n$ for which $a \in C_{\bar{i}}$. If $\bar{i}(a, n+1) = i_0...i_n$, $n \in N$, then by $i(a, n+1)$ we denote the number i_n. For every subset F of C and for every integer $n \in N$, we denote by $\mathrm{st}(F, n)$ the union of all sets $C_{\bar{i}}$, $\bar{i} \in L_n$, such that $C_{\bar{i}} \cap F \neq \emptyset$. If $F = \{a\}$ we set $\mathrm{st}(a, n) = \mathrm{st}(F, n)$. Obviously $\mathrm{st}(a, n) = C_{\bar{i}(a,n)}$.

A *partition* of a space X is a set D of closed non-empty subsets of X such that (α) if $F_1 . F_2 \in D$ and $F_1 \neq F_2$, then $F_1 \cap F_2 = \emptyset$, and (β) the union of all ellements of D is X. The *natural projection* of X onto D is the map p defined as follows: if $x \in X$, then $p(x) = F$, where F is the uniquely determined element of D containing x. The *quotient space* of the partition D is the set D with a topology which is the minimal (with respect to the open sets) for which the map p is continuous. (We observe that we use the same notation for a partition of a space and for the corresponding quotient space). The partition D is called *upper semi-continuous* iff for every $F \in D$ and for every open subset U of X containing F there exists an open subset V of X which is union of elements of D such that $F \subseteq V \subseteq U$.

I. Representations of spaces corresponding to a given basis of open sets.

In the sequel, n is a fixed integer of $\mathbb{N} \setminus \{0\}$.

1. Definition. Let $\mathbb{B}$ be a family of open sets of $X \in \mathbb{R}^n(\mathbb{M})$. It is possible that for distinct elements U and V of $\mathbb{B}$ we have $U = V$. We say that $\mathbb{B}$ has *the property of boundary intersections* iff for every integer k, $1 \leq k \leq n$, and for every mutually distinct elements $V_1, ..., V_k$ of $\mathbb{B}$ we have

$$\bigcap \{\mathrm{Bd}(V_i) : i = 1, ..., k\} \in \mathbb{R}^{n-k}(\mathbb{M}).$$

It is not difficult to prove the following two lemmas.

2. Lemma. *Let $X \in \mathbb{R}^n(\mathbb{M})$ and $\mathbb{B}$ be a basis for open sets of X. Then there exists a countable locally finite open covering π of X such that for every $U \in \pi$ we have $\mathrm{Bd}(U) \subseteq \mathrm{Bd}(V_0) \cup ... \cup \mathrm{Bd}(V_m)$ for some elements $V_0, ..., V_m$ of $\mathbb{B}$.*

3. Lemma. *Let $X \in \mathbb{R}^n(\mathbb{M})$, F be a closed subset of X, $F \in \mathbb{R}^k(\mathbb{M})$, $0 \leq k \leq n$, $x \in F$ and V_0 be an open neighbourhood of x in X. Then there exists an open set V of X such that: (α) $x \in V \subseteq V_0$, (β) $\mathrm{Bd}(V) \in \mathbb{R}^{n-1}(\mathbb{M})$ and (γ) $F \cap \mathrm{Bd}(V) \in \mathbb{R}^{k-1}(\mathbb{M})$.*

The Lemmas 2 and 3 are used for the proof of the following lemma, which is also stated without proof.

4. Lemma. *Let $X \in \mathbb{R}^n(\mathbb{M})$, K and Q be disjoint closed subsets of X and F_i, $i = 0, ..., n-1$, be a closed subset of X such that $F_i \in \mathbb{R}^i(\mathbb{M})$ and $F_0 \subseteq ... \subseteq F_{n-1}$. Then there exists an open subset U of X such that:*

(1) *The set U separates K and Q and $K \subseteq U$,*

(2) $\mathrm{Bd}(U) \in I\!R^{n-1}(I\!M)$, *and*

(3) $F_i \cap \mathrm{Bd}(U) \in I\!R^{i-1}(I\!M)$, $i = 0, ..., n-1$.

5. Theorem. *A space X belongs to $I\!R^n(I\!M)$ iff there exists a basis $I\!B$ for open sets of X having the property of boundary intersections.*

Proof. Obviously, it is sufficient to prove that if $X \in I\!R^n(I\!M)$, then X has a basis $I\!B$ for open sets with the property of boundary intersections. We can suppose that X is a metric space. Let $\{V_0, V_1, ...\}$ be a basis for open sets of X. For every $j \in N$, let V^j be an open set of X such that $\mathrm{Cl}(V_j) \subseteq V^j$ and $\mathrm{diam}(V^j) \leq 3\ \mathrm{diam}(V_j)$. We set $K^j = \mathrm{Cl}(V_j)$ and $Q^j = X \setminus V^j$. Obviously, $K^j \cap Q^j = \emptyset$.

Using Lemma 4 we can construct by induction an open subset U_j of X, $j \in N$, such that:

(1) The set U_j separates the closed subsets K^j and Q^j and $K^j \subseteq U_j$.

(2) $\mathrm{Bd}(U_j) \in I\!R^{n-1}(I\!M)$.

(3) If F_t^j, $j \geq 1$, $1 \leq t \leq n$, is the union of all sets of the form $\mathrm{Bd}(U_{i_1}) \cap ... \cap \mathrm{Bd}(U_{i_t})$, where $\{i_1, ..., i_t\} \subseteq \{0, ..., j-1\}$ and $|\{i_1, ..., i_t\}| = t$, then $F_t^j \cap \mathrm{Bd}(U_j) \in I\!R^{n-t-1}(I\!M)$.

It is easy to prove that the set $I\!B = \{U_0, U_1, ...\}$ is the required basis for open sets of X having the property of boundary intersections.

6. Definitions and Notations. Let X be a space. Suppose that for every $k \in N$ we have two closed subsets $A_0^k(X) \equiv A_0^k$ and $A_1^k(X) \equiv A_1^k$ of X such that $A_0^k \cup A_1^k = X$. (It is possible that either $A_0^k = \emptyset$ or $A_1^k = \emptyset$). By $\sigma_k(X) \equiv \sigma_k$ we denote the ordered closed cover $\{A_0^k, A_1^k\}$ of X. It is possible that for distinct indexes i and j, the ordered covers σ_i and σ_j of X coincide, that is, $A_0^i = A_0^j$ and $A_1^i = A_1^j$, while these covers are considered to be distinct elements of Σ. The ordered set $\Sigma = \{\sigma_0, \sigma_1, ...\}$ is called *basic system for* X iff for every $x \in X$ and for every open neighbourhood U of x in X there exists an integer $k \in N$ such that $x \in A_0^k \setminus A_1^k \subseteq A_0^k \subseteq U$.

In what follows of Section I, X is a fixed space and $\Sigma = \{\sigma_0, \sigma_1, ...\}$ is a fixed basic system for X, where $\sigma_k = \{A_0^k, A_1^k\}$, $k = 0, 1, ...$.

For every integer $k \in N$, we set $\mathrm{Fr}(\sigma_k) = A_0^k \cap A_1^k$. Also, we set

$$\mathrm{Fr}(\Sigma) = \bigcup\{\mathrm{Fr}(\sigma_k) : k = 0, 1, ...\}.$$

For every $\bar{i} = i_1 ... i_k \in L_k$, $k > 0$, we set $X_{\bar{i}} = A_{i_1}^0 \cap ... \cap A_{i_k}^{k-1}$. Also, we set $X_\emptyset = X$. It is easy to see that $X_{\bar{j}} \subseteq X_{\bar{i}}$, if $\bar{i} \leq \bar{j}$, and $X = \bigcup\{X_{\bar{i}} : \bar{i} \in L_k\}$, for every $k \in N$.

We define a subset $S(X,\Sigma) \equiv S$ of C as follows: a point a of C belongs to S iff $X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots \neq \emptyset$. For every $a \in S$ the set $X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots$ is a singleton. Indeed, let $x, y \in X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots$ and $x \neq y$. Since Σ is a basic system for X, there exists an integer $k \in N$ such that $x \in A_0^k \setminus A_1^k$ and $y \notin A_0^k \setminus A_1^k$, that is, $x \in A_0^k$, $y \notin A_0^k$ and $x \notin A_1^k$, $y \in A_1^k$. Since, either $X_{\bar{i}(a,k+1)} = X_{\bar{i}(a,k)} \cap A_0^k$ or $X_{\bar{i}(a,k+1)} = X_{\bar{i}(a,k)} \cap A_1^k$ we have that either $y \notin X_{\bar{i}(a,k+1)}$ or $x \notin X_{\bar{i}(a,k+1)}$, which is a contradiction. We define a map $q(X,\Sigma) \equiv q$ of S into X as follows: if $X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots = \{x\}$, then we set $q(a) = x$. Also we set $D(X,\Sigma) \equiv D = \{q^{-1}(x) : x \in X\}$. By $h(X,\Sigma) \equiv h$ we denote the map of D into X defined as follows: $h(d) = x$ iff $d = q^{-1}(x)$. Obviously, D is a partition of S. By $p(X,\Sigma) \equiv p$ we denote the natural projection of S onto D.

7. Lemma. *The following properties are true:*

(1) $q(C_{\bar{i}} \cap S) = X_{\bar{i}}$, $\bar{i} \in L$.

(2) *For every* $x \in X \setminus \mathrm{Fr}(\Sigma)$, *the set* $q^{-1}(x)$ *is a singleton.*

(3) *For every* $x \in \mathrm{Fr}(\Sigma)$, *the set* $q^{-1}(x)$ *is compact.*

(4) *Let* $N(x)$ *be the set of all elements* k *of* N, *for which* $x \in \mathrm{Fr}(\sigma_k)$ *and let* $a \in q^{-1}(x)$. *Then, the set* $q^{-1}(x)$ *consists of all points* b *of* C *for which* $i(a,k+1) = i(b,k+1)$ *for every* $k \in N \setminus N(x)$.

(5) *The map* q *is continuous.*

(6) *The map* q *is closed.*

(7) *The set* D *is an upper semi-continuous partition of* S.

(8) *The map* h *is a homeomorphism of* D *onto* X *and* $h \circ p = q$.

(9) *The set* $h^{-1}(A_0^k \setminus A_1^k)$, $k \in N$, *is the set of all elements of* D *which are contained in the set* $\bigcup\{C_{\bar{i}0} : \bar{i} \in L_k\}$.

(10) *The set* $h^{-1}(A_1^k \setminus A_0^k)$, $k \in N$, *is the set of all elements of* D *which are contained in the set* $\bigcup\{C_{\bar{i}1} : \bar{i} \in L_k\}$.

(11) *The set* $h^{-1}(\mathrm{Fr}(\sigma_k))$, $k \in N$, *is the set of all elements of* D, *which intersect both sets* $\bigcup\{C_{\bar{i}0} : \bar{i} \in L_k\}$ *and* $\bigcup\{C_{\bar{i}1} : \bar{i} \in L_k\}$.

(12) *If* $\{k_1, \ldots, k_m\}$ *is a subset of* N, *then the set* $h^{-1}(\mathrm{Fr}(\sigma_{k_1}) \cap \ldots \cap \mathrm{Fr}(\sigma_{k_m}))$ *is the set of all elements of* D, *which intersect all of the sets:* $\bigcup\{C_{\bar{i}0} : \bar{i} \in L_{k_1}\}, \ldots, \bigcup\{C_{\bar{i}0} : \bar{i} \in L_{k_m}\}, \bigcup\{C_{\bar{i}1} : \bar{i} \in L_{k_1}\}, \ldots, \bigcup\{C_{\bar{i}1} : \bar{i} \in L_{k_m}\}$.

Proof. (1). Let $a \in S$. By the definitions of S and q, $\{q(a)\} = X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots$. If $a \in C_{\bar{i}}$, $\bar{i} \in L_k$, then $\bar{i}(a,k) = \bar{i}$ and hence $q(a) \in X_{\bar{i}}$, that is, $q(C_{\bar{i}} \cap S) \subseteq X_{\bar{i}}$. Let $x \in X_{\bar{i}}$, $\bar{i} \in L_k$. For every integer m, $0 \leq m \leq k$, we denote by $\bar{i}_m$ the unique element of L_m for which $\bar{i}_m \leq \bar{i}$. Obviously, $x \in X_{\bar{i}_m}$. Since

$X_{\bar{i}} = X_{\bar{i}0} \cup X_{\bar{i}1}$ we have $x \in X_{\bar{i}0} \cup X_{\bar{i}1}$. By $\bar{i}_{k+1}$ we denote one of the elements $\bar{i}0$ and $\bar{i}1$ of L_{k+1} for which $x \in X_{\bar{i}_{k+1}}$. By induction, for every integer $m \geq k$, we construct an element $\bar{i}_m \in L_m$ such that $\bar{i}_m \leq \bar{i}_{m+1}$ and $x \in X_{\bar{i}_m}$. Then $C_{\bar{i}_{m+1}} \subseteq C_{\bar{i}_m}$ and $C_{\bar{i}0} \cap C_{\bar{i}1} \cap \ldots \neq \emptyset$. Obviously, this intersection is a singleton $\{a\}$. Since $\bar{i}(a, m) = \bar{i}_m$ and $x \in X_{\bar{i}_0} \cap X_{\bar{i}_1} \cap \ldots \neq \emptyset$ we have $a \in S$ and $q(a) = x$, that is, $q(C_{\bar{i}} \cap S) \supseteq X_{\bar{i}}$. Hence $q(C_{\bar{i}} \cap S) = X_{\bar{i}}$.

(2). By property (1), $q^{-1}(x) \neq \emptyset$. Let $a, b \in q^{-1}(x)$, $a \neq b$. Let k be the minimal integer for which there exists $\bar{j}_1, \bar{j}_2 \in L_k$, $\bar{j}_1 \neq \bar{j}_2$, such that $a \in C_{\bar{j}_1}$ and $b \in C_{\bar{j}_2}$. Let $\bar{i} \in L_{k-1}$ such that $a, b \in C_{\bar{i}}$. Obviously, $\{\bar{j}_1, \bar{j}_2\} = \{\bar{i}0, \bar{i}1\}$. By property (1), $x \in X_{\bar{i}0} \cap X_{\bar{i}1} = (X_{\bar{i}} \cap A_0^{k-1}) \cap (X_{\bar{i}} \cap A_1^{k-1})$. Hence $x \in A_0^{k-1} \cap A_1^{k-1} = \mathrm{Fr}(\sigma^{k-1})$, which is a contradiction. Hence $q^{-1}(x)$ is a singleton.

(3). It is sufficient to prove that $\mathrm{Cl}(q^{-1}(x)) \subseteq q^{-1}(x)$. Let $a \in \mathrm{Cl}(q^{-1}(x))$. Then, for every integer $k \in N$, $q^{-1}(x) \cap C_{\bar{i}(a,k)} \neq \emptyset$, that is, $x \in X_{\bar{i}(a,k)}$. Hence $\{x\} = X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots$ and therefore $a \in S$ and $q(a) = x$, that is, $a \in q^{-1}(x)$. Thus, $\mathrm{Cl}(q^{-1}(x)) \subseteq q^{-1}(x)$ and hence $q^{-1}(x)$ is compact.

(4). Let $b \in q^{-1}(x)$. Then $\{x\} = X_{\bar{i}(a,0)} \cap X_{\bar{i}(a,1)} \cap \ldots = A^0_{i(a,1)} \cap A^1_{i(a,2)} \cap \ldots = A^0_{i(b,1)} \cap A^0_{i(b,2)} \cap \ldots$. Let $m \in N \setminus N(x)$. Then $x \in A^m_{i(a,m+1)}$ and $x \notin A^m_{1-i(a,m+1)}$. Since $x \in A^m_{i(b,m+1)}$, $i(a, m+1) = i(b, m+1)$. Conversely, let $b \in C$ and $i(a, m+1) = i(b, m+1)$ for all $m \in N \setminus N(x)$. Then $A^m_{i(b,m+1)} = A^m_{i(a,m+1)}$, $m \in N \setminus N(x)$. Since $x \in A^k_{i(a,k+1)} \cap A^k_{1-i(a,k+1)}$, $k \in N(x)$, it follows that $x \in A^k_{i(b,k+1)}$, because either $i(b, k+1) = i(a, k+1)$ or $i(b, k+1) = 1 - i(a, k+1)$. Hence $\{x\} = A^0_{i(b,1)} \cap A^1_{i(b,2)} \cap \ldots = X_{\bar{i}(b,0)} \cap X_{\bar{i}(b,1)} \cap \ldots$. Thus $b \in S$ and $q(b) = x$.

(5). Let $q(a) = x$ and U be an open neighbourhood of x in X. There exists an integer $m \in N$ such that $x \in A_0^m \setminus A_1^m \subseteq A_0^m \subseteq U$. Let $\bar{i} \in L_{m+1}$ and $x \in X_{\bar{i}}$. Since $x \in A_0^m \subseteq U$ and $x \notin A_1^m$ we have $X_{\bar{i}} \subseteq A_0^m \subseteq U$. Then the set $V = C_{\bar{i}} \cap S$ is an open neighbourhood of a in S for which $q(V) \subseteq U$ (see property (1)). Hence q is continuous.

(6). Let F be a closed subset of S. We prove that $q(F)$ is closed in X. Let $x \notin q(F)$. Then $q^{-1}(x) \cap F = \emptyset$. Since $q^{-1}(x)$ is compact, there exists an integer m such that $\mathrm{st}(q^{-1}(x), m) \cap \mathrm{st}(F, m) = \emptyset$. The union K of all sets $X_{\bar{i}}$, $\bar{i} \in L_m$, for which $C_{\bar{i}} \subseteq \mathrm{st}(F, m)$, contains $q(F)$ and does not contain x. Hence the set $U = X \setminus K$ is an open neighbourhood of x in X for which $U \cap q(F) = \emptyset$, that is, $q(F)$ is closed. Thus q is closed.

(7). It is sufficient to prove that the natural projection p of S onto D is closed. (See [K], Ch. 3, Theorem 12), that is, for every closed subset F of S the set $p^{-1}(p(F))$ is closed. (See [K], Ch. 3, Theorem 10). It is easy to see that

$p^{-1}(p(F)) = q^{-1}(q(F))$. By properties (5) and (6) the set $q^{-1}(q(F))$ is closed. Hence p is closed and D is an upper semi-continuous partition.

(8). It follows by properties (5), (6) and (7).

(9). Let $d \in D$ and $d \subseteq \bigcup\{C_{\bar{i}0} : \bar{i} \in L_k\}$. We prove that $h(d) = x \in A_0^k \setminus A_1^k$. Suppose that $x \notin A_0^k \setminus A_1^k$ and let $\bar{i}$ be an element of L_k for which $x \in X_{\bar{i}}$. Then $x \in X_{\bar{i}} \cap A_1^k = X_{\bar{i}1}$. Hence, by property (1), $q^{-1}(x) \cap C_{\bar{i}1} = d \cap C_{\bar{i}1} \neq \emptyset$, which is a contradiction. Conversely, let $h(d) = x \in A_0^k \setminus A_1^k$, $k \in N$. We prove that $h^{-1}(x) = d \subseteq \bigcup\{C_{\bar{i}0} : \bar{i} \in L_k\}$. Indeed, in the opposite case, there exists an element $\bar{i} \in L_k$ such that $d \cap C_{\bar{i}1} \neq \emptyset$. Then $h(d) = x \in X_{\bar{i}1}$. This means that $x \in A_1^k$, that is, $x \notin A_0^k \setminus A_1^k$, which is a contradiction.

(10). The proof is similar to the proof of property (9).

(11). The proof follows by properties (9) and (10).

(12). The proof follows by property (11).

8. **Definition.** A pair (S, D), where S is a subset of C and D is an upper semi-continuous partition of S whose elements are compact, is called *a representation*. Obviously, if X is a space and Σ is a basic system for X, then the pair $(S(X, \Sigma), D(X, \Sigma))$ is a representation. This representation is called *the representation of X corresponding to the basic system Σ*.

II. The main Lemma.

1. **Definitions and Notations.** Let $\Re$ be a family of representations, the cardinality of which is less than or equal to the continuum. It is possible that for two distinct elements (S_1, D_1) and (S_2, D_2) of $\Re$, $S_1 = S_2$ and $D_1 = D_2$. We suppose that for every element $\zeta = (S, D) \in \Re$ there exists a space $X(\zeta) \in I\!R^n(I\!M)$ (we recall that n is a fixed integer of $N \setminus \{0\}$) and a basic system $\Sigma(\zeta) \equiv \{\sigma_0(\zeta), \sigma_1(\zeta), ...\}$ for $X(\zeta)$ such that (S, D) is the representation of $X(\zeta)$ corresponding to the basic system $\Sigma(\zeta)$. Moreover, we suppose that the basic system $\Sigma(\zeta)$ has the following property calling *the property of boundary intersections*: for every integer k, $1 \leq k \leq n$, and for every mutually distinct integers $j_1, ..., j_k$ of N (that is, $|\{j_1, ..., j_k\}| = k$) we have

$$\bigcap\{\mathrm{Fr}(\sigma_{j_i}(\zeta)) : i = 1, ..., k\} \in I\!R^{n-k}(I\!M).$$

For every representation $\zeta = (S, D)$, the subset S of C is denoted also by $S(\zeta)$ and the partition D of S is denoted also by $D(\zeta)$. If $\zeta \in \Re$, then the map $h(X(\zeta), \Sigma(\zeta))$ is denoted also by h_ζ.

Since the cardinality of $\Re$ is less than or equal to the continuum, for every element $\bar{i} \in L$ there exists a subfamily $\Re(\bar{i})$ of $\Re$ such that: (α) $\Re(\emptyset) = \Re$, (β) $\Re(\bar{i}) \cap \Re(\bar{j}) = \emptyset$, if $\bar{i}, \bar{j} \in L_k$, $\bar{i} \neq \bar{j}$, $k \in N$, (γ) $\Re(\bar{i}) = \Re(\bar{i}0) \cup \Re(\bar{i}1)$, $\bar{i} \in L$, and (δ) for distinct elements $\zeta_1, \zeta_2 \in \Re$ there exist an integer $k \in N$ and elements $\bar{i}, \bar{j} \in L_k$, $\bar{i} \neq \bar{j}$, such that $\zeta_1 \in \Re(\bar{i})$ and $\zeta_2 \in \Re(\bar{j})$.

For every integer $k \in N$, we set

$$U_k^C = \bigcup \{C_{\bar{i}0} : \bar{i} \in L_k\}.$$

If $\zeta = (S, D)$ is a representation, then we denote by U_k^S the set $U_k^C \cap S$ and by U_k^D the set of all elements of D, which are contained in the set U_k^S. Also, we denote by $\overline{U}_k^D$ the set of all elements of D which intersect the set U_k^S. We set $\mathrm{Fr}(U_k^D) = \overline{U}_k^D \setminus U_k^D$. It easy to see that if $\zeta \in \Re$, then $\mathrm{Fr}(U_k^{D(\zeta)}) = h_\zeta^{-1}(\mathrm{Fr}(\sigma_k(\zeta)))$. (See property 11 of Lemma 7.I). Also, the ordered set $\mathbb{B}(D(\zeta)) \equiv \{U_0^{D(\zeta)}, U_1^{D(\zeta)}, ...\}$ is an ordered basis for open sets of $D(\zeta)$.

For every $\zeta \in \Re$ we denote by $D(\zeta)(0)$ the set of all elements d of $D(\zeta)$ for which there exist mutually distinct integers $j_1, ..., j_n$ of N such that

$$d \in \bigcap \{\mathrm{Fr}(U_{j_i}^{D(\zeta)}) : i = 1, ..., n\}.$$

Since $\Sigma(\zeta)$ has the property of boundary intersections and

$$\mathrm{Fr}(U_{j_i}^{D(\zeta)}) = h_\zeta^{-1}(\mathrm{Fr}(\sigma_{j_i}(\zeta))),$$

$i = 1, ..., n$, the set $D(\zeta)(0)$ is countable.

We consider an ordered set

$$\overrightarrow{D}(\zeta)(0) \equiv \{d_0^{D(\zeta)}, d_1^{D(\zeta)}, ...\}$$

such that: (α) for every $d \in D(\zeta)(0)$ there exists uniquely determined integer $i \in N$, for which $d = d_i^{D(\zeta)}$ and (β) if for some $i \in N$ there is no element $d \in D(\zeta)(0)$ for which $d_i^{D(\zeta)} = d$, then $d_i^{D(\zeta)} = \emptyset$. We observe that, in general, $\emptyset \in \overrightarrow{D}(\zeta)(0)$, while $\emptyset \notin D(\zeta)(0)$. Also, if $d_k^{D(\zeta)} \neq \emptyset$ and $d_k^{D(\zeta)} = d_i^{D(\zeta)}$, then $i = k$.

For every subset C' of C and for every subfamily $\Re'$ of $\Re$ we set

$$J(C' \times \Re') = \{(a, \zeta) \in C' \times \Re' : \ a \in S(\zeta)\}.$$

Let $\{U_0, ..., U_m\}$ be an ordered set of subsets of a space X and $\{V_0, ..., V_m\}$ be an ordered set of subsets of a space Y. We say that *the ordered sets* $\{U_0, ..., U_m\}$ *and*

$\{V_0, \dots, V_m\}$ *have the same structure* iff for every $i_1, \dots, i_k \in N$, $0 \le i_1, \dots, i_k \le m$ we have $U_{i_1} \cap \dots \cap U_{i_k} \ne \emptyset$ iff $V_{i_1} \cap \dots \cap V_{i_k} \ne \emptyset$.

2. Lemma. *For every integer $k \in N$, for every element $\overline{\alpha}$ of Λ_{k+1} and for every $m \in N$, $0 \le m \le k$, there exist:*

(1) *An integer $n(\Re) \ge 0$.*

(2) *An integer $n(\overline{\alpha}) \ge k+1$.*

(3) *An integer $n(\overline{\alpha}, m) \ge 0$.*

(4) *A subset $\Re(\overline{\alpha})$ of $\Re$. (It is possible that $\Re(\overline{\alpha}) = \emptyset$ for some $\overline{\alpha} \in \Lambda_{k+1}$).*

(5) *A subset $d(\overline{\alpha}, k)$ of $J(C \times \Re(\overline{\alpha}))$. (It is possible that $d(\overline{\alpha}, k) = \emptyset$ for some $\overline{\alpha} \in \Lambda_{k+1}$).*

(6) *A subset $U(\overline{\alpha}, m)$ of $J(C \times \Re(\overline{\alpha}))$. (It is possible that $U(\overline{\alpha}, m) = \emptyset$ for some $\overline{\alpha} \in \Lambda_{k+1}$ and some m, $0 \le m \le k$),*

such that:

(7) *$n(\overline{\alpha}) \ge n(\overline{\beta})$ if $\overline{\alpha} \ge \overline{\beta}$.*

(8) *$n(\overline{\alpha}, m) \le n(\overline{\alpha})$.*

(9) *$\Re = \bigcup\{\Re(\overline{\alpha}) : \overline{\alpha} \in \Lambda_1\}$.*

(10) *If $\overline{\alpha}_1$, $\overline{\alpha}_2 \in \Lambda_{k+1}$, $\overline{\alpha}_1 \ne \overline{\alpha}_2$, then $\Re(\overline{\alpha}_1) \cap \Re(\overline{\alpha}_2) = \emptyset$. If $k > 0$, $\overline{\beta} \in \Lambda_k$, $\overline{\beta} \le \overline{\alpha}$ and $\Re(\overline{\beta}) = \Re(\overline{\alpha})$, then the set $\Re(\overline{\alpha})$ is a singleton.*

(11) *If $\overline{\beta} \in \Lambda_k$, $k > 0$, then*

$$\Re(\overline{\beta}) = \bigcup\{\Re(\overline{\alpha}) : \overline{\alpha} \in \Lambda_{k+1}, \overline{\beta} \le \overline{\alpha}\}.$$

(12) *There exists an element $\overline{i}(\overline{\alpha}) \in L_k$ such that $\Re(\overline{\alpha}) \subseteq \Re(\overline{i}(\overline{\alpha}))$.*

(13) *If $k+1 \ge n(\Re)$ and $\zeta, \chi \in \Re(\overline{\alpha})$, then the set*

$$\{U_0^{D(\zeta)}, \dots, U_{n(\overline{\alpha})}^{D(\zeta)}, \overline{U}_0^{D(\zeta)}, \dots, \overline{U}_{n(\overline{\alpha})}^{D(\zeta)}, D(\zeta) \setminus U_0^{D(\zeta)}, \dots, D(\zeta) \setminus U_{n(\overline{\alpha})}^{D(\zeta)}, D(\zeta) \setminus \overline{U}_0^{D(\zeta)}, \dots, D(\zeta) \setminus \overline{U}_{n(\overline{\alpha})}^{D(\zeta)}, \operatorname{Fr}(U_0^{D(\zeta)}), \dots, \operatorname{Fr}(U_{n(\overline{\alpha})}^{D(\zeta)}), D(\zeta) \setminus \operatorname{Fr}(U_0^{D(\zeta)}), \dots, D(\zeta) \setminus \operatorname{Fr}(U_{n(\overline{\alpha})}^{D(\zeta)})\}$$

has the same structure with the set

$$\{U_0^{D(\chi)}, \dots, U_{n(\overline{\alpha})}^{D(\chi)}, \overline{U}_0^{D(\chi)}, \dots, \overline{U}_{n(\overline{\alpha})}^{D(\chi)}, D(\chi) \setminus U_0^{D(\chi)}, \dots, D(\chi) \setminus U_{n(\overline{\alpha})}^{D(\chi)}, D(\chi) \setminus \overline{U}_0^{D(\chi)}, \dots, D(\chi) \setminus \overline{U}_{n(\overline{\alpha})}^{D(\chi)}, \operatorname{Fr}(U_0^{D(\chi)}), \dots, \operatorname{Fr}(U_{n(\overline{\alpha})}^{D(\chi)}), D(\chi) \setminus \operatorname{Fr}(U_0^{D(\chi)}), \dots, D(\chi) \setminus \operatorname{Fr}(U_{n(\overline{\alpha})}^{D(\chi)})\}.$$

(14) *If $\zeta, \chi \in \Re(\overline{\alpha})$, then $d_k^{D(\zeta)} \ne \emptyset$ iff $d_k^{D(\chi)} \ne \emptyset$.*

(15) *If $\zeta \in \Re(\overline{\alpha})$ and $d_k^{D(\zeta)} \ne \emptyset$, then*

$$d(\overline{\alpha}, k) \cap (C \times \{\zeta\}) = d_k^{D(\zeta)} \times \{\zeta\}.$$

(16) *If* $\zeta, \chi \in \Re(\overline{\alpha})$ *and* $d_k^{D(\zeta)} \neq \emptyset$, *then* $d_k^{D(\zeta)} \in \mathrm{Fr}(U_i^{D(\zeta)})$ *iff* $d_k^{D(\chi)} \in \mathrm{Fr}(U_i^{D(\chi)})$ *for every* $i \in N$.

(17) *If* $k > 0$, $\overline{\beta} \in \Lambda_k$, $\overline{\beta} \leq \overline{\alpha}$, $\zeta, \chi \in \Re(\overline{\alpha})$ *and* $d_m^{D(\zeta)} \neq \emptyset$, *then* $d_m^{D(\zeta)} \in U_i^{D(\zeta)}$, *where* $0 \leq i \leq n(\overline{\beta})$, *iff* $d_m^{D(\chi)} \in U_i^{D(\chi)}$.

(18) *If* $\zeta \in \Re(\overline{\alpha})$ *and* $d_m^{D(\zeta)} \neq \emptyset$, *then* $d_m^{D(\zeta)} \in U_{n(\overline{\alpha},m)}^{D(\zeta)}$.

(19) *If* $k > 0$, $\overline{\beta} \in \Lambda_k$, $\overline{\beta} \leq \overline{\alpha}$, $\zeta \in \Re(\overline{\alpha})$, $d_m^{D(\zeta)} \neq \emptyset$ *and* $d_m^{D(\zeta)} \in U_i^{D(\zeta)}$, *where* $0 \leq i \leq n(\overline{\beta})$, *then* $U_{n(\overline{\alpha},m)}^{D(\zeta)} \subseteq U_i^{D(\zeta)}$.

(20) *If* $k > 0$, $\overline{\beta} \in \Lambda_k$, $\overline{\beta} \leq \overline{\alpha}$, $\zeta \in \Re(\overline{\alpha})$, $d_m^{D(\zeta)} \neq \emptyset$ *and* $d_m^{D(\zeta)} \notin \overline{U}_i^{D(\zeta)}$, *where* $0 \leq i \leq n(\overline{\beta})$, *then* $U_{n(\overline{\alpha},m)}^{D(\zeta)} \cap \overline{U}_i^{D(\zeta)} = \emptyset$.

(21) *If* $\zeta \in \Re(\overline{\alpha})$, $m_1, m_2 \in N$, $0 \leq m_1,\ m_2 \leq k$, $m_1 \neq m_2$, $d_{m_1}^{D(\zeta)} \neq \emptyset$ *and* $d_{m_2}^{D(\zeta)} \neq \emptyset$, *then* $\overline{U}_{n(\overline{\alpha},m_1)}^{D(\zeta)} \cap \overline{U}_{n(\overline{\alpha},m_2)}^{D(\zeta)} = \emptyset$.

(22) *If* $\zeta \in \Re(\overline{\alpha})$ *and* $d_m^{D(\zeta)} \neq \emptyset$, *then*

$$U(\overline{\alpha}, m) = J(U_{n(\overline{\alpha},m)}^{C} \times \Re(\overline{\alpha})).$$

(23) *If* $k > 0$, $\overline{\beta} \in \Lambda_k$, $\overline{\beta} \leq \overline{\alpha}$, $\zeta \in \Re(\overline{\alpha})$, $d_m^{D(\zeta)} \neq \emptyset$ *and* $0 \leq m \leq k-1$, *then* $\overline{U}_{n(\overline{\alpha},m)}^{D(\zeta)} \subseteq U_{n(\overline{\beta},m)}^{D(\zeta)}$.

Proof. Let $n(\Re)$ be an arbitrary integer of N. We prove the lemma by induction on integer k. Let $k = 0$. For every $\zeta \in \Re$, we denote by $n(\zeta) \geq 1$ an integer of N such that $d_0^{D(\zeta)} \in U_{n(\zeta)}^{D(\zeta)}$. Also, if the set $\Re$ is not a singleton, then we denote by $\Re_1$ and $\Re_2$ two disjoint non-empty subsets of $\Re$, the union of which is the set $\Re$.

In the set $\Re$ we define an equivalence relation $''\sim''$. We say that two elements ζ and χ of $\Re$ are equivalent iff the following conditions are satisfied: (α) either $d_0^{D(\zeta)} \neq \emptyset$ and $d_0^{D(\chi)} \neq \emptyset$, or $d_0^{D(\zeta)} = \emptyset$ and $d_0^{D(\chi)} = \emptyset$, (β) $n(\zeta) = n(\chi)$, (γ) if $d_0^{D(\zeta)} \neq \emptyset$, then, for every $i \in N$, either $d_0^{D(\zeta)} \in \mathrm{Fr}(U_i^{D(\zeta)})$ and $d_0^{D(\chi)} \in \mathrm{Fr}(U_i^{D(\chi)})$ or $d_0^{D(\zeta)} \notin \mathrm{Fr}(U_i^{D(\zeta)})$ and $d_0^{D(\chi)} \notin \mathrm{Fr}(U_i^{D(\chi)})$, (δ) if $1 \geq n(\Re)$, then the set

$$\{U_0^{D(\zeta)}, \ldots, U_{n(\zeta)}^{D(\zeta)}, \overline{U}_0^{D(\zeta)}, \ldots, \overline{U}_{n(\zeta)}^{D(\zeta)}, D(\zeta) \setminus U_0^{D(\zeta)}, \ldots, D(\zeta) \setminus U_{n(\zeta)}^{D(\zeta)}, D(\zeta) \setminus \overline{U}_0^{D(\zeta)}, \ldots,$$
$$D(\zeta) \setminus \overline{U}_{n(\zeta)}^{D(\zeta)}, \mathrm{Fr}(U_0^{D(\zeta)}), \ldots, \mathrm{Fr}(U_{n(\zeta)}^{D(\zeta)}), D(\zeta) \setminus \mathrm{Fr}(U_0^{D(\zeta)}), \ldots, D(\zeta) \setminus \mathrm{Fr}(U_{n(\zeta)}^{D(\zeta)})\}$$

has the same structure with the set

$$\{U_0^{D(\chi)}, \ldots, U_{n(\chi)}^{D(\chi)}, \overline{U}_0^{D(\chi)}, \ldots, \overline{U}_{n(\chi)}^{D(\chi)}, D(\chi) \setminus U_0^{D(\chi)}, \ldots, D(\chi) \setminus U_{n(\chi)}^{D(\chi)}, D(\chi) \setminus \overline{U}_0^{D(\chi)}, \ldots,$$
$$D(\chi) \setminus \overline{U}_{n(\chi)}^{D(\chi)}, \mathrm{Fr}(U_0^{D(\chi)}), \ldots, \mathrm{Fr}(U_{n(\chi)}^{D(\chi)}), D(\chi) \setminus \mathrm{Fr}(U_0^{D(\chi)}), \ldots, D(\chi) \setminus \mathrm{Fr}(U_{n(\chi)}^{D(\chi)})\}$$

and (ε) if the set $\Re$ is not a singleton, then the elements ζ and χ belong to the same set $\Re_1$ or $\Re_2$.

Since for every $\zeta \in \Re$ the basic system $\Sigma(\zeta)$ has the property of boundary intersections, the set of all equivalence classes of the above relation are countable. Hence there exists an one-to-one correspondence between this set of equivalence classes and a subset Λ_1' of Λ_1. For every $\overline{\alpha} \in \Lambda_1'$, we denote by $\Re(\overline{\alpha})$ the equivalence class corresponding to $\overline{\alpha}$. If $\overline{\alpha} \notin \Lambda_1'$, then we set $\Re(\overline{\alpha}) = \emptyset$.

We define the set $d(\overline{\alpha}, 0)$ as follows: if for some $\zeta \in \Re(\overline{\alpha})$ (and, hence, by property (α) of the definition of the relation $'' \sim ''$, for every $\zeta \in \Re(\overline{\alpha})$) we have $d_0^{D(\zeta)} \neq \emptyset$, then we set

$$d(\overline{\alpha}, 0) = \bigcup \{(d_0^{D(\zeta)} \times \{\zeta\}) : \zeta \in \Re(\overline{\alpha})\}.$$

If for some $\zeta \in \Re(\overline{\alpha})$ (and, hence, for every $\zeta \in \Re(\overline{\alpha})$) we have $d_0^{D(\zeta)} = \emptyset$ or if $\Re(\overline{\alpha}) = \emptyset$, then we set $d(\overline{\alpha}, 0) = \emptyset$.

We set $n(\overline{\alpha}) = n(\overline{\alpha}, 0) = n(\zeta)$, where $\zeta \in \Re(\overline{\alpha})$. By property ($\beta$) of the definition of the relation $'' \sim ''$, the integer $n(\overline{\alpha}) = n(\overline{\alpha}, 0)$ is independent from element ζ of $\Re(\overline{\alpha})$.

We define the set $U(\overline{\alpha}, 0)$ setting

$$U(\overline{\alpha}, 0) = J(U^C_{n(\overline{\alpha}, 0)} \times \Re(\overline{\alpha})).$$

Obviously, properties (7)$-$(10), (12)$-$(16), (18) and (22) of the lemma are satisfied for $k = 0$. Properties (11), (17), (19) $-$ (21) and (23) concern $k > 0$.

Suppose that for every integer k, $k < r$, $r > 0$, for every $\overline{\alpha} \in \Lambda_{k+1}$ and for every $m \in N$, $0 \leq m \leq k$, we have construct an integer $n(\overline{\alpha})$, an integer $n(\overline{\alpha}, m)$ a subset $\Re(\overline{\alpha})$ of $\Re$, a subset $d(\overline{\alpha}, k)$ of $J(C \times \Re(\overline{\alpha}))$ and a subset $U(\overline{\alpha}, m)$ of $J(C \times \Re(\overline{\alpha}))$ such that properties (7) $-$ (23) of the lemma are satisfied for $k < r$.

Now, for every $\overline{\alpha} \in \Lambda_{r+1}$ and for every $m \in N$, $0 \leq m \leq r$, we define an integer $n(\overline{\alpha})$, an integer $n(\overline{\alpha}, m)$, a subset $\Re(\overline{\alpha})$ of $\Re$, a subset $d(\overline{\alpha}, k)$ of $J(C \times \Re(\overline{\alpha}))$ and a subset $U(\overline{\alpha}, m)$ of $J(C \times \Re(\overline{\alpha}))$ such that properties (7) $-$ (23) are satisfied for $k \leq r$. Let $\overline{\alpha} \in \Lambda_{r+1}$. Let $\overline{\beta} \in \Lambda_r$ be the uniquely determined element of Λ_r for which $\overline{\beta} \leq \overline{\alpha}$. If $\Re(\overline{\beta}) = \emptyset$, then we set $\Re(\overline{\alpha}) = \emptyset$.

Suppose that $\Re(\overline{\beta}) \neq \emptyset$. If the set $\Re(\overline{\beta})$ is not a singleton then we denote by $\Re_1(\overline{\beta})$ and $\Re_2(\overline{\beta})$ two disjoint non-empty subsets of $\Re$, the union of which is the set $\Re(\overline{\beta})$. For every $\zeta \in \Re(\overline{\beta})$ we consider the elements $d_0^{D(\zeta)}, \ldots, d_r^{D(\zeta)}$ of $\overrightarrow{D}(\zeta)(0)$. For every m, $0 \leq m \leq r$, we denote by $n(\overline{\beta}, m, \zeta)$ an element of N

such that: (α) $d_m^{D(\zeta)} \in U_{n(\overline{\beta},m,\zeta)}^{D(\zeta)}$, (β) if $0 \leq m_1,\ m_2 \leq r$, $m_1 \neq m_2$, $d_{m_1}^{D(\zeta)} \neq \emptyset$ and $d_{m_2}^{D(\zeta)} \neq \emptyset$, then $\overline{U}_{n(\overline{\beta},m_1,\zeta)}^{D(\zeta)} \cap \overline{U}_{n(\overline{\beta},m_2,\zeta)}^{D(\zeta)} = \emptyset$, (γ) if $d_m^{D(\zeta)} \in U_i^{D(\zeta)}$, $0 \leq i \leq n(\overline{\beta})$, then $U_{n(\overline{\beta},m,\zeta)}^{D(\zeta)} \subseteq U_i^{D(\zeta)}$, (δ) if $d_m^{D(\zeta)} \notin \overline{U}_i^{D(\zeta)}$, $0 \leq i \leq n(\overline{\beta})$, then $U_{n(\overline{\beta},m,\zeta)}^{D(\zeta)} \cap \overline{U}_i^{D(\zeta)} = \emptyset$, and (ε) if $d_m^{D(\zeta)} \neq \emptyset$, $0 \leq m < r$, then $\overline{U}_{n(\overline{\beta},m,\zeta)}^{D(\zeta)} \subseteq U_{n(\overline{\beta},m)}^{D(\zeta)}$. The existence of the integers $n(\overline{\beta},m,\zeta)$ are easily proved.

In the set $\Re(\overline{\beta})$ we define an equivalence relation "$\sim$". We say that the elements ζ and χ of $\Re(\overline{\beta})$ are equivalent iff the following conditions are satisfied: (α) for every m, $0 \leq m \leq r$, either $d_m^{D(\zeta)} \neq \emptyset$ and $d_m^{D(\chi)} \neq \emptyset$ or $d_m^{D(\zeta)} = \emptyset$ and $d_m^{D(\chi)} = \emptyset$, (β) for every m, $0 \leq m \leq r$, $n(\overline{\beta},m,\zeta) = n(\overline{\beta},m,\chi)$, (γ) for every m, $0 \leq m \leq r$, if $d_m^{D(\zeta)} \neq \emptyset$, then for every $i \in N$, either $d_m^{D(\zeta)} \in \mathrm{Fr}(U_i^{D(\zeta)})$ and $d_m^{D(\chi)} \in \mathrm{Fr}(U_i^{D(\chi)})$ or $d_m^{D(\zeta)} \notin \mathrm{Fr}(U_i^{D(\zeta)})$ and $d_m^{D(\chi)} \notin \mathrm{Fr}(U_i^{D(\chi)})$, (δ) for every m, $0 \leq m \leq r$, if $d_m^{D(\zeta)} \neq \emptyset$, then $d_m^{D(\zeta)} \in U_i^{D(\zeta)}$, $0 \leq i \leq n(\overline{\beta})$, iff $d_m^{D(\chi)} \in U_i^{D(\chi)}$, (ε) there exists an element $\overline{i} \in L_r$ such that $\zeta,\ \chi \in \Re(\overline{i})$, (ζ) If $r+1 \geq n(\Re)$, then the set

$$\{U_0^{D(\zeta)},\dots,U_{n(r,\zeta)}^{D(\zeta)},\overline{U}_0^{D(\zeta)},\dots,\overline{U}_{n(r,\zeta)}^{D(\zeta)},D(\zeta)\setminus U_0^{D(\zeta)},\dots,D(\zeta)\setminus U_{n(r,\zeta)}^{D(\zeta)},D(\zeta)\setminus \overline{U}_0^{D(\zeta)},\dots,$$
$$D(\zeta)\setminus \overline{U}_{n(r,\zeta)}^{D(\zeta)},\mathrm{Fr}(U_0^{D(\zeta)}),\dots,\mathrm{Fr}(U_{n(r,\zeta)}^{D(\zeta)}),D(\zeta)\setminus \mathrm{Fr}(U_0^{D(\zeta)}),\dots,D(\zeta)\setminus \mathrm{Fr}(U_{n(r,\zeta)}^{D(\zeta)})\}$$

has the same structure with the set

$$\{U_0^{D(\chi)},\dots,U_{n(r,\chi)}^{D(\chi)},\overline{U}_0^{D(\chi)},\dots,\overline{U}_{n(r,\chi)}^{D(\chi)},D(\chi)\setminus U_0^{D(\chi)},\dots,D(\chi)\setminus U_{n(r,\chi)}^{D(\chi)},D(\chi)\setminus \overline{U}_0^{D(\chi)},\dots,$$
$$D(\chi)\setminus \overline{U}_{n(r,\chi)}^{D(\chi)},\mathrm{Fr}(U_0^{D(\chi)}),\dots,\mathrm{Fr}(U_{n(r,\chi)}^{D(\chi)}),D(\chi)\setminus \mathrm{Fr}(U_0^{D(\chi)}),\dots,D(\chi)\setminus \mathrm{Fr}(U_{n(r,\chi)}^{D(\chi)})\},$$

where

$$n(r,\zeta) = \max\{n(\overline{\beta},0,\zeta),\dots,n(\overline{\beta},r,\zeta),r+1,n(\overline{\beta})\} = n(r,\chi) =$$
$$= \max\{n(\overline{\beta},0,\chi),\dots,n(\overline{\beta},r,\chi),r+1,n(\overline{\beta})\}$$

and (θ) if the set $\Re(\overline{\beta})$ is not a singleton, then the elements ζ and χ belong to the same set $\Re_1(\overline{\beta})$ and $\Re_2(\overline{\beta})$.

It is easy to see that the set of all equivalence classes of the above relation is countable. Hence there exists an one-to-one correspondence between the set of all equivalence classes and a subset $(\Lambda_{r+1}^{\overline{\beta}})'$ of the set $\Lambda_{r+1}^{\overline{\beta}}$ of all elements of Λ_{r+1}, which are larger than $\overline{\beta}$. For every $\overline{\alpha} \in (\Lambda_{r+1}^{\overline{\beta}})'$, we denote by $\Re(\overline{\alpha})$ the equivalence class corresponding to $\overline{\alpha}$. If $\overline{\alpha} \notin (\Lambda_{r+1}^{\overline{\beta}})'$, then we set $\Re(\overline{\alpha}) = \emptyset$.

Now, for every m, $0 \leq m \leq r$, we define the set $d(\overline{\alpha},r)$, the integer $n(\overline{\alpha},m)$ and the set $U(\overline{\alpha},m)$ as follows:

$$d(\overline{\alpha},r) = \bigcup\{d_r^{D(\zeta)} \times \{\zeta\} : \zeta \in \Re(\overline{\alpha})\},$$

if for some $\zeta \in \Re(\overline{\alpha})$ (and hence for every $\zeta \in \Re(\overline{\alpha})$) we have $d_r^{D(\zeta)} \neq \emptyset$, and $d(\overline{\alpha}, r) = \emptyset$ if for some $\zeta \in \Re(\overline{\alpha})$ (and hence for every $\zeta \in \Re(\overline{\alpha})$) we have $d_r^{D(\zeta)} = \emptyset$ or if $\Re(\overline{\alpha}) = \emptyset$.

We set $n(\overline{\alpha}, m) = n(\overline{\beta}, m, \zeta)$ if $\zeta \in \Re(\overline{\alpha})$ and $n(\overline{\alpha}, m)$ is an arbitrary element of N if $\Re(\overline{\alpha}) = \emptyset$. Obviously, the integer $n(\overline{\alpha}, m)$ is independent of the element $\zeta \in \Re(\overline{\alpha})$.

If $d(\overline{\alpha}, r) \neq \emptyset$, then we set

$$U(\overline{\alpha}, m) = J(U^C_{n(\overline{\alpha}, m)} \times \Re(\overline{\alpha}))$$

and $U(\overline{\alpha}, m) = \emptyset$ if $d(\overline{\alpha}, r) = \emptyset$ or if $\Re(\overline{\alpha}) = \emptyset$.

Finally, we set $n(\overline{\alpha}) = \max\{n(\overline{\alpha}, 0), ..., n(\overline{\alpha}, r), r+1, n(\overline{\beta})\}$.

Now, we prove the properties of the lemma for the case $k = r$. The properties (7) – (11) of the lemma are satisfied by the construction of the subsets $\Re(\overline{\alpha})$ of $\Re(\overline{\beta})$ and by the definition of the integer $n(\overline{\alpha})$. The properties (12), (13), (14), (16) and (17) follow, respectively, by the properties (ε) (ζ), (α), (γ) and (δ) of the definition of the equivalence relation "$\sim$" in the set $\Re(\overline{\beta})$. The properties (18), (19), (20), (21) and (23) follow, respectively, by the properties (α), (γ), (δ), (β) and (ε) of the definition of the integers $n(\overline{\beta}, m, \zeta)$ and the definition of the integer $n(\overline{\alpha}, m)$. The property (15) follows by the definition of the set $d(\overline{\alpha}, r)$. Finally, the property (22) follows by the definition of the set $U(\overline{\alpha}, m)$. The proof of the lemma is completed.

III. The construction of the space $T(\Re)$

1. Notations. By $T(\Re)(0)$ we denote the set of all non-empty sets of the form $d(\overline{\alpha}, k)$, $\overline{\alpha} \in \Lambda_{k+1}$, $k \in N$. If $0 \leq m \leq k$, then we set

$$d(\overline{\alpha}, m) = \bigcup\{d_m^{D(\zeta)} \times \{\zeta\} : \zeta \in \Re(\overline{\alpha})\}.$$

We observe that, in general, the sets $d(\overline{\alpha}, m)$ are not elements of $T(\Re)(0)$. For every $\overline{\alpha} \in \Lambda_{k+1}$, $k \in N$, we denote by $T(\Re)(\overline{\alpha})$ the set of all elements $d(\overline{\alpha}_1, k_1) \in T(\Re)(0)$, where $\overline{\alpha}_1 \in \Lambda_{k_1+1}$ and $\overline{\alpha}_1 \leq \overline{\alpha}$. Obviously, the set $T(\Re)(\overline{\alpha})$ is finite. By $T(\Re)$ we denote the union of the set $T(\Re)(0)$ and the set of all subsets of $J(C \times \Re)$ of the form $d \times \{\zeta\}$, where $\zeta \in \Re$ and $d \in D(\zeta) \setminus D(\zeta)(0)$.

For every $\overline{\alpha} \in \Lambda_{k+1}$, $k+1 \geq n(\Re)$, and for every $r \in N$, $0 \leq r \leq n(\overline{\alpha})$, we denote by $H(\overline{\alpha}, r)$ the set $J(U_r^C \times \Re(\overline{\alpha}))$. The set of all sets of this form is denoted by $\mathcal{U}$. For every $\overline{\alpha} \in \Lambda_{k+1}$, $k \in N$, for which the set $d(\overline{\alpha}, k) \neq \emptyset$, and for every integer $r \in N$, for which $k + r + 1 \geq n(\Re)$, we set

$$V(\overline{\alpha}, r) = \bigcup \{U(\overline{\gamma}, k) : \overline{\gamma} \in \Lambda_{k+r+1}, \overline{\alpha} \leq \overline{\gamma}\}.$$

By $\mathcal{V}$ we denote the set of all sets of the form $V(\overline{\alpha}, r)$.

For every $W \in \mathcal{U} \cup \mathcal{V}$ we denote by $O(W)$ the set of all elements of $T(\Re)$, which are contained in W and by $\mathrm{Fr}(W)$ the set of all elements d of $T(\Re)$ such that $d \cap W \neq \emptyset$ and $d \cap (J(C \times \Re) \setminus W) \neq \emptyset$. We denote by $O(\mathcal{U})$ (respectively, by $O(\mathcal{V})$) the set of all subsets $O(W)$, where $W \in \mathcal{U}$ (respectively, $W \in \mathcal{V}$). Also, we set $\mathbb{B}(T(\Re)) = O(\mathcal{U}) \cup O(\mathcal{V})$.

2. Remarks. Let $k \in N$, $\overline{\alpha} \in \Lambda_{k+1}$, $m \in N$ and $0 \leq m \leq k$. It is not dificult to prove the following propositions:

(1) If $d(\overline{\alpha}, k) \in T(\Re)(0)$ and $\overline{\alpha} \leq \overline{\gamma}$, then $\emptyset \neq d(\overline{\gamma}, k) \subseteq d(\overline{\alpha}, k)$. (See properties (11) and (15) of Lemma 2.II and the definition of the set $d(\overline{\alpha}, m)$).

(2) If $d_1, d_2 \in T(\Re)$, $d_1 \neq d_2$, then $d_1 \cap d_2 = \emptyset$. (See the definition of the set $\overrightarrow{D}(\zeta)(0)$, property (15) of Lemma 2.II and the definition of the elements of the set $T(\Re)$).

(3) The union of all elements of $T(\Re)$ is the set $J(C \times \Re)$.

(4) If $d(\overline{\alpha}, k) \in T(\Re)(0)$, $\overline{\alpha} \leq \overline{\gamma}$, then $d(\overline{\gamma}, k) \subseteq U(\overline{\gamma}, k)$. (See the definition of the sets $d(\overline{\alpha}, m)$ and properties (15), (18) and (22) of Lemma 2.II).

(5) If $d(\overline{\alpha}, k) \in T(\Re)(0)$, $r \in N$ and $k + r + 1 \geq n(\Re)$, then $d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}, r)$. (See the definitions of the sets $d(\overline{\alpha}, m)$ and $V(\overline{\alpha}, r)$ and properties (11), (15), (18) and (22) of Lemma 2.II).

(6) If $d(\overline{\alpha}, k) \in T(\Re)(0)$ and $\overline{\alpha} \leq \overline{\beta} \leq \overline{\gamma}$, then $U(\overline{\gamma}, k) \subseteq U(\overline{\beta}, k)$. (See properties (7), (8), (11), (15), (19) and (22) of Lemma 2.II).

(7) If $d(\overline{\alpha}, k) \in T(\Re)(0)$, $r \in N$ and $k + r + 1 \geq n(\Re)$, then $V(\overline{\alpha}, r) \subseteq U(\overline{\alpha}, k)$. (See the definition of the set $V(\overline{\alpha}, r)$ and the above proposition (6)).

(8) If $d(\overline{\alpha}, k) \in T(\Re)(0)$, $r \in N$ and $k+r+1 \geq n(\Re)$, then $V(\overline{\alpha}, r+1) \subseteq V(\overline{\alpha}, r)$. (See the definition of the set $V(\overline{\alpha}, r)$ and the above proposition (6)).

(9) If $d(\overline{\alpha}, m) \subseteq H(\overline{\beta}, i)$, where $\overline{\beta} \in \Lambda_{k_1+1}$, $k_1 < k$ and $0 \leq i \leq n(\overline{\beta})$, then $U(\overline{\alpha}, m) \subseteq H(\overline{\beta}, i)$. (See the definitions of the sets $d(\overline{\alpha}, m)$ and $H(\overline{\alpha}, r)$, properties (17) and (19) of Lemma 2.II and the above propositions (1) and (6)).

(10) If $d(\overline{\alpha}, m) \cap H(\overline{\beta}, i) = \emptyset$, where $\overline{\beta} \in \Lambda_{k_1+1}$, $k_1 < k$ and $0 \leq i \leq n(\overline{\beta})$, then $U(\overline{\alpha}, m) \cap H(\overline{\beta}, i) = \emptyset$. (See the definitions of the sets $d(\overline{\alpha}, m)$ and $H(\overline{\alpha}, r)$, properties (16), (17) and (20) of Lemma 2.II and the above propositions (1) and (6)).

(11) $U(\overline{\alpha}, m) = H(\overline{\alpha}, n(\overline{\alpha}, m))$. (See property (22) of Lemma 2.II and the definition of the set $H(\overline{\alpha}, r)$).

(12) $U(\overline{\alpha}, m_1) \cap U(\overline{\alpha}, m_2) = \emptyset$, where $0 \leq m_1, \ m_2 \leq k$ and $m_1 \neq m_2$. (See properties (21) and (22) of Lemma 2.II).

(13) If $k + 1 \geq n(\Re)$, $\zeta \in \Re(\overline{\alpha})$, $r \in N$, $0 \leq r \leq n(\overline{\alpha})$, $d \in U_r^{D(\zeta)}$ and $d \times \{\zeta\} \in T(\Re) \setminus T(\Re)(0)$, then $d \times \{\zeta\} \subseteq H(\overline{\alpha}, r)$. (See the definition of the set $H(\overline{\alpha}, r)$).

(14) The union of all elements of $I\!B(T(\Re))$ is the set $T(\Re)$.

(15) The set $I\!B(T(\Re))$ is countable.

3. Lemma. *Let $d = d(\overline{\alpha}, k) \in T(\Re)(0)$, where $k \in N$, $\overline{\alpha} \in \Lambda_{k+1}$, and $W \equiv V(\overline{\alpha}_1, r_1) \in \mathcal{V}$, where $\overline{\alpha}_1 \in \Lambda_{k_1+1}$, $k_1 \in N$, $r_1 \in N$ and $k_1 + r_1 + 1 \geq n(\Re)$. The following properties are true:*

(1) *If $d \subseteq W$, then there exists an integer $r \in N$ such that $V(\overline{\alpha}, r) \subseteq W$.*

(2) *If $d \cap W = \emptyset$, then there exists an integer $r \in N$ such that $V(\overline{\alpha}, r) \cap W = \emptyset$.*

Proof. (1). Let $d \subseteq W$. Since $d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}_1, r_1)$, by properties (15) and (22) of Lemma 2.II and the definition of the sets $V(\overline{\alpha}, r)$, we have $\Re(\overline{\alpha}) \subseteq \Re(\overline{\alpha}_1)$. If $\overline{\alpha} \leq \overline{\alpha}_1$ and $\overline{\alpha} \neq \overline{\alpha}_1$, then by property (10) of Lemma 2.II, the set $\Re(\overline{\alpha}_1)$ is a singleton. In this case the lemma is easily proved.

Hence we can suppose that $\overline{\alpha}_1 \leq \overline{\alpha}$ and therefore $k_1 \leq k$. If $k_1 = k$, then $\overline{\alpha}_1 = \overline{\alpha}$ and setting $r = r_1$ we have $d \subseteq V(\overline{\alpha}, r) = V(\overline{\alpha}_1, r_1) = W$. Let $\overline{\alpha}_1 \leq \overline{\alpha}$, $\overline{\alpha}_1 \neq \overline{\alpha}$. Then $k_1 < k$. If $n(\Re) \leq k_1 + r_1 + 1 < k$, then $d = d(\overline{\alpha}, k) \subseteq U(\overline{\gamma}, k_1) \subseteq V(\overline{\alpha}_1, r_1)$, where $\overline{\gamma} \in \Lambda_{k_1+r_1+1}$ and $\overline{\gamma} \leq \overline{\alpha}$. Hence $U(\overline{\alpha}, k) \subseteq U(\overline{\gamma}, k_1)$. (See Remarks 2 (9),(11)). Setting $r = 0$ we have $U(\overline{\alpha}, k) = V(\overline{\alpha}, 0) \subseteq U(\overline{\gamma}, k_1) \subseteq V(\overline{\alpha}_1, r_1)$.

Now, suppose that $k \leq k_1 + r_1 + 1$. Let $r = k_1 + r_1 + 1 - k \in N$. We prove that $V(\overline{\alpha}, r) \subseteq V(\overline{\alpha}_1, r_1)$. For this it sufficient to prove that if $\overline{\gamma} \in \Lambda_{k+r+1}$, $\overline{\gamma} \geq \overline{\alpha}$, then $U(\overline{\gamma}, k) \subseteq V(\overline{\alpha}_1, r_1)$. Let $\overline{\gamma} \in \Lambda_{k+r+1}$, $\overline{\gamma} \geq \overline{\alpha}$. There exists an element $\overline{\gamma}_1 \in \Lambda_{k_1+r_1+1}$ such that $\overline{\gamma} \geq \overline{\gamma}_1 \geq \overline{\alpha}$. Since $d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}_1, r_1)$ we have $d(\overline{\gamma}, k) \subseteq U(\overline{\gamma}_1, k_1)$. On the other hand, since $k + r + 1 = (k_1 + r_1 + 1) + 1$, by Remarks 2 (9), we have $U(\overline{\gamma}, k) \subseteq U(\overline{\gamma}_1, k_1) \subseteq V(\overline{\alpha}_1, r_1)$.

(2). Let $d \cap W = \emptyset$. Suppose that $\Re(\overline{\alpha}) \cap \Re(\overline{\alpha}_1) = \emptyset$. Setting $r = n(\Re)$ we have $V(\overline{\alpha}, r) \cap V(\overline{\alpha}_1, r_1) = \emptyset$. Suppose that $\Re(\overline{\alpha}_1) \cap \Re(\overline{\alpha}) \neq \emptyset$. Let $\overline{\alpha} \leq \overline{\alpha}_1$, $\overline{\alpha} \neq \overline{\alpha}_1$. Then $k < k_1$ and $\Re(\overline{\alpha}_1) \subseteq \Re(\overline{\alpha})$. For every $\overline{\gamma} \in \Lambda_{k_1+r_1+1}$, $\overline{\gamma} \geq \overline{\alpha}_1 \geq \overline{\alpha}$, by Remarks 2 (12), we have $U(\overline{\gamma}, k_1) \cap U(\overline{\gamma}, k) = \emptyset$. From this and by the definition of the elements of the set $\mathcal{V}$ we have $V(\overline{\alpha}, r) \cap V(\overline{\alpha}_1, r_1) = \emptyset$, where $r = k_1 + r_1 - k$.

Now, let $\overline{\alpha}_1 \leq \overline{\alpha}$. Then $k_1 \leq k$. Let $n(\Re) \leq k_1 + r_1 + 1 \leq k$. Since $d(\overline{\alpha}, k) \cap V(\overline{\alpha}_1, r_1) = \emptyset$ we have $d(\overline{\alpha}, k) \cap U(\overline{\gamma}, k_1) = \emptyset$, where $\overline{\gamma} \in \Lambda_{k_1+r_1+1}$ and $\overline{\gamma} \leq \overline{\alpha}$. Hence $U(\overline{\alpha}, k) \cap U(\overline{\gamma}, k_1) = \emptyset$. (See Remarks 2 (10), (11)). Setting $r = 0$ we have $V(\overline{\alpha}, 0) \cap V(\overline{\alpha}_1, r_1) = U(\overline{\alpha}, k) \cap U(\overline{\gamma}, k_1) = \emptyset$.

Let $k < k_1 + r_1 + 1$. We set $r = k_1 + r_1 + 1 - k \in N$ and prove that $V(\overline{\alpha}, r) \cap V(\overline{\alpha}_1, r_1) = \emptyset$. For this it is sufficient to prove that if $\overline{\gamma} \in \Lambda_{k+r+1}$, then $U(\overline{\gamma}, k) \cap V(\overline{\alpha}_1, r_1) = \emptyset$. Let $\overline{\gamma} \in \Lambda_{k+r+1}$, $\overline{\gamma} \geq \overline{\alpha}$. There exists an element $\overline{\gamma}_1 \in \Lambda_{k_1+r_1+1}$ such that $\overline{\gamma} \geq \overline{\gamma}_1 \geq \overline{\alpha}$. Since $d(\overline{\alpha}, k) \cap V(\overline{\alpha}_1, r_1) = \emptyset$ we have $d(\overline{\gamma}, k) \cap U(\overline{\gamma}_1, k_1) = \emptyset$. On the other hand, since $k+r+1 = (k_1+r_1+1)+1$, we have $U(\overline{\gamma}, k) \cap U(\overline{\gamma}_1, k_1) = \emptyset$. (See Remarks 2 (10), (11)). Hence $U(\overline{\gamma}, k) \cap V(\overline{\gamma}_1, r_1) = \emptyset$.

4. Lemma. *Let* $d = d(\overline{\alpha}, k) \in T(\Re)(0)$, *where* $k \in N$, $\overline{\alpha} \in \Lambda_{k+1}$, *and* $W = H(\overline{\alpha}_1, r_1) \in \mathcal{U}$, *where* $\overline{\alpha}_1 \in \Lambda_{k_1+1}$, $k_1 + 1 \geq n(\Re)$ *and* $0 \leq r_1 \leq n(\overline{\alpha}_1)$. *The following properties are true:*

(1) *If* $d \subseteq W$, *then there exists an integer* $r \in N$ *such that* $V(\overline{\alpha}, r) \subseteq W$.

(2) *If* $d \cap W = \emptyset$, *then there exists an integer* $r \in N$ *such that* $V(\overline{\alpha}, r) \cap W = \emptyset$.

Proof. (1). Let $d \subseteq W$. Since $d(\overline{\alpha}, k) \subseteq H(\overline{\alpha}_1, r_1)$, by property (15) of Lemma 2.II and the definition of the sets $H(\overline{\alpha}, r)$, we have $\Re(\overline{\alpha}) \subseteq \Re(\overline{\alpha}_1)$.

If $\overline{\alpha} \leq \overline{\alpha}_1$ and $\overline{\alpha} \neq \overline{\alpha}_1$, then, $\Re(\overline{\alpha}_1)$ is a singleton. In this case the lemma is easily proved.

Let $\overline{\alpha} = \overline{\alpha}_1$. Then $k = k_1$ and $\Re(\overline{\alpha}) = \Re(\overline{\alpha}_1)$. For every $\overline{\gamma} \in \Lambda_{k_1+2}$, $\gamma \geq \overline{\alpha}_1$, we have $d(\overline{\gamma}, k) \subseteq d(\overline{\alpha}, k)$ (see Remarks 2 (1)), $d(\overline{\gamma}, k) \subseteq U(\overline{\gamma}, k)$ (see Remarks 2 (4)) and $U(\overline{\gamma}, k) \subseteq H(\overline{\alpha}_1, r_1)$ (see Remarks 2 (9)). Setting $r = 1$ we have

$$V(\overline{\alpha}, r) = \bigcup\{U(\overline{\gamma}, k) : \overline{\gamma} \in L_{k_1+r+1}, \overline{\gamma} \geq \overline{\alpha}_1\} \subseteq H(\overline{\alpha}_1, r_1).$$

Suppose that $\overline{\alpha}_1 \leq \overline{\alpha}$, $\overline{\alpha}_1 \neq \overline{\alpha}$. Then $k_1 < k$. Let r be an integer of N such that $k + r + 1 \geq n(\Re)$. Then $d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}, r) \subseteq U(\overline{\alpha}, k) \subseteq H(\overline{\alpha}_1, r_1)$. (See Remarks 2 (5), (7), (9)).

(2). Let $d \cap W = \emptyset$. Suppose that $\Re(\overline{\alpha}) \cap \Re(\overline{\alpha}_1) = \emptyset$. Setting $r = n(\Re)$ we have $V(\overline{\alpha}, r) \cap H(\overline{\alpha}_1, r_1) = \emptyset$. Suppose that $\Re(\overline{\alpha}) \cap \Re(\overline{\alpha}_1) \neq \emptyset$. Let $\overline{\alpha} \leq \overline{\alpha}_1$. Then $k \leq k_1$ and $\Re(\overline{\alpha}_1) \subseteq \Re(\overline{\alpha})$. For every $\overline{\gamma} \in \Lambda_{(k_1+1)+1}$, $\overline{\gamma} \geq \overline{\alpha}_1 \geq \overline{\alpha}$, we have $d(\overline{\gamma}, k) \subseteq d(\overline{\alpha}, k)$ (see Remarks 2 (1)) and hence $d(\overline{\gamma}, k) \cap H(\overline{\alpha}_1, r_1) = \emptyset$. By Remarks 2 (10) we have $U(\overline{\gamma}, k) \cap H(\overline{\alpha}_1, r_1) = \emptyset$. If $\overline{\gamma} \in \Lambda_{(k_1+1)+1}$, $\overline{\gamma} \geq \overline{\alpha}$ and $\overline{\gamma} \not\geq \overline{\alpha}_1$, then $\Re(\overline{\gamma}) \cap \Re(\overline{\alpha}_1) = \emptyset$ and hence $U(\overline{\gamma}, k) \cap H(\overline{\alpha}_1, r_1) = \emptyset$. Thus, $V(\overline{\alpha}, r) \cap H(\overline{\alpha}_1, r_1) = \emptyset$. Let $\overline{\alpha}_1 \leq \overline{\alpha}$ and $\overline{\alpha}_1 \neq \overline{\alpha}$. Then $k_1 < k$. Setting $r = 0$ we have $U(\overline{\alpha}, k) = V(\overline{\alpha}, 0)$ and $V(\overline{\alpha}, 0) \cap H(\overline{\alpha}_1, r_1) = \emptyset$. (See Remarks 2 (10)).

5. Lemma. *The set $\mathbb{B}(T(\Re))$ is a basis for the open sets of a topology on $T(\Re)$.*

Proof. It is sufficient to prove that: (α) for every $d \in T(\Re)$ there exists $W \in \mathcal{U} \cup \mathcal{V}$ such that $d \in O(W)$ and (β) if $W_1, W_2 \in \mathcal{U} \cup \mathcal{V}$ and $d \in O(W_1) \cap O(W_2)$, then there exists $W \in \mathcal{U} \cup \mathcal{V}$ such that $d \in O(W) \subseteq O(W_1) \cap O(W_2)$.

Property (α) follows by Remarks 2 (14). We prove property (β). Suppose that $d = d(\overline{\alpha}, k)$, where $\overline{\alpha} \in \Lambda_{k+1}$. By Lemma 3 (1) and Lemma 4 (1) it follows that there exist integers $r_1, r_2 \in N$ such that $k + r_1 + 1 \geq n(\Re)$, $k + r_2 + 1 \geq n(\Re)$, $d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}, r_1) \subseteq W_1$ and $d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}, r_2) \subseteq W_2$. Let $r = \max\{r_1, r_2\}$. Then by Remarks 2 (8) we have

$$d(\overline{\alpha}, k) \subseteq V(\overline{\alpha}, r) \subseteq V(\overline{\alpha}, r_1) \cap V(\overline{\alpha}, r_2) \subseteq W_1 \cap W_2.$$

Hence $d \in O(V(\overline{\alpha}, r)) \subseteq O(W_1) \cap O(W_2)$.

Now, suppose that $d = d' \times \{\zeta\} \in T(\Re) \setminus T(\Re)(0)$. If $W_1 = V(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k \in N$, $r \in N$ and $k + r + 1 \geq n(\Re)$, then by $\overline{\gamma}_1$ we denote the element of Λ_{k+r+1} for which $\zeta \in \Re(\overline{\gamma}_1)$. Setting $r_1 = n(\overline{\gamma}_1, k)$ we have $d' \times \{\zeta\} \subseteq J(U^C_{r_1} \times \Re(\overline{\gamma}_1)) \subseteq W_1$. If $W_1 = H(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k \in N$, $r \in N$, $0 \leq r \leq n(\overline{\alpha})$ and $k + 1 \geq n(\Re)$, then by $\overline{\gamma}_1$ we denote the element $\overline{\alpha}$ and by r_1 we denote the integer r. Hence $d' \times \{\zeta\} \subseteq J(U^C_{r_1} \times \Re(\overline{\gamma}_1)) \subseteq W_1$.

Similarly, there exists an element $\overline{\gamma}_2 \in \Lambda$ and an integer $r_2 \in N$ such that

$$d' \times \{\zeta\} \subseteq J(U^C_{r_2} \times \Re(\overline{\gamma}_2)) \subseteq W_2.$$

Let $r_0 \in N$ such that $d' \in U^{D(\zeta)}_{r_0} \subseteq U^{D(\zeta)}_{r_1} \cap U^{D(\zeta)}_{r_2}$. Let $k_0 \in N$ and $\overline{\gamma}_0 \in \Lambda_{k_0+1}$ such that $\zeta \in \Re(\overline{\gamma}_0)$, $k_0 + 1 \geq n(\Re)$, $0 \leq r_0 \leq n(\overline{\gamma}_0)$, $\overline{\gamma}_0 \geq \overline{\gamma}_1$ and $\overline{\gamma}_0 \geq \overline{\gamma}_2$. Then

$$d' \times \{\zeta\} \subseteq H(\overline{\gamma}_0, r_0) \subseteq J(U^C_{r_1} \times \Re(\overline{\gamma}_1)) \cap J(U^C_{r_2} \times \Re(\overline{\gamma}_2)) \subseteq W_1 \cap W_2.$$

Thus, $d \in O(H(\overline{\gamma}_0, r_0)) \subseteq O(W_1) \cap O(W_2)$.

6. Remark. In what follows, $T(\Re)$ denotes the topological space for which $\mathbb{B}(T(\Re))$ is a basis for the open sets.

7. Corollary. *If $d = d(\overline{\alpha}, k) \in T(\Re)(0)$, $\overline{\alpha} \in \Lambda_{k+1}$, then the set*

$$\mathbb{B}(d) \equiv \{O(V(\overline{\alpha}, r)) : r \in N \text{ and } k + r + 1 \geq n(\Re)\}$$

is a basis for open neighbourhoods of $d(\overline{\alpha}, k)$ in $T(\Re)$. If $d = d' \times \{\zeta\} \in T(\Re) \setminus T(\Re)(0)$, then the set

$$\mathbb{B}(d) \equiv \{O(H(\overline{\alpha}, r)) : \overline{\alpha} \in \Lambda_{k+1},\ k + 1 \geq n(\Re),\ \zeta \in \Re(\overline{\alpha}),\ d' \in U^{D(\zeta)}_r,\ 0 \leq r \leq n(\overline{\alpha})\}$$

is a basis for open neighbourhoods of $d' \times \{\zeta\}$ in $T(\Re)$.

Proof. The proof of this corollary follows immediately from the proof of Lemma 5.

8. Lemma. *The space $T(\Re)$ is Hausdorff.*

Proof. Let $d_1, d_2 \in T(\Re)$, $d_1 \neq d_2$. We shall prove that there exists $O_1 \in \mathbb{B}(d_1)$ and $O_2 \in \mathbb{B}(d_2)$ such that $O_1 \cap O_2 = \emptyset$. We consider the following cases: (α) $d_1 = d(\overline{\alpha}_1, k_1)$, $d_2 = d(\overline{\alpha}_2, k_2)$, where $\overline{\alpha} \in \Lambda_{k_1+1}$ and $\overline{\alpha}_2 \in \Lambda_{k_2+1}$, (β) $d_1 = d \times \{\zeta\} \in T(\Re) \setminus T(\Re)(0)$, $d_2 = d(\overline{\alpha}, k)$, where $\overline{\alpha} \in \Lambda_{k+1}$, and (γ) $d_1 = d'_1 \times \{\zeta_1\} \in T(\Re) \setminus T(\Re)(0)$ and $d_2 = d'_2 \times \{\zeta_2\} \in T(\Re) \setminus T(\Re)(0)$.

Consider the first case. Without loss of generality we can suppose that $k_1 \geq k_2$. If $\overline{\alpha}_1 \not\geq \overline{\alpha}_2$, then for every $O_1 \in \mathbb{B}(d_1)$ and $O_2 \in \mathbb{B}(d_2)$ we have $O_1 \cap O_2 = \emptyset$. Let $\overline{\alpha}_1 \geq \overline{\alpha}_2$. Since $d_1 \neq d_2$ we have $\overline{\alpha}_1 \neq \overline{\alpha}_2$ and hence $k_1 > k_2$. Let $r_1, r_2 \in N$ such that $k_1 + r_1 + 1 = k_2 + r_2 + 1 \geq n(\Re)$. We prove that $V(\overline{\alpha}_1, r_1) \cap V(\overline{\alpha}_2, r_2) = \emptyset$. Indeed, let $\overline{\gamma} \in \Lambda_{k_1+r_1+1}$ and $\overline{\gamma} \geq \overline{\alpha}_1$. It is sufficient to prove that $U(\overline{\gamma}, k_1) \cap U(\overline{\gamma}, k_2) = \emptyset$. But this follows by Remarks 2 (12).

Now, we condider the second case. Let $\zeta \notin \Re(\overline{\alpha})$ and let $r_1 \in N$ such that $d \in U_{r_1}^{D(\zeta)}$. There exist an integer $k_1 \in N$ and an element $\overline{\alpha}_1 \in \Lambda_{k_1+1}$ such that $\zeta \in \Re(\overline{\alpha}_1)$, $0 \leq r_1 \leq n(\overline{\alpha}_1)$, $k_1 > k$ and $k_1 + 1 \geq n(\Re)$. If $O_1 = O(H(\overline{\alpha}_1, r_1))$ and $O_2 \in \mathbb{B}(d_2)$, then we have $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$. Let $\zeta \in \Re(\overline{\alpha})$. Then $d \cap d_k^{D(\zeta)} = \emptyset$. Since $D(\zeta)$ is a Hausdorff space, there exist integers $r_1, i \in N$ such that $d \in U_{r_1}^{D(\zeta)}$, $d_k^{D(\zeta)} \in U_i^{D(\zeta)}$ and $U_{r_1}^{D(\zeta)} \cap U_i^{D(\zeta)} = \emptyset$. Let $k_1 \in N$, $k_1 + 1 \geq n(\Re)$, $k_1 > \max\{k, i, r_1\}$ and let $\overline{\gamma}_1 \in \Lambda_{k_1}$, $\overline{\gamma} \in \Lambda_{k_1+1}$ such that $\overline{\gamma} \geq \overline{\gamma}_1 \geq \overline{\alpha}$ and $\zeta \in \Re(\overline{\gamma})$. Then $n(\overline{\gamma}_1) \geq k_1$. We prove that $H(\overline{\gamma}, r_1) \cap V(\overline{\alpha}, r) = \emptyset$, where $r = k_1 - k$. It is sufficient to prove that $H(\overline{\gamma}, r_1) \cap U(\overline{\gamma}, k) = \emptyset$.

By property (13) of Lemma 2.II we have $U_{r_1}^{D(\chi)} \cap U_i^{D(\chi)} = \emptyset$ for every $\chi \in \Re(\overline{\gamma})$. This means that $H(\overline{\gamma}, r_1) \cap H(\overline{\gamma}, i) = \emptyset$. By property (17) of Lemma 2.II we have $d_k^{D(\chi)} \in U_i^{D(\chi)}$ for every $\chi \in \Re(\overline{\gamma})$. By property (19) of Lemma 2.II, for every $\chi \in \Re(\overline{\gamma})$, we have $U_{n(\overline{\gamma},k)}^{D(\chi)} \subseteq U_i^{D(\chi)}$. This means that $U(\overline{\gamma}, k) \subseteq H(\overline{\gamma}, i)$. Hence $H(\overline{\gamma}, r_1) \cap U(\overline{\gamma}, k) = \emptyset$. Setting $O_1 = O(H(\overline{\gamma}, r_1))$ and $O_2 = O(V(\overline{\alpha}, r))$ we have $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.

Finally, we consider the third case. If $\zeta_1 \neq \zeta_2$, then there exist integers $k, r_1, r_2 \in N$ and elements $\overline{\alpha}_1, \overline{\alpha}_2 \in \Lambda_{k+1}$ such that $k + 1 \geq \max\{n(\Re), r_1, r_2\}$, $\overline{\alpha}_1 \neq \overline{\alpha}_2$, $\zeta_1 \in \Re(\overline{\alpha}_1)$, $\zeta_2 \in \Re(\overline{\alpha}_2)$, $d'_1 \in U_{r_1}^{D(\zeta_1)}$, $d'_2 \in U_{r_2}^{D(\zeta_2)}$. Then we have $r_1 \leq n(\overline{\alpha}_1)$, $r_2 \leq n(\overline{\alpha}_2)$, $d_1 \subseteq H(\overline{\alpha}_1, r_1)$, $d_2 \subseteq H(\overline{\alpha}_2, r_2)$ and $H(\overline{\alpha}_1, r_1) \cap H(\overline{\alpha}_2, r_2) = \emptyset$.

Setting $O_1 = O(H(\overline{\alpha}_1, r_1))$, $O_2 = O(H(\overline{\alpha}_2, r_2))$ we have $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.

Now, let $\zeta_1 = \zeta_2 = \zeta$. Then $d'_1 \neq d'_2$. Since the space $D(\zeta)$ is Hausdorff, there exist $r_1, r_2 \in N$ such that $d'_1 \in U_{r_1}^{D(\zeta)}$, $d'_2 \in U_{r_2}^{D(\zeta)}$ and $U_{r_1}^{D(\zeta)} \cap U_{r_2}^{D(\zeta)} = \emptyset$. Let $k \in N$, $k+1 \geq \max\{n(\Re), r_1, r_2\}$ and let $\overline{\gamma} \in \Lambda_{k+1}$ and $\zeta \in \Re(\overline{\gamma})$. Then $n(\overline{\gamma}) \geq \max\{r_1, r_2\}$. By property (13) of Lemma 2.II, we have $U_{r_1}^{D(\chi)} \cap U_{r_2}^{D(\chi)} = \emptyset$ for every $\chi \in \Re(\overline{\gamma})$. This means that $H(\overline{\gamma}, r_1) \cap H(\overline{\gamma}, r_2) = \emptyset$. Setting $O_1 = O(H(\overline{\gamma}, r_1))$ and $O_2 = O(H(\overline{\gamma}, r_2))$ we have $d_1 \in O_1$, $d_2 \in O_2$ and $O_1 \cap O_2 = \emptyset$.

9. Lemma. *Let $W \in \mathcal{U} \cup \mathcal{V}$. For every point d of the boundary* $\mathrm{Bd}(O(W))$ *of the set $O(W)$ in $T(\Re)$, we have $d \cap W \neq \emptyset$ and $d \cap (J(C \times \Re) \setminus W) \neq \emptyset$, that is,* $\mathrm{Bd}(O(W)) \subseteq \mathrm{Fr}(W)$.

Proof. Let $d \in \mathrm{Bd}(O(W))$. If $d \in T(\Re)(0)$, then by Lemmas 3 and 4 we have $d \not\subseteq W$ and $d \cap W \neq \emptyset$ and hence $d \cap (T(\Re) \setminus W) \neq \emptyset$. Let $d \in T(\Re) \setminus T(\Re)(0)$, that is, $d = d' \times \{\zeta\}$. Since $d \not\subseteq W$ it is sufficient to prove that $d \cap W \neq \emptyset$. Let $W = H(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k+1 \geq n(\Re)$ and $0 \leq r \leq n(\overline{\alpha})$. We prove that $d' \in \mathrm{Cl}(U_r^{D(\zeta)})$. Indeed, in the opposite case, there exists an integer $i \in N$ such that $U_r^{D(\zeta)} \cap U_i^{D(\zeta)} = \emptyset$ and $d' \in U_i^{D(\zeta)}$. Let $k_1 \in N$ and $k_1 \geq \max\{k, i, r\}$. Let $\overline{\gamma} \in \Lambda_{k_1+1}$ and $\zeta \in \Re(\overline{\gamma})$. Then $n(\overline{\gamma}) \geq k_1$. We prove that $O(H(\overline{\gamma}, i)) \cap O(H(\overline{\gamma}, r)) = \emptyset$.

Indeed, in the opposite case, let $d_1 \in O(H(\overline{\gamma}, i)) \cap O(H(\overline{\gamma}, r))$. There exists $\zeta' \in \Re(\overline{\gamma})$ such that $d_1 \cap (C \times \{\zeta'\}) = d'_1 \in D(\zeta')$. Then $d'_1 \in U_i^{D(\zeta')} \cap U_r^{D(\zeta')} \neq \emptyset$. By property (13) of Lemma 2.II, this is a contradiction, because $\zeta, \zeta' \in \Re(\overline{\gamma})$ and $U_r^{D(\zeta)} \cap U_i^{D(\zeta)} = \emptyset$. Hence, $d' \in \mathrm{Cl}(U_r^{D(\zeta)})$.

On the other hand, $\zeta \in \Re(\overline{\alpha})$. Indeed, if $\zeta \notin \Re(\overline{\alpha})$, then there exist integers $i, k_1 \in N$ and an element $\overline{\gamma} \in \Lambda_{k_1+1}$ such that $d' \in U_i^{D(\zeta)}$, $\zeta \in \Re(\overline{\gamma})$, $k_1 + 1 \geq n(\Re)$, $k_1 \geq i$ and $\Re(\overline{\gamma}) \cap \Re(\overline{\alpha}) = \emptyset$. Then $d \in O(H(\overline{\gamma}, i))$ and $H(\overline{\gamma}, i) \cap W = \emptyset$, that is, $d \notin \mathrm{Bd}(O(W))$, which is contradiction. Hence $\zeta \in \Re(\overline{\alpha})$.

Now, we prove that $d \cap W \neq \emptyset$. Since $W \cap (C \times \{\zeta\}) = U_r^{S(\zeta)} \times \{\zeta\}$, it is sufficient to prove that $d' \cap U_r^{S(\zeta)} \neq \emptyset$. Indeed, in the opposite case, $d' \notin \overline{U}_r^{D(\zeta)}$ and since $\mathrm{Cl}(U_r^{D(\zeta)}) \subseteq \overline{U}_r^{D(\zeta)}$ we have $d' \notin \mathrm{Cl}(U_r^{D(\zeta)})$. But this is impossible. Let $W = V(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k + r + 1 \geq n(\Re)$. Let $\overline{\gamma} \in \Lambda_{k+r+1}$ and $\zeta \in \Re(\overline{\gamma})$. Then $U(\overline{\gamma}, k) \subseteq V(\overline{\alpha}, r)$ and $U(\overline{\gamma}, k) = H(\overline{\gamma}, n(\overline{\gamma}, k)) = W_1 \in \mathcal{U}$. We prove that $d \in \mathrm{Bd}(O(W_1))$. Indeed, it is sufficient to prove that if $\overline{\gamma}_1 \in \Lambda_{k_1+1}$, where $k_1 \geq k + r$, $\zeta \in \Re(\overline{\gamma})$, $r_1 \in N$, $0 \leq r_1 \leq n(\overline{\gamma}_1)$ and $d \in O(H(\overline{\gamma}_1, r_1))$, then $O(H(\overline{\gamma}_1, r_1)) \cap O(W_1) \neq \emptyset$. This follows by the relations: $O(H(\overline{\gamma}_1, r_1)) \cap O(W) \neq \emptyset$, $W \cap (C \times \Re(\overline{\gamma}_1)) = W_1$ and $H(\overline{\gamma}_1, r_1) \subseteq C \times \Re(\overline{\gamma})$. Hence $d \cap W_1 \neq \emptyset$ and therefore

$d \cap W \neq \emptyset$.

10. Theorem. *The space $T(\Re)$ is separable metrizable.*

Proof. By Lemma 5, Lemma 8 and Remarks 2 (15) it is sufficient to prove that the space $T(\Re)$ is regular. Let $d \in O(W)$, where $W \in \mathcal{U} \cup \mathcal{V}$. We prove that there exists an element $W_1 \in \mathcal{U} \cup \mathcal{V}$ such that $d \in O(W_1) \subseteq \mathrm{Cl}(O(W_1)) \subseteq O(W)$.

Let $d = d(\overline{\alpha}, k) \in T(\Re)(0)$. Without loss of generality, we can suppose that $W = V(\overline{\alpha}, r) \in \mathcal{V}$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k + r + 1 \geq n(\Re)$. (See Corollary 7). We prove that the set $W_1 = V(\overline{\alpha}, r+1)$ is the required element of $\mathcal{U} \cup \mathcal{V}$. By Lemma 9 and Remarks 2 (8), it is sufficient to prove that if $d_1 \in T(\Re)$ and $d_1 \cap V(\overline{\alpha}, r+1) \neq \emptyset$, then $d_1 \subseteq W$.

Let d_1 has the above property. First we suppose that $d_1 = d_1' \times \{\zeta\}$. Let $\overline{\beta} \in \Lambda_{k+r+1}$, $\overline{\gamma} \in \Lambda_{k+r+2}$, $\overline{\beta} \leq \overline{\gamma}$ and $\zeta \in \Re(\overline{\gamma})$. Obviously, $U(\overline{\beta}, k) \subseteq V(\overline{\alpha}, r)$ and $U(\overline{\gamma}, k) \subseteq V(\overline{\alpha}, r+1)$. Also, $U(\overline{\beta}, k) \cap (C \times \{\zeta\}) = U^{S(\zeta)}_{n(\overline{\beta}, k)} \times \{\zeta\}$ and $U(\overline{\gamma}, k) \cap (C \times \{\zeta\}) = U^{S(\zeta)}_{n(\overline{\gamma}, k)} \times \{\zeta\}$. Since $d_1 \cap V(\overline{\alpha}, r+1) \neq \emptyset$, we have $d_1' \cap U^{S(\zeta)}_{n(\overline{\gamma}, k)} \neq \emptyset$, that is, $d_1' \in \overline{U}^{D(\zeta)}_{n(\overline{\gamma}, k)}$. By property (23) of Lemma 2.II we have $d_1' \in U^{D(\zeta)}_{n(\overline{\beta}, k)}$, that is, $d_1' \subseteq U^{S(\zeta)}_{n(\overline{\beta}, k)}$. Hence $d_1' \times \{\zeta\} \subseteq U(\overline{\beta}, k) \subseteq V(\overline{\alpha}, r) = W$, that is, $d_1 \subseteq W$.

Let $d_1 \in T(\Re)(0)$. Then $d_1 = d(\overline{\alpha}_1, k_1)$, where $\overline{\alpha}_1 \in \Lambda_{k_1+1}$. If $k_1 \leq k + r + 1$, then for every $\overline{\gamma} \in \Lambda_{(k+r+1)+1}$ we have $U(\overline{\gamma}, k) \cap U(\overline{\gamma}, k_1) = \emptyset$. (See Remarks 2 (12)). This means that $d_1 \cap V(\overline{\alpha}, r+1) = \emptyset$, which is a contradiction. Hence we can suppose that $k_1 > k + r + 1$. Let $\overline{\gamma} \in \Lambda_{k+r+2}$, $\overline{\beta} \in \Lambda_{k+r+1}$ such that $\overline{\alpha}_1 \geq \overline{\gamma} \geq \overline{\beta}$. Since $d_1 \cap V(\overline{\alpha}, r+1) \neq \emptyset$, there exists an element $\zeta \in \Re(\overline{\alpha}_1)$ such that $d^{D(\zeta)}_{k_1} \cap U^{S(\zeta)}_{n(\overline{\gamma}, k)} \neq \emptyset$, that is, $d^{D(\zeta)}_{k_1} \in \overline{U}^{D(\zeta)}_{n(\overline{\gamma}, k)}$. By property (23) of Lemma 2.II, we have $\overline{U}^{D(\zeta)}_{n(\overline{\gamma}, k)} \subseteq U^{D(\zeta)}_{n(\overline{\beta}, k)}$, that is, $d^{D(\zeta)}_{k_1} \in U^{D(\zeta)}_{n(\overline{\beta}, k)}$. By property (17) of Lemma 2.II, for every $\chi \in \Re(\overline{\alpha}_1)$, we have $d^{D(\chi)}_{k_1} \in U^{D(\chi)}_{n(\overline{\beta}, k)}$, that is, $d^{D(\chi)}_{k_1} \subseteq U^{S(\chi)}_{n(\overline{\beta}, k)}$. Thus, for every $\chi \in \Re(\overline{\alpha}_1)$, we have $d^{D(\chi)}_{k_1} \times \{\chi\} \subseteq U(\overline{\beta}, k) \subseteq V(\overline{\alpha}, r) = W$. Hence $d_1 \subseteq W$.

Now, let $d = d' \times \{\zeta\} \in T(\Re) \setminus T(\Re)(0)$. Without loss of generality, we can suppose that $W = H(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k + 1 \geq n(\Re)$, $0 \leq r \leq n(\overline{\alpha})$, $\zeta \in \Re(\overline{\alpha})$ and $d' \in U^{D(\zeta)}_r$. There exists an integer $r_1 \in N$ such that $d' \in U^{D(\zeta)}_{r_1} \subseteq \overline{U}^{D(\zeta)}_{r_1} \subseteq U^{D(\zeta)}_r$ and $d^{D(\zeta)}_m \notin \overline{U}^{D(\zeta)}_{r_1}$ for every m, $0 \leq m \leq k$. Let $k_1 \in N$, $k_1 > k$, $k_1 \geq r_1$, $\overline{\gamma} \in \Lambda_{k_1+1}$, $\overline{\gamma} \geq \overline{\alpha}$ and $\zeta \in \Re(\overline{\gamma})$. We prove that $d \in O(H(\overline{\gamma}, r_1)) \subseteq \mathrm{Cl}(O(H(\overline{\gamma}, r_1))) \subseteq O(H(\overline{\alpha}, r))$. Since $H(\overline{\gamma}, r_1) \subseteq H(\overline{\alpha}, r)$, by Lemma 9, it is sufficient to prove that if $d_1 \in T(\Re)$ and $d_1 \cap H(\overline{\gamma}, r_1) \neq \emptyset$, then $d_1 \subseteq H(\overline{\alpha}, r)$.

Let d_1 has the above property. Suppose that $d_1 = d_1' \times \{\chi\} \in T(\Re) \setminus T(\Re)(0)$.

Since $d_1 \cap H(\overline{\gamma}, r_1) \neq \emptyset$, we have $\chi \in \Re(\overline{\gamma})$ and $d'_1 \cap U_{r_1}^{S(\chi)} \neq \emptyset$, that is, $d'_1 \in \overline{U}_{r_1}^{D(\chi)}$. Since $\overline{U}_{r_1}^{D(\zeta)} \subseteq U_r^{D(\zeta)}$, by property (13) of Lemma 2.II, we have $\overline{U}_{r_1}^{D(\chi)} \subseteq U_r^{D(\chi)}$. This means that $d_1 \subseteq H(\overline{\alpha}, r)$.

Now, suppose that $d_1 = d(\overline{\alpha}_2, k_2) \in T(\Re)(0)$, where $\overline{\alpha}_2 \in \Lambda_{k_2+1}$. Since $d \cap H(\overline{\gamma}, r_1) \neq \emptyset$, there exists an element $\chi' \in \Re(\overline{\gamma}) \cap \Re(\overline{\alpha}_2)$ such that $d_{k_2}^{D(\chi')} \cap U_{r_1}^{S(\chi')} \neq \emptyset$, that is, $d_{k_2}^{D(\chi')} \in \overline{U}_{r_1}^{D(\chi')}$. If $k_2 \leq k$, then $\overline{\alpha}_2 \leq \overline{\gamma}$ and hence $\Re(\overline{\gamma}) \subseteq \Re(\overline{\alpha}_2)$. Since, for every $\chi \in \Re(\overline{\gamma})$, $\overline{U}_{r_1}^{D(\chi)} = U_{r_1}^{D(\chi)} \cup \mathrm{Fr}(U_{r_1}^{D(\chi)})$, by properties (16) and (17) of Lemma 2.II, we have $d_{k_2}^{D(\chi)} \in \overline{U}_{r_1}^{D(\chi)}$ and hence $d_{k_2}^{D(\zeta)} \in \overline{U}_{r_1}^{D(\zeta)}$, which is a contradiction. Hence $k < k_2$, $\overline{\alpha} \leq \overline{\alpha}_2$ and $\Re(\overline{\alpha}_2) \subseteq \Re(\overline{\alpha})$. Since $\overline{U}_{r_1}^{D(\zeta)} \subseteq U_r^{D(\zeta)}$ and $\zeta \in \Re(\overline{\gamma})$, by property (13) of Lemma 2.II, we have $\overline{U}_{r_1}^{D(\chi)} \subseteq U_r^{D(\chi)}$ for every $\chi \in \Re(\overline{\gamma})$. Since $\chi' \in \Re(\overline{\gamma})$ and $d_{k_2}^{D(\chi')} \in \overline{U}_{r_1}^{D(\chi')} \subseteq U_r^{D(\chi')}$, by property (17) of Lemma 2. II, for every $\chi \in \Re(\overline{\alpha}_2)$, we have $d_{k_2}^{D(\chi)} \in U_r^{D(\chi)}$, that is, $d_{k_2}^{D(\chi)} \subseteq U_r^{S(\chi)}$. Hence, $d_{k_2}^{D(\chi)} \times \{\chi\} \subseteq U_r^{S(\chi)} \times \{\chi\} \subseteq H(\overline{\alpha}, r)$. This means that $d_1 \subseteq H(\overline{\alpha}, r)$.

IV. The rationality of $T(\Re)$.

1. Notations. Let X be a space and $\Sigma = \{\sigma_0, \sigma_1, ...\}$ be a basic system for X, where $\sigma_i = \{A_0^i, A_1^i\}$. Let $\widetilde{X}$ be a subspace of X. We set $\widetilde{A}_0^i = A_0^i \cap \widetilde{X}$, $\widetilde{A}_1^i = A_1^i \cap \widetilde{X}$, $\widetilde{\sigma}_i = \{\widetilde{A}_0^i, \widetilde{A}_1^i\}$ and $\widetilde{\Sigma} = \{\widetilde{\sigma}_0, \widetilde{\sigma}_1, ...\}$. It is easy to see that $\widetilde{\Sigma}$ is a basic system for the space $\widetilde{X}$. Therefore we can use the notations $\mathrm{Fr}(\widetilde{\sigma}_i)$, $\mathrm{Fr}(\widetilde{\Sigma})$, $\widetilde{X}_{\overline{i}}$, $\overline{i} \in L$, $S(\widetilde{X}, \widetilde{\Sigma}) \equiv \widetilde{S}$, $D(\widetilde{X}, \widetilde{\Sigma}) \equiv \widetilde{D}$, $q(\widetilde{X}, \widetilde{\Sigma}) \equiv \widetilde{q}$, $p(\widetilde{X}, \widetilde{\Sigma}) \equiv \widetilde{p}$, and $h(\widetilde{X}, \widetilde{\Sigma}) \equiv \widetilde{h}$, which are given in Section I.

If f is a map of a set Y into a set Z and $Q \subseteq Y$, then by $f|_Q$ we denote the restriction of f onto Q.

2. Lemma. *The following properties are true:*

(1) $\widetilde{X}_{\overline{i}} = X_{\overline{i}} \cap \widetilde{X}$, $\overline{i} \in L$.

(2) $\widetilde{S} = q^{-1}(\widetilde{X}) \subseteq S$.

(3) $\widetilde{q} = q|_{\widetilde{S}}$.

(4) $\widetilde{D} = \{q^{-1}(x) : x \in \widetilde{X}\} \subseteq D$.

(5) $\widetilde{p} = p|_{\widetilde{S}}$.

(6) $\widetilde{h} = h|_{\widetilde{D}}$.

This lemma is not dificult to be proved.

3. Notations. Let $\Re$ be a family of representations considered in Section 1.II. Let $\{r^1, ..., r^t\}$ be a fixed subset of N, where $0 \leq t \leq n$, such that $|\{r^1, ..., r^t\}| = t$. Hence, if $t = 0$, then $\{r^1, ..., r^t\} = \emptyset$.

Let $\zeta \equiv (S, D) \in \Re$. According to our assumptions (see Section 1.II), there exists a space $X(\zeta) \in I\!R^n(I\!M)$ and a basic system $\Sigma(\zeta) \equiv \{\sigma_0(\zeta), \sigma_1(\zeta), ...\}$ for $X(\zeta)$ such that (S, D) is the representation of $X(\zeta)$ corresponding to the basic system $\Sigma(\zeta)$. The pair (S, D) is denoted also by $(S(\zeta), D(\zeta))$. We set

$$\widetilde{X}(\zeta) = \bigcap\{\mathrm{Fr}(\sigma_{r^i}(\zeta)) : i = 1, ..., t\} \text{ if } t > 0 \text{ and } \widetilde{X}(\zeta) = X(\zeta) \text{ if } t = 0.$$

Setting $X(\zeta) = X$, $\Sigma(\zeta) = \Sigma$ and $\widetilde{X}(\zeta) = \widetilde{X}$, we can consider the ordered cover $\widetilde{\sigma}_i$ of $\widetilde{X}$, the basic system $\widetilde{\Sigma}$ for $\widetilde{X}$, the subset $\widetilde{S}$ of C, the partition $\widetilde{D}$ of $\widetilde{S}$ and the map $\widetilde{h}$ of $\widetilde{D}$ onto $\widetilde{X}$. In order to show that the above notions depend on ζ, we use the notations $\widetilde{\sigma}_i(\zeta)$, $\widetilde{\Sigma}(\zeta)$, $\widetilde{S}(\zeta)$, $\widetilde{D}(\zeta)$ and $\widetilde{h}_\zeta$ instead of notations $\widetilde{\sigma}_i$, $\widetilde{\Sigma}$, $\widetilde{S}$, $\widetilde{D}$ and $\widehat{h}$, respectively.

The pair $\widetilde{\zeta} \equiv (\widetilde{S}(\zeta), \widetilde{D}(\zeta))$ is a representation of $\widetilde{X}(\zeta)$ corresponding to basic system $\widetilde{\Sigma}(\zeta)$ for $\widetilde{X}(\zeta)$. The family of all representations $\widetilde{\zeta}$ is denoted by $\widetilde{\Re}$. If ζ_1, ζ_2 are distinct elements of $\Re$, then we consider $\widetilde{\zeta}_1$ and $\widetilde{\zeta}_2$ to be distinct elements of $\widetilde{\Re}$. The element ζ of $\Re$ and the element $\widetilde{\zeta}$ of $\widetilde{\Re}$ are considered to correspond to each other. We observe that the cardinality of $\widetilde{\Re}$ is less than or equal to the continuum.

For the family $\widetilde{\Re}$ we use all notations of Section 1.II, that is, if the element $\widetilde{\zeta} \equiv (\widetilde{S}(\zeta), \widetilde{D}(\zeta)) \in \widetilde{\Re}$ corresponds to the element $\zeta \equiv (S(\zeta), D(\zeta)) \in \Re$, then $X(\widetilde{\zeta}) = \widetilde{X}(\zeta)$, $\Sigma(\widetilde{\zeta}) = \widetilde{\Sigma}(\zeta)$, $\sigma_i(\widetilde{\zeta}) = \widetilde{\sigma}_i(\zeta)$, $S(\widetilde{\zeta}) = \widetilde{S}(\zeta)$, $D(\widetilde{\zeta}) = \widetilde{D}(\zeta)$, $h_{\widetilde{\zeta}} = \widetilde{h}_\zeta$, $U_k^{S(\widetilde{\zeta})} = U_k^C \cap \widetilde{S}(\zeta) = U_k^C \cap S(\widetilde{\zeta})$, $U_k^{D(\widetilde{\zeta})}$ is the set of all elements of $D(\widetilde{\zeta})$ containing in the set $U_k^{S(\widetilde{\zeta})}$ and $\overline{U}_k^{D(\widetilde{\zeta})}$ is the set of all elements of $D(\widetilde{\zeta})$ which intersect the set $U_k^{S(\widetilde{\zeta})}$. Also $\mathrm{Fr}(U_k^{D(\widetilde{\zeta})}) = \overline{U}_k^{D(\widetilde{\zeta})} \setminus U_k^{D(\widetilde{\zeta})}$. By Lemma 7.I and Lemma 2 it follows that the ordered set $I\!B(D(\widetilde{\zeta})) = \{U_0^{D(\widetilde{\zeta})}, U_1^{D(\widetilde{\zeta})}, ...\}$ is an ordered basis for open sets of $D(\widetilde{\zeta})$ and that the set $\overline{U}_k^{D(\widetilde{\zeta})}$ is the set of all elements $d \in D(\widetilde{\zeta})$ such that $d \cap (\bigcup\{C_{\bar{i}0} : \bar{i} \in L_k\}) \neq \emptyset$. We observe that: ($\alpha$) $U_k^{S(\widetilde{\zeta})} \subseteq U_k^{S(\zeta)}$, ($\beta$) $U_k^{D(\zeta)} \cap D(\widetilde{\zeta}) = U_k^{D(\widetilde{\zeta})}$ and (γ) $\mathrm{Fr}(U_k^{D(\zeta)}) \cap D(\widetilde{\zeta}) = \mathrm{Fr}(U_k^{D(\widetilde{\zeta})})$.

We denote by $D(\widetilde{\zeta})(0)$ the set of all elements d of $D(\widetilde{\zeta})$ for which there exist mutually distinct integers $j_1, ..., j_n$ of N (that is, $|\{j_1, ..., j_n\}| = n$) such that

$$d \in \bigcap\{\mathrm{Fr}(U_{j_i}^{D(\widetilde{\zeta})}) : i = 1, ..., n\}.$$

We observe that in this case, since $\Sigma(\zeta)$ has the property of boundary intersections, we have $\{r^1, ..., r^t\} \subseteq \{j_1, ..., j_n\}$. From the above it follows that $D(\widetilde{\zeta})(0) = D(\zeta)(0) \cap D(\widetilde{\zeta})$.

We denote by

$$\overrightarrow{D}(\widetilde{\zeta})(0) \equiv \{d_0^{D(\widetilde{\zeta})}, d_1^{D(\widetilde{\zeta})}, \ldots\}$$

an ordered set such that: (α) for every $d \in D(\widetilde{\zeta})(0)$ there exists uniquely determined integer $i \in N$ for which $d = d_i^{D(\widetilde{\zeta})}$), (β) if for some $i \in N$ there is no element $d \in D(\widetilde{\zeta})(0)$ for which $d_i^{D(\zeta)} = d$, then $d_i^{D(\widetilde{\zeta})} = \emptyset$, and (γ) if for some integer $i \in N$, $d_i^{D(\widetilde{\zeta})} \neq \emptyset$, then $d_i^{D(\widetilde{\zeta})} = d_i^{D(\zeta)}$.

We observe that for every $\widetilde{\zeta} \in \widetilde{\Re}$ by the property of boundary intersections of the basic system $\Sigma(\zeta)$, it follows that $X(\widetilde{\zeta}) \in I\!R^{n-t}(I\!M)$.

For every element $\overline{i} \in L$ we denote by $\widetilde{\Re}(\overline{i})$ the set of all elements $\widetilde{\zeta} \in \widetilde{\Re}$ for which $\zeta \in \Re(\overline{i})$. Obviously, subfamilies $\widetilde{\Re}(\overline{i})$ of $\widetilde{\Re}$ have properties (α)-(δ) mentioned for subfamilies $\Re(\overline{i})$ of $\Re$. (See Section 1.II).

For every subset C' of C and for every subfamily $\widetilde{\Re}'$ of $\widetilde{\Re}$ we set

$$J(C' \times \widetilde{\Re}') = \{(a, \widetilde{\zeta}) \in C' \times \widetilde{\Re}' : \ a \in S(\widetilde{\zeta})\}.$$

We define a map F of the set $J(C \times \widetilde{\Re})$ into the set $J(C \times \Re)$ as follows: if $(a, \widetilde{\zeta}) \in J(C \times \widetilde{\Re})$, then we set $F(a, \widetilde{\zeta}) = (a, \zeta)$. We observe that F is an one-to-one map of $J(C \times \widetilde{\Re})$ into $J(C \times \Re)$. Also, if $A \subseteq S(\widetilde{\zeta}) \subseteq S(\zeta)$, then $F^{-1}(A \times \{\zeta\}) = A \times \{\widetilde{\zeta}\}$.

4. **Lemma.** *For every integer $k \in N$, for every element $\overline{\alpha}$ of Λ_{k+1} and for every $m \in N$, $0 \leq m \leq k$, we denote by:*

(1) *$n(\widetilde{\Re})$ the integer $\max\{n(\Re), r^1, \ldots, r^t\} + 1$ if $t > 0$ and $n(\widetilde{\Re}) = n(\Re)$ if $t = 0$.*

(2) *$\widetilde{\Re}(\overline{\alpha})$ the set of all elements $\widetilde{\zeta} \in \widetilde{\Re}$ for which $\zeta \in \Re(\overline{\alpha})$.*

(3) *$\widetilde{d}(\overline{\alpha}, k)$ the set $F^{-1}(d(\overline{\alpha}, k))$, and*

(4) *$\widetilde{U}(\overline{\alpha}, m)$ the set $F^{-1}(U(\overline{\alpha}, m))$.*

Then, the properties (7)-(23) of Lemma 2.II are satisfied if we replace the integer $n(\Re)$, by the integer $n(\widetilde{\Re})$, the symbols $\Re$, ζ and χ by $\widetilde{\Re}$, $\widetilde{\zeta}$ and $\widetilde{\chi}$, respectively, and the sets $d(\overline{\alpha}, k)$ and $U(\overline{\alpha}, m)$ by the sets $\widetilde{d}(\overline{\alpha}, k)$ and $\widetilde{U}(\overline{\alpha}, m)$, respectively. (The numbers $n(\overline{\alpha})$ and $n(\overline{\alpha}, m)$ are not changed).

Proof. It is sufficient to prove the case $t > 0$.

(7)-(12). Obviously, these properties are true.

(13). Let $k + 1 \geq n(\widetilde{\Re})$ and $\widetilde{\zeta}, \widetilde{\chi} \in \widetilde{\Re}(\overline{\alpha})$. Obviously, $k + 1 \geq n(\Re)$. Let

$$\widetilde{A} = \{U_0^{D(\widetilde{\zeta})}, \ldots, U_{n(\overline{\alpha})}^{D(\widetilde{\zeta})}, \overline{U}_0^{D(\widetilde{\zeta})}, \ldots, \overline{U}_{n(\overline{\alpha})}^{D(\widetilde{\zeta})}, D(\widetilde{\zeta}) \setminus U_0^{D(\widetilde{\zeta})}, \ldots, D(\widetilde{\zeta}) \setminus U_{n(\overline{\alpha})}^{D(\widetilde{\zeta})}, D(\widetilde{\zeta}) \setminus \overline{U}_0^{D(\widetilde{\zeta})}, \ldots,$$
$$D(\widetilde{\zeta}) \setminus \overline{U}_{n(\overline{\alpha})}^{D(\widetilde{\zeta})}, \mathrm{Fr}(U_0^{D(\widetilde{\zeta})}), \ldots, \mathrm{Fr}(U_{n(\overline{\alpha})}^{D(\widetilde{\zeta})}), D(\widetilde{\zeta}) \setminus \mathrm{Fr}(U_0^{D(\widetilde{\zeta})}), \ldots, D(\widetilde{\zeta}) \setminus \mathrm{Fr}(U_{n(\overline{\alpha})}^{D(\widetilde{\zeta})})\}.$$

Let $\widetilde{B}$ be the set, which is obtained by $\widetilde{A}$ replacing the element $\widetilde{\zeta}$ by $\widetilde{\chi}$. Also, let A and B be the sets, which are obtained by the sets $\widetilde{A}$ and $\widetilde{B}$ replacing the elements $\widetilde{\zeta}$ and $\widetilde{\chi}$ by the elements ζ and χ, respectively. If $\widetilde{A}_i$, $i \in N$, is an element of $\widetilde{A}$, then by $\widetilde{B}_i$, A_i and B_i we denote the corresponding element of $\widetilde{B}$, A and B, respectively.

Since $\zeta, \chi \in \Re(\overline{\alpha})$, by property (13) of Lemma 2.II, the set A has the same structure with the set B. We observe that

$$D(\widetilde{\zeta}) = \bigcap\{\mathrm{Fr}(U_{r^i}^{D(\zeta)}) : i = 1, ..., t\}$$

and

$$D(\widetilde{\chi}) = \bigcap\{\mathrm{Fr}(U_{r^i}^{D(\chi)}) : i = 1, ..., t\}$$

Now, let $\widetilde{A}_1, ..., \widetilde{A}_r$ be elements of $\widetilde{A}$ such that $\widetilde{A}_1 \cap ... \cap \widetilde{A}_r \neq \emptyset$. Then $(A_1 \cap D(\widetilde{\zeta})) \cap ... \cap (A_r \cap D(\widetilde{\zeta})) \neq \emptyset$. (See Section 3). Hence

$$A_1 \cap ... \cap A_r \cap \mathrm{Fr}(U_{r^1}^{D(\zeta)}) \cap ... \cap \mathrm{Fr}(U_{r^t}^{D(\zeta)}) \neq \emptyset.$$

Since A has the same structure with B we have

$$B_1 \cap ... \cap B_r \cap \mathrm{Fr}(U_{r^1}^{D(\chi)}) \cap ... \cap \mathrm{Fr}(U_{r^t}^{D(\chi)}) \neq \emptyset,$$

that is, $(B_1 \cap D(\widetilde{\chi})) \cap ... \cap (B_r \cap D(\widetilde{\chi})) \neq \emptyset$. This means that $\widetilde{B}_1 \cap ... \cap \widetilde{B}_r \neq \emptyset$. Similarly, we prove that if $\widetilde{B}_1 \cap ... \cap \widetilde{B}_r \neq \emptyset$, then $\widetilde{A}_1 \cap ... \cap \widetilde{A}_r \neq \emptyset$. Hence the set $\widetilde{A}$ has the same structure with the set $\widetilde{B}$.

(14). Let $\widetilde{\zeta}, \widetilde{\chi} \in \widetilde{\Re}(\overline{\alpha})$ and $d_k^{D(\widetilde{\zeta})} \neq \emptyset$. Then $\zeta, \chi \in \Re(\overline{\alpha})$ and $d_k^{D(\widetilde{\zeta})} = d_k^{D(\zeta)} \neq \emptyset$ (see the definition of the ordered set $\overrightarrow{D}(\widetilde{\zeta})(0)$, property (γ)) By property (14) of Lemma 2.II, $d_k^{D(\chi)} \neq \emptyset$. Since $d_k^{D(\widetilde{\zeta})} = d_k^{D(\zeta)} \in \bigcap\{\mathrm{Fr}(U_{r^i}^{D(\zeta)}) : i = 1, ..., t\}$, by property (16) of Lemma 2.II, we have that $d_k^{D(\chi)} \in \bigcap\{\mathrm{Fr}(U_{r^i}^{D(\chi)}) : i = 1, ..., t\}$, that is, $d_k^{D(\widetilde{\chi})} \in D(\widetilde{\chi})(0)$. By the definition of the ordered set $\overrightarrow{D}(\widetilde{\chi})(0)$, $d_k^{D(\widetilde{\chi})} = d_k^{D(\chi)}$ and hence $d_k^{D(\widetilde{\chi})} \neq \emptyset$.

(15). Let $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$ and $d_k^{D(\widetilde{\zeta})} \neq \emptyset$. Then $\zeta \in \Re(\overline{\alpha})$ and $d_k^{D(\widetilde{\zeta})} = d_k^{D(\zeta)} \neq \emptyset$. We have

$$\begin{aligned}\widetilde{d}(\overline{\alpha}, k) \cap (C \times \{\widetilde{\zeta}\}) &= F^{-1}(d(\overline{\alpha}, k)) \cap F^{-1}((C \times \{\zeta\})) = F^{-1}(d(\overline{\alpha}, k) \cap (C \times \{\zeta\})) \\ &= F^{-1}(d_k^{D(\widetilde{\zeta})} \times \{\zeta\}) = d_k^{D(\widetilde{\zeta})} \times \{\widetilde{\zeta}\}.\end{aligned}$$

(See property (15) of Lemma 2.II and properties of the map F in Section 3).

(16). Let $\widetilde{\zeta}, \widetilde{\chi} \in \widetilde{\Re}(\overline{\alpha})$, $d_k^{D(\widetilde{\zeta})} \neq \emptyset$ and $d_k^{D(\widetilde{\zeta})} \in \mathrm{Fr}(U_i^{D(\widetilde{\zeta})})$, $i \in N$. Then $\zeta, \chi \in \Re(\overline{\alpha})$, $d_k^{D(\zeta)} = d_k^{D(\zeta)} \neq \emptyset$ and $d_k^{D(\zeta)} \in \mathrm{Fr}(U_i^{D(\zeta)}) \cap D(\widetilde{\zeta})$. By properties (14) and (16) of Lemma 2.II, we have $d_k^{D(\chi)} \neq \emptyset$ and $d_k^{D(\chi)} \in \mathrm{Fr}(U_i^{D(\chi)}) \cap D(\widetilde{\chi})$. Hence $d_k^{D(\chi)} \in D(\widetilde{\chi})(0)$ and $d_k^{D(\widetilde{\chi})} = d_k^{D(\chi)}$. Thus $d_k^{D(\widetilde{\chi})} \in \mathrm{Fr}(U_i^{D(\chi)})$.

Similarly we can prove properties (17)-(23).

5. Notations. The sets $T(\Re)(0)$, $T(\Re)$, $d(\overline{\alpha}, m)$, $H(\overline{\alpha}, r)$, $V(\overline{\alpha}, r)$, $\mathcal{U}$, $\mathcal{V}$, $O(W)$ for $W \in \mathcal{U} \cup \mathcal{V}$, $O(\mathcal{U})$, $O(\mathcal{V})$ and $\mathbb{B}(T(\Re))$ (See Notations 1.III) conserning the family $\Re$, for the family $\widetilde{\Re}$ will be denoted by $T(\widetilde{\Re})(0)$, $T(\widetilde{\Re})$, $\widetilde{d}(\overline{\alpha}, m)$, $\widetilde{H}(\overline{\alpha}, r)$, $\widetilde{V}(\overline{\alpha}, r)$, $\widetilde{\mathcal{U}}$, $\widetilde{\mathcal{V}}$, $O(\widetilde{W})$ for $\widetilde{W} \in \widetilde{\mathcal{U}} \cup \widetilde{\mathcal{V}}$, $O(\widetilde{\mathcal{U}})$, $O(\widetilde{\mathcal{V}})$ and $\mathbb{B}(T(\widetilde{\Re}))$, respectively.

All results of Section III, related to the above sets concerning the family $\Re$, are also true for the corresponding sets concerning the family $\widetilde{\Re}$. In the constuction of the family $\widetilde{\Re}$ we had a fixed subset $\{r^1, ..., r^t\}$ of N. Let $\{r^1, ..., r^t, r^{t+1}, ..., r^{t_1}\}$ be a subset of N such that $0 \leq t < t_1 \leq n$ and $|\{r^1, ..., r^{t_1}\}| = t_1$. The corresponding family $\widetilde{\Re}$ constructed for the fixed subset $\{r^1, ..., r^{t_1}\}$ of N will be denoted by $\widehat{\Re}$. Also, in all notations concerning this family, the symbol "$\sim$" will be replaced by the symbol "$\wedge$".

By Φ we denote a map of the space $T(\widehat{\Re})$ into the space $T(\widetilde{\Re})$ defined as follows: If $\overline{\alpha} \in \Lambda_{k+1}$ and $\widehat{d}(\overline{\alpha}, k) \in T(\widehat{\Re})(0)$, then we set $\Phi(\widehat{d}(\overline{\alpha}, k)) = \widetilde{d}(\overline{\alpha}, k)$. If $d \times \{\widehat{\zeta}\} \in T(\widehat{\Re}) \setminus T(\widehat{\Re})(0)$, then we set $\Phi(d \times \{\widehat{\zeta}\}) = d \times \{\widetilde{\zeta}\} \in T(\widetilde{\Re})$. We observe that $\widetilde{d}(\overline{\alpha}, k) \in T(\widetilde{\Re})(0)$, that is, $\widetilde{d}(\overline{\alpha}, k) \neq \emptyset$. Indeed, if $\widehat{\zeta} \in \widehat{\Re}(\overline{\alpha})$, then we have $\widehat{d}(\overline{\alpha}, k) \cap (C \times \{\widehat{\zeta}\}) = d_k^{D(\widehat{\zeta})} \times \{\widehat{\zeta}\}$, where $d_k^{D(\widehat{\zeta})} \neq \emptyset$. Then, by the definition of the ordered set $\overrightarrow{D}(\widehat{\zeta})(0)$, we have $d_k^{D(\zeta)} = d_k^{D(\widehat{\zeta})}$. Since $\{r^1, ..., r^t\} \subseteq \{r^1, ..., r^{t_1}\}$, $d_k^{D(\zeta)} \in D(\widetilde{\zeta})$ and hence $d_k^{D(\zeta)} = d_k^{D(\widetilde{\zeta})} \neq \emptyset$. Since $\widetilde{d}(\overline{\alpha}, k) \cap (C \times \{\widetilde{\zeta}\}) = d_k^{D(\widetilde{\zeta})} \times \{\widetilde{\zeta}\}$ we have $\widetilde{d}(\overline{\alpha}, k) \neq \emptyset$.

By $\widehat{F}$ we denote the map of the set $J(C \times \widehat{\Re})$ into the set $J(C \times \widetilde{\Re})$, which is defined as follows: if $(a, \widehat{\zeta}) \in J(C \times \widehat{\Re})$, then we set $\widehat{F}(a, \widehat{\zeta}) = (a, \widetilde{\zeta})$. Obviously, this map is one-to-one and $\widehat{F}(A \times \{\widehat{\zeta}\}) = A \times \{\widetilde{\zeta}\}$, where $A \subseteq S(\widehat{\zeta}) \subseteq S(\widetilde{\zeta})$.

6. Lemma. *The map Φ is a homeomorphism of the space $T(\widehat{\Re})$ into a subset of the space $T(\widetilde{\Re})$.*

Proof. It is not difficult to see that the map Φ is one-to-one. Let $\Phi(\widehat{d}(\overline{\alpha}, k)) = \widetilde{d}(\overline{\alpha}, k)$. Let r be an integer of N such that $k + r + 1 \geq n(\widehat{\Re}) \geq n(\widetilde{\Re})$. Consider the sets $\widehat{V}(\overline{\alpha}, r)$ and $\widetilde{V}(\overline{\alpha}, r)$. Then, $\widehat{d}(\overline{\alpha}, k) \subseteq \widehat{V}(\overline{\alpha}, r)$ and $\widetilde{d}(\overline{\alpha}, k) \subseteq \widetilde{V}(\overline{\alpha}, r)$.

Let $\widehat{d}(\overline{\alpha}_1, k_1) \in T(\widehat{\Re})(0)$, $\widehat{d}(\overline{\alpha}_1, k_1) \neq \widehat{d}(\overline{\alpha}, k)$ and $\widehat{d}(\overline{\alpha}_1, k_1) \subseteq \widehat{V}(\overline{\alpha}, r)$. Then, there exists an element $\overline{\gamma} \in \Lambda_{k+r+1}$ such that $\overline{\alpha}_1 \geq \overline{\gamma} \geq \overline{\alpha}$ and for every $\widehat{\zeta} \in \widehat{\Re}(\overline{\alpha}_1)$

we have $d_{k_1}^{D(\widehat{\zeta})} \subseteq U^C_{n(\overline{\gamma},k)}$. Then $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha}_1)$ and $d_{k_1}^{D(\widetilde{\zeta})} \subseteq U^C_{n(\overline{\gamma},k)}$. This means that

$$\Phi(\widehat{d}(\overline{\alpha}_1, k_1)) = \widetilde{d}(\overline{\alpha}_1, k_1) \subseteq \widetilde{V}(\overline{\alpha}, r).$$

Let $d \times \{\widehat{\zeta}\} \subseteq \widehat{V}(\overline{\alpha}, r)$. Let $\overline{\gamma} \in \Lambda_{k+r+1}$ and $\widehat{\zeta} \in \widehat{\Re}(\overline{\gamma})$. Then $\overline{\gamma} \geq \overline{\alpha}$ and $d \subseteq U^C_{n(\overline{\gamma},k)}$. This means that $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\gamma})$ and hence $\Phi(d \times \{\widehat{\zeta}\}) = d \times \{\widetilde{\zeta}\} \subseteq \widetilde{V}(\overline{\alpha}, r)$. Thus, $\Phi(O(\widehat{V}(\overline{\alpha}, r))) \subseteq O(\widetilde{V}(\overline{\alpha}, r))$. By Corollary 7.III, we have that the map Φ is continuous at the point $\widehat{d}(\overline{\alpha}, k)$ of $T(\widehat{\Re})$. Similarly we can prove that

$$\Phi^{-1}(\Phi(T(\widehat{\Re})) \cap O(\widetilde{V}(\overline{\alpha}, r))) \subseteq O(\widehat{V}(\overline{\alpha}, r)).$$

This means that the map Φ^{-1} of $\Phi(T(\widehat{\Re}))$ onto $T(\widehat{\Re})$ is continuous at the point $\widetilde{d}(\overline{\alpha}, k)$.

Now, let $\Phi(d \times \{\widehat{\zeta}\}) = d \times \{\widetilde{\zeta}\}$. Consider the sets $\widehat{H}(\overline{\alpha}, r)$ and $\widetilde{H}(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $k + 1 \geq n(\widehat{\Re})$, $\widehat{\zeta} \in \widehat{\Re}(\overline{\alpha})$, $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$, $0 \leq r \leq n(\overline{\alpha})$ and $d \subseteq U^C_r$. Then $d \times \{\widehat{\zeta}\} \subseteq \widehat{H}(\overline{\alpha}, r)$ and $d \times \{\widetilde{\zeta}\} \subseteq \widetilde{H}(\overline{\alpha}, r)$. Let $\widehat{d}(\overline{\alpha}_1, k_1) \in T(\widehat{\Re})(0)$ and $\widehat{d}(\overline{\alpha}_1, k_1) \subseteq \widehat{H}(\overline{\alpha}, r)$. Hence $\widehat{\Re}(\overline{\alpha}_1) \subseteq \widehat{\Re}(\overline{\alpha})$. If $\overline{\alpha}_1 \leq \overline{\alpha}$, then $\widehat{\Re}(\overline{\alpha})$ is a singleton. In this case it is easy to prove that $\Phi(\widehat{d}(\overline{\alpha}_1, k_1)) = \widetilde{d}(\overline{\alpha}_1, k_1) \subseteq \widetilde{H}(\overline{\alpha}, r)$. Therefore, we can suppose that $\overline{\alpha} \leq \overline{\alpha}_1$. Obviously, for every $\widehat{\zeta} \in \widehat{\Re}(\overline{\alpha}_1)$ we have $d_{k_1}^{D(\widehat{\zeta})} \subseteq U^C_r$. This means that $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha}_1)$ and $d_{k_1}^{D(\widetilde{\zeta})} \subseteq U^C_r$, that is, $\Phi(\widehat{d}(\overline{\alpha}_1, k_1)) = \widetilde{d}(\overline{\alpha}_1, k_1) \subseteq \widetilde{H}(\overline{\alpha}, r)$.

Let $d' \times \{\widehat{\zeta}'\} \subseteq \widehat{H}(\overline{\alpha}, r)$. Therefore, $\widehat{\zeta}' \in \widehat{\Re}(\overline{\alpha})$ and $d' \subseteq U^C_r$. Then $\widetilde{\zeta}' \in \widetilde{\Re}(\overline{\alpha})$ and hence $d' \times \{\widetilde{\zeta}'\} \subseteq \widetilde{H}(\overline{\alpha}, r)$, that is, $\Phi(d' \times \{\widehat{\zeta}'\}) = d' \times \{\widetilde{\zeta}'\} \subseteq \widetilde{H}(\overline{\alpha}, r)$. By Corollary 7.III, we have that the map Φ is continuous at the point $d \times \{\widehat{\zeta}\}$ of $T(\widehat{\Re})$.

Similarly, we can prove that

$$\Phi^{-1}(\Phi(T(\widehat{\Re})) \cap O(\widetilde{H}(\overline{\alpha}, r))) \subseteq O(\widehat{H}(\overline{\alpha}, r)).$$

Hence the map Φ^{-1} is continuous at the point $d \times \{\widetilde{\zeta}\}$ of $\Phi(T(\widehat{\Re}))$. Thus, Φ is a homeomorphism of the space $T(\widehat{\Re})$ onto the subspace $\Phi(T(\widehat{\Re}))$ of the space $T(\widetilde{\Re})$.

7. Lemma. *The set $\Phi(T(\widehat{\Re}))$ is a closed subset of $T(\widetilde{\Re})$.*

Proof. Let $d \in T(\widetilde{\Re}) \setminus \Phi(T(\widehat{\Re}))$. We prove that there exists an element $\widetilde{W} \in \widetilde{\mathcal{U}} \cup \widetilde{\mathcal{V}}$ such that

$$d \in O(\widetilde{W}) \subseteq T(\widetilde{\Re}) \setminus \Phi(T(\widehat{\Re})).$$

Let $d = d' \times \{\widetilde{\zeta}\} \in T(\widetilde{\Re}) \setminus T(\widetilde{\Re})(0)$. We prove that $d' \notin D(\widehat{\zeta})$. Indeed, let $d' \in D(\widehat{\zeta})$. If $d' \notin D(\widehat{\zeta})(0)$, then $d' \times \{\widehat{\zeta}\} \in T(\widehat{\Re})$ and $\Phi(d' \times \{\widehat{\zeta}\}) = d' \times \{\widetilde{\zeta}\}$, which is impossible. If $d' \in D(\widehat{\zeta})(0)$, then $d' = d_k^{D(\widehat{\zeta})}$, for some $k \in N$. Let $\overline{\alpha} \in \Lambda_{k+1}$

and $\widehat{\zeta} \in \widehat{\Re}(\overline{\alpha})$. Then $\widehat{d}(\overline{\alpha}, k) \in T(\widehat{\Re})$ and $\Phi(\widehat{d}(\overline{\alpha}, k)) = \widetilde{d}(\overline{\alpha}, k) \in T(\widetilde{\Re})$. Since $\widetilde{d}(\overline{\alpha}, k) \cap (C \times \{\widetilde{\zeta}\}) = d_k^{D(\widetilde{\zeta})} \times \{\widetilde{\zeta}\}$ and $d_k^{D(\widetilde{\zeta})} = d_k^{D(\widehat{\zeta})}$, we have $d \cap \widetilde{d}(\overline{\alpha}, k) \neq \emptyset$, which is a contradiction. Hence, $d' \notin D(\widehat{\zeta})$.

There exists an integer $r \in N$ such that $d' \in U_r^{D(\widetilde{\zeta})}$ and $U_r^{D(\widetilde{\zeta})} \cap D(\widehat{\zeta}) = \emptyset$. Let $k \in N$, $k+1 \geq n(\widehat{\Re})$, $\overline{\alpha} \in \Lambda_{k+1}$, $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$ and $0 \leq r \leq n(\overline{\alpha})$. We set $\widetilde{W} = \widetilde{H}(\overline{\alpha}, r)$ and prove that

$$O(\widetilde{H}(\overline{\alpha}, r)) \cap \Phi(T(\widehat{\Re})) = \emptyset$$

.

Indeed, in the opposite case, there exists an element $d_1 \in O(\widetilde{H}(\overline{\alpha}, r)) \cap \Phi(T(\widehat{\Re}))$. Let $d_1 = d_1' \times \{\widetilde{\chi}\} \in T(\widetilde{\Re}) \setminus T(\widetilde{\Re})(0)$. Then $d_1' \in U_r^{D(\widetilde{\chi})}$ and $\Phi(d_1' \times \{\widehat{\chi}\}) = d_1' \times \{\widetilde{\chi}\}$. This means that $d_1' \in D(\widehat{\chi})$ and hence $U_r^{D(\widetilde{\chi})} \cap D(\widehat{\chi}) \neq \emptyset$. Since $\widetilde{\zeta}, \widetilde{\chi} \in \widetilde{\Re}(\overline{\alpha})$ and since

$$D(\widehat{\zeta}) = \bigcap \{\mathrm{Fr}(U_{r^i}^{D(\widetilde{\zeta})}) : i = 1, ..., t_1\}$$

and

$$D(\widehat{\chi}) = \bigcap \{\mathrm{Fr}(U_{r^i}^{D(\widetilde{\chi})}) : i = 1, ..., t_1\},$$

by property (13) of Lemma 4, this is a contradiction.

Let $d_1 = \widetilde{d}(\overline{\alpha}_1, k_1) \in T(\widetilde{\Re})(0)$. Let $\widetilde{\chi} \in \widetilde{\Re}(\overline{\alpha}_1)$. Then

$$\widetilde{d}(\overline{\alpha}_1, k_1) \cap (C \times \{\widetilde{\chi}\}) = d_{k_1}^{D(\widetilde{\chi})} \times \{\widetilde{\chi}\}$$

and hence $d_{k_1}^{D(\widetilde{\chi})} \in U_r^{D(\widetilde{\chi})}$. On the other hand, $\Phi(\widehat{d}(\overline{\alpha}_1, k_1)) = \widetilde{d}(\overline{\alpha}_1, k_1)$. This means that $d_{k_1}^{D(\widehat{\chi})} = d_{k_1}^{D(\widetilde{\chi})} \in D(\widehat{\chi})$, and hence $U_r^{D(\widetilde{\chi})} \cap D(\widehat{\chi}) \neq \emptyset$. As in the above this is a contradiction.

Now, suppose that $d = \widetilde{d}(\overline{\alpha}, k)$. Let $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$. We prove that $d_k^{D(\widetilde{\zeta})} \notin D(\widehat{\zeta})$. Indeed, in the opposite case, $d_k^{D(\widetilde{\zeta})} = d_k^{D(\widehat{\zeta})}$ and $\widehat{d}(\overline{\alpha}, k) \in T(\widehat{\Re})(0)$ and hence $\Phi(\widehat{d}(\overline{\alpha}, k)) = \widetilde{d}(\overline{\alpha}, k)$, which is a contradiction. Hence $d_k^{D(\widetilde{\zeta})} \notin D(\widehat{\zeta})$.

Let $r \in N$ such that $k + r + 1 > n(\widehat{\Re})$. Since

$$D(\widehat{\zeta}) = \bigcap \{\mathrm{Fr}(U_{r^i}^{D(\zeta)}) : i = 1, ..., t_1\},$$

there exists an integer $i(\zeta) \in N$, $1 \leq i(\zeta) \leq t_1$, such that $d_k^{D(\zeta)} \notin \mathrm{Fr}(U_{r^{i(\zeta)}}^{D(\zeta)})$. Then, by properties, (19) and (20) of Lemma 2.II, $U_{n(\overline{\gamma}, k)}^{D(\zeta)} \cap \mathrm{Fr}(U_{r^{i(\zeta)}}^{D(\widetilde{\zeta})}) = \emptyset$, where $\overline{\gamma} \in \Lambda_{k+r+1}$, $\overline{\gamma} \geq \overline{\alpha}$ and $\zeta \in \Re(\overline{\gamma})$, that is, $U_{n(\overline{\gamma}, k)}^{D(\zeta)} \cap D(\widehat{\zeta}) = \emptyset$.

We set $\widetilde{W} = \widetilde{V}(\overline{\alpha}, r)$ and prove that $O(\widetilde{V}(\overline{\alpha}, r)) \cap \Phi(T(\widehat{\Re})) = \emptyset$. Indeed, in the opposite case, there exists $d_1 \in O(\widetilde{V}(\overline{\alpha}, r)) \cap \Phi(T(\widehat{\Re}))$. Let $d_1 = d_1' \times \{\widetilde{\chi}\} \in$

$T(\widehat{\Re})\setminus T(\widetilde{\Re})(0)$ and let $\widetilde{\chi} \in \widetilde{\Re}(\overline{\gamma})$, where $\overline{\gamma} \in \Lambda_{k+r+1}$. Then, $\overline{\gamma} \geq \overline{\alpha}$ and $d_1' \in U^{D(\widetilde{\chi})}_{n(\overline{\gamma},k)}$, that is, $d_1' \notin D(\widehat{\chi})$. On the other hand,

$$\Phi(d_1' \times \{\widehat{\chi}\}) = d_1' \times \{\widetilde{\chi}\}.$$

This means that $d_1' \in D(\widehat{\chi})$, which is a contradiction.

Let $d_1 = \widetilde{d}(\overline{\alpha}, k_1) \in T(\widetilde{\Re})(0)$ and let $\widetilde{\chi} \in \widetilde{\Re}(\overline{\alpha}_1)$. Then $\widetilde{d}(\overline{\alpha}_1, k_1) \cap (C \times \{\widetilde{\chi}\}) = d_{k_1}^{D(\widetilde{\chi})} \times \{\widetilde{\chi}\}$ and hence $d_{k_1}^{D(\widetilde{\chi})} \in U^{D(\widetilde{\chi})}_{n(\overline{\gamma},k)}$, where $\overline{\gamma} \in \Lambda_{k+r+1}$ and $\widetilde{\chi} \in \widetilde{\Re}(\overline{\gamma})$. Therefore, $d_{k_1}^{D(\widetilde{\chi})} \notin D(\widehat{\chi})$. On the other hand, $\Phi(\widehat{d}(\overline{\alpha}, k_1)) = \widetilde{d}(\overline{\alpha}_1, k_1)$ and hence $\widehat{d}(\overline{\alpha}_1, k_1) \cap (C \times \{\widehat{\chi}\}) = d_{k_1}^{D(\widehat{\chi})} \times \{\widehat{\chi}\}$, that is, $d_{k_1}^{D(\widehat{\chi})} = d_{k_1}^{D(\widetilde{\chi})} \in D(\widehat{\chi})$, which is a contradiction.

8. Lemma. *Let* $\{r^1, ..., r^{t_1}\} = \{r^1, ..., r^t, r^{t+1}\}$, *where* $r^{t+1} \in N \setminus \{r^1, .., r^t\}$. *Let* $\overline{\alpha} \in \Lambda_{k+1}$, $k + 1 \geq n(\widetilde{\Re})$ *and* $0 \leq r^{t+1} \leq n(\overline{\alpha})$. *Then* $\mathrm{Fr}(\widetilde{W}) \setminus T(\widetilde{\Re})(\overline{\alpha}) \subseteq \Phi(T(\widehat{\Re}))$, *where* $\widetilde{W} = \widetilde{H}(\overline{\alpha}, r^{t+1})$.

Proof. Let $d \in \mathrm{Fr}(\widetilde{W}) \setminus T(\widetilde{\Re})(\overline{\alpha})$. Then $d \cap \widetilde{W} \neq \emptyset$ and $d \cap (J(C \times \widetilde{\Re}) \setminus \widetilde{W}) \neq \emptyset$. Let $d = d' \times \{\widetilde{\zeta}\} \in T(\widetilde{\Re}) \setminus T(\widetilde{\Re})(0)$. Then $d' \notin D(\widetilde{\zeta})(0)$. We prove that $d' \in D(\widehat{\zeta})$. Since $\widetilde{H}(\overline{\alpha}, r^{t+1}) = J(U^C_{r^t+1} \times \widetilde{\Re}(\overline{\alpha}))$, we have $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$, $d' \cap U^C_{r^t+1} \neq \emptyset$ and $d' \cap (C \setminus U^C_{r^t+1}) \neq \emptyset$. This means that $d' \in \mathrm{Fr}(U^{D(\widetilde{\zeta})}_{r^t+1}) \subseteq \mathrm{Fr}(U^{D(\zeta)}_{r^t+1})$. Hence, if $t = 0$, then $d' \in D(\widehat{\zeta})$.

Since $d' \in D(\widetilde{\zeta})$, for $t > 0$, we have that $d' \in \bigcap\{\mathrm{Fr}(U^{D(\zeta)}_{r^t}) : i = 1, ..., t\}$. Hence,

$$d' \in \bigcap\{\mathrm{Fr}(U^{D(\zeta)}_{r^i}) : i = 1, \ldots, t+1\} = D(\widehat{\zeta}).$$

Since $D(\widehat{\zeta})(0) \subseteq D(\widetilde{\zeta})(0)$ we have $d' \notin D(\widehat{\zeta})(0)$ and hence $d' \times \{\widehat{\zeta}\} \in T(\widehat{\Re})$. Obviously, $\Phi(d' \times \{\widehat{\zeta}\}) = d' \times \{\widetilde{\zeta}\}$. Thus, $d = d' \times \{\widetilde{\zeta}\} \in \Phi(T(\widehat{\Re}))$.

Now, let $d = \widetilde{d}(\overline{\alpha}_1, k_1)$. Since $d \cap \widetilde{W} \neq \emptyset$, we have $\widetilde{\Re}(\overline{\alpha}) \cap \widetilde{\Re}(\overline{\alpha}_1) \neq \emptyset$. This means that either $\overline{\alpha}_1 \geq \overline{\alpha}$ or $\overline{\alpha}_1 \leq \overline{\alpha}$. If $\overline{\alpha}_1 \leq \overline{\alpha}$, then $d \in T(\widetilde{\Re})(\overline{\alpha})$. Hence $\overline{\alpha}_1 \geq \overline{\alpha}$. Let $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha}_1)$. By Lemma 4.IV, we have $d_{k_1}^{D(\widetilde{\zeta})} \cap U^C_{r^t+1} \neq \emptyset$ and $d_{k_1}^{D(\widetilde{\zeta})} \cap (C \setminus U^C_{r^t+1}) \neq \emptyset$. This means that $d_{k_1}^{D(\widetilde{\zeta})} \in \mathrm{Fr}(U^{D(\widetilde{\zeta})}_{r^t+1}) \subseteq \mathrm{Fr}(U^{D(\zeta)}_{r^t+1})$. Hence if $t = 0$, then $d_{k_1}^{D(\widetilde{\zeta})} \in D(\widehat{\zeta})$. For $t > 0$, since

$$d_{k_1}^{D(\widetilde{\zeta})} \in D(\widetilde{\zeta}) = \bigcap\{\mathrm{Fr}(U^{D(\zeta)}_{r^i}) : i = 1, ..., t\},$$

we have

$$d_{k_1}^{D(\widetilde{\zeta})} \in \bigcap\{\mathrm{Fr}(U^{D(\zeta)}_{r^i}) : i = 1, ..., t+1\} = D(\widehat{\zeta}).$$

Hence, $d_{k_1}^{D(\widehat{\zeta})} \neq \emptyset$, $\widehat{d}(\overline{\alpha}, k_1) \in T(\widehat{\Re})$ and $\Phi(\widehat{d}(\overline{\alpha}_1, k_1)) = \widetilde{d}(\overline{\alpha}_1, k_1)$. Thus $\widetilde{d}(\overline{\alpha}_1, k_1) \in \Phi(T(\widehat{\Re}))$.

9. Lemma. *Let $t = 0$ and $|\{r^1, ..., r^{t_1}\}| = t_1 = n$. Then $\Phi(T(\widehat{\Re})) \subseteq T(\widetilde{\Re})(0) = T(\Re)(0)$.*

Proof. Let $d \in T(\widehat{\Re})$. Let $\widehat{\zeta} \in \widehat{\Re}$ and $d' \in D(\widehat{\zeta})$ such that $d' \times \{\widehat{\zeta}\} = d \cap (C \times \{\widehat{\zeta}\}) \neq \emptyset$. Then,

$$d' \in D(\widehat{\zeta}) = \bigcap\{\mathrm{Fr}(U_{r^i}^{D(\zeta)}) : i = 1, ..., n\} \subseteq D(\zeta)(0).$$

Since $D(\widehat{\zeta})(0) = D(\zeta)(0) \cap D(\widehat{\zeta})$ we have $d' \in D(\widehat{\zeta})(0)$. Hence there exists an integer k such that $d' = d_k^{D(\widehat{\zeta})}$. If $\overline{\alpha} \in \Lambda_{k+1}$ and $\widehat{\zeta} \in \widehat{\Re}(\overline{\alpha})$, then $d = \widehat{d}(\overline{\alpha}, k)$. Hence, $\Phi(d) = \Phi(\widehat{d}(\overline{\alpha}, k)) = \widetilde{d}(\overline{\alpha}, k) = d(\overline{\alpha}, k) \in T(\Re)(0)$. Thus, $\Phi(T(\widehat{\Re})) \subseteq T(\Re)(0)$.

10. Corollary. *If $|\{r^1, ..., r^{t_1}\}| = t_1 = n$, then the space $T(\widehat{\Re})$ is countable.*

11. Theorem. *The space $T(\widetilde{\Re})$ belongs to the family $I\!R^{n-t}(I\!M)$.*

Proof. We prove the theorem by induction on integer $n-t$. Let $n-t = 0$. Then $t = n$ and by Corollary 10, the space $T(\widetilde{\Re})$ belongs to the family $I\!M = I\!R^0(I\!M)$.

Suppose that for every subset $\{r^1, ..., t^{t_1}\}$ of N for which $|\{r^1, ..., r^{t_1}\}| = t_1$ and $0 \leq n - t_1 < n - t$, we have proved that the space $T(\widetilde{\Re})$ belongs to $I\!R^{n-t_1}(I\!M)$.

Now, we prove that for every subset $\{r^1, .., r^t\}$ of N for which $|\{r^1, ..., r^t\}| = t$, the space $T(\widetilde{\Re})$ belongs to $I\!R^{n-t}(I\!M)$. By Corollary 7.III it is sufficient to prove that

$$\mathrm{Bd}(O(\widetilde{H}(\overline{\alpha}, r))) \in I\!R^{n-t-1}(I\!M),$$

where $\overline{\alpha} \in \Lambda_{k+1}$, $k + 1 \geq n(\widetilde{\Re})$ and $0 \leq r \leq n(\overline{\alpha})$, and

$$\mathrm{Bd}(O(\widetilde{V}(\overline{\alpha}, r))) \in I\!R^{n-t-1}(I\!M),$$

where $\overline{\alpha} \in \Lambda_{k+1}$ and $k + r + 1 \geq n(\widetilde{\Re})$.

Let $\overline{\alpha} \in \Lambda_{k+1}$, $k + 1 \geq n(\widetilde{\Re})$ and $0 \leq r \leq n(\overline{\alpha})$. Suppose that $r \in \{r^1, ..., r^t\}$. We prove that in this case $O((\widetilde{H}(\overline{\alpha}, r)) = \emptyset$. Indeed, let $d \in O(\widetilde{H}(\overline{\alpha}, r))$, that is, $d \subseteq \widetilde{H}(\overline{\alpha}, r)$. Let $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$ and $d' \in D(\widetilde{\zeta})$ such that $d \cap (C \times \{\widetilde{\zeta}\}) = d' \times \{\widetilde{\zeta}\}$. Since $d \subseteq \widetilde{H}(\overline{\alpha}, r)$ we have $d' \in U_r^{D(\widetilde{\zeta})}$ and hence $d' \in U_r^{D(\zeta)}$.

On the other hand we have $d' \in D(\widetilde{\zeta}) = \bigcap\{\mathrm{Fr}(U_{r^i}^{D(\zeta)}) : i = 1, ..., t\}$ and, since $r \in \{r^1, ..., t^t\}$, we have $d' \in \mathrm{Fr}(U_r^{D(\zeta)})$. Since $U_r^{D(\zeta)} \cap \mathrm{Fr}(U_r^{D(\zeta)}) = \emptyset$, this is a contradiction. Hence, $O(\widetilde{H}(\overline{\alpha}, r)) = \emptyset$ and $\mathrm{Bd}(O(\widetilde{H}(\overline{\alpha}, r))) = \emptyset \in I\!R^{n-t-1}(I\!M)$.

Thus, we can suppose that $r \notin \{r^1, ..., r^t\}$. For the subset $\{r^1, ..., r^t, r^{t+1}\}$ of N, where $r^{t+1} = r$ we construct the space $T(\widehat{\Re})$. Since $0 \leq n-(t+1) < n-t$, by induction, the space $T(\widehat{\Re})$ belongs to $I\!R^{n-t-1}(I\!M)$ and hence $\Phi(T(\widehat{\Re})) \in I\!R^{n-t-1}(I\!M)$. (See Lemma 6).

By Lemma 9.III we have $\mathrm{Bd}(O(\widetilde{H}(\overline{\alpha}, r))) \subseteq \mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r))$.

By Lemma 8, $\mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r)) \setminus T(\widetilde{\Re})(\overline{\alpha}) \subseteq \Phi(T(\widehat{\Re}))$. Let $H_1 = \mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r)) \cap \Phi(T(\widehat{\Re}))$ and $H_2 = \mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r)) \setminus \Phi(T(\widehat{\Re}))$. The set H_1 is a closed subset of $\mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r))$ and belongs to the family $I\!R^{n-t-1}(I\!M)$. The set H_2, as a finite subset of $T(\widetilde{\Re})$, is also closed in $\mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r))$ and belongs to the family $I\!R^{n-t-1}(I\!M)$. Since $\mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r)) = H_1 \cup H_2$, we have $\mathrm{Fr}(\widetilde{H}(\overline{\alpha}, r)) \in I\!R^{n-t-1}(I\!M)$ and hence $\mathrm{Bd}(O(\widetilde{H}(\overline{\alpha}, r))) \in I\!R^{n-t-1}(I\!M)$.

Now, let $\overline{\alpha} \in \Lambda_{k+1}$ and $k + r + 1 \geq n(\widetilde{\Re})$. We prove that $\mathrm{Bd}(O(\widetilde{V}(\overline{\alpha}, r))) \in I\!R^{n-t-1}(I\!M)$. By Lemma 9.III, it is sufficient to prove that

$$\mathrm{Fr}(\widetilde{V}(\overline{\alpha}, r)) \in I\!R^{n-t-1}(I\!M)$$

and for this, it is sufficient to prove that

$$\mathrm{Fr}(\widetilde{V}(\overline{\alpha}, r)) \subseteq \bigcup \{\mathrm{Fr}(H(\overline{\gamma}, n(\overline{\gamma}, k))) : \overline{\gamma} \in \Lambda_{k+r+1}, \overline{\gamma} \geq \overline{\alpha}\}.$$

We have

$$\begin{aligned} \widetilde{V}(\overline{\alpha}, r) &= \bigcup \{\widetilde{U}(\overline{\gamma}, k) : \overline{\gamma} \in \Lambda_{k+r+1}, \overline{\gamma} \geq \overline{\alpha}\} \\ &= \bigcup \{\widetilde{H}(\overline{\gamma}, n(\overline{\gamma}, k)) : \overline{\gamma} \in \Lambda_{k+r+1}, \overline{\gamma} \geq \overline{\alpha}\}. \end{aligned}$$

Let $d \in \mathrm{Fr}(\widetilde{V}(\overline{\alpha}, r))$. Then there exists an element $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\alpha})$ and $a \in C$ such that $(a, \widetilde{\zeta}) \in d \cap \widetilde{V}(\overline{\alpha}, r)$ and $d \cap (J(C \times \widetilde{\Re}) \setminus \widetilde{V}(\overline{\alpha}, r)) \neq \emptyset$. Let $\widetilde{\zeta} \in \widetilde{\Re}(\overline{\gamma})$, where $\overline{\gamma} \in \Lambda_{k+r+1}$ and $\overline{\gamma} \geq \overline{\alpha}$. Then $(a, \widetilde{\zeta}) \in d \cap \widetilde{H}(\overline{\gamma}, n(\overline{\gamma}, k))$ and $d \cap (J(C \times \widetilde{\Re}) \setminus H(\overline{\gamma}, n(\overline{\gamma}, k))) \neq \emptyset$, that is, $d \in \mathrm{Fr}(\widetilde{H}(\overline{\gamma}, n(\overline{\gamma}, k)))$. Hence

$$\mathrm{Fr}(\widetilde{V}(\overline{\alpha}, r)) \subseteq \bigcup \{\mathrm{Fr}(\widetilde{H}(\overline{\gamma}, n(\overline{\gamma}, k))) : \overline{\gamma} \in \Lambda_{k+r+1}, \overline{\gamma} \geq \overline{\alpha}\}.$$

12. Corollary. *The space $T(\Re)$ belongs to the family $I\!R^n(I\!M)$.*

V. Universal spaces

1. Notations. Let $\zeta_1 \equiv (S_1, D_1)$ and $\zeta_2 \equiv (S_2, D_2)$ are two representations and let $m \in N$. We say that ζ_1 *and* ζ_2 *are* $m-$*equivalent* and write $\zeta_1 \overset{m}{\sim} \zeta_2$ iff for every element $d \in D_1$ there exists an element $d' \in D_2$ such that $\mathrm{st}(d, m) = \mathrm{st}(d', m)$

and, conversely, for every $d \in D_2$ there exists $d' \in D_1$ such that $\mathrm{st}(d,m) = \mathrm{st}(d',m)$. It is easy to see that the relation "$\overset{m}{\sim}$" is an equivalence relation in the family of all representations. Obviously, the number of equivalence classes are finite.

2. Lemma. *Let $\mathbb{E}$ be a family of representations such that:*

(1) *For every $\zeta_1, \zeta_2 \in \mathbb{E}$ and for every $m \in N$, $\zeta_1 \overset{m}{\sim} \zeta_2$.*

(2) *For every $\zeta \equiv (S, D) \in \mathbb{E}$ the set $\Sigma(\zeta) \equiv \{\sigma_0(\zeta), \sigma_1(\zeta), \dots\}$, where $\sigma_k(\zeta) = \{\overline{U}_k^D, D \setminus U_k^D\}$, $k \in N$, is a basic system for the space D and ζ is the representation of D corresponding to the basic system $\Sigma(\zeta)$. Then we have:*

(3) *The pair $\zeta(\mathbb{E}) \equiv (S(\mathbb{E}), D(\mathbb{E}))$, where $S(\mathbb{E}) = \bigcup\{S(\zeta) : \zeta \in \mathbb{E}\}$ and $D(\mathbb{E}) = \bigcup\{D(\zeta) : \zeta \in \mathbb{E}\}$ is a representation.*

(4) *The set $\Sigma(\mathbb{E}) = \{\sigma_0(\mathbb{E}), \sigma_1(\mathbb{E}), \dots\}$, where $\sigma_k(\mathbb{E}) = \{\overline{U}_k^{D(\mathbb{E})}, D(\mathbb{E}) \setminus U_k^{D(\mathbb{E})}\}$, $k \in N$, is a basic system for the space $D(\mathbb{E})$.*

(5) *The pair $\zeta(\mathbb{E})$ is the representation of $D(\mathbb{E})$ corresponding to the basic system $\Sigma(\mathbb{E})$.*

Proof. (3). First, we observe that the set $S(\mathbb{E})$ is a subset of C and $D(\mathbb{E})$ is a set of subsets of $S(\mathbb{E})$, the union of all elements of which is the set $S(\mathbb{E})$.

Now, we prove that $D(\mathbb{E})$ is a partition of $S(\mathbb{E})$, that is, if d_1, d_2 are distinct elements of $D(\mathbb{E})$, then $d_1 \cap d_2 = \emptyset$. Indeed, let d_1, d_2 be distinct elements of $D(\mathbb{E})$, that is $d_1 \neq d_2$. There exist elements (S_1, D_1) and (S_2, D_2) of $\mathbb{E}$ such that $d_1 \in D_1$ and $d_2 \in D_2$. Suppose that $d_2 \cap d_1 \neq \emptyset$. If $d_2 \not\subseteq d_1$, then there exists an integer $m_0 \in N$ such that $d_2 \cap \mathrm{st}(d_1, m) \neq \emptyset$ and $d_2 \not\subseteq \mathrm{st}(d_1, m_0)$ for every $m \geq m_0$. Since $(S_1, D_1) \overset{m}{\sim} (S_2, D_2)$, for every $m \geq m_0$, there exists an element $d_1^m \in D_1$ such that $\mathrm{st}(d_2, m) = \mathrm{st}(d_1^m, m)$. This means that $d_1^m \cap \mathrm{st}(d_1, m) \neq \emptyset$ and $d_1^m \not\subseteq \mathrm{st}(d_1, m_0)$, that is, D_1 is not upper semi-continuous, which is a contradiction. Similarly, if $d_1 \not\subseteq d_2$, then D_2 is not upper semi-continuous. Hence $d_2 \cap d_1 = \emptyset$.

We prove that $D(\mathbb{E})$ is an upper semi-continuous partition of $S(\mathbb{E})$, that is, for every $d \in D(\mathbb{E})$ and for every $m \in N$, there exists an integer $k \in N$ such that if $d' \cap \mathrm{st}(d, k) \neq \emptyset$, where $d' \in D(\mathbb{E})$, then $d' \subseteq \mathrm{st}(d, m)$. Suppose that $D(\mathbb{E})$ is not upper semi-continuous. Then, there exists an element $d \in D(\mathbb{E})$, an integer $m \in N$ and for every $k \in N$, there exists an element $d^k \in D(\mathbb{E})$ such that $d^k \cap \mathrm{st}(d, k) \neq \emptyset$ and $d^k \not\subseteq \mathrm{st}(d, m)$.

Let (S', D') and (S_k, D_k), $k \in N$, be elements of $\mathbb{E}$ such that $d \in D'$ and $d^k \in D_k$. Since $(S', D') \overset{k}{\sim} (S_k, D_k)$, there exists an element d'_k of D' such that $\mathrm{st}(d^k, k) = \mathrm{st}(d'_k, k)$. Then $\mathrm{st}(d'_k, k) \cap \mathrm{st}(d, k) \neq \emptyset$ and hence $d'_k \cap \mathrm{st}(d, k) \neq \emptyset$. Also, for every $k \geq m$, we have $\mathrm{st}(d^k, k) \not\subseteq \mathrm{st}(d, m)$, that is, $\mathrm{st}(d'_k, k) \not\subseteq \mathrm{st}(d, m)$ and hence $d'_k \not\subseteq \mathrm{st}(d,m)$. This means that D' is not upper semi-continuous, which is a contradiction. Hence $D(\mathbb{E})$ is an upper semi-continuous partition.

(4). Let $d \in D(\mathbb{E})$ and $m_0 \in N$. It is sufficient to prove that there exists an integer $k \in N$ such that $d \in U_k^{D(\mathbb{E})}$ and every element of $\overline{U}_k^{D(\mathbb{E})}$ is contained in $\mathrm{st}(d,m_0)$. There exists an element $(S,D) \in \mathbb{E}$ such that $d \in D$. Since the set $\Sigma(\zeta)$ is a basic system for D, there exists an integer $k \in N$ such that $d \in U_k^D$ and every element of $\overline{U}_k^D$ is contained in $\mathrm{st}(d,m_0)$. We prove that $d \in U_k^{D(\mathbb{E})}$ and every element of $\overline{U}_k^{D(\mathbb{E})}$ is contained in $\mathrm{st}(d,m_0)$. By the definition of the sets U_k^C, U_k^D and $U_k^{D(\mathbb{E})}$ it follows that $U_k^D \subseteq U_k^{D(\mathbb{E})}$ and hence $d \in U_k^{D(\mathbb{E})}$.

Let $d' \in \overline{U}_k^{D(\mathbb{E})}$. Suppose that $d' \not\subseteq \mathrm{st}(d,m_0)$. Let $(S',D') \in \mathbb{E}$ and $d' \in D'$. Since $(S',D') \overset{m}{\sim} (S,D)$, for every $m \in N$, there exists an element $d^0 \in D$ such that $\mathrm{st}(d',m_1) = \mathrm{st}(d^0,m_1)$, where $m_1 = \max\{m_0,k\}$. Since $d' \in \overline{U}_k^{D(\mathbb{E})}$, we have $d' \cap U_k^C \neq \emptyset$ and hence $\mathrm{st}(d',m_1) \cap U_k^C \neq \emptyset$. Then $\mathrm{st}(d^0,m_1) \cap U_k^C \neq \emptyset$ and hence $d^0 \cap U_k^C \neq \emptyset$, which means that $d^0 \in \overline{U}_k^D$. Since $d' \not\subseteq \mathrm{st}(d,m_0)$, we have $\mathrm{st}(d',m_1) \not\subseteq \mathrm{st}(d,m_0)$. Hence $\mathrm{st}(d^0,m_1) \not\subseteq \mathrm{st}(d,m_0)$ and therefore $d^0 \not\subseteq \mathrm{st}(d,m_0)$. This is a contradiction. Thus $d' \subseteq \mathrm{st}(d,m_0)$ and therefore the set $\Sigma(\mathbb{E})$ is a basic system for the space $D(\mathbb{E})$.

(5). Let $S(D(\mathbb{E}),\Sigma(\mathbb{E}))$ and $D(D(\mathbb{E}),\Sigma(\mathbb{E}))$ be the subset of C and the partition of $S(D(\mathbb{E}),\Sigma(\mathbb{E}))$, respectively, constructed in Section I for the basic system $\Sigma(\mathbb{E})$ of $D(\mathbb{E})$. We prove that $S(\mathbb{E}) = S(D(\mathbb{E}),\Sigma(\mathbb{E}))$ and $D(\mathbb{E}) = D(D(\mathbb{E}),\Sigma(\mathbb{E}))$.

First, we prove by induction on integer k that the set $(D(\mathbb{E}))_{\bar{i}}$, $\bar{i} \in L_k$, is the set of all elements of $D(\mathbb{E})$ which intersect the set $C_{\bar{i}}$. Indeed, this is true if $\bar{i} = \emptyset \in L_0$. Suppose that this statement is true if $k \leq k_0$. Let $\bar{j}_0 \in L_{k_0+1}$. Then there exists an element $\bar{i}_0 \in L_{k_0}$ such that either $\bar{j}_0 = \bar{i}_0 0$ or $\bar{j}_0 = \bar{i}_0 1$. Hence either $(D(\mathbb{E}))_{\bar{j}_0} = (D(\mathbb{E}))_{\bar{i}_0} \cap \overline{U}_{k_0}^{D(\mathbb{E})}$ or $(D(\mathbb{E}))_{\bar{j}_0} = (D(\mathbb{E}))_{\bar{i}_0} \cap (D(\mathbb{E}) \setminus U_{k_0}^{D(\mathbb{E})})$.

Let $(D(\mathbb{E}))_{\bar{j}_0} = (D(\mathbb{E}))_{\bar{i}_0} \cap \overline{U}_{k_0}^{D(\mathbb{E})}$ and let $d \in (D(\mathbb{E}))_{\bar{j}_0}$. Then $d \in (D(\mathbb{E}))_{\bar{i}_0}$ and by induction, $d \cap C_{\bar{i}_0} \neq \emptyset$. On the other hand, $d \in \overline{U}_{k_0}^{D(\mathbb{E})}$, which means that

$$d \cap (\bigcup\{C_{\bar{i}0} : \bar{i} \in L_{k_0}\}) \neq \emptyset.$$

Let $a \in d \cap C_{\bar{i}_0}$. If $a \in C_{\bar{i}_0 0} = C_{\bar{j}_0}$, then $d \cap C_{\bar{j}_0} \neq \emptyset$. Let $a \in C_{\bar{i}_0 1}$. Then, $d \in \mathrm{Fr}(U_{k_0}^{D(\mathbb{E})}) = \mathrm{Fr}(\sigma_{k_0}(\mathbb{E}))$. Let b be a point of C, $b \neq a$, for which the $\mathrm{k^{th}}$ digit in the ternary expansion coincides with the corresponding digit of a for all $k \in N$ except $k = k_0 + 1$. Then $b \in C_{\bar{i}_0 0}$ and by property (4) of Lemma 7.I, $b \in d$. This means that $d \cap C_{\bar{j}_0} \neq \emptyset$. Similarly, we prove that if $D(\mathbb{E})_{\bar{j}_0} = (D(\mathbb{E}))_{\bar{i}_0} \cap (D(\mathbb{E}) \setminus U_{k_0}^{D(\mathbb{E})})$, then $d \in (D(\mathbb{E}))_{\bar{j}_0}$ iff $d \cap C_{\bar{j}_0} \neq \emptyset$.

For the proof of the equalities

$$S(\mathbb{E}) = S(D(\mathbb{E}), \Sigma(\mathbb{E}))$$

and

$$D(\mathbb{E}) = D(D(\mathbb{E}), \Sigma(\mathbb{E}))$$

it is sufficient to prove that for every $d \in D(\mathbb{E})$ we have $(q(D(\mathbb{E}), \Sigma(\mathbb{E}))^{-1}(d) = d \subseteq S(\mathbb{E})$. Let $a \in S(D(\mathbb{E}), \Sigma(\mathbb{E}))$ and let $q(D(\mathbb{E}), \Sigma(\mathbb{E}))(a) = d$. Then,

$$\{d\} = \bigcap\{(D(\mathbb{E}))_{\bar{i}(a,k)} : k \in N\}.$$

By the above, $d \cap C_{\bar{i}(a,k)} \neq \emptyset$, for every $k \in N$, which means that $a \in d$. Conversely, let $a \in d$. Then, $d \cap C_{\bar{i}(a,k)} \neq \emptyset$, for every $k \in N$, that is,

$$\{d\} = \bigcap\{(D(\mathbb{E}))_{\bar{i}(a,k)} : k \in N\},$$

which means that $a \in (q(D(\mathbb{E}), \Sigma(\mathbb{E})))^{-1}(d)$. Thus, the pair $\zeta(\mathbb{E})$ is the representation of $D(\mathbb{E})$ corresponding to the basic system $\Sigma(\mathbb{E})$.

3. Lemma. *Let $\mathbb{E}$ be the family of representations of Lemma 2. Suppose that:*

(1) *For every subset $s \subseteq N$ with $|s| = t \leq n$ and for every $\zeta \in \mathbb{E}$ we have*

$$\bigcap\{\mathrm{Fr}(U_k^{D(\zeta)}) \in \mathbb{R}^{n-t}(\mathbb{M}) : k \in s\}.$$

(We recall again that n is fixed).

(2) *There exists a countable subset S^0 of S such that for $\zeta \in \mathbb{E}$ and for every subset $s \subseteq N$ with $|s| = n$ we have*

$$\bigcap\{\mathrm{Fr}(U_k^{D(\zeta)}) : k \in s\} \subseteq S^0.$$

Then, for every $s \subseteq N$ with $|s| = t \leq n$ we have

$$\bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) \in \mathbb{R}^{n-t}(\mathbb{M}) : k \in s\}.$$

Proof. By Lemma 2 the pair $(S(\mathbb{E}), D(\mathbb{E}))$ is a representation. First we observe that for every $s \in N$ with $|s| = t \leq n$ we have

(3) $$\bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) : k \in s\} = \bigcup\{\bigcap\{\mathrm{Fr}(U_k^{D(\zeta)}) : k \in s\} : \zeta \in \mathbb{E}\}.$$

This follows immediately by the definition of the sets $\mathrm{Fr}(U_k^{D(\zeta)})$ and $\mathrm{Fr}(U_k^{D(\mathbb{E})})$.

We prove the lemma by induction on integer $n-t$. Let $n-t=0$, that is, $t=n$. Let $s \subseteq N$ and $|s|=n$. By property (2) and relation (3) it follows that

$$\bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) : k \in s\} \subseteq S^0$$

and hence

$$\bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) : k \in s\} \in \mathbb{R}^0(\mathbb{M}).$$

Suppose that the lemma has been proved for all integers $n-t'$, $0 \leq n-t' < n-t$. We prove the lemma for the integer $n-t$. Let $s \subseteq N$ and $|s|=t$. Consider the set

$$D^s(\mathbb{E}) \equiv \bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) : k \in s\}.$$

Since $D^s(\mathbb{E})$ is a subspace of $D(\mathbb{E})$ and the set $\{U_k^{D(\mathbb{E})} : k \in N\}$ is a basis for open sets of $D(\mathbb{E})$ (see the definition of the basic system and Lemma 2), the set $\{D^s(\mathbb{E}) \cap U_k^{D(\mathbb{E})} : k \in N\}$ is a basis for open sets of $D^s(\mathbb{E})$. For the proof of the lemma it is sufficient to prove that for every $r \in N$,

$$\mathrm{Bd}_{D^s(\mathbb{E})}(D^s(\mathbb{E}) \cap U_r^{D(\mathbb{E})}) \in \mathbb{R}^{n-t-1}(\mathbb{M}).$$

Let $r \in N$. First we suppose that $r \in s$. Then $D^s(\mathbb{E}) \subseteq \mathrm{Fr}(U_r^{D(\mathbb{E})})$ and hence

$$D^s(\mathbb{E}) \cap U_r^{D(\mathbb{E})} \subseteq \mathrm{Fr}(U_r^{D(\mathbb{E})}) \cap U_r^{D(\mathbb{E})} = \emptyset$$

Thus

$$\mathrm{Bd}_{D^s(\mathbb{E})}(D^s(\mathbb{E}) \cap U_r^{D(\mathbb{E})}) \in \mathbb{R}^{n-t-1}(\mathbb{M}).$$

Now, let $r \notin s$. Let $s_1 = s \cup \{r\}$. Then $|s_1| = t+1$ and by induction,

$$\bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) : k \in s_1\} \in \mathbb{R}^{n-t-1}(\mathbb{M}).$$

Since

$$\mathrm{Bd}_{D^s(\mathbb{E})}(D^s(\mathbb{E}) \cap U_k^{D(\mathbb{E})}) \subseteq \mathrm{Bd}(U_k^{D(\mathbb{E})}) \subseteq \mathrm{Fr}(U_k^{D(\mathbb{E})})$$

for every $k \in N$, we have

$$\mathrm{Bd}_{D^s(\mathbb{E})}(D^s(\mathbb{E}) \cap U_r^{D(\mathbb{E})}) \subseteq \bigcap\{\mathrm{Fr}(U_k^{D(\mathbb{E})}) : k \in s_1\} \in \mathbb{R}^{n-t-1}(\mathbb{M}).$$

4. **Corollary.** *If $\mathbb{E}$ is the family of Lemma 3, then $D(\mathbb{E})$ is an element of $\mathbb{R}^n(\mathbb{M})$ containing topologically every space D for every $\zeta \equiv (S, D) \in \mathbb{E}$.*

Proof. Since the set $\{U_k^{D(\mathbb{E})} : k \in N\}$ is a basis for open sets of $D(\mathbb{E})$, by the relation

$$\mathrm{Bd}(U_k^{D(\mathbb{E})}) \subseteq \mathrm{Fr}(U_k^{D(\mathbb{E})}) \in \mathbb{R}^{n-1}(\mathbb{M})$$

for every $k \in N$, we have that $D(\mathbb{E}) \in \mathbb{R}^n(\mathbb{M})$.

Let $\zeta \equiv (S, D) \in \mathbb{E}$. It is easy to see that the map $e_\zeta^{\mathbb{E}}$ of D into $D(\mathbb{E})$ for which $e_\zeta^{\mathbb{E}}(d) = d \in D(\mathbb{E})$, for every $d \in D$, is a homeomorphism of D into $D(\mathbb{E})$.

The map $e_\zeta^{\mathbb{E}} : D \to D(\mathbb{E})$ is called *the natural embedding of D into $D(\mathbb{E})$*.

5. Theorem. *In the family of all spaces having rational dimension $\leq n$, $n = 1, 2, \ldots$, there exists a universal element.*

Proof. For every element X of the family $\mathbb{R}^n(\mathbb{M})$ of all spaces having rational dimension $\leq n$, we denote by $\Sigma(X)$ a basic system for X with the property of boundary intersections. The existence of such a basic system follows by Theorem 5.I. Indeed, if $\mathbb{B}(X) = \{U_0^X, U_1^X, \ldots\}$ is a basis for open sets of X having the property of boundary intersections, then it is easy to see that the set $\Sigma(X) \equiv \{\sigma^0, \sigma^1, \ldots\}$, where $\sigma^i = \{\mathrm{Cl}(U_i^X), X \setminus U_i^X\}$, is a basic system for X having the property of boundary intersections. Let $(S(X, \Sigma(X)), D(X, \Sigma(X)))$ be the representation of X corresponding to the basic system $\Sigma(X)$ constructed in Section 1.I. The family of all such representations is denoted by $\mathbb{R}e^n(\mathbb{M})$.

In the family $\mathbb{R}e^n(\mathbb{M})$ we define an equivalence relation $''\sim''$. We say that two elements ζ_1 and ζ_2 of $\mathbb{R}e^n(\mathbb{M})$ are equivalent and we write $\zeta_1 \sim \zeta_2$ iff for every $m \in N$, $\zeta_1 \overset{m}{\sim} \zeta_2$ and $D(\zeta_1)(0) = D(\zeta_2)(0)$. It is easy to see that the cardinality of the set $E.C.\mathbb{R}e^n(\mathbb{M})$ of all equivalence classes of the relation $''\sim''$ is less than or equal to the continuum.

By $\Re$ we denote the family of all representations of the form $(S(\mathbb{E}), D(\mathbb{E}))$, where $\mathbb{E} \in E.C.\mathbb{R}e^n(\mathbb{M})$. (See Lemma 2). If $\zeta \equiv (S(\mathbb{E}), D(\mathbb{E})) \in \Re$, then by $X(\zeta)$ we denote the space $D(\mathbb{E}) \in \mathbb{R}^n(\mathbb{M})$ (see Corollary 4) and by $\Sigma(\zeta)$ we denote the basic system $\Sigma(\mathbb{E}) \equiv \{\sigma^0(\zeta), \sigma^1(\zeta), \ldots\}$ of $D(\mathbb{E})$, where $\sigma^k(\zeta) \equiv \sigma_k(\mathbb{E}) = \{\overline{U}_k^{D(\mathbb{E})}, D(\mathbb{E}) \setminus U_k^{D(\mathbb{E})}\}$. (See Lemma 2). By Lemma 2 the pair ζ is the representation of $X(\zeta)$ corresponding to the basic system $\Sigma(\zeta)$.

Let $T(\Re)$ be the space constructed in Section III. Since $\Sigma(\zeta)$ has the property of boundary intersections (see Lemma 3), by Corollary 12.IV we have $T(\Re) \in \mathbb{R}^n(\mathbb{M})$. We prove that the space $T(\Re)$ is the required universal element of $\mathbb{R}^n(\mathbb{M})$.

Let $\zeta \in \Re$. We construct a map e_ζ of $D(\zeta)$ into $T(\Re)$ as follows: if $d \in D(\zeta) \setminus D(\zeta)(0)$, then by the definition of the set $T(\Re)$ we have $d \times \{\zeta\} \in T(\Re) \setminus T(\Re)(0)$.

In this case $e_\zeta(d) = d \times \{\zeta\}$. Let $d \in D(\zeta)(0)$. Then there exists an integer $k \in N$ such that $d = d_k^{D(\zeta)}$. If $\overline{\alpha} \in \Lambda_{k+1}$ and $\zeta \in \Re(\overline{\alpha})$, then $d(\overline{\alpha}, k) \in T(\Re)(0) \subseteq T(\Re)$. In this case we set $e_\zeta(d) = d(\overline{\alpha}, k)$.

We prove that e_ζ is an embedding of $D(\zeta)$ into $T(\Re)$. Obviously, e_ζ is one-to-one. We prove the continuity of e_ζ. Let $e_\zeta(d) = d'$ and $O(W)$, $W \in \mathcal{U} \cup \mathcal{V}$, be an open neighbourhood of d' in $T(\Re)$. If $d \in D(\zeta) \setminus D(\zeta)(0)$, that is, $d' \in T(\Re) \setminus T(\Re)(0)$, then we can suppose that $W = H(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $\zeta \in \Re(\overline{\alpha})$, $k + 1 \geq n(\Re)$ and $0 \leq r \leq n(\overline{\alpha})$. (See Corollary 7. III). Obviously, $d \in U_r^{D(\zeta)}$ and $d' \notin T(\Re)(\overline{\alpha})$. Hence, the set

$$U \equiv U_r^{D(\zeta)} \setminus e_\zeta^{-1}(T(\Re)(\overline{\alpha}))$$

is an open neighbourhood of d in $D(\zeta)$. It easy to verify that $e_\zeta(U) \subseteq O(W)$.

If $d \in D(\zeta)(0)$, that is, $d' \in T(\Re)(0)$, then we can suppose that $W = V(\overline{\alpha}, r)$, where $\overline{\alpha} \in \Lambda_{k+1}$, $\zeta \in \Re(\overline{\alpha})$, $k + r + 1 \geq n(\Re)$. Let $\overline{\gamma} \in \Lambda_{k+r+1}$ and $\zeta \in \Re(\overline{\gamma})$. Then $d \in U_{n(\overline{\gamma},k)}^{D(\zeta)}$ and it is easy to verify that $e_\zeta(U_{n(\overline{\gamma},k)}^{D(\zeta)}) \subseteq O(W)$. Hence, e_ζ is continuous.

We prove the continuity of e_ζ^{-1}. Let $U_r^{D(\zeta)}$ be an open neighbourhood of d. Let $d' \in T(\Re) \setminus T(\Re)(0)$. Let $k \in N$ and $k + 1 \geq \max\{r, n(\Re)\}$ and let $\overline{\alpha} \in \Lambda_{k+1}$ such that $\zeta \in \Re(\overline{\alpha})$. Then, $H(\overline{\alpha}, r)$ is an open neighbourhood of d' in $T(\Re)$ such that $e_\zeta^{-1}(O(H(\overline{\alpha}, r))) \subseteq U_r^{D(\zeta)}$.

Let $d' \in T(\Re)(0)$. There exists an integer $k \in N$ such that $d = d_k^{D(\zeta)}$. Let $r_1 \in N$ such that $k + r_1 > r$, $k + r_1 + 1 \geq n(\Re)$, $\overline{\gamma} \in \Lambda_{k+r_1+1}$ and $\zeta \in \Re(\overline{\gamma})$. If $\overline{\beta} \in \Lambda_{k+r_1}$ and $\overline{\beta} \leq \overline{\gamma}$, then $0 \leq r \leq n(\overline{\beta})$. By property (19) of Lemma 2.II we have $U_{n(\overline{\gamma},k)}^{D(\zeta)} \subseteq U_r^{D(\zeta)}$. It is easy to verify that

$$e_\zeta^{-1}(O(V(\overline{\alpha}, r_1))) \subseteq U_r^{D(\zeta)}.$$

This means that e_ζ^{-1} is continuous and hence e_ζ is an embedding of $D(\zeta)$ into $T(\Re)$.

Now, let $X \in I\!R^n(I\!M)$. Then the map $(h(X, \Sigma(X)))^{-1}$ is an embedding of X into $D(X, \Sigma(X))$. (See Section I). Let $I\!E \in E.C.I\!Re^n(I\!M)$ such that $\zeta(X) \equiv (S(X, \Sigma(X)), D(X, \Sigma(X))) \in I\!E$ and let $e_{\zeta(X)}^{I\!E}$ the natural embedding of $D(X, \Sigma(X))$ into $D(I\!E)$. (See Section 4). Let $\zeta \equiv (S(I\!E), D(I\!E))$ and let e_ζ be the embedding of $D(I\!E)$ into the space $T(\Re)$. The map $e_X \equiv e_\zeta \circ e_{\zeta(X)}^{I\!E} \circ (h(X, \Sigma(X)))^{-1}$ is an embedding of X into $T(\Re)$. Thus, $T(\Re)$ is a universal elemnt of the family $I\!R^n(I\!M)$.

6. Definition. We say that a universal element T for a family Sp of spaces has *the property of boundary intersections with respect to subfamily* $(\mathrm{Sp})_1$ *of* Sp iff for every $X \in \mathrm{Sp}$ there exists an embedding i_X of X into T such that if Y and Z are distinct elements of Sp and $Y \in (\mathrm{Sp})_1$, then the set $i_Y(Y) \cap i_Z(Z)$ is finite. (See, for example, [I$_3$]).

7. Theorem. *In the family $\mathbb{R}^n(\mathbb{M})$ there exists a universal element having the property of finite intersections with respect to a given subfamily of $\mathbb{R}^n(\mathbb{M})$ the cardinality of which is less than or equal to the continuum.*

Proof. Let $\mathbb{R}$ be a fixed subfamily of $\mathbb{R}^n(\mathbb{M})$. For every $X \in \mathbb{R}^n(\mathbb{M})$ let $\Sigma(X)$ and $(S(X, \Sigma(X)), D(X, \Sigma(X)))$ be the basic system for X and the representation of X, respectively, constructed in the proof of Theorem 5. As in Theorem 5, by $\mathbb{R}^n(\mathbb{M})$ we denote the family of all representations $(S(X, \Sigma(X)), D(X, \Sigma(X)))$.

By $\mathfrak{R}_1$ we denote the family of all representations of the form

$$(S(\mathbb{E}), D(\mathbb{E})),$$

where $\mathbb{E} \in E.C.\mathbb{R}e^n(\mathbb{M})$.(In the proof of Theorem 5, this family is denoted by $\mathfrak{R}$). By $\mathfrak{R}_2$ we denote the family of all representations of the form

$$(S(X, \Sigma(X)), D(X, \Sigma(X))),$$

where $X \in \mathbb{R}$.

We set $\mathfrak{R} = \mathfrak{R}_1 \cup \mathfrak{R}_2$. If $\zeta_1 \in \mathfrak{R}_1$ and $\zeta_2 \in \mathfrak{R}_2$, then ζ_1 and ζ_2 we consider as distinct elements of $\mathfrak{R}$. Obviously, the cardinality of $\mathfrak{R}$ is less than or equal to the continuum.

For every $\zeta \equiv (S(X, \Sigma(X)), D(X, \Sigma(X))) \in \mathfrak{R}_2$ we denote by $X(\zeta)$ the space X and by $\Sigma(\zeta)$ the basic system $\Sigma(X)$ for X.

If $\zeta \equiv (S(\mathbb{E}), D(\mathbb{E})) \in \mathfrak{R}_1$, then, as in the proof of Theorem 5, by $X(\zeta)$ we denote the space $D(\mathbb{E}) \in \mathbb{R}^n(\mathbb{M})$ and by $\Sigma(\zeta)$ we denote the basic system $\Sigma(\mathbb{E})$ for $D(\mathbb{E})$.

Let $T(\mathfrak{R})$ be the space constructed in Section III. If $X \in \mathbb{R}$, then the pair $\zeta \equiv (S(X, \Sigma(X)), D(X, \Sigma(X))) \in \mathfrak{R}_2 \subseteq \mathfrak{R}$. Hence the map $e_X \equiv e_\zeta \circ (h(X, \Sigma(X))^{-1}$ is an embedding of X into $T(\mathfrak{R})$, where e_ζ is the embedding of $D(\zeta)$ into $T(\mathfrak{R})$ constructed in the proof of Theorem 5.

If $X \notin \mathbb{R}$, then by e_X we denote the embedding of X into $T(\mathfrak{R})$ constructed in the proof of Theorem 5.

For the proof of the Theorem it is sufficient to prove that $T(\mathfrak{R})$ has the property of finite intersections with respect to subfamily $\mathbb{R} \subseteq \mathbb{R}^n(\mathbb{M})$.

Let Y and Z are distinct elements of $I\!R^n(I\!M)$ such that $Y \in I\!R$. Let $\zeta_1 = (S(Y\Sigma(Y)), D(Y\Sigma(Y))$ and $\zeta_2 = (S(Z,\Sigma(Z)), D(Z,\Sigma(Z)))$ if $Z \in I\!R$ and $\zeta_2 = (S(I\!E), D(I\!E))$ if $Z \notin I\!R$, where $(S(Z,\Sigma(Z)), D(Z,\Sigma(Z))) \in I\!E \in E.C.I\!Re^n(I\!M)$. Then ζ_1 and ζ_2 are distinct elements of $\Re$. There exists an integer $k \in N$ and elements $\overline{\alpha}_1, \overline{\alpha}_2 \in \Lambda_{k+1}$, $\overline{\alpha}_1 \neq \overline{\alpha}_2$, such that $\zeta_1 \in \Re(\overline{\alpha}_1)$ and $\zeta_2 \in \Re(\overline{\alpha}_2)$. It is easy to verify that

$$\epsilon_Y(Y) \cap \epsilon_Z(Z) \subseteq T(\Re)(\overline{\alpha}_1) \cup T(\Re)(\overline{\alpha}_2).$$

Hence $T(\Re)$ has the property of finite intersections with respect to $I\!R$.

References

[H]. W. Hurewicz, *Normalbereiche und Dimensionstheorie*, Math. Ann. 96, 736-764 (1927).

[I_1]. S. D. Iliadis, *Rim-finite spaces and the property of universality*, Houston J. of Math., 12 (1986), pp. 55-78.

[I_2]. S. D. Iliadis,*The rim-type of spaces and the property of universality*, Houston J. Math. 13 (1987), pp. 373-388.

[I_3]. S. D. Iliadis, *Rational spaces of a given rim-type and the property of universality*, Topology Proceedings, Vol. 11 (1986), pp. 65-113.

[I_4]. S. D. Iliadis, *Rational spaces and the property of universality*, Fund. Math., 131 (1988), pp. 167-184.

[I_5]. S. D. Iliadis, *Universal spaces for some families of rim-scattered spaces*, Tsukuba J. of Math., No. 1(1992) pp. 123-160.

[I-Z]. S. D. Iliadis and S. S. Zafiridou, *Planar rational compacta and universality*, Fund. Math. 141 (1992), pp. 109-118.

[K]. J. L. Kelley, *General Topology*, Graduate Texts in Mathematics, No 27.

[Ku]. K. Kuratowski, *Topology*, Vol. I, II, New-York, 1966, 1968.

[Me]. K. Menger, *Dimensionstheorie*, Leipzig 1928.

[M-T_1]. J. C. Mayer and E. D. Tymchatyn, *Universal rational spaces*, Dissertationes Mathematicae CCXCIII, Warszawa 1990.

[M-T_2]. J. C. Mayer and E. D. Tymchatyn, *Containing spaces for planar rational compacta*, Dissertationes Mathematicae CCC, Warszawa 1990.

[N]. J. Nagata, *Modern Dimension Theory*, (Helderman Verlag, Berlin, revised and extented edition, 1983).

[Nö]. G. Nöbeling, *Über regular-eindimensionale Räume*, Math. Ann. 104, No. 1(1931), 81-91.

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