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ASYMPTOTIC ANALYSIS OF INTERFACIAL CRACKS

by

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ASYMPTOTIC ANALYSIS OF INTERFACIAL CRACKS

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Abstract

One of the most significant sources of failure in adhesive joints, thin films and composite materials is the propagation of interfacial cracks between the constituent materials. In the last few decades significant progress has been made towards the understanding of the mechanics of the interface crack within the framework of linear elasticity. The oscillatory character of the singular elastic crack-tip stress field and the coupling of the opening and shearing modes are important features that distinguish fracture mechanics from the mechanics of cracks in homogeneous media. The elasto-plastic analysis of the interface cracks has also attracted a lot of attention recently.

In the context of this thesis an elastic-plastic asymptotic solution of the problem of a plane strain crack lying along the interface between an incompressible elastic-plastic power-law hardening material and a rigid substrate is developed. The elastoplastic asymptotic stress field expansion which is assumed to be separable in r and θ, where (r,θ) are polar coordinates at the crack tip, consists of two terms and is of the general form

$$\boldsymbol{\sigma}(r,\theta)=\left(\frac{r}{J}\right)^{s}\boldsymbol{\sigma}^{(0)}(\theta)+Q r^{t}\boldsymbol{\sigma}^{(1)}(\theta)+\cdots \quad \text{as } r\rightarrow 0, \qquad \text{where } s=-\frac{1}{n+1}<t .$$

The leading and second order terms in the stress and displacement field expansions are derived from the solution of two eigenvalue problems, non-linear and linear respectively. The elastoplastic asymptotic solution of the interfacial crack problem is studied via a perturbation of the elastic solution, i.e., for $n=1+\varepsilon$ where ε is a small parameter and n the strain hardening exponent. It is shown that both the leading and second order terms in the stress expansion are singular and branch from the mode-I and mode-II of the linear elastic solution respectively.

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Contents

1. BASIC CONCEPTS OF FRACTURE MECHANICS

1.1 Introduction

The mechanical design of engineering structures involves usually an analysis of the stress and displacement fields in conjunction with a criterion for the prediction of failure. Accidents, involving both human and material losses, showed that the failure of a broad class of structures made of high-strength materials cannot be predicted accurately by conventional design criteria. A plausible explanation of these failures is that material deficiencies in the form of pre-existing flaws could initiate cracks that lead to failure of the structure. In cases of low temperature and in conditions of triaxial stress that may exist at a sharp flaw, the plastic deformation is suppressed and fracture can be truly brittle, resulting in low stress fracture even in high strength materials. The occurrence of low stress fracture was the major reason that gave impetus to the development of a new philosophy in structural design based on Fracture Mechanics.

In order to put the subject of this thesis on a theoretical basis, this chapter is devoted to presenting a brief overview of the most related topics of fracture mechanics. Basic concepts of Linear-Elastic and Elastic-Plastic Fracture Mechanics are presented. Furthermore, a theoretical correlation of the fracture in homogeneous media with interfacial fracture is attempted.

1.2 Linear Elastic Fracture Mechanics (LEFM)

The notion of the stress concentration at the vicinity of a flaw, resulting in a propagating crack and eventually in failure set the ground for linear elastic fracture mechanics (LEFM).

For the case of a traction-free crack of length $2a$ in an infinite plate loaded by a uniaxial stress σ in the direction normal to the crack, the stress components near the crack tip are given by

$$\sigma_{rr}\left(r,\theta\right)=\frac{K_I}{\sqrt{2\pi r}}\left(\frac{5}{4}\cos\frac{\theta}{2}-\frac{1}{4}\cos\frac{3\theta}{2}\right)+\text{h.o.t. in } r\,, \tag{1.1}$$

$$\sigma_{\theta\theta}\left(r,\theta\right)=\frac{K_I}{\sqrt{2\pi r}}\left(\frac{3}{4}\cos\frac{\theta}{2}+\frac{1}{4}\cos\frac{3\theta}{2}\right)+\text{h.o.t. in } r\,, \tag{1.2}$$

$$\sigma_{r\theta}\left(r,\theta\right)=\frac{K_I}{\sqrt{2\pi r}}\left(\frac{1}{4}\sin\frac{\theta}{2}+\frac{1}{4}\sin\frac{3\theta}{2}\right)+\text{h.o.t. in } r\,, \tag{1.3}$$

where (r,θ) are crack tip polar coordinates as shown in Fig. 1, h.o.t. means "higher order terms", and $K_I = \sigma\sqrt{\pi a}$ is the so-called "mode-I stress intensity factor".

Similarly, the corresponding crack tip stress field when the plate is subjected to a macroscopic shear stress τ is of the form

$$\sigma_{rr}(r,\theta) = \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{5}{4}\sin\frac{\theta}{2} + \frac{3}{4}\sin\frac{3\theta}{2}\right) + \text{h.o.t. in } r, \tag{1.4}$$

$$\sigma_{\theta\theta}(r,\theta) = \frac{K_{II}}{\sqrt{2\pi r}}\left(-\frac{3}{4}\sin\frac{\theta}{2} - \frac{3}{4}\sin\frac{3\theta}{2}\right) + \text{h.o.t. in } r, \tag{1.5}$$

$$\sigma_{r\theta}(r,\theta) = \frac{K_{II}}{\sqrt{2\pi r}}\left(\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right) + \text{h.o.t. in } r, \tag{1.6}$$

where now $K_{II} = \tau\sqrt{\pi a}$ is the so-called "mode-II stress intensity factor" (Westergaard, 1939 Williams, 1952, 1957).

The value of the out of plane stress σ_{zz} depends on whether plane stress or plane strain is assumed, i.e., $\sigma_{zz} = 0$ in plane stress and $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ in plane strain. Equations (1.1) -(1.6) apply to all crack-tip stress fields and the stress intensity factors depend on the magnitude and type of the applied loads as well as the geometry under consideration. The terms shown in these equations are the leading terms in an infinite series which describes the solution in the near tip region.

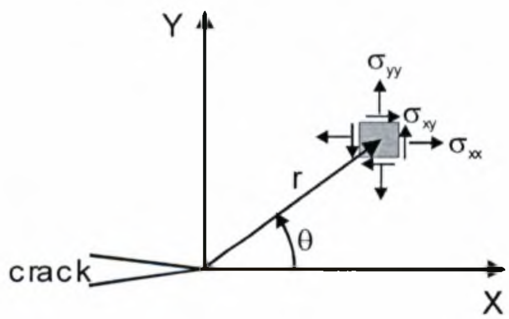

Fig. 1. Definition of stress components and coordinate systems

The quantity K_I which appears in (1.1)-(1.3) plays a central role in "Linear Elastic Fracture Mechanics (LEFM)". Note that

$$K_I = \lim_{r\to 0}\left[\sigma_{\theta\theta}(r,\theta=0)\sqrt{2\pi r}\right] \quad \text{and} \quad K_{II} = \lim_{r\to 0}\left[\sigma_{r\theta}(r,\theta=0)\sqrt{2\pi r}\right].$$

As mentioned above, the K – fields are not the full solution to the problem, but the leading (and dominant) term in asymptotic crack tip solution. If the K – field describes accurately the exact solution in the crack tip region over a distance that is larger than the "fracture process zone"[1], then we can talk about "K – dominance". On the other hand, in a ductile metal, plastic deformation takes place in the crack tip region. When the size of this plastic zone is much smaller than the region of K – dominance, the asymptotic elastic K – solutions are still valid and LEFM can be used (Broek, 1984, Gdoutos, 1993).

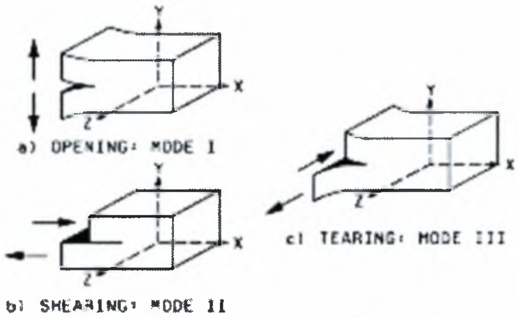

Fig. 2 Independent modes of crack displacements.

There are three "modes" of fracture as shown in Fig. 2, which are distinguished form one another by the relative motion of the upper and lower crack surfaces. The corresponding elastic stress intensity factors are denoted by K_I, K_{II} and K_{III}. The K's are proportional to the magnitude of the applied load and depend on the type of loading and the geometry of the structure under consideration.

When a crack advances, the total potential energy Π of the structure decreases and the "energy release rate" is defined as

$$\wp = -\frac{\partial \Pi}{\partial a},$$

where a is the crack length. In the presence of crack tip K – fields, the energy release rate is related to the stress intensity factors by the following equation

$$\wp = \frac{K_I^2 + K_{II}^2}{E'} + \frac{K_{III}^2}{2G},$$

[1] The region over which the microstructural fracture processes take place.

where G is the elastic shear modulus, E the Young's modulus, ν the Poisson's ratio, $E' = E$ for plane stress, and $E' = \frac{E}{1-\nu^2}$ for plane strain (Irwin, 1957).

In LEFM mode I crack growth initiation takes place when the energy release rate reaches a critical value, say $\wp_c$. This is equivalent to the mode I stress intensity factor reaching a critical value

$$K_{Ic} = \sqrt{\wp_c E'} ,$$

which is the "fracture toughness" of the material.

In some cases, more than one term are needed in the asymptotic expansion of the crack tip solution in order to have an accurate description of the crack tip stresses. The mode I crack tip solution can be written as

$$\sigma_{ij} = \frac{K_I}{\sqrt{2\pi r}} f_{ij}(\theta) + T\,\delta_{1j}\,\delta_{i1} + O\left(r^{1/2}\right), \tag{1.7}$$

where T is the magnitude of the second order term and δ_{ij} is the Kronecker delta. Like K_I, the parameter T is proportional to the magnitude of the applied load and depends on the type of loading and the geometry of the structure under consideration. Note that the second order term in (1.7) is a constant stress $\sigma_{xx} = T$ parallel to the crack line. From a classical fracture mechanics point of view the first singular term is assumed to control the behaviour at the crack tip, and is the only term necessary to consider. However, as will be discussed in the following, this is not always the case and the T – stress can play also an important role.

1.3 Non-linear Fracture Mechanics

LEFM works well in ductile materials when the size of the crack tip plastic zone is smaller than the crack size and all specimen dimensions, and well embedded within the region of K – dominance ("small scale yielding" SSY). This is usually the case in materials for which fracture occurs at stresses well below the yield stress.

When the size of the plastic zone is large compared to the crack size or the dimensions of the specimen Elastic-Plastic Fracture Mechanics (EPFM) must be used. Wells (Wells, 1961) proposed that the opening of the blunted crack due to plastic strains could be used as an alternative fracture mechanics parameter in the cases where the LEFM is not applicable. The idea behind this proposal is that fracture would occur once the opening reached a critical value. This value would then represent the fracture toughness of the material and replace the critical stress intensity factor.

Another major contribution to the elastic-plastic fracture mechanics was Rice's (Rice, 1968) proposal of the J – integral as a parameter to characterise the crack tip loading for the case of non-linear material behaviour. The J – integral is defined for a hyperelastic material as:

$$J = \int_{\Gamma} \left(\bar{U}\, dy - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x} ds \right), \tag{1.8}$$

where $\bar{U}$ is the strain energy density $\left(\boldsymbol{\sigma} = \frac{\partial \bar{U}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \right)$, $\mathbf{T}$ and $\mathbf{u}$ are the traction and displacement vectors respectively, and Γ is the integration path that starts on lower crack face, goes through the materials and ends on the upper crack face as shown in Fig. 3.

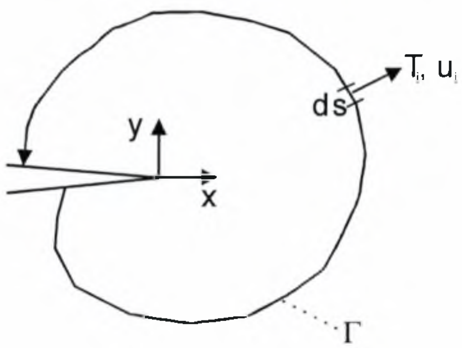

Fig. 3. Arbitrary contour Γ for the evaluation of the J – integral.

Rice showed that J is independent of the particular path Γ used for its evaluation and equals the energy release rate, i.e., $J = \wp = -\frac{\partial \Pi}{\partial a}$.

We consider next a nonlinear elastic material with a constitutive equation of the form

$$\varepsilon_{ij} = \frac{s_{ij}}{2G} + \frac{p}{3K}\delta_{ij} + \frac{3}{2}\alpha\,\varepsilon_0 \left(\frac{\sigma_e}{\sigma_0} \right)^{n-1} \frac{s_{ij}}{\sigma_0}, \tag{1.9}$$

where ε_{ij} is the strain tensor, σ_0 a reference stress usually taken to be equal the yield stress, n the strain hardening exponent, α a dimensionless constant, $\sigma_e = \sqrt{\frac{3}{2} s_{ij}\, s_{ij}}$ the von Mises equivalent stress, and s_{ij} the deviatoric stress tensor, p the hydrostatic pressure, $\varepsilon_0 = \sigma_0 / E$, K and G the bulk and shear modulus respectively. The first two terms on the right hand side of the above equation are the usual "elastic" strains and the last term corresponds to a "plastic" strain with "power law hardening" (J_2 – deformation theory of plasticity). Equation (1.9) cannot be inverted analytically to determine the stress tensor $\boldsymbol{\sigma}$ in terms of the strain tensor $\boldsymbol{\varepsilon}$; of course, numerical inversion is always possible. Using (1.9), we can determine the complementary elastic strain energy density $\bar{U}^c(\boldsymbol{\sigma})$ as follows:

$$\bar{U}^c(\boldsymbol{\sigma}) = \int_0^{\boldsymbol{\sigma}} \varepsilon_{ij}(\boldsymbol{\sigma})\, d\sigma_{ij} = \frac{\sigma_e^2}{6G} + \frac{p^2}{2K} + \frac{1}{n+1}\alpha\, \varepsilon_0\, \sigma_0 \left(\frac{\sigma_e}{\sigma_0}\right)^{n+1}. \tag{1.10}$$

The corresponding elastic strain energy density $\bar{U}(\boldsymbol{\varepsilon})$ is determined from the relationship

$$\bar{U}(\boldsymbol{\varepsilon}) = \int_0^{\boldsymbol{\varepsilon}} \sigma_{ij}(\boldsymbol{\varepsilon})\, d\varepsilon_{ij} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) : \boldsymbol{\varepsilon} - \bar{U}^c(\boldsymbol{\sigma}(\boldsymbol{\varepsilon})). \tag{1.11}$$

However, since the relation $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon})$ is not known analytically, the above calculation of $\bar{U}$ in terms of $\boldsymbol{\varepsilon}$ is not possible. However, it is indeed possible to find $\bar{U}$ in terms of $\boldsymbol{\sigma}$ as follows:

$$\bar{U}(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) - \bar{U}^c(\boldsymbol{\sigma}) = \frac{\sigma_e^2}{6G} + \frac{p^2}{2K} + \frac{n}{n+1}\alpha\, \varepsilon_0\, \sigma_0 \left(\frac{\sigma_e}{\sigma_0}\right)^{n+1}. \tag{1.12}$$

Hutchinson (Hutchinson, 1968) and Rice and Rosengren (Rice and Rosengren, 1968) showed that, for a material that obeys (1.9), the near tip fields of a mode I crack are of the form

$$\frac{\sigma_{ij}}{\sigma_0} = \left(\frac{J}{\alpha\, \varepsilon_0\, \sigma_0\, I_n\, r}\right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}(\theta, n) + \text{h.o.t. in } r, \tag{1.13}$$

$$\frac{\varepsilon_{ij}}{\alpha\, \varepsilon_0} = \left(\frac{J}{\alpha\, \varepsilon_0\, \sigma_0\, I_n\, r}\right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}(\theta, n) + \text{h.o.t. in } r, \tag{1.14}$$

and

$$\frac{u_i}{\alpha\, \varepsilon_0} = \left(\frac{J}{\alpha\, \varepsilon_0\, \sigma_0\, I_n}\right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}}\, \tilde{u}_i(\theta, n) + \text{h.o.t. in } r, \tag{1.15}$$

where $(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\varepsilon}}, \tilde{\mathbf{u}})$ are known dimensionless functions, and I_n a dimensionless constant that depends on n, determined so that $\max_{\theta\in[0,\pi]} \tilde{\sigma}_e = \max_{\theta\in[0,\pi]} \sqrt{\frac{3}{2}\tilde{s}_{ij}\, \tilde{s}_{ij}} = 1$.

The asymptotic solution (1.13)-(1.15) is known as the HRR solution and can be used as an approximate asymptotic mode I crack tip solution for an elastoplastic material that obeys the usual J_2 – flow theory of plasticity (as opposed to J_2 – deformation theory). This approximation is possible because in the crack tip region i) there is no unloading, and ii) the loading is almost "proportional", as has been shown by detailed finite element calculations based on J_2 – flow theory.

Equations (1.13)-(1.15) show that the J – integral is essentially a "plastic stress intensity factor" that controls the magnitude of the crack tip fields and, therefore, can be used as a "fracture parameter" in the sense crack growth initiation occurs when it reaches a critical value, say J_{Ic}. In general, the value of J depends on the applied loads and the specimen geometry, and the fracture criterion takes the form

$$J = J_{Ic} \, .$$

The HRR solution determines the leading term in the asymptotic expansion of the crack tip solution (Mianny, 1998). However, in certain cases more terms are needed for an accurate description of the crack tip fields. Higher order terms were calculated by Li and Wang (Li and Wang, 1986) and Sharma and Aravas (Sharma and Aravas, 1991). The last two authors showed that the crack tip fields can be written in the form

$$\frac{\sigma_{ij}}{\sigma_0} = \left(\frac{J}{a\varepsilon_0\sigma_0 I_n r} \right)^{\frac{1}{n+1}} \tilde{\sigma}_{ij}^{(1)}(\theta,n) + Q\left(\frac{r}{J/\sigma_0} \right)^{t} \tilde{\sigma}_{ij}^{(2)}(\theta,n) + \cdots, \tag{1.16}$$

$$\frac{\varepsilon_{ij}}{\varepsilon_0} = \left(\frac{J}{a\,\varepsilon_0\,\sigma_0\, I_n\, r} \right)^{\frac{n}{n+1}} \tilde{\varepsilon}_{ij}^{(1)}(\theta,n) + Q\left(\frac{r}{J/\sigma_0} \right)^{t} \left(\frac{J}{a\,\varepsilon_0\,\sigma_0\, I_n\, r} \right)^{\frac{n-1}{n+1}} \tilde{\varepsilon}_{ij}^{(2)}(\theta,n) + \cdots, \tag{1.17}$$

$$\frac{u_i}{a\,\varepsilon_0} = \left(\frac{J}{a\,\varepsilon_0\,\sigma_0\, I_n} \right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}}\, \tilde{u}_i^{(1)}(\theta,n) + Q\left(\frac{r}{J/\sigma_0} \right)^{t} \left(\frac{J}{a\,\varepsilon_0\,\sigma_0\, I_n} \right)^{\frac{n-1}{n+1}} r^{\frac{2}{n+1}}\, \tilde{u}_i^{(2)}(\theta,n) + \cdots, \tag{1.18}$$

where $\left(\tilde{\boldsymbol{\sigma}}^{(2)}, \tilde{\boldsymbol{\varepsilon}}^{(2)}, \tilde{\mathbf{u}}^{(2)}\right)$ are known dimensionless functions, and the parameter Q that controls the magnitude of the second term in the expansion of the solution depends on the applied loads and the specimen geometry. Sharma and Aravas showed that in the special cases where $1 < n < 1.6$,

$$t = \frac{n-2}{n-1} < 0 \qquad \text{and} \qquad Q = \left(\alpha\,\varepsilon_0\, I_n\right)^{\frac{n-2}{n-1}} .$$

Details on the calculation of the above asymptotic expansions are presented in Chapter 2 of this Thesis.

1.4. Interfacial Fracture Mechanics

One of the most significant sources of failure in adhesive joints, thin films and composite materials is the propagation of interfacial cracks between the constituent materials. In the last few decades there has been significant progress made towards an understanding of the mechanics of the interface crack within the framework of linear elasticity. The concept of the stress intensity factor (SIF) has proved to be very successful for the homogeneous crack in linear-elastic fracture mechanics (LEFM), and has motivated the development of a similar approach for the interfacial crack.

Williams (Williams, 1959) was the first to develop the asymptotic crack tip solution for a crack lying along the straight interface of two linear elastic materials. He showed that the asymptotic crack tip solution is non variable-separable in (r,θ) and involves "oscillatory singularities" of the form

$$\sigma_{ij}, \varepsilon_{ij} \sim r^{-1/2}(\sin,\cos)(\varepsilon \ln r), \qquad u_i \sim r^{1/2}(\sin,\cos)(\varepsilon \ln r), \tag{1.19}$$

where r is the radius from the crack tip and ε is the oscillation index defined as:

$$\varepsilon = \frac{1}{2\pi}\ln\left(\frac{G_1+\kappa_1 G_2}{G_2+\kappa_2 G_1}\right), \tag{1.20}$$

where $\kappa_j = 3-4\nu_j$ for plane strain, $\kappa_j = (3-\nu_j)/(1+\nu_j)$ for plane stress, and G_j and ν_j $(j=1, 2)$ are the shear moduli and Poisson's ratios of the constituent materials.

By introducing an intrinsic material length scale ℓ_0, Rice (Rice, 1988) defined a complex stress intensity factor $K = K_1 + iK_2$ that has the usual units as

$$\sigma_{yy} + i\sigma_{xy} = \left(\frac{r}{\ell_0}\right)^{i\varepsilon} \frac{K}{\sqrt{2\pi r}} + \text{h.o.t. in } r, \tag{1.21}$$

so that

$$K = \lim_{r\to 0}\left[\left(\sigma_{yy}+i\sigma_{xy}\right)_{\theta=0}\sqrt{2\pi r}\left(\frac{r}{\ell_0}\right)^{-i\varepsilon}\right]. \tag{1.22}$$

Note that in plane strain problems, if both materials are incompressible $(\nu_1=\nu_2=1/2)$, then $\kappa_1=\kappa_2=1$, $\varepsilon=0$, the crack tip oscillations disappear and the crack tip stresses have the usual $\frac{1}{\sqrt{r}}$ singularity.

Under conditions of plane strain, in the case of two **incompressible** materials, or one incompressible and the other rigid, the asymptotic crack tip solution is of the form:

$$\sigma_{xx} = \frac{1}{4\sqrt{2\pi r}}\left[K_I\left(3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\right)+K_{II}\left(-7\sin\frac{\theta}{2}-\sin\frac{5\theta}{2}\right)\right], \tag{1.23}$$

$$\sigma_{yy} = \frac{1}{4\sqrt{2\pi r}}\left[K_I\left(5\cos\frac{\theta}{2}-\cos\frac{5\theta}{2}\right)+K_{II}\left(-\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}\right)\right], \tag{1.24}$$

$$\sigma_{xy} = \frac{1}{4\sqrt{2\pi r}}\left[K_I\left(-\sin\frac{\theta}{2}+\sin\frac{5\theta}{2}\right)+K_{II}\left(3\cos\frac{\theta}{2}+\cos\frac{5\theta}{2}\right)\right]. \tag{1.25}$$

A different approach to the interfacial crack problem was presented by Comninou (Comninou, 1977), who considered the possibility of contact of the crack faces (crack closure) near the tip. The "contact zone model" as it is known, predicts the usual square root stress singularity and its magnitude is completely defined by a shearing mode stress intensity factor. The size of the contact zone is strongly dependent to the remote type of loading, being small compared to the crack size (small-scale contact) when remote tension is applied, but approaching the crack length in the case of remote shear. In the former case, Comninou's solution is valid for distances from the tip smaller than the contact zone size, whereas beyond this distance the traditional solution of the complex stress intensity factor gains validity.

Generally, it was early understood that the elastoplastic analysis of the interface crack seamed more promising.

Finite element numerical results showed that the influence of plastic effects would dominate close to the crack tip and the structure of the resulting stress fields was of a more complex character than for the HRR fields for a homogeneous material due to a coupling between the angular and radial dependence of the stress component.

The corresponding solution for an interfacial crack along the straight interface between two nonlinear materials was considered by Sharma and Aravas (Sharma and Aravas, 1993), who developed a two-term asymptotic expansion of the solution of the HRR-type. In the present thesis, a description of the derivation of the above elastoplastic asymptotic solution is presented. The solution is examined for values of the hardening exponent n near $n=1$ (linear elastic solution). The methodology, along with the corresponding results and conclusions remain to be presented in the next chapters.

2. ASYMPTOTIC ELASTOPLASTIC SOLUTION FOR INTERFACIAL CRACKS

2.1 Introduction

In this chapter is given the formulation of the problem of a plane strain crack lying on the interface between an elastic-plastic power-law hardening material and a rigid substrate as described in Sharma and Aravas (Sharma and Aravas, 1991, 1993). The asymptotic solution is derived from a set of five non-linear differential equations and a linear eigenproblem, which in the next chapter will be used in order to analyze the solution for values of the hardening exponent n near $n = 1$ about the linear problem.

2.2 Problem Formulation

The problem under consideration is that of a crack lying along the surface of a homogeneous deformable matter and a rigid substrate. The constitutive law governing the deformation of the matter is that of the J_2 deformation theory for a Ramberg- Osgood uniaxial stress-strain behavior, namely

$$\varepsilon_{ij} = \frac{1+\nu}{E} s_{ij} + \frac{1-2\nu}{3E} \sigma_{kk}\, \delta_{ij} + \frac{3}{2} \alpha\, \varepsilon_0 \left(\frac{\sigma_e}{\sigma_0} \right)^{n-1} \frac{s_{ij}}{\sigma_0}, \tag{2.1}$$

where the two first terms on the right-hand side correspond to the elastic part of the deformation and the third part represents the strain hardening of the material.

The equilibrium equations corresponding to plain problems using polar coordinates are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{2.2}$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0. \tag{2.3}$$

The kinematical equations of the problem in terms of strains and displacement are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \tag{2.4}$$

$$\varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \tag{2.5}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right). \tag{2.6}$$

Finally, the compatibility equation resulting from (2.4)-(2.6) is

$$\left(\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)\varepsilon_{rr} + \left(\frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right)\varepsilon_{\theta\theta} - \left(\frac{2}{r^2}\frac{\partial}{\partial \theta} + \frac{2}{r}\frac{\partial^2}{\partial r \partial \theta}\right)\varepsilon_{r\theta} = 0. \tag{2.7}$$

2.3 Asymptotic solution and hierarchy of the problem

The problem is formulated considering a polar coordinate system which is centred at the crack tip as shown in Fig. 2.1, and looking for an asymptotic solution to the problem as $r \to 0$. The expansion of the asymptotic solution in terms of stresses is in the form

$$\boldsymbol{\sigma}(r,\theta) = \left(\frac{r}{J}\right)^s \boldsymbol{\sigma}^{(0)}(\theta) + Q r^t \boldsymbol{\sigma}^{(1)}(\theta) + \cdots \text{as } r \to 0, \tag{2.8}$$

where $\boldsymbol{\sigma}^{(0)}$ and $\boldsymbol{\sigma}^{(1)}$ are normalized angular functions, $s < t < \cdots$, J is the $J-$integral, and Q is a parameter that controls the magnitude of the second term and depends on the type and magnitude of the applied loads and on the geometry under consideration.

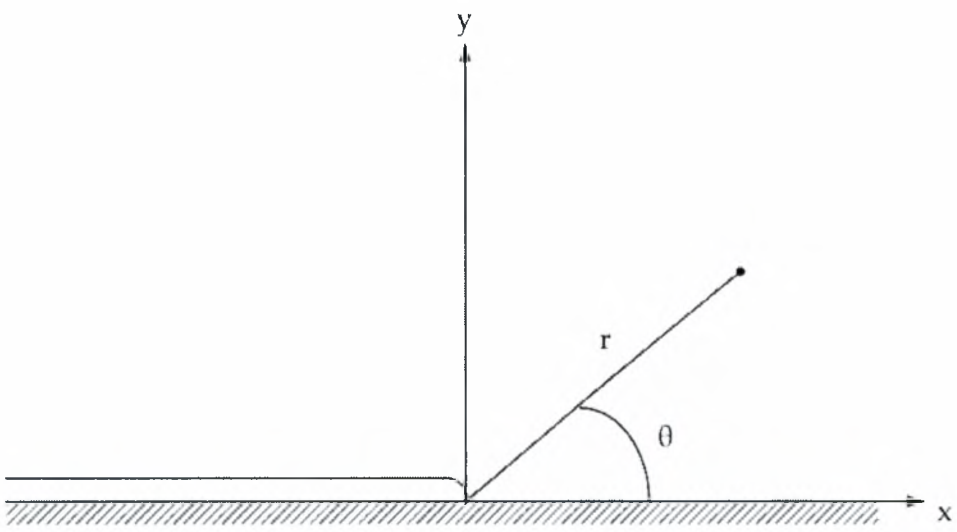

Fig. 2.1 Schematic representation of the crack tip region.

The corresponding expansion for the Mises equivalent stress is

$$\frac{\sigma_e^{(0)}(r,\theta)}{\sigma_0} = r^s \sigma_e^{(0)} + r^t \sigma_e^{(1)} + \cdots as \text{ r} \to 0, \tag{2.9}$$

where

$$\sigma_e^{(0)} = \left(\frac{3}{2} s_{ij}^{(0)} s_{ij}^{(0)}\right)^{1/2} \quad \text{and} \quad \sigma_e^{(1)} = \frac{3}{2}\frac{s_{ij}^{(0)} s_{ij}^{(1)}}{\sigma_e^{(0)}}. \tag{2.10}$$

Plain strain conditions are assumed and the plain of deformation is described by crack-tip polar coordinates r and θ.

The boundary conditions of the problem turn to be

$$u_r(r,0) = 0, \qquad u_\theta(r,0) = 0, \tag{2.11}$$

$$\sigma_{\theta\theta}(r,\pi) = 0, \qquad \sigma_{r\theta}(r,\pi) = 0. \tag{2.12}$$

For the solution to be variable-separable, the J – integral argument results in the leading order exponent s being equal to $-1/(n+1)$ while for elasticity to enter the asymptotic solution no sooner than the third term the second order exponent must be less than $(n-2)/(n+1)$ and remains to be determined from the eigenproblem in the next chapter.

The corresponding to the stress expansion (2.8) strain and displacement expansions are

$$\frac{\boldsymbol{\varepsilon}(r,\theta)}{\alpha\varepsilon_0} = r^{sn}\boldsymbol{\varepsilon}^{(0)}(\theta) + r^{s(n-1)+t}\boldsymbol{\varepsilon}^{(1)}(\theta) + \cdots as \text{ r} \to 0, \tag{2.13}$$

$$\frac{\mathbf{u}(r,\theta)}{\alpha\varepsilon_0} = r^{sn+1}\mathbf{u}^{(0)}(\theta) + r^{s(n-1)+t+1}\mathbf{u}^{(1)}(\theta) + \cdots as \text{ r} \to 0. \tag{2.14}$$

After the J-integral argument is applied the dimensionless stress expansion can be written in the normalized form as

$$\frac{\boldsymbol{\sigma}(r,\theta)}{\sigma_0} = \left(\frac{J}{\alpha\,\varepsilon_0\,\sigma_0\,I_n}\right)^{\frac{1}{n+1}} \tilde{\boldsymbol{\sigma}}^{(0)}(\theta) + Q\left(\frac{r}{J/\sigma_0}\right)^t \tilde{\boldsymbol{\sigma}}^{(1)}(\theta) + \cdots, \tag{2.15}$$

and the corresponding displacement expansion is of the form

$$\frac{\mathbf{u}(r,\theta)}{\alpha\,\varepsilon_0} = \left(\frac{J}{\alpha\,\varepsilon_0\,\sigma_0\,I_n}\right)^{\frac{n}{n+1}} r^{\frac{1}{n+1}}\tilde{\mathbf{u}}^{(0)}(\theta) + \frac{Q}{(J/\sigma_0)^t}\left(\frac{J}{\alpha\,\varepsilon_0\,\sigma_0\,I_n}\right)^{\frac{n-1}{n+1}} r^{-\frac{n-1}{n+1}+t+1}\tilde{\mathbf{u}}^{(1)}(\theta) + \cdots, \tag{2.16}$$

where I_n is defined in Cartesian coordinates as

$$I_n = \int_0^{\pi} \left[\frac{n}{n+1} \tilde{\sigma}^{(0)n+1} \cos\theta - n_i \tilde{\sigma}_{ij}^{(0)} \left(\frac{1}{n+1} \tilde{u}_j^{(0)} \cos\theta - \frac{d\tilde{u}_j^{(0)}}{d\theta} \sin\theta \right) \right] d\theta . \tag{2.17}$$

The expansions (2.15) and (2.16), are next substituted into the governing equations (2.2) -(2.6), the strains are eliminated and terms having like powers of r are collected and the hierarchy of the problem is obtained. The leading order problem defines $\tilde{\boldsymbol{\sigma}}^{(0)}$and $\tilde{\mathbf{u}}^{(0)}$ and consists of five non-linear ordinary differentials equations of the form

$$(s+1)\tilde{\sigma}_{rr}^{(0)} - \tilde{\sigma}_{\theta\theta}^{(0)} + \frac{d\tilde{\sigma}_{r\theta}^{(0)}}{d\theta} = 0 , \tag{2.18}$$

$$\frac{d\tilde{\sigma}_{\theta\theta}^{(0)}}{d\theta} + (s+2)\tilde{\sigma}_{r\theta}^{(0)} = 0 , \tag{2.19}$$

$$(sn+1)\tilde{u}_r^{(0)} - \frac{3}{2}\tilde{\sigma}_e^{(0)n-1}\,\tilde{s}_{rr}^{(0)} = 0 , \tag{2.20}$$

$$\frac{d\tilde{u}_\theta^{(0)}}{d\theta} + \tilde{u}_r^{(0)} - \frac{3}{2}\tilde{\sigma}_e^{(0)n-1}\,\tilde{s}_{\theta\theta}^{(0)} = 0 , \tag{2.21}$$

$$\frac{d\tilde{u}_r^{(0)}}{d\theta} + sn\tilde{u}_\theta^{(0)} - 3\tilde{\sigma}_e^{(0)n-1}\,\tilde{\sigma}_{r\theta}^{(0)} = 0 , \tag{2.22}$$

with boundary conditions

$$\tilde{u}_r^{(0)}(0) = \tilde{u}_\theta^{(0)}(0) = 0 , \tag{2.23}$$

and

$$\tilde{\sigma}_{\theta\theta}^{(0)}(\pi) = \tilde{\sigma}_{r\theta}^{(0)}(\pi) = 0 . \tag{2.24}$$

The second order problem turns to be a linear eigenvalue problem from the solution of which the corresponding eigenvalue t and eigenfunctions $\tilde{\boldsymbol{\sigma}}^{(1)}$ and $\tilde{\mathbf{u}}^{(1)}$ are defined and is of the form

$$(t+1)\tilde{\sigma}_{rr}^{(1)} - \tilde{\sigma}_{\theta\theta}^{(1)} + \frac{d\tilde{\sigma}_{r\theta}^{(1)}}{d\theta} = 0 , \tag{2.25}$$

$$\frac{d\tilde{\sigma}_{\theta\theta}^{(1)}}{d\theta} + (t+2)\tilde{\sigma}_{r\theta}^{(1)} = 0 , \tag{2.26}$$

$$\left[s(n-1)+t+1\right]\tilde{u}_r^{(1)} - \frac{3}{2}\tilde{\sigma}_e^{(0)^{n-1}} \left[\tilde{s}_{rr}^{(1)} + \frac{3}{2}(n-1)\frac{\tilde{s}_{kl}^{(0)}\tilde{\sigma}_{kl}^{(1)}}{\tilde{\sigma}_e^{(0)^2}}\tilde{s}_{rr}^{(0)} \right] = 0 , \tag{2.27}$$

$$\tilde{u}_r^{(1)} + \frac{d\tilde{u}_\theta^{(1)}}{d\theta} - \frac{3}{2}\tilde{\sigma}_e^{(0)^{n-1}} \left[\tilde{s}_{\theta\theta}^{(1)} + \frac{3}{2}(n-1)\frac{\tilde{s}_{kl}^{(0)}\tilde{\sigma}_{kl}^{(1)}}{\tilde{\sigma}_e^{(0)^2}}\tilde{s}_{\theta\theta}^{(0)} \right] = 0 , \tag{2.28}$$

$$\frac{d\tilde{u}_r^{(1)}}{d\theta}+\left[s(n-1)+t\right]\tilde{u}_\theta^{(1)}-3\tilde{\sigma}_e^{(0)^{n-1}}\left[\tilde{\sigma}_{r\theta}^{(1)}+\frac{3}{2}(n-1)\frac{\tilde{s}_{kl}^{(0)}\tilde{\sigma}_{kl}^{(1)}}{\tilde{\sigma}_e^{(0)^2}}\tilde{\sigma}_{r\theta}^{(0)}\right]=0, \tag{2.29}$$

with the corresponding boundary conditions

$$\tilde{u}_r^{(1)}(0)=\tilde{u}_\theta^{(1)}(0)=0 \tag{2.30}$$

and

$$\tilde{\sigma}_{\theta\theta}^{(1)}(\pi)=\tilde{\sigma}_{r\theta}^{(1)}(\pi)=0. \tag{2.31}$$

It is important to notice that the linear eigenproblem (2.25)-(2.29) corresponds to the condition $s<t<(n-2)/(n+1)<0$, which is the case for the second term being singular and be determined to within the constant Q.

For $s<t=(n-2)/(n+1)<0$ the corresponding second order eigenproblem is of different form and can be found in detail in Sharma and Aravas (Sharma and Aravas, 1991). Finally, the condition $s=t$ corresponds to the linear elastic and perfectly plastic case (i.e., $n=1$ and $n=\infty$) respectively.

As already mentioned, the leading term describing the stress field derives from the solution of (2.18)-(2.22) and involves only the $J-$integral as an arbitrary constant, the mode-mix on the interface can then be determined by the asymptotic solution and is found to be of Mode-I tension. The second term is also singular for all values of the hardening exponent exhibiting a Mode-II like stress field on the crack tip. The analytical form of the two terms in the stress expansion, for values of the hardening exponent slightly greater than the linear elastic case, is obtained in the next chapter.

3. ASYMPTOTIC SOLUTION VIA A PERTURBATION METHOD

3.1 Introduction

In this chapter a methodology is developed for the analytical determination of the asymptotic solution discussed in Chapter 2 for small deviations from linearity, i.e., for values of the hardening exponent n slightly larger than unity.

3.2 The Leading order eigenproblem

As discussed in Chapter 2, the leading order problem in the crack tip asymptotic expansion of the solution for crack along the interface between a nonlinear material and a rigid substrate is of the form

$$(s+1)\tilde{\sigma}_{rr}^{(0)} - \tilde{\sigma}_{\theta\theta}^{(0)} + \frac{d\tilde{\sigma}_{r\theta}^{(0)}}{d\theta} = 0, \tag{3.1}$$

$$\frac{d\tilde{\sigma}_{\theta\theta}^{(0)}}{d\theta} + (s+2)\tilde{\sigma}_{r\theta}^{(0)} = 0, \tag{3.2}$$

$$(sn+1)\tilde{u}_r^{(0)} - \frac{3}{2}\tilde{\sigma}_e^{(0)n-1}\tilde{s}_{rr}^{(0)} = 0, \tag{3.3}$$

$$\frac{d\tilde{u}_\theta^{(0)}}{d\theta} + \tilde{u}_r^{(0)} - \frac{3}{2}\tilde{\sigma}_e^{(0)n-1}\tilde{s}_{\theta\theta}^{(0)} = 0, \tag{3.4}$$

$$\frac{d\tilde{u}_r^{(0)}}{d\theta} + sn\tilde{u}_\theta^{(0)} - 3\tilde{\sigma}_e^{(0)n-1}\tilde{\sigma}_{r\theta}^{(0)} = 0, \tag{3.5}$$

with boundary conditions

$$\tilde{u}_r^{(0)}(0) = \tilde{u}_\theta^{(0)}(0) = 0, \tag{3.6}$$

$$\tilde{\sigma}_{\theta\theta}^{(0)}(\pi) = \tilde{\sigma}_{r\theta}^{(0)}(\pi) = 0. \tag{3.7}$$

It should be noted that (3.3) is an algebraic equation whereas the rest of the above are differential equations.

The plane strain condition $\varepsilon_{zz} = 0$ of an incompressible material $(\nu=1/2)$ implies that $s_{zz} = 0$, so that $\sigma_{zz} = (\sigma_{xx} + \sigma_{yy})/2$ and $s_{rr} = -s_{\theta\theta} = (\sigma_{rr} - \sigma_{\theta\theta})/2$. Therefore,

$$\tilde{s}_{rr}^{(0)} = -\tilde{s}_{\theta\theta}^{(0)} = \left(\tilde{\sigma}_{rr}^{(0)} - \tilde{\sigma}_{\theta\theta}^{(0)}\right)/2. \tag{3.8}$$

In order to simplify the system of equations, we replace (3.4) by the sum (3.3)+(3.4):

$$(3.3)+(3.4) \quad \Rightarrow \quad (sn+2)\tilde{u}_r^{(0)} + \frac{d\tilde{u}_\theta^{(0)}}{d\theta} = 0. \tag{3.9}$$

We can now eliminate $\tilde{\sigma}_{rr}^{(0)}$ from the system of equations as follows. Solving (3.3) for $\tilde{s}_{rr}^{(0)}$ we find $\tilde{s}_{rr}^{(0)} = \frac{2}{3}(sn+1)\frac{\tilde{u}_r^{(0)}}{\tilde{\sigma}_e^{(0)n-1}}$. Combining this result with (3.8), we conclude that

$$\tilde{\sigma}_{rr}^{(0)} = 2\tilde{s}_{rr}^{(0)} + \tilde{\sigma}_{\theta\theta}^{(0)}, \qquad \tilde{s}_{rr}^{(0)} = \frac{2}{3}(sn+1)\frac{\tilde{u}_r^{(0)}}{\tilde{\sigma}_e^{(0)n-1}}. \tag{3.10}$$

Also, substituting (3.10) into (3.1), we find

$$\frac{d\tilde{\sigma}_{r\theta}^{(0)}}{d\theta} + s\tilde{\sigma}_{\theta\theta}^{(0)} + \frac{4}{3}(s+1)(sn+1)\frac{\tilde{u}_r^{(0)}}{\tilde{\sigma}_e^{(0)n-1}} = 0. \tag{3.11}$$

In summary, we have the following system of equations for $\left(\tilde{u}_r^{(0)}, \tilde{u}_\theta^{(0)}, \tilde{\sigma}_{\theta\theta}^{(0)}, \tilde{\sigma}_{r\theta}^{(0)}\right)$

$$(3.5) \quad \Rightarrow \quad \frac{d\tilde{u}_r^{(0)}}{d\theta} + sn\tilde{u}_\theta^{(0)} = 3\tilde{\sigma}_e^{n-1}\tilde{\sigma}_{r\theta}^{(0)}, \tag{3.12}$$

$$(3.9) \quad \Rightarrow \quad \frac{d\tilde{u}_\theta^{(0)}}{d\theta} + (sn+2)\tilde{u}_r^{(0)} = 0, \tag{3.13}$$

$$(3.2) \quad \Rightarrow \quad \frac{d\tilde{\sigma}_{\theta\theta}^{(0)}}{d\theta} + (s+2)\tilde{\sigma}_{r\theta}^{(0)} = 0, \tag{3.14}$$

$$(3.11) \quad \Rightarrow \quad \frac{d\tilde{\sigma}_{r\theta}^{(0)}}{d\theta} + s\tilde{\sigma}_{\theta\theta}^{(0)} = -\frac{4}{3}(s+1)(sn+1)\frac{\tilde{u}_r^{(0)}}{\tilde{\sigma}_e^{(0)n-1}}, \tag{3.15}$$

together with the boundary conditions (3.6) and (3.7). The stress component $\tilde{\sigma}_{rr}^{(0)}$ is determined from (3.10).

The above system of ordinary differential equations can be written in matrix form as

$$\frac{d}{d\theta}\begin{Bmatrix}\tilde{u}_r^{(0)}\\ \tilde{u}_\theta^{(0)}\\ \tilde{\sigma}_{\theta\theta}^{(0)}\\ \tilde{\sigma}_{r\theta}^{(0)}\end{Bmatrix} + \begin{bmatrix}0 & sn & 0 & 0\\ sn+2 & 0 & 0 & 0\\ 0 & 0 & 0 & s+2\\ 0 & 0 & s & 0\end{bmatrix}\begin{Bmatrix}\tilde{u}_r^{(0)}\\ \tilde{u}_\theta^{(0)}\\ \tilde{\sigma}_{\theta\theta}^{(0)}\\ \tilde{\sigma}_{r\theta}^{(0)}\end{Bmatrix} = \begin{bmatrix}0 & 0 & 0 & 3\tilde{\sigma}_e^{(0)n-1}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -\frac{4}{3}\frac{(s+1)(sn+1)}{\tilde{\sigma}_e^{(0)n-1}} & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}\tilde{u}_r^{(0)}\\ \tilde{u}_\theta^{(0)}\\ \tilde{\sigma}_{\theta\theta}^{(0)}\\ \tilde{\sigma}_{r\theta}^{(0)}\end{Bmatrix}, \tag{3.16}$$

where the right hand side is the non-linear part of the equation.

The perturbation of the leading term is next performed by letting the hardening exponent n be $n = 1+\varepsilon$. For $n = 1+\varepsilon$ we find

$$\tilde{\sigma}_e^{(0)n-1} = \tilde{\sigma}_e^{(0)\varepsilon} = 1 + \varepsilon\ln\tilde{\sigma}_e^{(0)} + \frac{1}{2}\varepsilon^2\left(\ln\tilde{\sigma}_e^{(0)}\right)^2 + O\left(\varepsilon^3\right),$$

$$\frac{1}{\tilde{\sigma}_e^{(0)n-1}} = \frac{1}{\tilde{\sigma}_e^{(0)\varepsilon}} = \tilde{\sigma}_e^{(0)-\varepsilon} = 1 - \varepsilon \ln \tilde{\sigma}_e^{(0)} + \frac{1}{2}\varepsilon^2 \left(\ln \tilde{\sigma}_e^{(0)}\right)^2 + O\left(\varepsilon^3\right),$$

$$-\frac{4}{3}\frac{(s+1)(sn+1)}{\tilde{\sigma}_e^{(0)n-1}} = -\frac{4}{3}(s+1)(s+1+\varepsilon s)\left[1 - \varepsilon \ln \tilde{\sigma}_e^{(0)} + \frac{1}{2}\varepsilon^2\left(\ln \tilde{\sigma}_e^{(0)}\right)^2 + O\left(\varepsilon^3\right)\right] =$$

$$= -\frac{4}{3}(s+1)^2 + \varepsilon\frac{4}{3}(s+1)\left[(s+1)\ln \tilde{\sigma}_e^{(0)} - s\right] + \varepsilon^2\frac{4}{3}(s+1)\ln \tilde{\sigma}_e^{(0)}\left(s - \frac{s+1}{2}\ln \tilde{\sigma}_e^{(0)}\right) + O\left(\varepsilon^3\right).$$

Substituting the above expressions in (3.16), we find

$$\frac{d}{d\theta}\begin{Bmatrix} \tilde{u}_r^{(0)} \\ \tilde{u}_\theta^{(0)} \\ \tilde{\sigma}_{\theta\theta}^{(0)} \\ \tilde{\sigma}_{r\theta}^{(0)} \end{Bmatrix} - \begin{bmatrix} 0 & -s & 0 & 3 \\ -(s+2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(s+2) \\ -\frac{4}{3}(s+1)^2 & 0 & -s & 0 \end{bmatrix}\begin{Bmatrix} \tilde{u}_r^{(0)} \\ \tilde{u}_\theta^{(0)} \\ \tilde{\sigma}_{\theta\theta}^{(0)} \\ \tilde{\sigma}_{r\theta}^{(0)} \end{Bmatrix} =$$

$$= \varepsilon\begin{bmatrix} 0 & -s & 0 & 3\ln \tilde{\sigma}_e^{(0)} \\ -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{3}(s+1)\left[(s+1)\ln \tilde{\sigma}_e^{(0)} - s\right] & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \tilde{u}_r^{(0)} \\ \tilde{u}_\theta^{(0)} \\ \tilde{\sigma}_{\theta\theta}^{(0)} \\ \tilde{\sigma}_{r\theta}^{(0)} \end{Bmatrix} +$$

$$+ \varepsilon^2\begin{bmatrix} 0 & 0 & 0 & \frac{3}{2}\left(\ln \tilde{\sigma}_e^{(0)}\right)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{3}(s+1)\ln \tilde{\sigma}_e^{(0)}\left(s - \frac{s+1}{2}\ln \tilde{\sigma}_e^{(0)}\right) & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \tilde{u}_r^{(0)} \\ \tilde{u}_\theta^{(0)} \\ \tilde{\sigma}_{\theta\theta}^{(0)} \\ \tilde{\sigma}_{r\theta}^{(0)} \end{Bmatrix} + O\left(\varepsilon^3\right). \quad (3.17)$$

In general, the leading order non-linear eigenproblem (3.1)-(3.7) has a solution to within a "multiplicative" constant in the sense that, if $\left(\tilde{\mathbf{u}}^{(0)}, \tilde{\boldsymbol{\sigma}}^{(0)}\right)$ is a solution, then $\left(c^n\,\tilde{\mathbf{u}}^{(0)}, c\,\tilde{\boldsymbol{\sigma}}^{(0)}\right)$ is a solution as well.

There is also the possibility of a degenerate case, in which the solution has more than one arbitrary constants (e.g., simultaneous mode-I and mode-II solutions leading to an arbitrary mode mix at the crack tip). To allow for this possibility, we set

$$\tilde{\sigma}_{\theta\theta}^{(0)}(0) = \tilde{A}_I \qquad \text{and} \qquad \tilde{\sigma}_{r\theta}^{(0)}(0) = \tilde{A}_{II}, \quad (3.18)$$

where $\tilde{A}_I$ and $\tilde{A}_{II}$ are the aforementioned two arbitrary constants that control the magnitude of the leading order solution and define the so-called "mode mix".

We can normalize the solution by dividing through by $\tilde{A}_I$ and end up with the normalized problem for $\left(\mathbf{u}^{(0)}, \boldsymbol{\sigma}^{(0)}\right)$ (without tilde) that corresponds to

$$\sigma_{\theta\theta}^{(0)}(0) = 1 \qquad \text{and} \qquad \sigma_{r\theta}^{(0)}(0) = \frac{\tilde{A}_{II}}{\tilde{A}_I} = A_{II}, \tag{3.19}$$

provided that $\tilde{A}_I = \tilde{\sigma}_{\theta\theta}^{(0)}(0) \neq 0$.

Remark: We treat separately the special case in which $\tilde{\sigma}_{\theta\theta}^{(0)}(0) = 0$ and $\tilde{\sigma}_{r\theta}^{(0)}(0) = \tilde{A}_{II} \neq 0$. Using the methodology described in the following, it is shown in Appendix A that it is impossible to find a solution in that case. □

The corresponding complete solution $\left(\tilde{\mathbf{u}}^{(0)}, \tilde{\boldsymbol{\sigma}}^{(0)}\right)$ can be written as

$$\tilde{\boldsymbol{\sigma}}^{(0)} = \tilde{A}_I\, \boldsymbol{\sigma}^{(0)} \qquad \text{and} \qquad \tilde{\mathbf{u}}^{(0)} = \tilde{A}_I^n\, \mathbf{u}^{(0)}. \tag{3.20}$$

For $n = 1 + \varepsilon$, the normalized (i.e., without tilde) solution $\boldsymbol{\sigma}^{(0)}$ and $\mathbf{u}^{(0)}$ can be written as

$$\boxed{\boldsymbol{\sigma}^{(0)} = \boldsymbol{\Sigma}^{(0)} + \varepsilon\, \boldsymbol{\Sigma}^{(1)} + \varepsilon^2\, \boldsymbol{\Sigma}^{(2)} + O\left(\varepsilon^3\right)} \qquad \text{and} \qquad \boxed{\mathbf{u}^{(0)} = \mathbf{U}^{(0)} + \varepsilon\, \mathbf{U}^{(1)} + \varepsilon^2\, \mathbf{U}^{(2)} + O\left(\varepsilon^3\right)}.$$

Note also that

$$\tilde{A}_I^n = \tilde{A}_I^{1+\varepsilon} = \tilde{A}_I\, \tilde{A}_I^\varepsilon = \tilde{A}_I \left[1 + \varepsilon \ln \tilde{A}_I + \frac{1}{2}\varepsilon^2 \left(\ln \tilde{A}_I\right)^2 + O\left(\varepsilon^3\right)\right].$$

Then, equations (3.20) become

$$\tilde{\boldsymbol{\sigma}}^{(0)} = \tilde{A}_I\, \boldsymbol{\sigma}^{(0)}(n) \quad \Rightarrow \quad \boxed{\tilde{\boldsymbol{\sigma}}^{(0)} = \tilde{A}_I\, \boldsymbol{\Sigma}^{(0)} + \varepsilon\, \tilde{A}_I\, \boldsymbol{\Sigma}^{(1)} + \varepsilon^2\, \tilde{A}_I\, \boldsymbol{\Sigma}^{(2)} + O\left(\varepsilon^3\right)}$$

and

$$\tilde{\mathbf{u}}^{(0)} = \tilde{A}_I^n\, \mathbf{u}^{(0)}(n) = \tilde{A}_I \left[1 + \varepsilon \ln \tilde{A}_I + \frac{1}{2}\varepsilon^2 \left(\ln \tilde{A}_I\right)^2 + O\left(\varepsilon^3\right)\right]\left[\mathbf{U}^{(0)} + \varepsilon\, \mathbf{U}^{(1)} + \varepsilon^2\, \mathbf{U}^{(2)} + O\left(\varepsilon^3\right)\right] \quad \Rightarrow$$

$$\boxed{\tilde{\mathbf{u}}^{(0)} = \tilde{A}_I\, \mathbf{U}^{(0)} + \varepsilon\, \tilde{A}_I \left[\left(\ln \tilde{A}_I\right)\mathbf{U}^{(0)} + \mathbf{U}^{(1)}\right] + \varepsilon^2\, \tilde{A}_I \left[\frac{1}{2}\left(\ln \tilde{A}_I\right)^2 \mathbf{U}^{(0)} + \left(\ln \tilde{A}_I\right)\mathbf{U}^{(1)} + \mathbf{U}^{(2)}\right] + O\left(\varepsilon^3\right)}.$$

3.3 Asymptotic solution of the leading order problem

Equation (3.17) can be written in a compact form as

$$\frac{d\mathbf{x}(\theta)}{d\theta} - \mathbf{F}(s) \cdot \mathbf{x}(\theta) = \varepsilon\, \mathbf{G}\left(\mathbf{x}(\theta), s\right) \cdot \mathbf{x}(\theta), \tag{3.21}$$

where

$$\mathbf{x} = \left\{u_r^{(0)}, u_\theta^{(0)}, \sigma_{\theta\theta}^{(0)}, \sigma_{r\theta}^{(0)}\right\}, \tag{3.22}$$

are the normalized unknown stress and displacement components of the asymptotic solution,

$$\mathbf{F}(s)=\begin{bmatrix} 0 & -s & 0 & 3 \\ -(s+2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(s+2) \\ -\frac{4}{3}(s+1)^2 & 0 & -s & 0 \end{bmatrix}, \tag{3.23}$$

and

$$\mathbf{G}(\mathbf{x}(\theta),s)=\begin{bmatrix} 0 & -s & 0 & 3\ln\sigma_e^{(0)} \\ -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{3}(s+1)\left[(s+1)\ln\sigma_e^{(0)}-s\right] & 0 & 0 & 0 \end{bmatrix}+ \\ +\varepsilon\begin{bmatrix} 0 & 0 & 0 & \frac{3}{2}\left(\ln\sigma_e^{(0)}\right)^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{3}(s+1)\ln\sigma_e^{(0)}\left(s-\frac{s+1}{2}\ln\sigma_e^{(0)}\right) & 0 & 0 & 0 \end{bmatrix}+O(\varepsilon^2). \tag{3.24}$$

The differential equation (3.21) can be written in an integral equation form as (Boyce and Di Prima, 1977)

$$\mathbf{x}(\theta)=\boldsymbol{\Psi}(\theta,s)\cdot\boldsymbol{\Psi}^{-1}(0,s)\cdot\mathbf{x}(0)+\varepsilon\boldsymbol{\Psi}(\theta,s)\cdot\int_0^\theta\boldsymbol{\Psi}^{-1}(\phi,s)\cdot\mathbf{G}(\mathbf{x}(\phi),s)\cdot\mathbf{x}(\phi)d\phi, \tag{3.25}$$

where

$$\boldsymbol{\Psi}(\theta,s)=\begin{bmatrix} a\sin(s\theta) & -a\cos(s\theta) & \cos[(2+s)\theta] & \sin[(2+s)\theta] \\ b\cos(s\theta) & b\sin(s\theta) & -\sin[(2+s)\theta] & \cos[(2+s)\theta] \\ -c\sin(s\theta) & c\cos(s\theta) & -d\cos[(2+s)\theta] & -d\sin[(2+s)\theta] \\ \cos(s\theta) & \sin(s\theta) & -d\sin[(2+s)\theta] & d\cos[(2+s)\theta] \end{bmatrix},$$

with $a=\frac{3}{2(1+s)}$, $c=\frac{2+s}{s}$, $b=ac$, and $d=\frac{1}{a}$.

The first term in the right hand side is the solution of the linear homogeneous problem $\frac{d\mathbf{x}(\theta)}{d\theta}-\mathbf{F}(s)\mathbf{x}(\theta)=\mathbf{0}$, and the second term is a partial solution of the non-homogeneous problem (3.21). The columns of matrix $\boldsymbol{\Psi}$ are the fundamental solutions of the homogeneous version of equation (3.21) (Boyce and Di Prima, 1977).

The integral equation form (3.25) of the problem is convenient for the development of an asymptotic expansion of the solution $\mathbf{x}(\theta)$ in ε.

The corresponding boundary conditions are

$$u_r^{(0)}(0) = u_\theta^{(0)}(0) = 0 , \qquad \sigma_{\theta\theta}^{(0)}(\pi) = \sigma_{r\theta}^{(0)}(\pi) = 0 .$$

The solution is normalized so that

$$\sigma_{\theta\theta}^{(0)}(0) = 1 , \qquad \sigma_{r\theta}^{(0)}(0) = A_{II} . \tag{3.26}$$

We look for an asymptotic expansion of the solution in the form

$$\mathbf{x} = \mathbf{x}^{(0)} + \varepsilon\,\mathbf{x}^{(1)} + \varepsilon^2\,\mathbf{x}^{(2)} + O\left(\varepsilon^3\right), \tag{3.27}$$

or, equivalently,

$$\boldsymbol{\sigma}^{(0)} = \boldsymbol{\Sigma}^{(0)} + \varepsilon\,\boldsymbol{\Sigma}^{(1)} + \varepsilon^2\,\boldsymbol{\Sigma}^{(2)} + O\left(\varepsilon^3\right) \quad \text{and} \quad \mathbf{u}^{(0)} = \mathbf{U}^{(0)} + \varepsilon\,\mathbf{U}^{(1)} + \varepsilon^2\,\mathbf{U}^{(2)} + O\left(\varepsilon^3\right). \tag{3.28}$$

Then

$$\sigma_e^{(0)} = \sqrt{\frac{3}{2} s_{ij}^{(0)} s_{ij}^{(0)}} = \Sigma_e^{(0)} + \varepsilon\,\Sigma_e^{(1)} + O\left(\varepsilon^2\right), \qquad \Sigma_e^{(0)} = \sqrt{\frac{3}{2} S_{ij}^{(0)} S_{ij}^{(0)}} = \sqrt{3\left(S_{rr}^{(0)^2} + \Sigma_{r\theta}^{(0)^2}\right)}, \tag{3.29}$$

$$\Sigma_e^{(1)} = \frac{3}{2}\frac{S_{ij}^{(0)} S_{ij}^{(1)}}{\Sigma_e^{(0)}} = \frac{3\left(S_{rr}^{(0)} S_{rr}^{(1)} + \Sigma_{r\theta}^{(0)}\Sigma_{r\theta}^{(1)}\right)}{\Sigma_e^{(0)}}, \qquad \ln\sigma_e^{(0)} = \ln\Sigma_e^{(0)} + \varepsilon\frac{\Sigma_e^{(1)}}{\Sigma_e^{(0)}} + O\left(\varepsilon^2\right),$$

where $\mathbf{S}^{(0)}$ and $\mathbf{S}^{(1)}$ are the deviatoric parts of $\boldsymbol{\Sigma}^{(0)}$ and $\boldsymbol{\Sigma}^{(1)}$ respectively.

Also, the normalization conditions (3.26) become

$$\sigma_{\theta\theta}^{(0)}(0) = \Sigma_{\theta\theta}^{(0)}(0) + \varepsilon\,\Sigma_{\theta\theta}^{(1)}(0) + \varepsilon^2\,\Sigma_{\theta\theta}^{(2)}(0) + O\left(\varepsilon^3\right) = 1$$

and $$\sigma_{r\theta}^{(0)}(0) = \Sigma_{r\theta}^{(0)}(0) + \varepsilon\,\Sigma_{r\theta}^{(1)}(0) + \varepsilon^2\,\Sigma_{r\theta}^{(2)}(0) + O\left(\varepsilon^3\right) = A_{II} ,$$

so that

$$\Sigma_{\theta\theta}^{(0)}(0) = 1 , \qquad \Sigma_{\theta\theta}^{(1)}(0) = \Sigma_{\theta\theta}^{(2)}(0) = \cdots = 0$$

and $$\Sigma_{r\theta}^{(0)}(0) = A_{II} , \qquad \Sigma_{r\theta}^{(1)}(0) = \Sigma_{r\theta}^{(2)}(0) = \cdots = 0 .$$

The matrix $\mathbf{G}\left(\mathbf{x}(\theta), s\right)$ on the right hand side of (3.21) becomes

$$\mathbf{G} = \mathbf{G}^{(0)} + \varepsilon\,\mathbf{G}^{(1)} + O\left(\varepsilon^2\right), \tag{3.30}$$

where $$\mathbf{G}^{(0)}\left(\mathbf{x}^{(0)}, s\right) = \begin{bmatrix} 0 & -s & 0 & 3\ln\Sigma_e^{(0)} \\ -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{3}(s+1)\left[(s+1)\ln\Sigma_e^{(0)} - s\right] & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{G}^{(1)}\left(\mathbf{x}^{(0)},s\right)=\begin{bmatrix} 0 & 0 & 0 & \frac{3}{2}\left(\ln\Sigma_e^{(0)}\right)^2+3\frac{\Sigma_e^{(1)}}{\Sigma_e^{(0)}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4}{3}(s+1)\ln\Sigma_e^{(0)}\left(s-\frac{s+1}{2}\ln\Sigma_e^{(0)}\right)+\frac{4}{3}(s+1)^2\frac{\Sigma_e^{(1)}}{\Sigma_e^{(0)}} & 0 & 0 & 0 \end{bmatrix}.$$

The leading order stress exponent $s=-\frac{1}{n+1}$ can be written as

$$s=-\frac{1}{n+1}=-\frac{1}{2}+\frac{1}{4}\varepsilon-\frac{1}{8}\varepsilon^2+O\left(\varepsilon^3\right). \tag{3.31}$$

Substituting(3.27), (3.30) and (3.31) into (3.25), and collecting terms having like powers of ε, we have arrive at the following sequence of problems:

$$O(1):\quad \mathbf{x}^{(0)}(\theta)=\mathbf{A}\left(\theta,-\frac{1}{2}\right)\cdot\mathbf{x}^{(0)}(0), \tag{3.32}$$

where $\mathbf{A}(\theta,s)=\boldsymbol{\Psi}(\theta,s)\cdot\boldsymbol{\Psi}^{-1}(0,s)$, and $\mathbf{x}^{(0)}(0)=\{0\quad 0\quad 1\quad A_{II}\}$,

$$O(\varepsilon):\quad \mathbf{x}^{(1)}(\theta)=\frac{1}{4}\left[\frac{\partial\mathbf{A}(\theta,s)}{\partial s}\right]_{s=-1/2}\cdot\mathbf{x}^{(0)}(0)+\int_0^\theta\mathbf{B}\left(\theta,\phi,-\frac{1}{2}\right)\cdot\mathbf{G}^{(0)}\left(\mathbf{x}^{(0)}(\phi),-\frac{1}{2}\right)\cdot\mathbf{x}^{(0)}(\phi)\,d\phi, \tag{3.33}$$

where $\mathbf{B}(\theta,\phi,s)=\boldsymbol{\Psi}(\theta,s)\cdot\boldsymbol{\Psi}^{-1}(\phi,s)$,

$$\begin{aligned} O\left(\varepsilon^2\right):\quad \mathbf{x}^{(2)}(\theta)=&\left[-\frac{1}{8}\frac{\partial\mathbf{A}(\theta,s)}{\partial s}+\frac{1}{32}\frac{\partial^2\mathbf{A}(\theta,s)}{\partial s^2}\right]_{s=-1/2}\cdot\mathbf{x}^{(0)}(0)+\\ &+\frac{1}{4}\left\{\frac{\partial}{\partial s}\left[\int_0^\theta\mathbf{B}(\theta,\phi,s)\cdot\mathbf{G}^{(0)}\left(\mathbf{x}^{(0)}(\phi),s\right)\cdot\mathbf{x}^{(0)}(\phi)\,d\phi\right]\right\}_{s=-1/2}+\\ &+\int_0^\theta\mathbf{B}\left(\theta,\phi,-\frac{1}{2}\right)\cdot\left[\mathbf{G}^{(0)}\left(\mathbf{x}^{(0)}(\phi),-\frac{1}{2}\right)\cdot\mathbf{x}^{(1)}(\phi)+\mathbf{G}^{(1)}\left(\mathbf{x}^{(0)}(\phi),-\frac{1}{2}\right)\cdot\mathbf{x}^{(0)}(\phi)\right]d\phi. \end{aligned} \tag{3.34}$$

Also, equation (3.10) that determines $\sigma_{rr}^{(0)}$ implies

$$S_{rr}^{(0)}=\frac{U_r^{(0)}}{3},\qquad \Sigma_{rr}^{(0)}=2S_{rr}^{(0)}+\Sigma_{\theta\theta}^{(0)}, \tag{3.35}$$

$$S_{rr}^{(1)}=\frac{1}{3}\left[U_r^{(1)}-U_r^{(0)}\left(\frac{1}{2}+\ln\Sigma_e^{(0)}\right)\right],\qquad \Sigma_{rr}^{(1)}=2S_{rr}^{(1)}+\Sigma_{\theta\theta}^{(1)}, \tag{3.36}$$

$$S_{rr}^{(2)} = \frac{1}{3}\left[U_r^{(2)} - U_r^{(1)}\left(\frac{1}{2} + \ln\Sigma_e^{(0)}\right) + U_r^{(0)}\left(\frac{1}{4} + \frac{1}{2}\ln\Sigma_e^{(0)} + \frac{1}{2}\left(\ln\Sigma_e^{(0)}\right)^2 - \frac{\Sigma_e^{(1)}}{\Sigma_e^{(0)}}\right)\right],$$

$$\Sigma_{rr}^{(2)} = 2S_{rr}^{(2)} + \Sigma_{\theta\theta}^{(2)}. \qquad (3.37)$$

Equations (3.32)-(3.34) are evaluated by using Mathematica®.

The solution $\mathbf{x}^{(0)}(\theta)$ of the $O(1)$ problem is found to be

$$\mathbf{x}^{(0)}(\theta) = \begin{Bmatrix} \frac{3}{2}\sin\frac{\theta}{2}\sin\theta \\ -3\sin^3\frac{\theta}{2} \\ \cos^3\frac{\theta}{2} \\ \frac{1}{2}\cos\frac{\theta}{2}\sin\theta \end{Bmatrix} + A_{II}\begin{Bmatrix} \frac{3}{2}\sin\frac{\theta}{2}(1+3\cos\theta) \\ -\frac{9}{2}\sin\frac{\theta}{2}\sin\theta \\ -\frac{3}{2}\cos\frac{\theta}{2}\sin\theta \\ \frac{1}{2}\cos\frac{\theta}{2}(-1+3\cos\theta) \end{Bmatrix}, \qquad (3.38)$$

and no the boundary conditions $x_3^{(0)}(\pi)=0$ and $x_4^{(0)}(\pi)=0$ are automatically satisfied for all values of A_{II}.

The solution $\mathbf{x}^{(1)}(\theta)$ of the $O(\varepsilon)$ problem is very involved and is the boundary conditions on $\theta=\pi$ take the following form:

$$x_3^{(1)}(\pi)=0 \;\Rightarrow\; -\frac{A_{II}}{4} + \frac{3}{8}\int_0^\pi \sin\theta\left(A_{II} + 3A_{II}\cos 2\theta + \sin 2\theta\right)\ln\Sigma_e^{(0)}\,d\theta = 0, \qquad (3.39)$$

$$\text{where} \quad \Sigma_e^{(0)} = \frac{1}{2}\sqrt{\frac{3}{2}\left[1+5A_{II}^2 - \left(1-3A_{II}^2\right)\cos 2\theta + 4A_{II}\sin 2\theta\right]},$$

$$x_4^{(1)}(\pi)=0 \;\Rightarrow\; \frac{1}{6} + \frac{1}{16}\int_0^\pi \left(7A_{II}\cos\theta + 9A_{II}\cos 3\theta - \sin\theta + 3\sin 3\theta\right)\ln\Sigma_e^{(0)}\,d\theta = 0. \qquad (3.40)$$

It is found numerically that equations (3.39) and (3.40) have the common solution $A_{II}=0$. For $A_{II}=0$, the solutions $\mathbf{x}^{(0)}(\theta)$ and $\mathbf{x}^{(1)}(\theta)$ can be written as

$$\mathbf{x}^{(0)}(\theta)=\begin{Bmatrix} \frac{3}{2}\sin\frac{\theta}{2}\sin\theta \\ -3\sin^3\frac{\theta}{2} \\ \cos^3\frac{\theta}{2} \\ \frac{1}{2}\cos\frac{\theta}{2}\sin\theta \end{Bmatrix}, \quad \text{and} \quad \mathbf{x}^{(1)}(\theta)=\begin{Bmatrix} -\frac{3}{4}\sin\frac{\theta}{2}\sin\theta\left[2-\ln\left(3\sin^2\frac{\theta}{2}\right)\right] \\ \frac{3}{2}\sin^3\frac{\theta}{2}\left[3-\ln\left(3\sin^2\frac{\theta}{2}\right)\right] \\ -\cos^3\frac{\theta}{2}\ln\left(\cos\frac{\theta}{2}\right) \\ -\frac{1}{4}\cos\frac{\theta}{2}\sin\theta\left[1+2\ln\left(\cos\frac{\theta}{2}\right)\right] \end{Bmatrix}. \tag{3.41}$$

The stress component $\sigma_{rr}^{(0)}$ can now be determined by (3.35) and (3.36) as

$$\sigma_{rr}^{(0)}=\Sigma_{rr}^{(0)}+\varepsilon\,\Sigma_{rr}^{(1)}+O\left(\varepsilon^2\right),$$

where $\Sigma_{rr}^{(0)}=\frac{1}{2}\cos\frac{\theta}{2}(3-\cos\theta)$ and $\Sigma_{rr}^{(1)}=-\frac{3}{2}\sin\frac{\theta}{2}\sin\theta-\frac{1}{2}\cos\frac{\theta}{2}(3-\cos\theta)\ln\left(\cos\frac{\theta}{2}\right)$.

The solution $\mathbf{x}^{(2)}(\theta)$, as defined in (3.34), is even more involved and the boundary conditions on $\theta=\pi$

$$x_3^{(2)}(\pi)=0 \quad \text{and} \quad x_4^{(2)}(\pi)=0$$

are satisfied automatically.

3.4 The second order eigenproblem

The second order **linear** eigenproblem as derived in the previous chapter is

$$(t+1)\tilde{\sigma}_{rr}^{(1)}-\tilde{\sigma}_{\theta\theta}^{(1)}+\frac{d\tilde{\sigma}_{r\theta}^{(1)}}{d\theta}=0, \tag{3.42}$$

$$\frac{d\tilde{\sigma}_{\theta\theta}^{(1)}}{d\theta}+(t+2)\tilde{\sigma}_{r\theta}^{(1)}=0, \tag{3.43}$$

$$\left[s(n-1)+t+1\right]\tilde{u}_r^{(1)}-\frac{3}{2}\tilde{\sigma}_e^{(0)^{n-1}}\left[\tilde{s}_{rr}^{(1)}+\frac{3}{2}(n-1)\frac{\tilde{s}_{kl}^{(0)}\tilde{\sigma}_{kl}^{(1)}}{\tilde{\sigma}_e^{(0)^2}}\tilde{s}_{rr}^{(0)}\right]=0, \tag{3.44}$$

$$\tilde{u}_r^{(1)}+\frac{d\tilde{u}_\theta^{(1)}}{d\theta}-\frac{3}{2}\tilde{\sigma}_e^{(0)^{n-1}}\left[\tilde{s}_{\theta\theta}^{(1)}+\frac{3}{2}(n-1)\frac{\tilde{s}_{kl}^{(0)}\tilde{\sigma}_{kl}^{(1)}}{\tilde{\sigma}_e^{(0)^2}}\tilde{s}_{\theta\theta}^{(0)}\right]=0, \tag{3.45}$$

$$\frac{d\tilde{u}_r^{(1)}}{d\theta}+\left[s(n-1)+t\right]\tilde{u}_\theta^{(1)}-3\tilde{\sigma}_e^{(0)^{n-1}}\left[\tilde{\sigma}_{r\theta}^{(1)}+\frac{3}{2}(n-1)\frac{\tilde{s}_{kl}^{(0)}\tilde{\sigma}_{kl}^{(1)}}{\tilde{\sigma}_e^{(0)^2}}\tilde{\sigma}_{r\theta}^{(0)}\right]=0. \tag{3.46}$$

The corresponding boundary conditions are

$$\tilde{u}_r^{(1)}(0)=\tilde{u}_\theta^{(1)}(0)=0 \tag{3.47}$$

and $$\tilde{\sigma}_{\theta\theta}^{(1)}(\pi)=\tilde{\sigma}_{r\theta}^{(1)}(\pi)=0\,. \tag{3.48}$$

It should be noted that (3.44) is an algebraic equation whereas the rest of the above are differential equations.

In order to simplify the system, we replace (3.45) by (3.44)+(3.45):

$$\frac{d\tilde{u}_\theta^{(1)}}{d\theta}+\left[s(n-1)+t+2\right]\tilde{u}_r^{(1)}=0\,. \tag{3.49}$$

The plane strain condition yields

$$\tilde{s}_{rr}^{(1)}=-\tilde{s}_{\theta\theta}^{(1)}=\frac{\tilde{\sigma}_{rr}^{(1)}-\tilde{\sigma}_{\theta\theta}^{(1)}}{2}, \tag{3.50}$$

so that

$$\tilde{s}_{kl}^{(0)}\,\tilde{\sigma}_{kl}^{(1)}=2\,\tilde{s}_{r\theta}^{(0)}\,\tilde{\sigma}_{r\theta}^{(1)}+2\,\tilde{s}_{rr}^{(0)}\,\tilde{s}_{rr}^{(1)}\,. \tag{3.51}$$

We can now eliminate $\tilde{\sigma}_{rr}^{(1)}$ from the system of equations as follows. Using (3.51) into (3.44) and solving for $\tilde{s}_{rr}^{(1)}$, we find

$$\tilde{s}_{rr}^{(1)}=\frac{2\left[s(n-1)+t+1\right]}{3\tilde{\sigma}_e^{(0)^{n-3}}\left[\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}\right]}\tilde{u}_r^{(1)}-\frac{3(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}\tilde{\sigma}_{r\theta}^{(1)}\,. \tag{3.52}$$

Then, $\tilde{\sigma}_{rr}^{(1)}$ is found from (3.50):

$$\tilde{\sigma}_{rr}^{(1)}=2\,\tilde{s}_{rr}^{(1)}+\tilde{\sigma}_{\theta\theta}^{(1)}\,. \tag{3.53}$$

Finally, substituting (3.53) and (3.52) into (3.42) and (3.46) we find

$$\frac{d\tilde{\sigma}_{r\theta}^{(1)}}{d\theta}+\frac{4(t+1)\left[s(n-1)+t+1\right]}{3\tilde{\sigma}_e^{(0)^{n-3}}\left(\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}\right)}\tilde{u}_r^{(1)}+t\tilde{\sigma}_{\theta\theta}^{(1)}-\frac{6(t+1)(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}\tilde{\sigma}_{r\theta}^{(1)}=0\,, \tag{3.54}$$

and

$$\begin{aligned}&\frac{d\tilde{u}_r^{(1)}}{d\theta}+\left[s(n-1)+t\right]\tilde{u}_\theta^{(1)}+\left\{\left[s(n-1)+t+1\right]\frac{6(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}\right\}\tilde{u}_r^{(1)}+\\&+\left\{\frac{\tilde{\sigma}_e^{(0)^{n-3}}\left(3(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}\right)^2}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}-3\tilde{\sigma}_e^{(0)^{n-1}}\left[1+3(n-1)\frac{\tilde{\sigma}_{r\theta}^{(0)^2}}{\tilde{\sigma}_e^{(0)^2}}\right]\right\}\tilde{\sigma}_{r\theta}^{(1)}=0\,.\end{aligned} \tag{3.55}$$

In summary, the second order eigenproblem that defines $\left(\tilde{u}_r^{(1)},\tilde{u}_r^{(1)},\tilde{\sigma}_{\theta\theta}^{(1)},\tilde{\sigma}_{r\theta}^{(1)}\right)$ is

$$\frac{d\tilde{u}_r^{(1)}}{d\theta}+\left[s(n-1)+t\right]\tilde{u}_\theta^{(1)}+\left[s(n-1)+t+1\right]\frac{6(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}\tilde{u}_r^{(1)}+$$
$$+\left\{\frac{\tilde{\sigma}_e^{(0)^{n-3}}\left[3(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}\right]^2}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}-3\tilde{\sigma}_e^{(0)^{n-1}}\left[1+3(n-1)\frac{\tilde{\sigma}_{r\theta}^{(0)^2}}{\tilde{\sigma}_e^{(0)^2}}\right]\right\}\tilde{\sigma}_{r\theta}^{(1)}=0, \tag{3.56}$$

$$\frac{d\tilde{u}_\theta^{(1)}}{d\theta}+\left[s(n-1)+t+2\right]\tilde{u}_r^{(1)}=0, \tag{3.57}$$

$$\frac{d\tilde{\sigma}_{\theta\theta}^{(1)}}{d\theta}+(t+2)\tilde{\sigma}_{r\theta}^{(1)}=0, \tag{3.58}$$

$$\frac{d\tilde{\sigma}_{r\theta}^{(1)}}{d\theta}+\frac{4(t+1)\left[s(n-1)+t+1\right]}{3\tilde{\sigma}_e^{(0)^{n-3}}\left[\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}\right]}\tilde{u}_r^{(1)}+t\,\tilde{\sigma}_{\theta\theta}^{(1)}-\frac{6(t+1)(n-1)\tilde{s}_{rr}^{(0)}\tilde{\sigma}_{r\theta}^{(0)}}{\tilde{\sigma}_e^{(0)^2}+3(n-1)\tilde{s}_{rr}^{(0)^2}}\tilde{\sigma}_{r\theta}^{(1)}=0. \tag{3.59}$$

The stress component $\tilde{\sigma}_{rr}^{(1)}$ is computed by (3.52) and (3.53).

In general, the second order linear eigenproblem has a solution to within a multiplicative constant, i.e., if $\left(\tilde{\mathbf{u}}^{(1)},\tilde{\boldsymbol{\sigma}}^{(1)}\right)$ is a solution, then $\left(a\tilde{\mathbf{u}}^{(1)},a\tilde{\boldsymbol{\sigma}}^{(1)}\right)$ is a solution as well.

There is also the possibility of a degenerate case, in which the solution has more than one arbitrary constants (e.g. simultaneous mode-I and mode-II solutions). To allow for this possibility, we set

$$\tilde{\sigma}_{\theta\theta}^{(1)}(0)=B_I \qquad \text{and} \qquad \tilde{\sigma}_{r\theta}^{(1)}(0)=B_{II}. \tag{3.60}$$

The constants B_I and B_{II} control the magnitude of the second order solution.

The solution $\left(\tilde{\mathbf{u}}^{(1)},\tilde{\boldsymbol{\sigma}}^{(1)}\right)$ depends on the leading order solution $\tilde{\boldsymbol{\sigma}}^{(0)}$, on the hardening exponent n, and on the parameters B_I and B_{II} as follows

$$\tilde{\mathbf{u}}^{(1)}=\tilde{\mathbf{u}}^{(1)}\left(\tilde{\boldsymbol{\sigma}}^{(0)},B_I,B_{II}\right) \qquad \text{and} \qquad \tilde{\boldsymbol{\sigma}}^{(1)}=\tilde{\boldsymbol{\sigma}}^{(1)}\left(\tilde{\boldsymbol{\sigma}}^{(0)},B_I,B_{II}\right). \tag{3.61}$$

Also, if $\tilde{\boldsymbol{\sigma}}^{(0)}$ is replaced by $c\,\tilde{\boldsymbol{\sigma}}^{(0)}$ in the coefficients of the second order linear eigenproblem, then its solution is changing

$$\text{from } \left(\tilde{\mathbf{u}}^{(1)},\tilde{\boldsymbol{\sigma}}^{(1)}\right) \qquad \text{to } \left(\tilde{\mathbf{u}}^{(1)},c^{n-1}\tilde{\boldsymbol{\sigma}}^{(1)}\right), \tag{3.62}$$

i.e.,

$$\tilde{\mathbf{u}}^{(1)}\left(c\,\tilde{\boldsymbol{\sigma}}^{(0)},B_I,B_{II}\right)=\tilde{\mathbf{u}}^{(1)}\left(\tilde{\boldsymbol{\sigma}}^{(0)},B_I,B_{II}\right) \text{ and } \hat{\boldsymbol{\sigma}}^{(1)}\left(c\,\tilde{\boldsymbol{\sigma}}^{(0)},B_I,B_{II}\right)=c^{n-1}\hat{\boldsymbol{\sigma}}^{(1)}\left(\tilde{\boldsymbol{\sigma}}^{(0)},B_I,B_{II}\right). \tag{3.63}$$

Let $\left(\mathbf{u}^{(1)},\boldsymbol{\sigma}^{(1)}\right)$ (without tilde) be the solution $\left(\tilde{\mathbf{u}}^{(1)},\tilde{\boldsymbol{\sigma}}^{(1)}\right)$ that corresponds to the normalized solution $\boldsymbol{\sigma}^{(0)}$ (without tilde), i.e.,

$$\mathbf{u}^{(1)}=\tilde{\mathbf{u}}^{(1)}\left(\boldsymbol{\sigma}^{(0)},B_I,B_{II}\right) \quad \text{and} \quad \boldsymbol{\sigma}^{(1)}=\tilde{\boldsymbol{\sigma}}^{(1)}\left(\boldsymbol{\sigma}^{(0)},B_I,B_{II}\right), \tag{3.64}$$

where we recall that $\sigma_{\theta\theta}^{(0)}(0)=1$ and $\sigma_{r\theta}^{(0)}(0)=A_{II}\equiv\dfrac{\tilde{A}_{II}}{\tilde{A}_I}$.

Now, we can identify $\tilde{A}_I$ with the aforementioned "multiplicative" constant c of the leading order problem, and write its solution in the form

$$\left(\tilde{\mathbf{u}}^{(0)},\tilde{\boldsymbol{\sigma}}^{(0)}\right)=\left(\tilde{A}_I^n\,\mathbf{u}^{(0)},\tilde{A}_I\,\boldsymbol{\sigma}^{(0)}\right). \tag{3.65}$$

According to (3.63), the corresponding solution of the second order problem is

$$\tilde{\mathbf{u}}^{(1)}=\tilde{\mathbf{u}}^{(1)}\left(\tilde{A}_I\,\boldsymbol{\sigma}^{(0)},B_I,B_{II}\right)=\tilde{\mathbf{u}}^{(1)}\left(\boldsymbol{\sigma}^{(0)},B_I,B_{II}\right) \quad \Rightarrow \quad \tilde{\mathbf{u}}^{(1)}=\mathbf{u}^{(1)}, \tag{3.66}$$

and

$$\tilde{\boldsymbol{\sigma}}^{(1)}=\tilde{\boldsymbol{\sigma}}^{(1)}\left(\tilde{A}_I\,\boldsymbol{\sigma}^{(0)},B_I,B_{II}\right)=\tilde{A}_I^{n-1}\,\tilde{\boldsymbol{\sigma}}^{(1)}\left(\boldsymbol{\sigma}^{(0)},B_I,B_{II}\right) \quad \Rightarrow \quad \tilde{\boldsymbol{\sigma}}^{(1)}=\tilde{A}_I^{n-1}\,\boldsymbol{\sigma}^{(1)}. \tag{3.67}$$

For $n=1+\varepsilon$, the solution $\left(\mathbf{u}^{(1)},\boldsymbol{\sigma}^{(1)}\right)$ can be written as

$$\boldsymbol{\sigma}^{(1)}=\hat{\boldsymbol{\Sigma}}^{(0)}+\varepsilon\,\hat{\boldsymbol{\Sigma}}^{(1)}+\varepsilon^2\,\hat{\boldsymbol{\Sigma}}^{(2)}+O\left(\varepsilon^3\right) \quad \text{and} \quad \mathbf{u}^{(1)}=\hat{\mathbf{U}}^{(0)}+\varepsilon\,\hat{\mathbf{U}}^{(1)}+\varepsilon^2\,\hat{\mathbf{U}}^{(2)}+O\left(\varepsilon^3\right).$$

Note that

$$\tilde{A}_I^{n-1}=\tilde{A}_I^{\varepsilon}=1+\varepsilon\,\ln\tilde{A}_I+\frac{1}{2}\varepsilon^2\left(\ln\tilde{A}_I\right)^2+O\left(\varepsilon^3\right).$$

Therefore, (3.66) and (3.67) become

$$\boxed{\tilde{\mathbf{u}}^{(1)}=\hat{\mathbf{U}}^{(0)}+\varepsilon\,\hat{\mathbf{U}}^{(1)}+\varepsilon^2\,\hat{\mathbf{U}}^{(2)}+O\left(\varepsilon^3\right)}, \tag{3.68}$$

And

$$\tilde{\boldsymbol{\sigma}}^{(1)}=\left[1+\varepsilon\ln\tilde{A}_I+\frac{1}{2}\varepsilon^2\left(\ln\tilde{A}_I\right)^2+O\left(\varepsilon^3\right)\right]\left[\hat{\boldsymbol{\Sigma}}^{(0)}+\varepsilon\,\hat{\boldsymbol{\Sigma}}^{(1)}+\varepsilon^2\,\hat{\boldsymbol{\Sigma}}^{(2)}+O\left(\varepsilon^3\right)\right] \quad \Rightarrow$$

$$\boxed{\tilde{\boldsymbol{\sigma}}^{(1)}=\hat{\boldsymbol{\Sigma}}^{(0)}+\varepsilon\left[\left(\ln\tilde{A}_I\right)\hat{\boldsymbol{\Sigma}}^{(0)}+\hat{\boldsymbol{\Sigma}}^{(1)}\right]+\varepsilon^2\left[\frac{1}{2}\left(\ln\tilde{A}_I\right)^2\hat{\boldsymbol{\Sigma}}^{(0)}+\left(\ln\tilde{A}_I\right)\hat{\boldsymbol{\Sigma}}^{(1)}+\hat{\boldsymbol{\Sigma}}^{(2)}\right]+O\left(\varepsilon^3\right)} \tag{3.69}$$

3.5 Asymptotic solution of the second order problem

Recall that for $n=1+\varepsilon$, we write the leading and second order normalized (i.e., without tilde) solution as

$$\boldsymbol{\sigma}^{(0)}=\boldsymbol{\Sigma}^{(0)}+\varepsilon\,\boldsymbol{\Sigma}^{(1)}+\varepsilon^2\,\boldsymbol{\Sigma}^{(2)}+O\left(\varepsilon^3\right) \quad \text{and} \quad \boldsymbol{\sigma}^{(1)}=\hat{\boldsymbol{\Sigma}}^{(0)}+\varepsilon\,\hat{\boldsymbol{\Sigma}}^{(1)}+\varepsilon^2\,\hat{\boldsymbol{\Sigma}}^{(2)}+O\left(\varepsilon^3\right).$$

Then

$$\sigma_e^{(0)} = \sqrt{\frac{3}{2} s_{ij}^{(0)} s_{ij}^{(0)}} = \Sigma_e^{(0)} + \varepsilon \Sigma_e^{(1)} + O\left(\varepsilon^2\right), \qquad \Sigma_e^{(0)} = \sqrt{\frac{3}{2} S_{ij}^{(0)} S_{ij}^{(0)}}, \qquad \Sigma_e^{(1)} = \frac{3}{2}\frac{S_{ij}^{(0)} S_{ij}^{(1)}}{\Sigma_e^{(0)}},$$

$$\sigma_e^{(0)\varepsilon} = 1 + \varepsilon \ln \sigma_e^{(0)} + \frac{1}{2}\varepsilon^2 \left(\ln \sigma_e^{(0)}\right)^2 + O\left(\varepsilon^3\right), \qquad \ln \sigma_e^{(0)} = \ln \Sigma_e^{(0)} + \varepsilon \frac{\Sigma_e^{(1)}}{\Sigma_e^{(0)}} + O\left(\varepsilon^2\right),$$

where

$$S_{rr}^{(0)} = \frac{U_r^{(0)}}{3}, \qquad S_{rr}^{(1)} = \frac{1}{3}\left[U_r^{(1)} - U_r^{(0)}\left(\frac{1}{2} + \ln \Sigma_e^{(0)}\right)\right].$$

Substituting the above expansions into the eigenproblem (3.56)-(3.59) and performing some algebra, the normalized problem can be written in the form

$$\frac{d\mathbf{y}(\theta)}{d\theta} - \mathbf{F}(t)\cdot\mathbf{y}(\theta) = \varepsilon\,\hat{\mathbf{G}}(\theta,t)\cdot\mathbf{y}(\theta), \tag{3.70}$$

where

$$\mathbf{y} = \left\{u_r^{(1)}, u_\theta^{(1)}, \sigma_{\theta\theta}^{(1)}, \sigma_{r\theta}^{(1)}\right\},$$

and

$$\mathbf{F}(t) = \begin{bmatrix} 0 & -t & 0 & 3 \\ -(t+2) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(t+2) \\ -\frac{4}{3}(t+1)^2 & 0 & -t & 0 \end{bmatrix}, \tag{3.71}$$

$$\mathbf{G}(\theta,t) = \mathbf{G}^{(0)}(\theta,t) + \varepsilon\,\mathbf{G}^{(1)}(\theta,t) + O\left(\varepsilon^2\right), \tag{3.72}$$

$$\mathbf{G}^{(0)}\left(\boldsymbol{\Sigma}^{(0)},t\right) = \begin{bmatrix} 6(t+1)X & \frac{1}{2} & 0 & 3R \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{3}(t+1)\left[2(t+1)Z+1\right] & 0 & 0 & 6(t+1)X \end{bmatrix},$$

where

$$X\left(\boldsymbol{\Sigma}^{(0)}\right) = \frac{S_{rr}^{(0)}\Sigma_{r\theta}^{(0)}}{\Sigma_e^{(0)^2}}, \quad R\left(\boldsymbol{\Sigma}^{(0)}\right) = m + 3\frac{\Sigma_{r\theta}^{(0)^2}}{\Sigma_e^{(0)^2}}, \quad Z\left(\boldsymbol{\Sigma}^{(0)}\right) = m + 3\frac{S_{rr}^{(0)^2}}{\Sigma_e^{(0)^2}}, \quad m\left(\boldsymbol{\Sigma}^{(0)}\right) = \ln \Sigma_e^{(0)},$$

and $$\mathbf{G}^{(1)}\left(\boldsymbol{\Sigma}^{(0)},\boldsymbol{\Sigma}^{(1)},t\right)=\begin{bmatrix} 3X\left[2(t+1)Y-1\right] & -\frac{1}{4} & 0 & 3T \\ -\frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{3}(t+1)\left[1+2Z-4(t+1)P\right] & 0 & 0 & 6(t+1)XY \end{bmatrix},$$

with

$$Y\left(\boldsymbol{\Sigma}^{(0)},\boldsymbol{\Sigma}^{(1)}\right)=\frac{\Sigma_{r\theta}^{(1)}}{\Sigma_{r\theta}^{(0)}}+\frac{S_{rr}^{(1)}}{S_{rr}^{(0)}}-2\frac{\Sigma_{e}^{(1)}}{\Sigma_{e}^{(0)}}-3\frac{S_{rr}^{(0)^2}}{\Sigma_{e}^{(0)^2}},$$

$$P\left(\boldsymbol{\Sigma}^{(0)},\boldsymbol{\Sigma}^{(1)}\right)=\frac{m^2}{2}+\frac{\Sigma_{e}^{(1)}}{\Sigma_{e}^{(0)}}+6\frac{S_{rr}^{(0)^2}}{\Sigma_{e}^{(0)^2}}\left(\frac{S_{rr}^{(1)}}{S_{rr}^{(0)}}-\frac{\Sigma_{e}^{(1)}}{\Sigma_{e}^{(0)}}+\frac{m}{2}\right)-Z^2,$$

and $$T\left(\boldsymbol{\Sigma}^{(0)},\boldsymbol{\Sigma}^{(1)}\right)=\frac{m^2}{2}+\frac{\Sigma_{e}^{(1)}}{\Sigma_{e}^{(0)}}+6\frac{S_{r\theta}^{(0)^2}}{\Sigma_{e}^{(0)^2}}\left(\frac{S_{r\theta}^{(1)}}{S_{r\theta}^{(0)}}-\frac{\Sigma_{e}^{(1)}}{\Sigma_{e}^{(0)}}+\frac{m}{2}\right)-9X^2.$$

Equation (3.70) can be written in an integral equation form as in the case of the leading order problem as

$$\mathbf{y}(\theta)=\boldsymbol{\Psi}(\theta,t)\cdot\boldsymbol{\Psi}^{-1}(0,t)\cdot\mathbf{y}(0)+\varepsilon\boldsymbol{\Psi}(\theta,t)\cdot\int_0^\theta\boldsymbol{\Psi}^{-1}(\phi,t)\cdot\hat{\mathbf{G}}(\phi,t)\cdot\mathbf{y}(\phi)d\phi \tag{3.73}$$

with boundary conditions

$$u_r^{(1)}(0)=u_\theta^{(1)}(0)=0, \qquad \sigma_{\theta\theta}^{(1)}(\pi)=\sigma_{r\theta}^{(1)}(\pi)=0, \tag{3.74}$$

and normalization conditions

$$\sigma_{\theta\theta}^{(1)}(0)=B_I, \qquad \sigma_{r\theta}^{(1)}(0)=B_{II}. \tag{3.75}$$

We look for an asymptotic expansion of the solution in the form

$$\mathbf{y}=\mathbf{y}^{(0)}+\varepsilon\,\mathbf{y}^{(1)}+\varepsilon^2\,\mathbf{y}^{(2)}+O\left(\varepsilon^3\right), \tag{3.76}$$

or, equivalently,

$$\boldsymbol{\sigma}^{(1)}=\hat{\boldsymbol{\Sigma}}^{(0)}+\varepsilon\,\hat{\boldsymbol{\Sigma}}^{(1)}+\varepsilon^2\,\hat{\boldsymbol{\Sigma}}^{(2)}+O\left(\varepsilon^3\right) \quad \text{and} \quad \mathbf{u}^{(1)}=\hat{\mathbf{U}}^{(0)}+\varepsilon\,\hat{\mathbf{U}}^{(1)}+\varepsilon^2\,\hat{\mathbf{U}}^{(2)}+O\left(\varepsilon^3\right). \tag{3.77}$$

Then, the normalization conditions of the second order problem become

$$\sigma_{\theta\theta}^{(1)}(0)=\hat{\Sigma}_{\theta\theta}^{(0)}(0)+\varepsilon\,\hat{\Sigma}_{\theta\theta}^{(1)}(0)+\varepsilon^2\,\hat{\Sigma}_{\theta\theta}^{(2)}(0)+O\left(\varepsilon^3\right)=B_I \qquad \Rightarrow$$

$$\hat{\Sigma}_{\theta\theta}^{(0)}(0)=B_I, \qquad \hat{\Sigma}_{\theta\theta}^{(1)}(0)=\hat{\Sigma}_{\theta\theta}^{(2)}(0)=\cdots=0,$$

$$\sigma_{r\theta}^{(1)}(0) = \hat{\Sigma}_{r\theta}^{(0)}(0) + \varepsilon\,\hat{\Sigma}_{r\theta}^{(1)}(0) + \varepsilon^2\,\hat{\Sigma}_{r\theta}^{(2)}(0) + O\left(\varepsilon^3\right) = B_{II} \qquad \Rightarrow$$

$$\hat{\Sigma}_{r\theta}^{(0)}(0) = B_{II}\,, \qquad \hat{\Sigma}_{r\theta}^{(1)}(0) = \hat{\Sigma}_{r\theta}^{(2)}(0) = \cdots = 0\,.$$

The stress exponent t of the second order term in the expansion is written as

$$t = -\frac{1}{2} + \varepsilon C + \varepsilon^2 D + O\left(\varepsilon^3\right). \tag{3.78}$$

Substituting the expansions (3.76) and (3.78) into the integral equation (3.73), and collecting terms having like powers of ε, we find

$$O(1): \quad \mathbf{y}^{(0)}(\theta) = \mathbf{A}\left(\theta, -\frac{1}{2}\right)\cdot\mathbf{y}^{(0)}(0), \tag{3.79}$$

where $\quad \mathbf{A}(\theta,t) = \mathbf{\Psi}(\theta,t)\cdot\mathbf{\Psi}^{-1}(0,t), \quad$ and $\quad \mathbf{y}^{(0)}(0) = \{0 \quad 0 \quad B_I \quad B_{II}\}$,

$$O(\varepsilon): \quad \mathbf{y}^{(1)}(\theta) = C\left[\frac{\partial \mathbf{A}(\theta,t)}{\partial t}\right]_{t=-1/2}\cdot\mathbf{y}^{(0)}(0) + \int_0^\theta \mathbf{B}\left(\theta,\phi,-\frac{1}{2}\right)\cdot\hat{\mathbf{G}}^{(0)}\left(\phi,-\frac{1}{2}\right)\cdot\mathbf{y}^{(0)}(\phi)\,d\phi\,, \tag{3.80}$$

where $\quad \mathbf{B}(\theta,\phi,t) = \mathbf{\Psi}(\theta,t)\cdot\mathbf{\Psi}^{-1}(\phi,t)$,

$$O\left(\varepsilon^2\right): \quad \begin{aligned} \mathbf{y}^{(2)}(\theta) = &\left[D\frac{\partial \mathbf{A}(\theta,t)}{\partial t} + \frac{1}{2}C^2\frac{\partial^2 \mathbf{A}(\theta,t)}{\partial t^2}\right]_{t=-1/2}\cdot\mathbf{y}^{(0)}(0) + \\ &+ C\left\{\frac{\partial}{\partial t}\left[\int_0^\theta \mathbf{B}(\theta,\phi,t)\cdot\hat{\mathbf{G}}^{(0)}(\phi,t)\cdot\mathbf{y}^{(0)}(\phi)\,d\phi\right]\right\}_{t=-1/2} + \\ &+ \int_0^\theta \mathbf{B}\left(\theta,\phi,-\frac{1}{2}\right)\cdot\left[\hat{\mathbf{G}}^{(0)}\left(\phi,-\frac{1}{2}\right)\cdot\mathbf{y}^{(1)}(\phi) + \hat{\mathbf{G}}^{(1)}\left(\phi,-\frac{1}{2}\right)\cdot\mathbf{y}^{(0)}(\phi)\right]d\phi\,. \end{aligned} \tag{3.81}$$

Also, equations (3.52) and (3.53) that determine $\sigma_{rr}^{(1)}$ imply

$$\hat{S}_{rr}^{(0)} = \frac{1}{3}\hat{U}_r^{(0)}, \qquad \hat{\Sigma}_{rr}^{(0)} = 2\,\hat{S}_{rr}^{(0)} + \hat{\Sigma}_{\theta\theta}^{(0)}, \tag{3.82}$$

$$\hat{S}_{rr}^{(1)} = \frac{1}{3}\hat{U}_r^{(1)} - \frac{1}{3}\hat{U}_r^{(0)}\left(\frac{1}{2} + \ln\Sigma_e^{(0)} + 3\frac{S_{rr}^{(0)2}}{\Sigma_e^{(0)2}}\right) - 3\,\hat{\Sigma}_{r\theta}^{(0)}\frac{S_{rr}^{(0)}\,\Sigma_{r\theta}^{(0)}}{\Sigma_e^{(0)2}}, \qquad \hat{\Sigma}_{rr}^{(1)} = 2\,\hat{S}_{rr}^{(1)} + \hat{\Sigma}_{\theta\theta}^{(1)}, \tag{3.83}$$

$$\hat{S}_{rr}^{(2)} = \frac{1}{3}\hat{U}_r^{(2)} - \frac{1}{3}\hat{U}_r^{(1)}\left(\frac{1}{2} + \ln\Sigma_e^{(0)} + 3\frac{S_{rr}^{(0)2}}{\Sigma_e^{(0)2}}\right) +$$

$$+\hat{U}_r^{(0)}\left(\frac{1}{8} - \frac{\Sigma_e^{(1)}}{3\Sigma_e^{(0)}} + \frac{S_{rr}^{(0)2}}{2\Sigma_e^{(0)2}} + 2\frac{S_{rr}^{(0)2}\Sigma_e^{(1)}}{\Sigma_e^{(0)3}} + 3\frac{S_{rr}^{(0)4}}{\Sigma_e^{(0)4}} - 2\frac{S_{rr}^{(0)}S_{rr}^{(1)}}{\Sigma_e^{(0)2}} + \frac{\ln\Sigma_e^{(0)}}{6} + \frac{S_{rr}^{(0)2}\ln\Sigma_e^{(0)}}{\Sigma_e^{(0)2}} + \frac{\left(\ln\Sigma_e^{(0)}\right)^2}{6}\right) -$$

$$-3\hat{\Sigma}_{r\theta}^{(1)}\frac{S_{rr}^{(0)}\Sigma_{r\theta}^{(0)}}{\Sigma_e^{(0)2}} + 3\frac{\hat{\Sigma}_{r\theta}^{(0)}}{\Sigma_e^{(0)2}}\left(2\frac{S_{rr}^{(0)}\Sigma_{r\theta}^{(0)}\Sigma_e^{(1)}}{\Sigma_e^{(0)}} + 3\frac{S_{rr}^{(0)3}\Sigma_{r\theta}^{(0)}}{\Sigma_e^{(0)2}} - \Sigma_{r\theta}^{(0)}S_{rr}^{(1)} - S_{rr}^{(0)}\Sigma_{r\theta}^{(1)}\right),$$

$$\hat{\Sigma}_{rr}^{(2)} = 2\hat{S}_{rr}^{(2)} + \hat{\Sigma}_{\theta\theta}^{(2)}. \qquad (3.84)$$

Equations (3.79)-(3.81) are evaluated again by using Mathematica®.

The solution of the $O(1)$ problem is

$$\mathbf{y}^{(0)}(\theta) = B_I\begin{Bmatrix} \frac{3}{2}\sin\frac{\theta}{2}\sin\theta \\ -3\sin^3\frac{\theta}{2} \\ \cos^3\frac{\theta}{2} \\ \frac{1}{2}\cos\frac{\theta}{2}\sin\theta \end{Bmatrix} + B_{II}\begin{Bmatrix} \frac{3}{2}\sin\frac{\theta}{2}(1+3\cos\theta) \\ -\frac{9}{2}\sin\frac{\theta}{2}\sin\theta \\ -\frac{3}{2}\cos\frac{\theta}{2}\sin\theta \\ -\frac{1}{2}\cos\frac{\theta}{2}(1-3\cos\theta) \end{Bmatrix}. \qquad (3.85)$$

The solution $\mathbf{y}^{(1)}(\theta)$ of the $O(\varepsilon)$ problem is very involved and the boundary conditions on $\theta = \pi$ take the following form:

$$y_3^{(1)}(\pi) = 0 \quad \Rightarrow \quad \frac{\pi}{16}\left(C - \frac{1}{4}\right)B_I = 0,$$

$$y_4^{(1)}(\pi) = 0 \quad \Rightarrow \quad \frac{\pi}{16}\left(C - \frac{1}{4}\right)B_{II} = 0.$$

The only non-trivial solution of the above equations is

$$C = \frac{1}{4}.$$

For this value of C, the solution of the $O(\varepsilon)$ problem takes the form

$$\mathbf{y}^{(1)}(\theta)=B_I\begin{Bmatrix}\frac{3}{4}\sin\frac{\theta}{2}\sin\theta\ln\left(3\sin^2\frac{\theta}{2}\right)\\ \frac{3}{2}\sin^3\frac{\theta}{2}\left[1-\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ -\cos^3\frac{\theta}{2}\ln\left(\cos\frac{\theta}{2}\right)\\ -\frac{1}{4}\cos\frac{\theta}{2}\sin\theta\left[1+2\ln\left(\cos\frac{\theta}{2}\right)\right]\end{Bmatrix}+B_{II}\begin{Bmatrix}\frac{3}{4}\sin\frac{\theta}{2}\left[1-\cos\theta+(1+3\cos\theta)\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ \frac{3}{4}\sin\frac{\theta}{2}\sin\theta\left[4-3\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ -\frac{1}{4}\cos\frac{\theta}{2}\sin\theta\left[1-6\ln\left(\cos\frac{\theta}{2}\right)\right]\\ \frac{1}{2}\cos\frac{\theta}{2}\left[1-\cos\theta+(1-3\cos\theta)\ln\left(\cos\frac{\theta}{2}\right)\right]\end{Bmatrix}.$$

The solution $\mathbf{y}^{(2)}(\theta)$, as defined in (3.81), is even more involved and the boundary conditions on $\theta=\pi$ are

$$y_3^{(2)}(\pi)=0 \quad\Rightarrow\quad \pi\left(D+\frac{1}{8}\right)B_I=0, \tag{3.86}$$

and

$$y_4^{(2)}(\pi)=0 \quad\Rightarrow\quad \pi\left(D+\frac{1}{16}\right)B_{II}=0. \tag{3.87}$$

If $B_{II}=0$, then the second order eigenvalue problem has the obvious solution

$$t=s,\qquad \tilde{\mathbf{u}}^{(1)}=c\,\tilde{\mathbf{u}}^{(0)}\qquad\text{and}\qquad \tilde{\boldsymbol{\sigma}}^{(1)}=c\,\tilde{\boldsymbol{\sigma}}^{(0)}/n,$$

which cannot be accepted since it violates the condition $s<t<0$. Therefore, equations (3.86) and (3.87) have the solution

$$D=-\frac{1}{16}\qquad\text{and}\qquad B_I=0. \tag{3.88}$$

For the values of $C=\frac{1}{4}$, $D=-\frac{1}{16}$ and $B_I=0$, the $O(1)$ and $O(\varepsilon)$ solutions take the from

$$\boxed{\mathbf{y}^{(0)}(\theta)=B_{II}\begin{Bmatrix}\frac{3}{2}\sin\frac{\theta}{2}(1+3\cos\theta)\\ -\frac{9}{2}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\\ -\frac{3}{2}\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\\ -\frac{1}{2}\cos\frac{\theta}{2}(1-3\cos\theta)\end{Bmatrix}},\qquad \boxed{\mathbf{y}^{(1)}(\theta)=B_{II}\begin{Bmatrix}\frac{3}{4}\sin\frac{\theta}{2}\left[1-\cos\theta+(1+3\cos\theta)\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ \frac{3}{4}\sin\frac{\theta}{2}\sin\theta\left[4-3\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ -\frac{1}{4}\cos\frac{\theta}{2}\sin\theta\left[1-6\ln\left(\cos\frac{\theta}{2}\right)\right]\\ \frac{1}{2}\cos\frac{\theta}{2}\left[1-\cos\theta+(1-3\cos\theta)\ln\left(\cos\frac{\theta}{2}\right)\right]\end{Bmatrix}}.$$

The stress component $\sigma_{rr}^{(1)}$ can now be determined by (3.82) and (3.83) as

$$\sigma_{rr}^{(1)}=\hat{\Sigma}_{rr}^{(0)}+\varepsilon\,\hat{\Sigma}_{rr}^{(1)}+O\left(\varepsilon^2\right),$$

where

$$\hat{\Sigma}_{rr}^{(0)} = \sin\frac{\theta}{2}\left(1+3\cos\theta\right) - \frac{3}{2}\cos\frac{\theta}{2}\sin\theta$$

and

$$\hat{\Sigma}_{rr}^{(1)} = \frac{1}{4}\sin\frac{\theta}{2}\left[-1+2\ln\left(\cos\frac{\theta}{2}\right) - \cos\theta\left(17+6\ln\left(\cos\frac{\theta}{2}\right)\right)\right].$$

Finally, for the same values of the constants C, D and B_I the second hardening exponent t take the form

$$\boxed{t = -\frac{1}{2} + \varepsilon\frac{1}{4} - \varepsilon^2\frac{1}{16} + O\left(\varepsilon^3\right)}.$$

4. CONCLUSIONS

4.1 Introduction

The stress and displacement field at the crack tip of a plain strain crack lying on the interface of an elastic-plastic material and a rigid substrate were assumed to be of the form

$$\frac{\boldsymbol{\sigma}(r,\theta)}{\sigma_0}=\left(\frac{J}{\alpha\,\varepsilon_0\,\sigma_0\,I_n}\right)^{\frac{1}{n+1}}\tilde{\boldsymbol{\sigma}}^{(0)}(\theta)+Q\left(\frac{r}{J/\sigma_0}\right)^{t}\tilde{\boldsymbol{\sigma}}^{(1)}(\theta)+\cdots,$$

$$\frac{\mathbf{u}(r,\theta)}{\alpha\varepsilon_0}=\left(\frac{J}{\alpha\,\varepsilon_0\,\sigma_0\,I_n}\right)^{\frac{n}{n+1}}r^{\frac{1}{n+1}}\tilde{\mathbf{u}}^{(0)}(\theta)+\frac{Q}{(J/\sigma_0)^{t}}\left(\frac{J}{\alpha\,\varepsilon_0\,\sigma_0\,I_n}\right)^{\frac{n-1}{n+1}}r^{-\frac{n-1}{n+1}+t+1}\tilde{\mathbf{u}}^{(1)}(\theta)+\cdots,$$

where $\tilde{\boldsymbol{\sigma}}^{(0)}(\theta)$ is normalized so that $\max\limits_{\theta\in[0,\pi]}\sqrt{\frac{3}{2}\tilde{s}_{ij}^{(0)}\tilde{s}_{ij}^{(0)}}=1$.

4.2 The leading order problem

For values of n near unity , i.e., $n=1+\varepsilon$, the first order solution is written in the form

$$\tilde{\boldsymbol{\sigma}}^{(0)}=\tilde{A}_I\,\boldsymbol{\sigma}^{(0)}=\tilde{A}_I\left(\boldsymbol{\Sigma}^{(0)}+\varepsilon\,\boldsymbol{\Sigma}^{(1)}+O\left(\varepsilon^2\right)\right), \tag{4.1}$$

$$\tilde{\mathbf{u}}^{(0)}=\tilde{A}_I^n\,\mathbf{u}^{(0)}=\tilde{A}_I\left\{\mathbf{U}^{(0)}+\varepsilon\left[\left(\ln\tilde{A}_I\right)\mathbf{U}^{(0)}+\mathbf{U}^{(1)}\right]+O\left(\varepsilon^2\right)\right\}, \tag{4.2}$$

where $\boldsymbol{\sigma}^{(0)}$ is normalized so that $\sigma_{\theta\theta}^{(0)}(0)=1$. The above solution is of the form

$$\boldsymbol{\Sigma}^{(0)}=\begin{Bmatrix}\Sigma_{rr}^{(0)}\\ \Sigma_{\theta\theta}^{(0)}\\ \Sigma_{r\theta}^{(0)}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{2}\cos\frac{\theta}{2}(3-\cos\theta)\\ \cos^3\frac{\theta}{2}\\ \frac{1}{2}\cos\frac{\theta}{2}\sin\theta\end{Bmatrix},\quad \boldsymbol{\Sigma}^{(1)}=\begin{Bmatrix}\Sigma_{rr}^{(1)}\\ \Sigma_{\theta\theta}^{(1)}\\ \Sigma_{r\theta}^{(1)}\end{Bmatrix}=\begin{Bmatrix}-\frac{3}{2}\sin\frac{\theta}{2}\sin\theta-\frac{1}{2}\cos\frac{\theta}{2}(3-\cos\theta)\ln\left(\cos\frac{\theta}{2}\right)\\ -\cos^3\frac{\theta}{2}\ln\left(\cos\frac{\theta}{2}\right)\\ -\frac{1}{4}\cos\frac{\theta}{2}\sin\theta\left[1+2\ln\left(\cos\frac{\theta}{2}\right)\right]\end{Bmatrix},$$

and

$$\mathbf{U}^{(0)}=\begin{Bmatrix}U_r^{(0)}\\ U_\theta^{(0)}\end{Bmatrix}=\begin{Bmatrix}\frac{3}{2}\sin\frac{\theta}{2}\sin\theta\\ -3\sin^3\frac{\theta}{2}\end{Bmatrix},\quad \mathbf{U}^{(1)}=\begin{Bmatrix}U_r^{(1)}\\ U_\theta^{(1)}\end{Bmatrix}=\begin{Bmatrix}-\frac{3}{4}\sin\frac{\theta}{2}\sin\theta\left[2-\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ \frac{3}{2}\sin^3\frac{\theta}{2}\left[3-\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\end{Bmatrix}.$$

Note that on $\theta=0$ we have $\sigma_{\theta\theta}^{(0)}=1$ and $\sigma_{r\theta}^{(0)}=0$ (or $\tilde{\sigma}_{\theta\theta}^{(0)}=\tilde{A}_I$, $\tilde{\sigma}_{r\theta}^{(0)}=0$), i.e., the leading term in the stress expansion branches from the mode I linear elastic solution.

Also, $\sigma_e^{(0)}=\sqrt{\frac{3}{2}s_{ij}^{(0)}s_{ij}^{(0)}}=\Sigma_e^{(0)}+\Sigma_e^{(1)}+O\left(\varepsilon^2\right)$

$$\Sigma_e^{(0)} = \sqrt{3\left(S_{rr}^{(0)2} + \Sigma_{r\theta}^{(0)2}\right)} \quad \Rightarrow \quad \boxed{\Sigma_e^{(0)} = \frac{\sqrt{3}}{2}\sin\theta} \tag{4.3}$$

and

$$\Sigma_e^{(1)} = \frac{3}{2}\frac{S_{ij}^{(0)}S_{ij}^{(1)}}{\Sigma_e^{(0)}} = \frac{3\left(S_{rr}^{(0)}S_{rr}^{(1)} + \Sigma_{r\theta}^{(0)}\Sigma_{r\theta}^{(1)}\right)}{\Sigma_e^{(0)}} \quad \Rightarrow$$

$$\boxed{\Sigma_e^{(1)} = -\frac{\sqrt{3}}{8}\sin\theta\left[3-\cos\theta+2\left(1+\cos\theta\right)\ln\left(\cos\frac{\theta}{2}\right)-2\left(1-\cos\theta\right)\ln\left(3\sin\frac{\theta}{2}\right)\right]}. \tag{4.4}$$

Figure 4.1 shows the angular variation of the stress and displacement components as well as the variation of the equivalent stress for both the linear elastic case $\left(\boldsymbol{\Sigma}^{(0)}, \mathbf{U}^{(0)}\right)$ and the correction term $\left(\boldsymbol{\Sigma}^{(1)}, \mathbf{U}^{(1)}\right)$. It is interesting to note that the components $\Sigma_e^{(0)}$ and $\Sigma_e^{(1)}$ of the von Mises equivalent stress both vanish along the interface and on the crack face (i.e., on $\theta = 0$ and $\theta = \pi$). Also, the stress component $\Sigma_{rr}^{(1)}$ approaches zero on $\theta = \pi$ with an infinite slope, i.e., $\left.\frac{d\Sigma_{rr}^{(1)}}{d\theta}\right|_{\theta=\pi} = -\infty$.

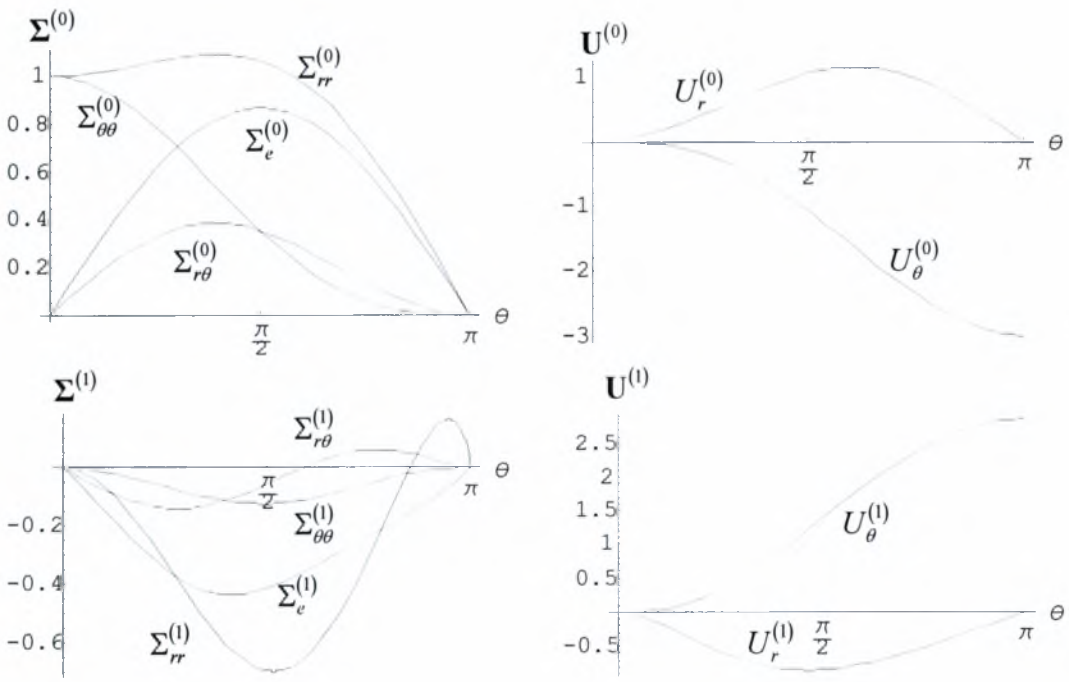

Fig. 4.1. Angular variation of the leading order stress and displacement components.

4.3 The second order problem

For values of n near unity , i.e., $n=1+\varepsilon$, the first order solution is written in the form

$$\tilde{\boldsymbol{\sigma}}^{(1)}=B_{II}\left\{\hat{\boldsymbol{\Sigma}}^{(0)}+\varepsilon\left[\left(\ln\tilde{A}_I\right)\hat{\boldsymbol{\Sigma}}^{(0)}+\hat{\boldsymbol{\Sigma}}^{(1)}\right]+\varepsilon^2\left[\frac{1}{2}\left(\ln\tilde{A}_I\right)^2\hat{\boldsymbol{\Sigma}}^{(0)}+\left(\ln\tilde{A}_I\right)\hat{\boldsymbol{\Sigma}}^{(1)}+\hat{\boldsymbol{\Sigma}}^{(2)}\right]+O\left(\varepsilon^3\right)\right\}, \quad (4.5)$$

$$\tilde{\mathbf{u}}^{(1)}=B_{II}\left[\hat{\mathbf{U}}^{(0)}+\varepsilon\hat{\mathbf{U}}^{(1)}+\varepsilon^2\hat{\mathbf{U}}^{(2)}+O\left(\varepsilon^3\right)\right]. \quad (4.6)$$

The above solution is of the form

$$\hat{\boldsymbol{\Sigma}}^{(0)}=\begin{Bmatrix}\hat{\Sigma}_{rr}^{(0)}\\ \hat{\Sigma}_{\theta\theta}^{(0)}\\ \hat{\Sigma}_{r\theta}^{(0)}\end{Bmatrix}=\begin{Bmatrix}\sin\frac{\theta}{2}(1+3\cos\theta)-\frac{3}{2}\cos\frac{\theta}{2}\sin\theta\\ -\frac{3}{2}\cos\frac{\theta}{2}\sin\theta\\ -\frac{1}{2}\cos\frac{\theta}{2}(1-3\cos\theta)\end{Bmatrix},$$

$$\hat{\boldsymbol{\Sigma}}^{(1)}=\begin{Bmatrix}\hat{\Sigma}_{rr}^{(1)}\\ \hat{\Sigma}_{\theta\theta}^{(1)}\\ \hat{\Sigma}_{r\theta}^{(1)}\end{Bmatrix}=\begin{Bmatrix}\frac{1}{4}\sin\frac{\theta}{2}\left[-1+2\ln\left(\cos\frac{\theta}{2}\right)-\cos\theta\left(17+6\ln\left(\cos\frac{\theta}{2}\right)\right)\right]\\ -\frac{1}{4}\cos\frac{\theta}{2}\sin\theta\left[1-6\ln\left(\cos\frac{\theta}{2}\right)\right]\\ \frac{1}{2}\cos\frac{\theta}{2}\left[1-\cos\theta+(1-3\cos\theta)\ln\left(\cos\frac{\theta}{2}\right)\right]\end{Bmatrix},$$

$$\hat{\mathbf{U}}^{(0)}=\begin{Bmatrix}\hat{U}_r^{(0)}\\ \hat{U}_\theta^{(0)}\end{Bmatrix}=\begin{Bmatrix}\frac{3}{2}\sin\frac{\theta}{2}(1+3\cos\theta)\\ -\frac{9}{2}\sin\frac{\theta}{2}\sin\theta\end{Bmatrix},\quad \hat{\mathbf{U}}^{(1)}=\begin{Bmatrix}\hat{U}_r^{(1)}\\ \hat{U}_\theta^{(1)}\end{Bmatrix}=\begin{Bmatrix}\frac{3}{4}\sin\frac{\theta}{2}\left[1-\cos\theta+(1+3\cos\theta)\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\\ \frac{3}{4}\sin\frac{\theta}{2}\sin\theta\left[4-3\ln\left(3\sin^2\frac{\theta}{2}\right)\right]\end{Bmatrix}.$$

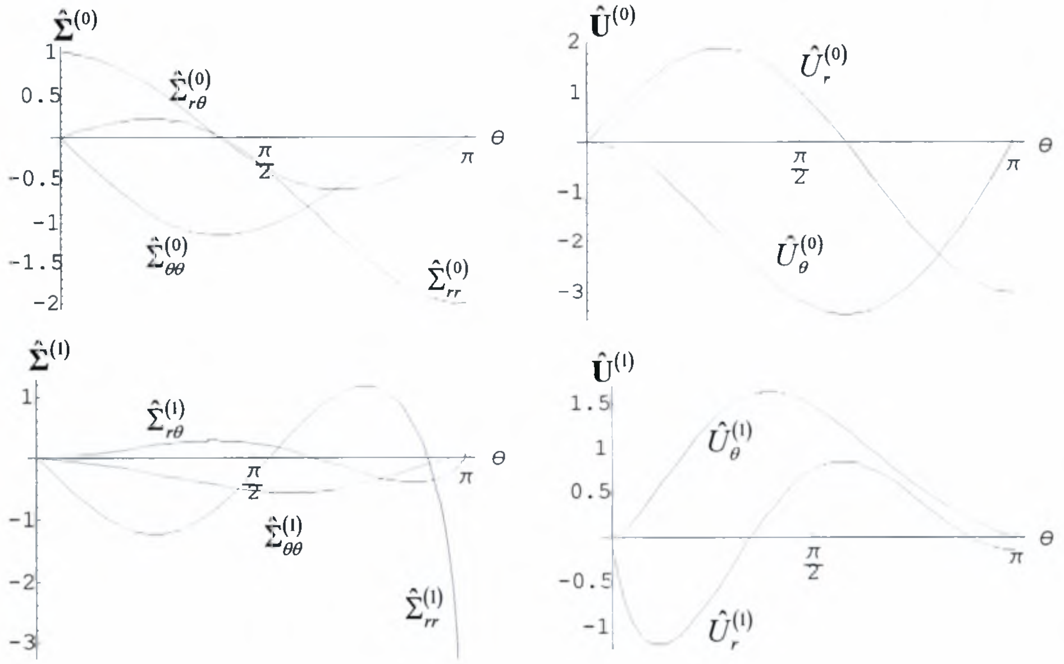

Fig.4.2 Angular variation of the second order stress and displacement components.

Note that on $\theta = 0$ we have $\tilde{\sigma}^{(1)}_{\theta\theta} = 0$ and $\tilde{\sigma}^{(1)}_{r\theta} = B_{II}$, i.e., the second order term in the stress expansion branches from the mode II linear elastic solution.

Figure 4.2 shows the angular variation of the stress and displacement components of both the linear elastic case $\left(\hat{\mathbf{\Sigma}}^{(0)}, \hat{\mathbf{U}}^{(0)}\right)$ and the correction term $\left(\hat{\mathbf{\Sigma}}^{(1)}, \hat{\mathbf{U}}^{(1)}\right)$. Note that the stress component $\hat{\Sigma}^{(1)}_{rr}$ is singular on the crack face $(\theta = \pi)$, i.e.,

$$\lim_{\theta \to \pi} \hat{\Sigma}^{(1)}_{rr} = -\infty .$$

This suggests that a boundary layer may develop near $\theta = \pi$. This behavior can be attributed to the fact that the von Mises equivalent stress vanishes on $\theta = \pi$, whereas the above solution is developed on the assumption that whole range $[0, \pi]$ is "at yield".

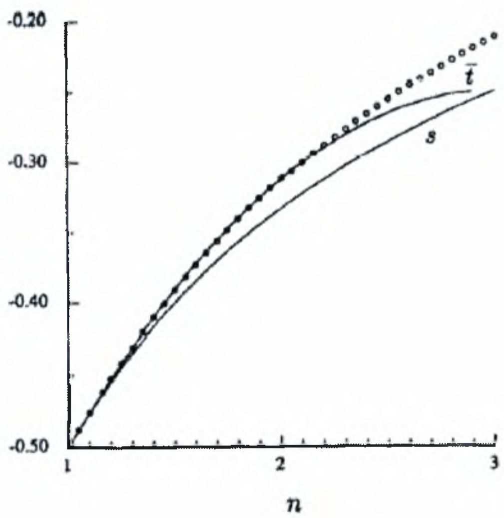

Fig. 4.3 Variation of s and t near $n = 1$.

We note also that the leading and second order stress exponents can be written as

$$s = -\frac{1}{n+1} = -\frac{1}{2} + \frac{1}{4}\varepsilon - \frac{1}{8}\varepsilon^2 + O\left(\varepsilon^3\right) \quad \text{and} \quad t = -\frac{1}{2} + \frac{1}{4}\varepsilon - \frac{1}{16}\varepsilon^2 + O\left(\varepsilon^3\right), \tag{4.7}$$

i.e., $t < s$ with

$$\boxed{t = s + O\left(\varepsilon^2\right) < 0}.$$

Fig. 4.3 shows the variation of the leading and second order exponent near $n = 1$. The open circles are the results of the numerical solution of the second order eigenproblem developed by Sharma and Aravas (1993) and $\bar{t} = -\frac{1}{2} + \frac{1}{4}\varepsilon - \frac{1}{16}\varepsilon^2$ is the variation of the second order exponent as predicted by the asymptotic solution (4.7) up to $O(\varepsilon^2)$. The fact that $t = s + O(\varepsilon^2) < 0$ shows that the second order term is singular in r and cannot be ignored as $r \to 0$.

APPENDIX A

Here we examine the possibility of finding a solution $\left(\tilde{\mathbf{u}}^{(0)},\tilde{\boldsymbol{\sigma}}^{(0)}\right)$ such that $\tilde{\sigma}_{\theta\theta}^{(0)}(0)=0$ and $\tilde{\sigma}_{r\theta}^{(0)}(0)=\tilde{A}_{II}\neq 0$. We can normalize the solution by dividing through by $\tilde{A}_{II}$ and end up with the normalized problem for $\left(\mathbf{u}^{(0)},\boldsymbol{\sigma}^{(0)}\right)$ (without tilde) that corresponds to

$$\sigma_{r\theta}^{(0)}(0)=1 \qquad \text{and} \qquad \sigma_{\theta\theta}^{(0)}(0)=0\,. \tag{1.1}$$

For $n=1+\varepsilon$, the normalized solution $\boldsymbol{\sigma}^{(0)}$ and $\mathbf{u}^{(0)}$ is written in the form

$$\boxed{\boldsymbol{\sigma}^{(0)}=\boldsymbol{\Sigma}^{(0)}+\varepsilon\,\boldsymbol{\Sigma}^{(1)}+\varepsilon^2\,\boldsymbol{\Sigma}^{(2)}+O\left(\varepsilon^3\right)} \qquad \text{and} \qquad \boxed{\mathbf{u}^{(0)}=\mathbf{U}^{(0)}+\varepsilon\,\mathbf{U}^{(1)}+\varepsilon^2\,\mathbf{U}^{(2)}+O\left(\varepsilon^3\right)}. \tag{1.2}$$

Note that

$$\tilde{A}_{II}^{n}=\tilde{A}_{II}^{1+\varepsilon}=\tilde{A}_{II}\,\tilde{A}_{II}^{\varepsilon}=\tilde{A}_{II}\left[1+\varepsilon\ln\tilde{A}_{II}+\frac{1}{2}\varepsilon^2\left(\ln\tilde{A}_{II}\right)^2+O\left(\varepsilon^3\right)\right].$$

Then

$$\tilde{\boldsymbol{\sigma}}^{(0)}=\tilde{A}_{II}\,\boldsymbol{\sigma}^{(0)}(n) \quad\Rightarrow\quad \boxed{\tilde{\boldsymbol{\sigma}}^{(0)}=\tilde{A}_{II}\,\boldsymbol{\Sigma}^{(0)}+\varepsilon\,\tilde{A}_{II}\,\boldsymbol{\Sigma}^{(1)}+\varepsilon^2\,\tilde{A}_{II}\,\boldsymbol{\Sigma}^{(2)}+O\left(\varepsilon^3\right)}$$

and

$$\tilde{\mathbf{u}}^{(0)}=\tilde{A}_{II}^{n}\,\mathbf{u}^{(0)}(n)=\tilde{A}_{II}\left[1+\varepsilon\ln\tilde{A}_{II}+\frac{1}{2}\varepsilon^2\left(\ln\tilde{A}_{II}\right)^2+O\left(\varepsilon^3\right)\right]\left[\mathbf{U}^{(0)}+\varepsilon\,\mathbf{U}^{(1)}+\varepsilon^2\,\mathbf{U}^{(2)}+O\left(\varepsilon^3\right)\right] \Rightarrow$$

$$\boxed{\tilde{\mathbf{u}}^{(0)}=\tilde{A}_{II}\,\mathbf{U}^{(0)}+\varepsilon\,\tilde{A}_{II}\left[\left(\ln\tilde{A}_{II}\right)\mathbf{U}^{(0)}+\mathbf{U}^{(1)}\right]+\varepsilon^2\,\tilde{A}_{II}\left[\frac{1}{2}\left(\ln\tilde{A}_{II}\right)^2\mathbf{U}^{(0)}+\left(\ln\tilde{A}_{II}\right)\mathbf{U}^{(1)}+\mathbf{U}^{(2)}\right]+O\left(\varepsilon^3\right)}.$$

We use again the compact form of the leading order problem as

$$\frac{d\mathbf{x}(\theta)}{d\theta}-\mathbf{F}(s)\cdot\mathbf{x}(\theta)=\varepsilon\,\mathbf{G}\left(\mathbf{x}(\theta),s\right)\cdot\mathbf{x}(\theta), \tag{1.3}$$

where $\mathbf{F}(s)$, $\mathbf{G}\left(\mathbf{x}(\theta),s\right)$, $\mathbf{x}(\theta)$ are given in Chapter 3. The differential equation can be written in an integral equation form as

$$\mathbf{x}(\theta)=\boldsymbol{\Psi}(\theta,s)\cdot\boldsymbol{\Psi}^{-1}(0,s)\cdot\mathbf{x}(0)+\varepsilon\,\boldsymbol{\Psi}(\theta,s)\cdot\int_0^\theta\boldsymbol{\Psi}^{-1}(\phi,s)\cdot\mathbf{G}\left(\mathbf{x}(\phi),s\right)\cdot\mathbf{x}(\phi)\,d\phi \tag{1.4}$$

where $\boldsymbol{\Psi}(\theta,s)$, $\mathbf{G}\left(\mathbf{x}(\theta),s\right)$ are also given in Chapter 3. The corresponding boundary conditions are

$$u_r^{(0)}(0)=u_\theta^{(0)}(0)=0\,, \qquad \sigma_{\theta\theta}^{(0)}(\pi)=\sigma_{r\theta}^{(0)}(\pi)=0\,.$$

The normalization (1.1) requires that

$$\Sigma_{\theta\theta}^{(0)}(0)=\Sigma_{\theta\theta}^{(1)}(0)=\Sigma_{\theta\theta}^{(2)}(0)=\cdots=0 \tag{1.5}$$

and $$\Sigma_{r\theta}^{(0)}(0)=1,\qquad \Sigma_{r\theta}^{(1)}(0)=\Sigma_{r\theta}^{(2)}(0)=\cdots=0. \tag{1.6}$$

The leading order stress exponent $s=-\dfrac{1}{n+1}$ can be written as

$$s=-\frac{1}{n+1}=-\frac{1}{2}+\frac{1}{4}\varepsilon-\frac{1}{8}\varepsilon^2+O\left(\varepsilon^3\right). \tag{1.7}$$

Substituting the expansions (1.2) and (1.7) into the integral equation (1.4) and collecting terms having like powers of ε, we arrive at the following sequence of problems:

$$O(1):\qquad \mathbf{x}^{(0)}(\theta)=\mathbf{A}\left(\theta,-\frac{1}{2}\right)\cdot\mathbf{x}^{(0)}(0), \tag{1.8}$$

where $\mathbf{A}(\theta,s)=\boldsymbol{\Psi}(\theta,s)\cdot\boldsymbol{\Psi}^{-1}(0,s)$, and $\mathbf{x}^{(0)}(0)=\{0\quad 0\quad 0\quad 1\}$,

$$O(\varepsilon):\mathbf{x}^{(1)}(\theta)=\frac{1}{4}\left[\frac{\partial\mathbf{A}(\theta,s)}{\partial s}\right]_{s=-1/2}\cdot\mathbf{x}^{(0)}(0)+\int_0^\theta\mathbf{B}\left(\theta,\phi,-\frac{1}{2}\right)\cdot\mathbf{G}^{(0)}\left(\mathbf{x}^{(0)}(\phi),-\frac{1}{2}\right)\cdot\mathbf{x}^{(0)}(\phi)\,d\phi, \tag{1.9}$$

where $\mathbf{B}(\theta,\phi,s)=\boldsymbol{\Psi}(\theta,s)\cdot\boldsymbol{\Psi}^{-1}(\phi,s)$, and $\mathbf{G}^{(0)}\left(\mathbf{x}^{(0)}(\phi),s\right)$ as defined in Chapter 3.

We notice that the boundary conditions $x_3^{(0)}(\pi)=0$ and $x_4^{(0)}(\pi)=0$ are automatically satisfied by the solution (1.8), which takes the form

$$\mathbf{x}^{(0)}(\theta)=\begin{Bmatrix}\frac{3}{2}\sin\frac{\theta}{2}(1+3\cos\theta)\\ -\frac{9}{2}\sin\frac{\theta}{2}\sin\theta\\ -\frac{3}{2}\cos\frac{\theta}{2}\sin\theta\\ \frac{1}{2}\cos\frac{\theta}{2}(-1+3\cos\theta)\end{Bmatrix}\qquad\text{and}\qquad \Sigma_e^{(0)}=\frac{1}{2}\sqrt{\frac{3}{2}(5+3\cos 2\theta)}. \tag{1.10}$$

The corresponding boundary condition of the of the $O(\varepsilon)$ problem are $x_3^{(1)}(\pi)=x_4^{(1)}(\pi)=0$. Substituting the expressions (1.10) in (1.9), we conclude that

$$x_3^{(1)}(\pi)=-\frac{1}{2}+\frac{3}{8}\int_0^\pi\sin\theta(1+3\cos 2\theta)\ln\Sigma_e^{(0)}\,d\theta=-\frac{1}{2},$$

and $$x_4^{(1)}(\pi)=\frac{1}{16}\int_0^\pi(7\cos\theta+9\cos 3\theta)\ln\Sigma_e^{(0)}\,d\theta=0,$$

i.e., it is impossible to satisfy the condition $x_3^{(1)}(\pi)=0$. In other words, a solution that satisfies the conditions

$$\tilde{\sigma}_{\theta\theta}^{(0)}(0)=0 \qquad \text{and} \qquad \tilde{\sigma}_{r\theta}^{(0)}(0)=\tilde{A}_{II}$$

does not exist.

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