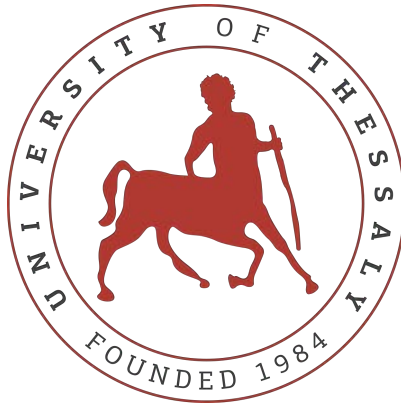




UNIVERSITY OF THESSALY
DOCTORAL THESIS

Modified theories of Gravity and Cosmology

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Abstract

Modified theories of Gravity and Cosmology

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The standard model of cosmology has been successful in describing observations at the astrophysical and cosmological scales for several decades. The Λ CDM model consists of general relativity as the underlying theory of gravitation, cold dark matter, the cosmological constant and the mechanism of inflation. Recent observations suggest that the Universe recently entered into a phase of accelerating expansion, which is assumed to be driven by dark energy. The simplest explanation is the introduction of the cosmological constant, which suffers from theoretical problems, namely the cosmological constant and the coincidence problem. In the last two decades, the increasing accuracy of observations revealed several differences between certain cosmological probes, two of the most important being the H_0 and σ_8 tensions. One main approach that has been pursued in order to resolve the aforementioned issues is to modify the gravitational framework.

Since general relativity is non-renormalizable, adding higher order curvature terms in the Lagrangian tends to improve its quantum behavior. The Gauss–Bonnet term is considered topologically invariant in 4D and it has been crucial in regulating the divergences in higher dimensional theories, such as string theory. While general relativity and quantum field theory, by many aspects, remain unreconciled, their partial synthesis (e.g, Hawking–Unruh–Davis effects) has led to the development of black hole thermodynamics and subsequently to applications in alternative cosmological approaches, such as the spacetime thermodynamic approach.

The aim of this thesis is to investigate the cosmological implications of the Gauss–Bonnet gravity in 4D during topological transitions, first at the spacetime foam level predicted by Euclidean quantum gravity and second at the Universe’s apparent horizon inside the spacetime conjecture cosmological framework.

Euclidean quantum gravity predicts solutions at the microscopic foam level, namely instantons, with different topology from the background. We started by asking: what happens to the variation of the Gauss–Bonnet term during the appearance of these topology alternating instantons? By assuming that the variation of the quantum fluctuations causes a change in the topology of spacetime, we have incorporated the topological variation procedure into the semiclassical approach under the background field method for the Einstein–Hilbert plus the Gauss–Bonnet term. Interestingly enough, we have obtained the Einstein equation for the background with an effective cosmological constant term of topological origin, which equals the density of instantons per four-volume.

Additionally, we employed quantum field theory techniques for describing the creation rate of instantons inside an evolving Hubble sphere, and we have obtained the differential equation describing the cosmological evolution of the novel spacetime foam topological dark energy sector. We have confronted the model’s predictions with observational datasets in comparison with Λ CDM and found the Spacetime Foam Topological Dark Energy model to be systematically preferred over Λ CDM. The model’s thermal history deviates from the concor-

dance one, as it exhibits an excess of dark energy over dark matter. An intriguing feature is that the equation of state parameter tends to zero at late times, suggesting a dark energy to dark matter conversion mechanism. While the model stays inside acceptable observational bounds regarding the Baryon Acoustic Oscillation and the sound horizon radius, a future analysis of its implication in the CMB spectrum could further constrain the model's adequacy to describe Nature. In summary, spacetime foam topological dark energy arises through a well described microphysical mechanism resolving the cosmological constant problem while performing better than the concordance cosmological model regarding observational fitting.

We continued by exploring the implications of the Gauss–Bonnet action term in the spacetime thermodynamic conjecture framework. In the case of general relativity, the corresponding entropy is the standard horizon area law described by the Bekenstein–Hawking relation, which yields the standard Friedmann equations. By the Noether charge method, Wald has calculated the entropy expression for the Gauss–Bonnet term to be proportional to the Euler characteristic of the horizon. Nevertheless, the Wald–Gauss–Bonnet topological entropy term causes a violation of the second law of thermodynamics during black hole horizon merging. Inspired by the holographic principle, in order to resolve the second law violation, we have assumed that the Universe's apparent horizon is topologically linked with the interior black hole horizons in a way that compensates for the entropy decrease during black hole merging. Hence, the apparent horizon becomes topologically dynamic, and when we apply the spacetime thermodynamic approach for the Wald–Gauss–Bonnet entropy, we obtain modified Friedmann equations that depend on the black hole formation and merging rate.

By employing the star formation rate best fit model of Madau and Dickinson, we were able to approximate the black hole formation and merging rate, thus we have obtained an analytical solution of the modified Friedmann equations that depend on one free parameter, the Gauss–Bonnet coupling constant, and other astrophysical observationally bounded parameters. From the modified Friedmann

equations we have extracted a novel dark energy correction term. The topological dark energy from black hole formation and merging may exhibit quintessence-like or phantom-like behavior, depending on the Gauss–Bonnet coupling sign. In each case, we have presented that the modified cosmological scenario reproduces the standard thermal history of the Universe.

Since the phantom behavior at intermediate redshifts is a known sufficient mechanism that can lead to the H_0 tension alleviation, we have shown that for particular regions of the parameter space, the scenario at hand does lead to higher H_0 values, exactly due to the suitable phantom behavior at intermediate redshifts. For completeness, we have verified this result by performing a confrontation with observational data from Supernovae Type Ia (SNIa) and Cosmic Chronometers (CC) measurements. We proceeded to the investigation of the growth of structures and of the evolution of the matter overdensity. Since in Wald-Gauss-Bonnet cosmology we do not obtain an effective Newton’s constant different from the standard one, under the assumption that the effective topological dark energy does not cluster, we were able to calculate the redshift behavior of the observable $f\sigma_8$. Interestingly enough, we saw that we obtained smaller values than the ones arising in Λ CDM cosmology, and thus the scenario at hand can alleviate the σ_8 tension too.

In summary, Wald-Gauss-Bonnet cosmology can relieve the H_0 and σ_8 tensions at the same time. Furthermore, as we proceed in the multimessenger astronomy era, datasets of increasing accuracy are expected, especially from the space based gravity wave detectors (e.g., LISA). Thus, the model’s predictions dependence on black hole formation and merging rate, enhances its falsifiability in the light of the expected measurements. The aforementioned characteristics can be considered advantages of the scenario at hand, and they place it as a promising candidate for the description of Nature.



Πανεπιστήμιο Θεσσαλίας

Διδακτορική Διατριβή

Τροποποιημένες θεωρίες βαρύτητας και κοσμολογία

Συγγραφέας:
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Η ανα χείρας διατριβή υποβλήθηκε ως μέρος των απαιτήσεων
προς απόκτηση Διδακτορικού Διπλώματος

από το

Τμήμα Φυσικής
Λαμία, Σεπτέμβριος 2025

Η έγκριση Διδακτορικής Διατριβής από το Τμήμα Φυσικής του Πανεπιστημίου Θεσσαλίας δεν υποδηλώνει την αποδοχή των γνώμών του συγγραφέα (ν. 5343/1932, άρθρο 202, παρ. 2)

Με ατομική μου ευθύνη και έχοντας επίγνωση της δυνατότητας επιβολής κυρώσεων από τη Γενική Συνέλευση του Τμήματος, δηλώνω υπεύθυνα ότι για τη συγγραφή της Διδακτορικής μου Διατριβής δεν χρησιμοποίησα ολόκληρο ή μέρος έργου άλλου δημιουργού ή τις ιδέες και αντιλήψεις άλλου δημιουργού χωρίς να γίνεται αναφορά στην πηγή προέλευσης (βιβλίο, άρθρο από εφημερίδα ή περιοδικό, ιστοσελίδα κ.τ.λ.). Επιπρόσθετα, δηλώνω ότι έλεγξα την πιθανότητα λογοκλοπής της τελικής έκδοσης της Διδακτορικής Διατριβής μέσω της ηλεκτρονικής πλατφόρμας του Πανεπιστημίου Θεσσαλίας και κατέθεσα τη σχετική ανάλυση και τα αποτελέσματά της σε ηλεκτρονική και έντυπη μορφή στη Γραμματεία του Τμήματος.

Πανεπιστήμιο Θεσσαλίας

Εκτεταμένη Περίληψη

Διδακτορική Διατριβή

Τροποποιημένες θεωρίες βαρύτητας και κοσμολογία

Το καθιερωμένο κοσμολογικό πρότυπο έχει καταφέρει επί δεκαετίες να περιγράψει επιτυχώς τις παρατηρήσεις σε αστροφυσικές και κοσμολογικές κλίμακες. Το μοντέλο Λ CDM αποτελείται από τη γενική σχετικότητα ως τη θεμελιώδη θεωρία βαρύτητας, την ψυχρή σκοτεινή ύλη, την κοσμολογική σταθερά και το μηχανισμό του πληθωρισμού. Πρόσφατες παρατηρήσεις υποδεικνύουν ότι το Σύμπαν έχει εισέλθει πρόσφατα σε φάση επιταχυνόμενης διαστολής, η οποία θεωρείται ότι προκαλείται από τη σκοτεινή ενέργεια. Η απλούστερη εξήγηση είναι η εισαγωγή της κοσμολογικής σταθεράς, η οποία ωστόσο παρουσιάζει θεωρητικά προβλήματα, συγκεκριμένα το πρόβλημα της κοσμολογικής σταθεράς και το πρόβλημα της σύμπτωσης. Τις τελευταίες δύο δεκαετίες, η αυξανόμενη ακρίβεια των παρατηρήσεων ανέδειξε διαφορές μεταξύ διαφόρων κοσμολογικών δεδομένων, με σημαντικότερες τις εντάσεις H_0 και σ_8 . Μία κύρια προσέγγιση για την επίλυση αυτών των ζητημάτων είναι η ανάπτυξη τροποποιημένων θεωριών βαρύτητας.

Ένα επιπλέον πλεονέκτημα της τροποποιημένης βαρύτητας είναι ότι καθώς η γενική σχετικότητα δεν είναι επανακανονικοποιήσιμη, η προσθήκη όρων καμπυλότητας ανώτερης τάξης στην Λαγκρανζιανή, βελτιώνει την συμπεριφορά της σε κβαντικό επίπεδο. Ο όρος **Gauss–Bonnet** θεωρείται τοπολογικά αναλλοίωτος σε τέσσερις διαστάσεις και έχει διαδραματίσει καίριο ρόλο στη ρύθμιση απειρισμών σε θεωρίες υψηλότερων διαστάσεων, όπως η θεωρία χορδών. Αν και η γενική σχετικότητα και η κβαντική θεωρία πεδίου παραμένουν σε μεγάλο βαθμό μη συμφιλιωμένες, η μερική σύνθεσή τους (π.χ. τα φαινόμενα **Hawking–Unruh–Davis**) οδήγησε στην ανάπτυξη της θερμοδυναμικής των μελανών οπών και σε εφαρμογές εναλλακτικών κοσμολογικών προσεγγίσεων, όπως η εικασία της θερμοδυναμικής του χωρόχρονου.

Στόχος αυτής της διατριβής είναι η διερεύνηση των κοσμολογικών συνεπειών της βαρύτητας **Gauss–Bonnet** στις τέσσερις διαστάσεις κατά τη διάρκεια τοπολογικών μεταβάσεων, πρώτον στο επίπεδο του χωροχρονικού αφρού όπως προβλέπεται από την Ευκλείδεια χβαντική βαρύτητα, και δεύτερον στον φαινομενικό ορίζοντα του Σύμπαντος στο πλαίσιο της θερμοδυναμικής εικασίας του χωρόχρονου.

Η Ευκλείδεια χβαντική βαρύτητα προβλέπει λύσεις στο μικροσκοπικό επίπεδο αφρού, γνωστές ως στιγμιόνια, τα οποία έχουν διαφορετική τοπολογία από το υποβάθρο. Εκκινώντας από την παραπάνω παρατήρηση θέσαμε το ερώτημα: πώς μεταβάλλεται ο όρος **Gauss–Bonnet** κατά την εμφάνιση στιγμιονίων διαφορετικής τοπολογίας; Θεωρώντας ότι οι χβαντικές διακυμάνσεις προκαλούν αλλαγή στην τοπολογία, ενσωματώσαμε τη διαδικασία τοπολογικής μεταβολής στη ημι-κλασική προσέγγιση, στο πλαίσιο της μεθόδου πεδίου υποβάθρου, για τον όρο **Einstein–Hilbert** συν τον **Gauss–Bonnet**. Προέκυψαν οι εξισώσεις **Einstein** για το υποβάθρο με έναν ενεργό όρο κοσμολογικής σταθεράς τοπολογικής προέλευσης, που ισούται με την πυκνότητα των στιγμιονίων ανά τετραδιάστατο όγκο.

Επιπλέον, εφαρμόσαμε τεχνικές χβαντικής θεωρίας πεδίου για να περιγράψουμε το ρυθμό δημιουργίας των στιγμιονίων μέσα σε μια εξελισσόμενη σφαίρα **Hubble** και καταλήξαμε σε μία διαφορική εξίσωση που περιγράφει την κοσμολογική εξέλιξη της προτεινόμενης σκοτεινής ενέργειας τοπολογικής φύσης. Συγκρίνοντας τις προβλέψεις του μοντέλου με τις αντίστοιχες του καθιερωμένου κοσμολογικού προτύπου Λ CDM βάσει των παρατηρήσεων, βρήκαμε ότι το σενάριο τοπολογικής σκοτεινής ενέργειας στο επίπεδο του χωροχρονικού αφρού προτιμάται συστηματικά από τα δεδομένα. Το μοντέλο εμφανίζει διαφορές στην θερμική ιστορία του Σύμπαντος συγκριτικά με το καθιερωμένο κοσμολογικό πρότυπο, καθώς παρουσιάζει περίσσεια σκοτεινής ενέργειας έναντι της σκοτεινής ύλης. Ένα ιδιαίτερα ενδιαφέρον χαρακτηριστικό είναι ότι η παράμετρος της καταστατικής εξίσωσης της σκοτεινής ενέργειας τείνει στο μηδέν σε μεγάλους χρόνους, υπονοώντας ότι το μοντέλο μπορεί να περιέχει κάποιον μηχανισμό μετατροπής της σκοτεινής ενέργειας σε σκοτεινή ύλη. Ενώ το μοντέλο παραμένει εντός αποδεκτών παρατηρησιακών ορίων ως προς τις ταλαντώσεις βαρυονικής ύλης (BAO) και την ακτίνα του ηχητικού ορίζοντα, μια μελλοντική ανάλυση των επιπτώσεων του στο φάσμα της κοσμικής μικροκυματικής ακτινοβολίας (CMB) θα μπορούσε να θέσει επιπλέον περιορισμούς στην επάρκειά του ως μοντέλο περιγραφής της φύσης. Συνοψίζοντας, η τοπολογική σκοτεινή ενέργεια στο επίπεδο του χωροχρονικού αφρού αναδύεται μέσω ενός καλά περιγεγραμμένου μικροφυσικού μηχανισμού, επιλύοντας το πρόβλημα τη κοσμολογικής σταθεράς και αποδίδοντας καλύτερα από το καθιερωμένο κοσμολογικό μοντέλο ως προς την περιγραφή των παρατηρησιακών δεδομένων.

Συνεχίσαμε διερευνώντας τις συνέπειες του όρου δράσης **Gauss–Bonnet** στο πλαίσιο της θερμοδυναμικής εικασίας του χωρόχρονου. Στο πλαίσιο αυτό, στην περίπτωση της γενικής σχετικότητας, η αντίστοιχη εντροπία είναι ο κλασικός νόμος εμβαδού του ορίζοντα, όπως περιγράφεται από τη σχέση **Bekenstein–Hawking**, που οδηγεί στις καθιερωμένες εξισώσεις **Friedmann**. Μέσω της μεθόδου φορτίου της **Noether**, ο **Wald** υπολόγισε ότι η έκφραση της εντροπίας για τον όρο **Gauss–Bonnet** είναι ανάλογη της χαρακτηριστικής **Euler** του ορίζοντα. Ωστόσο, ο τοπολογικός όρος εντροπίας **Wald–Gauss–Bonnet** προκαλεί παραβίαση του δεύτερου θερμοδυναμικού νόμου κατά τη συγχώνευση των οριζόντων μελανών οπών. Εμπνευσμένοι από την ολογραφική αρχή, προκειμένου να επιλυθεί αυτή η παραβίαση, θεωρήσαμε ότι ο φαινομενικός ορίζοντας του Σύμπαντος είναι τοπολογικά συνδεδεμένος με τους ορίζοντες των μελανών οπών στο εσωτερικό του, με τρόπο που αντισταθμίζει τη μείωση της εντροπίας κατά τη συγχώνευση των μελανών οπών. Συνεπώς, ο φαινομενικός ορίζοντας καθίσταται τοπολογικά δυναμικός και όταν εφαρμόζουμε τη θερμοδυναμική προσέγγιση του χωρόχρονου για την εντροπία **Wald–Gauss–Bonnet**, λαμβάνουμε τροποποιημένες εξισώσεις **Friedmann** που εξαρτώνται από τον ρυθμό σχηματισμού και συγχώνευσης των μελανών οπών.

Χρησιμοποιώντας το μοντέλο των **Madau** και **Dickinson** για τον ρυθμό σχηματισμού των άστρων, καταφέραμε να προσεγγίσουμε τον ρυθμό σχηματισμού και συγχώνευσης μελανών οπών. Με τον τρόπο αυτό αποκτήσαμε αναλυτική λύση των τροποποιημένων εξισώσεων **Friedmann**, οι οποίες εξαρτώνται από μία ελεύθερη παράμετρο, τη σταθερά σύζευξης **Gauss–Bonnet** και άλλες αστροφυσικές παραμέτρους περιορισμένες από παρατηρήσεις. Από τις τροποποιημένες εξισώσεις **Friedmann** εξάγαμε έναν νέο όρο διόρθωσης της σκοτεινής ενέργειας. Η τοπολογική σκοτεινή ενέργεια που προκύπτει από το σχηματισμό και τη συγχώνευση μελανών οπών μπορεί να παρουσιάσει συμπεριφορά τύπου πεμπτουσίας (*quintessence*) ή φαντάσματος (*phantom*), ανάλογα με το πρόσημο της σταθεράς **Gauss–Bonnet**. Σε κάθε περίπτωση, δείξαμε ότι το τροποποιημένο κοσμολογικό σενάριο αναπαράγει τη γνωστή θερμική ιστορία του Σύμπαντος.

Καθώς η φαντομική συμπεριφορά σε ενδιαμέσες τιμές της ερυθρομετατόπισης αποτελεί γνωστό μηχανισμό που μπορεί να αμβλύνει την ένταση H_0 , δείξαμε ότι για συγκεκριμένες περιοχές του παραμετρικού χώρου, το υπό εξέταση σενάριο οδηγεί πράγματι σε υψηλότερες τιμές του H_0 , ακριβώς λόγω της κατάλληλης φαντομικής συμπεριφοράς σε ενδιαμέσες τιμές της ερυθρομετατόπισης. Για λόγους πληρότητας, επαληθεύσαμε αυτό το αποτέλεσμα μέσω σύγκρισης με παρατηρησιακά δεδομένα από σουπερνόβα τύπου **Ia (SNIa)** και μετρήσεις κοσμικών χρονομέτρων (**CC**). Στη συνέχεια, εξετάσαμε την ανάπτυξη των δομών και την εξέλιξη της σηματοποίησης της ύλης. Δεδομένου ότι στην κοσμολογία **Wald–Gauss–**

–**Bonnet** δεν προκύπτει ενεργός σταθερά του Νεύτωνα διαφορετική από την καθιερωμένη, υπό την παραδοχή ότι η ενεργός τοπολογική σκοτεινή ενέργεια δεν συσσωρεύεται, καταφέραμε να υπολογίσουμε την εξέλιξη ως προς την ερυθρομετατόπιση της παρατηρήσιμης ποσότητας $f\sigma_8$. Διαπιστώσαμε ότι αποκτήσαμε μικρότερες τιμές από αυτές που προκύπτουν στην κοσμολογία Λ CDM, και επομένως το υπό εξέταση σενάριο μπορεί να αμβλύνει και την ένταση σ_8 .

Συνοπτικά, η κοσμολογία **Wald–Gauss–Bonnet** μπορεί να ανακουφίσει ταυτόχρονα τις εντάσεις H_0 και σ_8 . Επιπλέον, καθώς προχωράμε στην εποχή της πολυμηνυματικής αστρονομίας, αναμένονται ολοένα πιο ακριβή δεδομένα, ιδίως από διαστημικούς ανιχνευτές βαρυτικών κυμάτων (π.χ. **LISA**). Συνεπώς, η εξάρτηση των προβλέψεων του μοντέλου από τον ρυθμό σχηματισμού και συγχώνευσης μελανών οπών ενισχύει τη δυνατότητα διάψευσής του υπό το πρίσμα των μελλοντικών μετρήσεων. Τα παραπάνω χαρακτηριστικά συνιστούν πλεονεκτήματα του σεναρίου της τοπολογικής ενέργειας από σχηματισμό και συγχώνευση μελανών οπών και το καθιστούν υποσχόμενο υποψήφιο για την περιγραφή της Φύσης.

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Τριμελής επιτροπή

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2. Διονύσιος Βαβουγιός, Ομότιμος Καθηγητής, Τμήμα Φυσικής, Πανεπιστημίου Θεσσαλίας
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List of publications

During my PhD studies I have co-authored four peer-reviewed (or under peer-review) articles, one N-author review and one article published in conference proceedings. The articles are listed below in descending chronological order:

1. "Dark energy from topology change induced by microscopic Gauss-Bonnet worm-holes", Stylianos A. Tsilioukas, Charalambos Tzerefos, Emmanuel N. Saridakis, Phys.Rev.D 109 (2024) 8, 084010
2. "Dark Energy from topology change at the foam level", Stylianos A. Tsilioukas, Emmanuel N. Saridakis, PoS CORFU2023 (2024) 182
3. "Topological dark energy from black-hole formations and mergers through the gravity-thermodynamics approach", Stylianos A. Tsilioukas, Nicholas Petropoulos, Emmanuel N. Saridakis, Phys.Rev.D 111 (2025) 10, 103514
4. "Using Wald-Gauss-Bonnet topological dark energy to simultaneously alleviate the H_0 and σ_8 tensions", Stylianos A. Tsilioukas, Maria Petronikolou, Spyros Basilakos, Emmanuel N. Saridakis, arXiv:2501.15927v1 [gr-qc] – submitted to Eur.Phys.J.C
5. "Dark Energy from spacetime foam: Challenging the Λ CDM" Fotios K. Anagnostopoulos, Stylianos A. Tsilioukas, Emmanuel N. Saridakis, arXiv:2507.18389v1 [gr-qc]–submitted to Physical Review Letters
6. "The CosmoVerse White Paper: Addressing observational tensions in cosmology with systematics and fundamental physics", Eleonora Di Valentini et. al Phys.Dark Univ. 49 (2025) 101965 .

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Chapter 1.

Introduction

“Laws were made to be broken.”

— Christopher North, 1785–1854

1.1. Introduction

1.1.1. Historical development of astronomy, cosmology and gravity

The history of astronomy, cosmology and gravity can be seen as a history of tensions between theoretical predictions and observations. According to the epistemological perspective of Kuhn, as the tensions accumulate and increase, they challenge the dominant scientific paradigm and the crisis is resolved when a new theory appears which meets the observational requirements. The progress of cosmology, intertwined with the evolution of gravitational theories, up to the present-day state-of-the-art, challenges, is going to be presented in the light of scientific revolutions methodology, therefore forming a historical narrative.

The first attempts to observe celestial bodies in the sky and formulate theories about their behavior can be traced back to ancient times. The ancient Greek civilization was highly influential, as cosmology emerged as a subject among natural philosophers and by the 4th century B.C, the geocentric model with a spherical Earth at the center of the Universe had been established. The underlying theory was Aristotle’s natural laws, which stated that heavier bodies fall faster in order to return to their initial state, and

that the Sun and the planets orbit the Earth in perfect circles. The geocentric model was further developed by Ptolemaeus of Alexandria with the implementation of epicycles. The geocentric Ptolemaic model was compatible with Aristotelian natural philosophy and predicted the movement of celestial bodies, within the observational accuracy of the time, for the next 1000 years. In China, India and Persia, observations around 1100 A.C began to question Earth's lack of motion, but not geocentrism. As the accuracy of astronomical observations increased, their discrepancy with the Ptolemaic model also increased, challenging its validity. The crisis escalated in the 1500s, when Copernicus re-introduced the heliocentric model (2000 years after Aristarchus) assuming that the Earth, like the other planets, orbits around the Sun in perfect circles. Later on, the astronomers Brahe and Kepler modified the Copernican circular orbits with elliptical ones. According to Kepler's laws, the planets move with varying speeds in elliptical orbits with the Sun at one focus. The experiments of Galileo challenged the physics of Aristotle, the theoretical backbone of the geocentric model, and his astronomical observations further supported the heliocentric model.

In 1687 Sir Isaac Newton, in his celebrated *Philosophiae Naturalis Principia Mathematica* [1] introduced the three laws of classical mechanics and the law of gravitational attraction. The laws of Kepler could be derived from Newton's laws, highlighting the concreteness of theory and observations within the new paradigm of classical and celestial mechanics. The theory of electromagnetism introduced by Maxwell [2] seemed to be incompatible with the Galilean relativity of Newtonian mechanics. It was this inconsistency along with the null result of the Michelson Morley experiment that hinted Einstein to formulate the special theory of relativity [3], which postulates that the speed of light is the same for all inertial observers and therefore no signal can travel faster than light. The special theory of relativity revolutionized the absolute time and space of Newtonian mechanics, combining them into a dynamical space-time manifold. When Einstein looked at Newton's law of gravitation in the light of the special theory of relativity, a new inconsistency appeared. According to Newtonian gravity, when two bodies interact, a change in the mass of one body would be felt instantaneously by the second body, but an instantaneous signal would violate the special theory of relativity, as the gravitational field would travel faster than light. Furthermore, the special theory of relativity applied only to inertial frames and not accelerated ones. By implementing the equivalence principle — the idea that acceleration is equivalent to gravitation— into the emerging mathematical framework of differential geometry, Albert Einstein introduced a new theory of gravitation. The general theory of relativity [4] considers that the gravitational field is the curvature

of the four-dimensional spacetime fabric sourced by mass and energy-momentum. Therefore, massive bodies, like the Sun, wrap the fabric of spacetime around them, causing other bodies, like the planets, to move around them in curved spacetime trajectories. General relativity had Newtonian gravity as a weak field limit, predicted the precession of Mercury's orbit, a phenomenon which could not be explained in the framework of Newtonian mechanics. It is important to note that the precession of Mercury's orbit phenomenon was discovered in 1859 and there had been many attempts to explain it by minor corrections to Newton's Law of gravitation or by the gravitational effect of a hidden planet, named Vulcan, which was not yet discovered. Finally, the solution was a conceptually radical new theory of gravity, general relativity, and not a dark planet. This paradigm resembles the present debate in cosmology between modified gravity and/or dark sector for solving the observational tensions.

1.1.2. The development of quantum mechanics and quantum field theories

Although the study of the cosmos involves classical gravity as the dominant force on cosmological scales, one should take into account the revolution of quantum mechanics, because, as we will see later, there are several aspects of quantum cosmology and modified theories of gravity that originate from the quantum regime.

While the development of relativity has been highlighted as a one-man show, attributed primarily to Albert Einstein, quantum mechanics emerged as a collaborative scientific endeavor. The first indications of quantum theory arose with Max Planck's introduction of energy quanta to resolve the ultraviolet catastrophe in black-body radiation [5]. Einstein extended this concept by proposing the quantization of light to explain the photoelectric effect [6]. Louis de Broglie later introduced the wave-particle duality of matter, drawing an analogy to the quantum nature of electromagnetism [7]. Furthermore, Niels Bohr developed the first quantum model of the atom, successfully explaining the discrete spectral lines of hydrogen [8].

Max Born's probabilistic interpretation of the wave function laid the foundation for the modern formalism of quantum mechanics [9]. Erwin Schrödinger introduced a wave equation, now known as the Schrödinger equation, by applying operator methods to the classical Hamiltonian formalism, establishing a differential equation for the evolution of quantum states [10]. Werner Heisenberg formulated the uncertainty

principle, setting fundamental limits on the simultaneous measurement of conjugate variables [11]. He also developed matrix mechanics, an alternative formulation of quantum mechanics where observables are represented by time-evolving operators rather than wavefunctions [12].

The development of quantum mechanics was both rapid and controversial. Albert Einstein, along with Boris Podolsky and Nathan Rosen, identified a seeming contradiction between quantum mechanics and locality, formulating what became known as the Einstein–Podolsky–Rosen (EPR) paradox [13]. This paradox suggested that quantum mechanics allowed for instantaneous actions (Einstein’s famous quote was “spooky action from a distance”) that would violate the relativistic principle of locality. John Bell later quantified these non-local features by deriving a set of inequalities that distinguish quantum mechanics from local hidden variable theories [14].

Experimental tests of Bell’s inequalities in the 1960s, most notably those by Clauser, Horne, Shimony, and Holt (CHSH), confirmed the predictions of quantum mechanics [15]. Since then, experiments testing quantum non-locality have been conducted with increasing precision and at ever-greater distances, reinforcing the foundational implications of quantum mechanics.

Despite the challenges posed by the issue of locality, quantum mechanics was on the verge of reconciliation with special relativity. Oskar Klein and Walter Gordon independently extended Schrödinger’s formalism by applying it to the relativistic energy-momentum relation, resulting in the Klein–Gordon equation, the first equation of relativistic quantum mechanics [16, 17]. Paul Dirac took this further by seeking a first-order differential equation consistent with both quantum mechanics and special relativity. By introducing matrices to satisfy the algebraic constraints imposed by Lorentz invariance, he formulated the Dirac equation, which successfully described relativistic fermions and predicted the existence of antimatter [18].

In this framework, bosons were found to obey the Klein–Gordon equation, while fermions were governed by the Dirac equation. Advances in experimental techniques led to the discovery of new elementary particles, revealing the internal structure of the atomic nucleus and demonstrating how nuclear binding energy could be converted into usable energy.

As humanity entered World War II, research in nuclear physics accelerated, focusing on weaponizing the tremendous nuclear energy density. The United States of America won the race, through the Manhattan Project. At the brink of ending of World War II,

the United States of America detonated the first nuclear bombs over Hiroshima and Nagasaki in August 1945 [19], demonstrating their power to its new adversary the Soviet Union. This event not only marked the end of the war but also signaled the beginning of the nuclear arms race between the United States and the Soviet Union.

Since the motion of particles was described quantum mechanically, the forces acting between them also required a quantum description. This was the central aim of Quantum Field Theory (QFT), which proved to be an extremely fruitful framework throughout its development. Richard Feynman introduced the path-integral formulation of quantum mechanics, an approach equivalent to the operator formalism [20]. Alongside Murray Gell-Mann, he contributed to the formulation of Quantum Electrodynamics (QED), where the electromagnetic force was mediated by the photon as the fundamental gauge boson [21–23].

The quantization of the electromagnetic interaction was relatively simpler because, as we now understand *a posteriori*, QED is an Abelian gauge theory based on the group $U(1)$. The weak nuclear force, governed by the $SU(2)$ gauge group, was incorporated into QFT through the pioneering work of Steven Weinberg, Abdus Salam, and Sheldon Glashow, culminating in the electroweak unification, in which the mediators of the interaction are the W and Z bosons [24–26].

The strong nuclear force presented a more intricate structure, as it is a non-Abelian gauge theory based on $SU(3)$. This required the introduction of three types of charges, categorized as *colors*, leading to the development of Quantum Chromodynamics (QCD), where the fundamental force carriers are the gluons [27–29]. Notably, in the case of particle physics, theoretical predictions often preceded experimental discoveries. A striking example of this time delay is the experimental confirmation of the Higgs boson. The Higgs mechanism, introduced independently by Peter Higgs, François Englert, and Robert Brout in 1964, explains how particles acquire mass through interactions with the Higgs field [30,31]. The discovery of the Higgs boson, the last missing piece of the Standard Model (SM), was finally confirmed in 2012 through experiments conducted at the Large Hadron Collider (LHC) at CERN [32,33].

A crucial theoretical development that significantly influenced modern research in gravity was the Yang–Mills theory. In 1954, Chen Ning Yang and Robert Mills formulated a general framework for non-Abelian gauge theories, demonstrating how the fundamental gauge interactions (electromagnetic, weak, and strong) could be derived from a unified mathematical structure [34]. Despite its success, Yang–Mills theory

still faces profound mathematical challenges, such as the *existence* and *mass gap* problem, which remains one of the unsolved Millennium Prize Problems in mathematical physics [35].

The success of the Standard Model, combined with the mathematical rigor of Yang–Mills theory, has served as a guiding paradigm in research on Quantum Gravity (QG) and Grand Unified Theories (GUTs). As a result, a branch of modern theoretical gravity (MTG) has emerged, exploring the possibility of formulating gravity as a gauge theory. This approach has been particularly relevant in the context of torsion-based theories such as Teleparallel Gravity and $f(T)$ gravity [36].

1.1.3. Modern cosmology

Albert Einstein in his attempt to describe a static Universe introduced the cosmological constant Λ to the General Relativity (GR) field equations, in such a way, that it provided a repulsive force that could counteract the gravitational attraction allowing for a static solution. In 1929, the astronomer Edwin Hubble, after years of systematic observations, presented compelling evidence that all observable celestial objects are receding from each other [37]. This discovery was based on the observed redshift of light from distant galaxies, providing a direct indication of cosmic expansion. Hubble's observations led to the radical conclusion that the Universe is not static but is instead expanding, fundamentally altering our understanding of cosmology. After Hubble's discovery, Einstein referred to the introduction of the cosmological constant as its "biggest blunder". Nevertheless, the cosmological constant has revived in the end of the 20th century, after the discovery of the accelerated expansion of the Universe, and its nature still remains one of the biggest problems in theoretical physics.

The concept of an expanding Universe is naturally accommodated within the framework of GR. In particular, Alexander Friedmann first derived a set of dynamical equations describing an evolving Universe, based on Einstein's field equations [38,39]. Subsequently, Howard P. Robertson and Arthur G. Walker further refined these ideas, leading to the formulation of the Friedmann–Robertson–Walker (FRW) metric [40,41]. This metric describes a homogeneous and isotropic expanding spacetime and serves as the foundation for modern cosmological models.

By imposing the FRW metric on Einstein's equations of General Relativity, one obtains the three fundamental Friedmann equations. These equations govern the

evolution of the Universe's scale factor and describe how the expansion rate depends on the energy content of the cosmos.

On the foundation of this new picture of an expanding Universe, Georges Lemaître made an intriguing argument. Since the Universe is expanding, tracing this evolution backward in time implies that the Universe was once concentrated in a singular state, initiating its evolution after a primordial explosion. This idea formed the basis of what later became known as the Big Bang Theory [42].

The Big Bang model initially faced considerable criticism, particularly from proponents of the steady-state theory and from philosophical perspectives rooted in realism [43]. However, the Big Bang paradigm has become a cornerstone of modern cosmology, as it provides a framework for understanding the early Universe and offers a natural connection to high-energy physics.

According to the standard cosmological model, the Universe, immediately following this primordial event, existed in an extremely high-energy-density state. In this regime, quantum gravity (QG) effects, primordial gravitational waves, grand unified theory (GUT) phase transitions, and early particle production are expected to play significant roles [44, 45]. The relics of these high-energy phenomena, such as the cosmic microwave background (CMB) anisotropies and potential imprints of primordial gravitational waves, serve as observational probes of physics at energy scales far beyond those accessible by terrestrial particle accelerators [46].

1.1.4. The Discovery of the Cosmic Microwave Background

In 1965, Arno Penzias and Robert Wilson accidentally discovered an almost perfect black-body microwave radiation, which was measured to be homogeneous and isotropic in all directions of the sky [47]. The origin of this radiation was soon attributed to an early stage of the Universe, as predicted by the Big Bang Theory. Under the expansion of spacetime, the radiation emitted in the early Universe would become redshifted, cooling down to the observed microwave frequencies while maintaining its spatial homogeneity.

This groundbreaking discovery provided strong empirical support for the Big Bang model, confirming the existence of relic radiation from the early Universe. For their contribution, Penzias and Wilson were awarded the Nobel Prize in Physics in

1978. The radiation was named the Cosmic Microwave Background (CMB), and its properties have been extensively studied, offering crucial insights into the thermal history, composition, and large-scale structure of the Universe [46,48].

The detailed study of the CMB, particularly through precision measurements of its anisotropies, has led to profound advancements in cosmology. Observations by the COBE, WMAP, and Planck satellites have revealed small fluctuations in the CMB temperature, which correspond to the primordial density perturbations that seeded the formation of galaxies and cosmic structures [49–51]. These fluctuations align remarkably well with predictions from inflationary cosmology, further solidifying our understanding of the early Universe.

1.1.5. The Theory of Inflation

The theory of cosmic inflation is one of the most significant advancements in modern cosmology, providing a compelling explanation for several fundamental issues within the standard Big Bang paradigm. Originally proposed in the early 1980s, inflation posits that the Universe underwent a brief but extremely rapid exponential expansion within the first instants after the Big Bang. This scenario successfully addresses long-standing problems such as the initial singularity, the horizon and flatness problems, the isotropy of the Cosmic Microwave Background (CMB), the origin of large-scale-structure (LSS), and the absence of relic magnetic monopoles [52–56].

The foundations of inflationary cosmology were laid by Alexei Starobinsky in 1980, who showed that quantum effects in gravity could lead to an exponential expansion [52]. Around the same time, Kazanas and Guth independently formulated the idea that an early de Sitter-like phase could resolve key cosmological puzzles [53,54]. Guth's seminal work introduced the concept of "false vacuum" inflation, where the Universe becomes trapped in a metastable vacuum state, leading to rapid expansion. However, this original model suffered from the problem of graceful exit, as the transition from the false vacuum resulted in an inhomogeneous Universe. To address this, Albrecht, Steinhardt and Linde proposed the "new" and "chaotic" inflationary models, respectively, offering mechanisms for a smoother exit from the inflationary phase [55,56].

A central feature of inflation is that it predicts a nearly scale-invariant spectrum of primordial fluctuations, which later seed the formation of galaxies and cosmic

structures. These fluctuations arise from quantum fluctuations of the inflaton field, which become stretched to macroscopic scales due to the rapid expansion [57–61]. Inflation also predicts the presence of primordial gravitational waves, whose detection remains a key goal of modern observational cosmology [62, 63]. The existence of these tensor perturbations is quantified through the tensor-to-scalar ratio r , which can be constrained by CMB polarization experiments [64].

Observations of the Cosmic Microwave Background (CMB) by COBE [65], WMAP [50] and Planck [51] have provided strong empirical support for inflationary cosmology. The detected temperature anisotropies in the CMB match the predictions of inflation with remarkable precision, particularly in terms of the spectral index n_s and the tensor-to-scalar ratio r . The latest Planck results confirm that the scalar spectral index is slightly tilted, $n_s \approx 0.9649$, in excellent agreement with slow-roll inflationary models [51]. Furthermore, constraints on non-Gaussianity in the CMB indicate that the simplest single-field slow-roll models remain viable [51].

Modern developments in inflationary theory explore various models, including single-field slow-roll inflation, multi-field inflation, and alternatives such as warm inflation [66] and bouncing cosmologies [67]. Furthermore, inflation plays a crucial role in connecting cosmology with high-energy physics, as it is naturally embedded in theories beyond the Standard Model, including Grand Unified Theories (GUTs) [68], supergravity [69], and string theory [70]. Despite its successes, inflation remains an open research topic, with ongoing debates regarding the nature of the inflaton field, the exact mechanism of reheating [71], and the potential signatures of new physics beyond the standard slow-roll paradigm [72]. Future observations, particularly those targeting the primordial B-mode polarization of the CMB, such as BICEP/Keck [73], could provide deeper insights into the inflationary epoch and the fundamental physics underlying the early Universe.

1.1.6. Dark Matter

In the 1930s, while studying the Coma galaxy cluster, Zwicky observed that the velocities of galaxies within the cluster were far greater than could be accounted for by the visible mass alone [74]. He coined the term "dunkle Materie" (dark matter) to describe this unseen mass, proposing that it must dominate the cluster's gravitational dynamics. In the decades following Zwicky's work, the evidence for dark matter

continued to accumulate. In the 1970s, Vera Rubin and her collaborators provided compelling observational support for dark matter through their studies of galaxy rotation curves [75]. Rubin's work demonstrated that stars in the outer regions of spiral galaxies orbit at velocities inconsistent with the predictions of Newtonian mechanics based on visible matter alone. This discrepancy strongly suggested the presence of a massive, invisible component enveloping galaxies, now understood as a dark matter halo.

Theoretical developments in the 1970s and 1980s further solidified the case for dark matter. The work of Peebles and Ostriker [76] highlighted the necessity of dark matter for the stability and formation of large-scale structures in the universe. Additionally, the advent of cosmological simulations, such as those by Davis, Efstathiou, Frenk, and White [77], demonstrated that the observed distribution of galaxies and clusters could only be reproduced if dark matter constituted a significant fraction of the universe's total mass.

The nature of dark matter, however, remains one of the most profound mysteries in modern physics. Early candidates for dark matter included massive compact halo objects (MACHOs), such as brown dwarfs or black holes, but microlensing surveys in the 1990s ruled out these objects as the primary component of dark matter [78]. Instead, the focus shifted to weakly interacting massive particles (WIMPs), which arise naturally in many extensions of the Standard Model of particle physics, such as supersymmetry [79]. Despite extensive experimental efforts, including direct detection experiments like LUX [80] and indirect searches via gamma-ray observations [81], WIMPs have yet to be conclusively detected.

The cosmological evidence for dark matter is now overwhelming. Observations of the cosmic microwave background (CMB) by the Planck satellite [46] have precisely measured the density of dark matter, revealing that it constitutes approximately 27% of the universe's total energy content. Similarly, gravitational lensing studies, such as those conducted by the Hubble Space Telescope [82], have provided direct maps of dark matter distributions on large scales.

1.1.7. Late time cosmic acceleration

Observations of SNeIA supernovas, from different research groups and methods, supported that the transition to the late time accelerative phase took place in the previous

past [83–88]. The simplest explanation is the introduction of a positive cosmological constant Λ , along with Cold Dark Matter (CDM) plus inflation and Einsteins General Relativity GR as the accepted theory of gravitation, they constitute the standard model of Cosmology Λ CDM which proves to be quite efficient in describing the evolution of the universe at the background and perturbation level [89]. According to Λ CDM the universe has passed from two phases of acceleration, one at early times (inflation) and one at late times.

Despite its success, Λ CDM faces several challenges both at the theoretical and observational level. In the first class we have the cosmological constant problem, the problem of inflation and the non-renormalizability of general relativity. Specifically, the cosmological constant is plagued by two problems. Quantum field theoretical approximations predict a value of 120 orders of magnitude larger than the present time observed one $\Lambda_{0(obs)}$. This notorious discrepancy is known as the cosmological constant problem [90, 91]. According to Λ CDM during the evolution of the universe, for the vast first period the value of the fraction density Ω_Λ was zero, for the vast second period will be one and only for a very short interval, which coincides with the present, its value is $\Omega_{\Lambda 0} \sim 0.7$. This fine-tuning of the model is considered the coincidence problem [92]. A way to overcome the coincidence is to assume a dynamic cosmological constant, that is, a Λ that varies during the evolution of the universe [93–101].

Confronting theoretically the vacuum energy density $\rho_{\Lambda_{obs}}$ discrepancy plus its dynamic behavior has led to two main branches of modifications. The first is to retain General Relativity (GR) as the underlying theory of gravitation and introduce new fields, known as the dark energy (DE) sector, usually in the dynamical form of scalar fields, as quintessence, phantom and quintom [102, 103]. The second is to construct modified theories of gravity (MTG), which tend back to GR at a particular limit, but exhibit new degrees of freedom that can drive the accelerated expansion [104–106]. Furthermore, since GR is non-renormalizable, MTG may contribute to the quest for a quantum theory of gravity.

1.1.8. Tensions in Cosmology

The challenges of Λ CDM to meet the state-of-the-art observations have been named as Λ CDM tensions. In particular, since 1998, when the accelerated expansion of the Universe was confirmed, the increasing accuracy of observations has revealed several tensions of the Λ CDM concordance model. One of the most notable issues is the H_0 tension, which refers to the discrepancy between the measurements of the present value of the Hubble constant H_0 from early and late time observations. Specifically, the present value of the Hubble parameter as measured indirectly by the Planck collaboration [107] from the CMB and BAO using the inverse ladder method is $H_0 = 67.27 \pm 0.60 \text{ kms}^{-1}\text{Mpc}^{-1}$, while its value from local measurements by the SH0ES collaboration (R19) [108] using the standard distance ladder method is $H_0 = 74.03 \pm 1.42 \text{ kms}^{-1}\text{Mpc}^{-1}$. This tension, currently exceeding 5σ , suggests the possibility of unknown systematic errors or indicates new physics beyond the standard model of cosmology [109, 110]. Nevertheless, recent observations from the James Webb Telescope provided evidence that systematic errors in HST Cepheid measurements do not affect the H_0 tension [111].

Another observational issue is the σ_8 or growth tension, which refers to the late-time distribution of galaxies and matter. The Planck collaboration reports $\sigma_8 = 0.811 \pm 0.006$ assuming the Λ CDM paradigm [107], while observations from weak gravitational lensing and galaxy clustering by SDSS/BOSS [112, 113], as well as large-scale structure (LSS) observations, such as those from the KiDS-1000 collaboration, give $S_8 \equiv \sigma_8 \sqrt{\Omega_m/0.3} = 0.759^{+0.024}_{-0.021}$, corresponding to $\sigma_8 \approx 0.76 - 0.78$ for typical Ω_m values [114].

If the above tensions are not the result of systematics, then one should try to alleviate them by modifying the standard cosmological framework. In this spirit, many solutions have been proposed in the literature that can be categorized as early- or late-time solutions, solutions arising from modified gravity, early dark energy, interacting dark energy, running vacuum, decaying dark matter, string inspired models, etc. [110, 115–171].

1.1.9. Modified Theories of Gravity

While General Relativity (GR) has been remarkably successful in describing gravitational interactions, several open questions—such as the nature of dark energy and dark matter, the singularity problem, and the quantization of gravity—motivate the search for modified theories of gravity (MTG). In this section, we review some of the most significant modifications to GR, highlighting their theoretical foundations, key differences from Einstein’s theory, and their implications for cosmology and quantum gravity.

Higher-Dimensional Theories and String Gravity

One of the earliest modifications to GR comes from higher-dimensional theories, which suggest that our four-dimensional spacetime is embedded in a higher-dimensional manifold. Kaluza–Klein theory unifies gravity and electromagnetism by introducing an extra spatial dimension [172, 173]. String theory further extends this idea, postulating that fundamental particles are one-dimensional strings propagating in a higher-dimensional space [174]. These theories predict additional degrees of freedom, such as extra scalar and tensor fields, which modify gravitational dynamics at low energies [175].

Higher-Order Theories: $f(R)$, $f(G)$, and $f(Q)$ Gravity

A natural extension of GR involves modifying the Einstein–Hilbert action by replacing the Ricci scalar R with a general function $f(R)$. These $f(R)$ theories introduce new dynamical degrees of freedom, effectively incorporating a scalar field that mediates an additional force [176, 177]. Higher-order extensions such as $f(G)$ gravity, which modifies the action using the Gauss–Bonnet invariant G , arise naturally in string-inspired scenarios and higher-dimensional gravity [178]. Additionally, the recently developed $f(Q)$ gravity is based on non-metricity, describing how the length of vectors changes under parallel transport, providing a curvature-free alternative approach to gravity [179].

Teleparallel Gravity and $f(T)$ Theories

Teleparallel Gravity describes gravity using torsion instead of curvature, reformulating Einstein's equations using the Weitzenböck connection. In this framework, the gravitational interaction is encoded in the tetrad fields rather than the metric alone [180]. Extensions such as $f(T)$ gravity generalize the action by introducing a function of the torsion scalar T , leading to novel explanations for cosmic acceleration [36].

Horndeski Gravity and Galileon Models

Horndeski gravity is the most general scalar-tensor theory in four dimensions that leads to second-order field equations, ensuring the absence of Ostrogradsky instabilities [181, 182]. The theory includes kinetic interactions, non-minimal couplings, and derivative self-interactions, making it a broad framework for modified gravity models.

A subclass of Horndeski gravity is Galileon Gravity, which arises from higher-dimensional brane constructions and features a scalar field with Galilean symmetry in flat spacetime [183, 184]. Galileon interactions introduce screening mechanisms such as the Vainshtein effect, which allows them to modify gravity at cosmological scales while remaining compatible with solar system tests. These models have been extensively studied in cosmology, particularly in explaining late-time acceleration without a cosmological constant [185].

Finsler Geometry and Modified Gravity

Finsler geometry generalizes Riemannian geometry by allowing the spacetime interval to depend not only on position but also on velocity. This framework leads to anisotropic modifications of gravity, which have been proposed as alternatives to dark matter and dark energy [186]. While Finslerian theories offer novel insights into quantum gravity, their complete field-theoretic formulation remains an open challenge.

MOND and Emergent Gravity

Modified Newtonian Dynamics (MOND) was proposed as an alternative to dark matter by modifying Newton's second law at low accelerations [187]. Relativistic extensions, such as TeVeS (Tensor-Vector-Scalar gravity), provide a covariant formulation of MOND [188]. More recently, emergent gravity models propose that gravity arises from microscopic quantum information processes rather than being a fundamental force [189].

Spacetime thermodynamics conjecture

Beyond the aforementioned approaches to modified gravity theories, there exists a well-known conjecture that gravity can be described through the laws of thermodynamics [190–192]. This concept is inspired from black-hole thermodynamics, where a black hole (BH) is assigned a specific temperature and entropy, dependent on its horizon [193]. The “thermodynamics of spacetime” conjecture [190] draws an analogy from BH horizon thermodynamics to the Universe horizon at cosmological scales. In particular, if one applies the first law of thermodynamics in the apparent horizon one can obtain the Friedmann equations [194–196], and vice versa the Friedmann equations can be expressed as the first law of thermodynamics. This procedure has been applied successfully both in the case of general relativity and as well as in modified theories of gravity [197–208], where in the latter case one should use the corresponding modified entropy relation.

Modified Theories of Gravity provide a broad range of alternatives to GR, each addressing different theoretical and observational challenges. Higher-dimensional and higher-order theories emerge naturally from fundamental physics, while teleparallel and Finslerian approaches offer alternative geometrical interpretations. Horndeski and Galileon models extend scalar-tensor theories with novel interactions, while MOND and emergent gravity challenge the need for dark matter. The ongoing search for a viable modification of GR that remains compatible with observational data and quantum principles continues to be an active area of research.

1.1.10. Quantum Theory of Gravity

The other three fundamental interactions, electromagnetic, weak and strong are successfully described by the conventional perturbative Quantum Field Theory (QFT) as they are renormalizable [209]. GR is a classical theory, with a coupling G_N of inverse square mass units, causing non-renormalizable ultraviolet (UV) divergences as the perturbative expansion progresses to higher orders [210]. Various approaches have been proposed to overcome these issues, including string theory [174], loop quantum gravity [211], and asymptotically safe gravity [212].

Due to its experimental success at the macroscopic level, GR is often regarded as a low energy effective theory of a fundamental theory of quantum gravity, thus there should exist a scale where GR should tend to be more renormalizable. When higher-order curvature terms are included in the Lagrangian they tend to cancel out the divergences. This feature fueled the construction of higher order theories of gravity [52, 213–215] at the price of the appearance of the Ostrogradsky instability, whose treatment imposes several restrictions. On this ground perturbative techniques are applied to gravity such as the path integral approach that yields the Euclidean Quantum Gravity (EQG) theory [216]. Another important class of quantum gravity (QG) theories, such as string theory and M-theory, are based on adding correction terms to the gravitational action that would lead to enhanced renormalizability properties. From all the higher order terms, the Gauss–Bonnet (GB) term holds a special place in the zoo of MTG due to its topological significance. The GB term is the Euler density in 4D, thus according to the Chern–Gauss–Bonnet Theorem [217] a topological invariant in 4D.

1.1.11. The Gauss Bonnet term in string theory

In heterotic string theory, the Gauss–Bonnet (GB) term is included in the Lagrangian due to its role in renormalizing the theory. It is well-known that string theory is plagued with ultraviolet divergences that require regularization and renormalization. The Gauss–Bonnet term plays a crucial role in regulating these divergences by introducing a curvature-squared term into the action that cancels out the UV divergences that appear in the one-loop partition function [175, 218, 219]. Furthermore, it has been shown that the Gauss–Bonnet term is the unique higher-derivative curvature-squared

term that preserves the local supersymmetry of the heterotic string theory, making it a natural choice for inclusion in the Lagrangian [220].

The Gauss–Bonnet term also plays an important role in M-theory, a proposed theory of quantum gravity that unifies all known string theories. In M-theory, the low-energy effective action contains the same Gauss–Bonnet term as in heterotic string theory, but with a different coefficient that depends on the specific compactification of the extra dimensions. This coefficient is related to the Euler characteristic of the compactification manifold, which is a topological invariant that characterizes the topology of the extra dimensions. The inclusion of the Gauss–Bonnet term in the M-theory action is necessary to ensure consistency and unitarity of the theory, particularly at high energies. It has been shown that the Gauss–Bonnet term contributes to the beta-function of M-theory, which governs the running of the coupling constants with energy scale. The contribution of the Gauss–Bonnet term is essential in cancelling the divergences that appear in the beta-function at high energies, and hence renormalizing the theory [221]. Moreover, it has been suggested that the Gauss–Bonnet term may play a role in resolving singularities in the spacetime geometry, such as those that arise in black hole solutions in M-theory [222].

1.1.12. The Gauss–Bonnet term in the trace anomaly

A Conformal Field Theory (CFT) is a theory whose partition function remains unchanged under rescaling of the background metric (Weyl transformations), thus the trace of the stress tensor in a CFT must vanish. This is not the case for gravity, where the conformal invariance breaks down in the quantum level, therefore the gravitational stress tensor exhibits an anomaly [223–226]. In $d = 2$ gravity, the anomaly is proportional to the integral of the 2D Euler density, thus the Euler character. In $d = 4$ gravity, the trace anomaly is given by a combination of the 4D Euler density (the GB term) and the square of the Weyl tensor [227]. In higher even dimensions $d = 2n$ the trace anomaly consists of two parts, the first is always the Euler density of the dimension E_{2n} and the second is all the non-vanishing Weyl invariant scalar polynomials whose number grow with dimension [226]. There have been many models to derive a dynamical relaxing mechanism of the cosmological constant by the trace anomaly [228–230]. In the context of Euclidean Quantum Gravity, Hawking's Spacetime Foam model [231] relies on the trace anomaly for the calculation of the cosmological constant [232] For

the importance of the Weyl anomaly in cosmology, strings and supergravity see [233] for a review.

1.1.13. Quantum origin of dark energy

The cosmological constant has units of $[Length]^2$ but in classical physics a fixed length scale cannot be defined. Only as soon as $\hbar$ is brought to the scene, one can then define a length scale $L_{pl} = (\hbar G/c^3)^{1/2}$ and expect the dimensionless quantity $\lambda = \Lambda L_{pl}^2$ to be defined by a quantum theory of gravity [234]. On this ground, quantum origins of dark energy have been explored in various areas of quantum gravity theories [210]. Inspired by the holographic principle [235] as a profound characteristic of quantum theory, and black hole thermodynamics, theories of holographic dark energy have been developed over the recent years [207, 236–239].

1.1.14. Spacetime foam

The concept of spacetime foam was first introduced by Wheeler in the 1950s as a way to describe the fluctuating, turbulent nature of spacetime at the Planck scale [240, 241]. In this picture, the smooth fabric of spacetime envisioned in general relativity is replaced by a frothy, ever-changing landscape where quantum fluctuations give rise to transient topological features such as virtual black holes and wormholes. This notion challenges the classical understanding of spacetime and suggests that at extremely small scales, spacetime is far from smooth and may exhibit a complex, foam-like structure.

Spacetime foam has been explored within various approaches to quantum gravity. In Loop Quantum Gravity (LQG), spacetime is quantized, and its geometry is represented by spin networks, leading to a discrete structure at the Planck scale [242]. String theory introduces additional dimensions and posits that fundamental particles are one-dimensional strings whose vibrations determine particle properties; the interactions of these strings and branes can result in spacetime fluctuations consistent with foam-like behavior [243]. Causal Dynamical Triangulations (CDT) build spacetime from discrete building blocks while preserving causality, naturally leading to a spacetime foam structure emerging from the sum over geometries [244]. Additionally, Euclidean Quantum Gravity (EQG) employs a Euclidean signature metric to perform path integrals over all

possible geometries [245]. In EQG, spacetime foam arises from topological fluctuations at the Planck scale, including virtual black holes and wormholes, contributing to a complex spacetime geometry [231,246].

Observables associated with spacetime foam have been proposed, particularly in the context of interferometric experiments. Laser interferometers like LIGO and the proposed LISA are sensitive to minute distortions in spacetime and have been suggested as potential tools for detecting spacetime foam-induced noise [247,248]. In EQG, spacetime foam could induce quantum decoherence or modifications to the propagation of light that might be detectable with these instruments [249,250]. The Holometer experiment aimed to detect holographic noise predicted by certain spacetime foam models but reported null results, thereby placing constraints on these models [251]. Current interferometers have not observed definitive evidence of spacetime foam, and the sensitivity required to detect such effects may be beyond present technological capabilities. Criticisms focus on the challenges of distinguishing spacetime foam effects from other sources of noise and the theoretical uncertainties in predicting the magnitude of such effects.

In astrophysics and cosmology, spacetime foam might influence the propagation of high-energy photons over vast distances. Observations of gamma-ray bursts (GRBs) and active galactic nuclei (AGNs) have been used to search for energy-dependent time delays that could indicate a foamy spacetime structure [252,253]. In the context of EQG, spacetime foam could lead to modifications in fundamental constants or induce decoherence effects observable in astrophysical phenomena [249]. While some initial observations hinted at possible dispersion effects, subsequent analyses attributed these to source-intrinsic properties rather than spacetime foam. Current bounds from astrophysical observations have placed stringent limits on models predicting significant dispersion due to spacetime foam. Critics argue that without a clear understanding of the intrinsic emission mechanisms of distant astrophysical sources, it remains challenging to isolate potential spacetime foam effects.

1.1.15. Wormholes, spacetime foam and the cosmological constant

Based on Euclidean quantum gravity, Hawking [254] and Baum [255] proposed a variable scalar field that mimics the gravitational constant, where the wave function will be

proportional to $\exp(1/\Lambda)$ and it would dominate the path integral at the value $\Lambda \approx 0$. In the same vein, Coleman [246] proposed a mechanism where wormholes, topological fluctuations of space time, induce a distribution of the values of nature's constants, that smear Λ distribution to peak at zero. Coleman's mechanism was seriously criticized by Klebanov and Susskind [256] that would lead to giant wormholes (FKS catastrophe) and a cold universe. A similar proposition was the Giddings–Strominger wormhole solution [257] where a wormhole is coupled to an instanton.

In [258] Garattini proposed a semiclassical model of space time foam where Casimir-like quantum fluctuations give rise to an arbitrary number of wormholes, as black hole anti-black hole pairs, that drive the induced cosmological constant to zero as they grow.

In [259] Hebecker et al. reviewed Coleman's mechanism, the critic and the recent revival of interest in wormholes among the community in the context of string theory and the EPR=ER conjecture.

There has been a revival on the significance of spacetime wormholes especially in the context of EPR=ER conjecture, suggested by Maldacena and Susskind [260], where a pair of entangled black holes/particles (EPR) is connected by an Einstein–Rosen bridge (ER), which after a Wick rotation becomes traversable. The conjecture has further generalizations, such as that spacetime is constructed from quantum entanglement [261]. Initially Maldacena and Susskind, in order to address the APMS firewall paradox, they proposed that all Hawking particles are entangled with the mother black hole through EPR=ER wormholes. Furthermore, they argue that every pair of entangled particles form an EPR=ER wormhole.

1.1.16. Topology change

Topology change of the spacetime manifold has been a controversial and obscure subject among the scientific community. In [262] Geroch showed that topology change in Lorentzian spacetime manifolds would lead to the development of singularities as closed timelike geodesics. Based on the previous result, Anderson and DeWitt [263] argued that quantum field theory is inconsistent with such singularities. Gibbons in [264] shows that in the context of Euclidean quantum gravity, the Wick rotation of the time coordinate to the imaginary plane reflects a change of signature and thereby a topology change of the manifold describing a real tunneling geometry. Hebecker

et al. [259] review topology change censorship unlikely, since it is a well understood feature of string theory in 2D and 10D [265], [266], [267], [268]. In [269], [270] the evolution of the topology of the manifold was illustrated in loop quantum gravity with topspin network formalism with applications in cosmology.

1.1.17. Dynamic dark energy from wormholes

Since the approach to dark energy as quantum field vacuum energy overestimates it 120 orders of magnitude, one should explore a different scenario. Gibbons in [264], suggests that a positive cosmological constant provides the mutual repulsion for a pair of particles to separate. In order to circumvent the positive energy theorem, Garattini in [258] employed the black hole anti-black hole pair formation model of Modanese [271] where spacetime wormholes of this type can appear by the Casimir-like energy quantum fluctuations, without the presence of an external field. In [272] Spaans proposes a topological extension of GR where planck scale mini black holes correspond to spacetime foam and constitute dark energy as their creation induces an increase in the four-volume to embed them. There have been several proposals to link the cosmological constant to Planck scale spacetime fluctuations [231], [273], [274], [275], [258], [272], [276] [277]

Chapter 2.

General relativity

“Matter tells spacetime how to curve and then spacetime tells matter how to move”

— John Archibald Wheeler, 1911–2008

2.1. Einstein heuristics of general relativity

According to Newtonian gravity, a change in the source-mass would lead to an immediate change of the gravitational field, violating the second postulate of the Special Theory of Relativity (STR). In STR mass is energy, momentum included. Therefore, in a relativistic framework, that all observers can agree on, gravitation must be sourced by energy and momentum. Therefore, a source of gravity consistent with STR is the energy momentum tensor $T_{\mu\nu}$ which is a two-rank (0,2) tensor. On this ground, Einstein stated the two fundamental principles of the theory of general relativity

- 1. Principle of General Covariance:** an equation must preserve its form under general coordinate transformation.
- 2. (Weak) Equivalence Principle:** The local effects of gravity are indistinguishable from an accelerated observer in the absence of gravity.

Starting from the Poisson equation for the Newtonian potential

$$\nabla^2\Phi = 4\pi G\rho,$$

the classical source ρ is substituted from the energy momentum tensor $T_{\mu\nu}$, the left part of Poisson equation is tensorilized to a two rank (0,2) tensor, a suggestive notation would be

$$\left[\nabla^2, g \right]_{\mu\nu} \propto T_{\mu\nu},$$

for conservation of energy and momentum to be preserved

$$\nabla^\mu T_{\mu\nu} = 0,$$

a sort of derivative of the left part of the equation had to be also zero. To meet the above requirements, Einstein had to formulate General Relativity on the grounds of Riemannian Differential Geometry. The space time coordinates are represented by four vectors as in STR:

$$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3), \quad (2.1)$$

space time is modeled as a pseudo Riemannian Manifold with Lorentzian signature $diag(-, +, +, +)$ and a metric g , which is a non degenerate, symmetric (0,2) tensor field. The distance element ds^2 is expressed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (2.2)$$

Furthermore, Einstein assumed the Levi-Civita (LC) connection, which defines a covariant Derivative ∇_ν

$$\nabla_\nu Y^\mu = \partial_\nu Y^\mu + \Gamma_{\nu\rho}^\mu Y^\rho. \quad (2.3)$$

The Levi-Civita connection is uniquely defined by demanding:

1. the metric is covariant constant $\nabla^\mu g_{\mu\nu} = 0$

2. the torsion is zero $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$,

then its components are the Christoffel symbols

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right). \quad (2.4)$$

Equipped with the metric g and the LC connection Γ , the space time pseudo-Riemannian manifold is enriched with more structure, the Riemann tensor

$$R_{\rho\mu\nu}^{\sigma} = \partial_{\mu}\Gamma_{\nu\rho}^{\sigma} - \partial_{\nu}\Gamma_{\mu\rho}^{\sigma} + \Gamma_{\nu\rho}^{\lambda}\Gamma_{\mu\lambda}^{\sigma} - \Gamma_{\mu\rho}^{\lambda}\Gamma_{\nu\lambda}^{\sigma}, \quad (2.5)$$

and by its contractions $R_{\mu\nu} = R_{\mu\rho\nu}^{\rho}$ the Ricci tensor

$$R_{\mu\nu} = \partial_{\rho}\Gamma_{\nu\mu}^{\rho} - \partial_{\nu}\Gamma_{\rho\mu}^{\rho} + \Gamma_{\nu\mu}^{\lambda}\Gamma_{\rho\lambda}^{\rho} - \Gamma_{\rho\mu}^{\lambda}\Gamma_{\nu\lambda}^{\rho}, \quad (2.6)$$

and subsequently the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu}. \quad (2.7)$$

With the aforementioned mathematical hardware, Einstein was able to find the appropriate (0,2) tensor which is covariantly constant, the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}gR \Rightarrow \nabla^{\mu}G_{\mu\nu} = 0. \quad (2.8)$$

Schematically one can picture the transcendence from Poisson to general relativity as

$$\nabla^2\Phi = 4\pi G\rho \Rightarrow G_{\mu\nu} = k^2T_{\mu\nu} \quad (2.9)$$

By demand the GR equation must collapse to the Poisson equation in the weak field limit. Then the proportionality constant k is determined as

$$k^2 = \frac{8\pi G}{c^4} \Rightarrow \boxed{R_{\mu\nu} - \frac{1}{2}gR = \frac{8\pi G}{c^4}T_{\mu\nu}} \quad (2.10)$$

2.2. General relativity from a variational approach

2.2.1. The Einstein–Hilbert action

All fundamental physical theories are described by action principles. Therefore, Einstein pursued to formulate GR from the principle of least action and at the same time so did Hilbert. They found the correct action and published it around the same time, thus the action was named Einstein–Hilbert. In the four dimensional spacetime

manifold $\mathcal{M}$ with a metric of Lorentzian signature, the Einstein–Hilbert (EH) action is defined as

$$S_{EH} = \int d^4x \sqrt{-g} R.$$

We would like to obtain the Euler–Lagrange equations for the EH action. Since the dynamical field is the metric, we will vary the metric and calculate the corresponding change in the action

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu},$$

writing the Ricci scalar as $R = g^{\mu\nu} R_{\mu\nu}$ then the action varies as

$$\begin{aligned} \delta S_{EH} &= \delta \int d^4x \sqrt{-g} R \\ &= \int d^4x \delta \left(\sqrt{-g} g^{\mu\nu} R_{\mu\nu} \right) \\ &= \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int d^4x g^{\mu\nu} R_{\mu\nu} \delta (\sqrt{-g}) \end{aligned} \quad (2.11)$$

It is convenient to express the variation in terms of the inverse metric

$$g_{\rho\mu} \delta g^{\mu\nu} = \delta g^\mu_\rho \Rightarrow \delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma},$$

and by the identity for diagonalisable matrices $\log \det A = \text{tr} \log A$ one obtains

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}.$$

If one substitutes the above expressions to Eq. (2.11) one obtains

$$\delta S_{EH} = \int d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}, \quad (2.12)$$

It can be proved that the variation of the Ricci tensor becomes

$$\delta R_{\mu\nu} = \nabla_\rho \Gamma_{\mu\nu}^\rho - \nabla_\nu \Gamma_{\mu\rho}^\rho, \quad (2.13)$$

where

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2} g^{\rho\sigma} \left(\nabla_\mu g_{\nu\sigma} + \nabla_\nu g_{\mu\sigma} - \nabla_\sigma g_{\mu\nu} \right), \quad (2.14)$$

then the second term becomes

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu X^\mu \text{ where } X^\mu = g^{\rho\nu}\Gamma_{\rho\nu}^\mu - g^{\mu\nu}\Gamma_{\nu\rho}^\rho, \quad (2.15)$$

and the variation of the EH action yields

$$\delta S_{EH} = \int d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \right) + \int d^4x \sqrt{-g} \nabla_\mu X^\mu. \quad (2.16)$$

where the second term is the integral of a derivative, thus it results in a surface term, which can be ignored at first, but as we will see will play a significant role in the context of black-hole thermodynamics. Therefore, by ignoring the surface term and demanding extremization of the action $\delta S_{EH} \rightarrow 0$ when $\delta g^{\mu\nu} \rightarrow 0$ we obtain that the quantity inside the integral must be zero. Equivalently, by defining the functional derivative $\frac{\delta S}{\sqrt{-g}\delta g^{\mu\nu}}$ the extremization of the action corresponds to the demand that the functional derivative becomes zero, therefore

$$\frac{\delta S}{\sqrt{-g}\delta g^{\mu\nu}} = 0 \Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \quad (2.17)$$

which is the Einstein field equation in vacuum, equivalently written with the Einstein tensor $G_{\mu\nu} = 0$.

2.2.2. The Gibbons–Hawking–York boundary Term

We have seen that the metric variation of the Einstein–Hilbert action has the form

$$\delta S_{EH} = \int \sqrt{-g} d^4x G_{\alpha\beta} \delta g^{\alpha\beta} + \int \sqrt{-g} d^4x \nabla_\lambda X^\lambda, \quad (2.18)$$

where

$$\begin{aligned} X^\lambda &= g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda - g^{\mu\lambda} \delta \Gamma_{\mu\nu}^\nu \\ &= (g^{\lambda\alpha} g^{\nu\beta} - g^{\lambda\nu} g^{\alpha\beta}) \nabla_\nu \delta g_{\alpha\beta}. \end{aligned} \quad (2.19)$$

If one neglects the second (surface) term, then the first (bulk) term gives us as the Euler–Lagrange equations the vacuum Einstein equations $G_{\alpha\beta} = 0$. Nevertheless, when one performs the integral over a space-time region M bounded by the hypersurface ∂M (which we shall assume to be spacelike or timelike), upon use of the Gauss integral

formula one finds that the second (total derivative) term can be written as

$$\begin{aligned} \int_M \sqrt{-g} d^4x \nabla_\lambda X^\lambda &= \int_{\partial M} d\sigma_\lambda X^\lambda \\ &= \epsilon \int_{\partial M} d^n y \sqrt{h} N_\lambda X^\lambda, \end{aligned} \quad (2.20)$$

where N^λ is the normal vector to the boundary ∂M in M and h_{ab} is the induced metric on the boundary. Implementing Eq. (2.19) the boundary integrand can be written as

$$N_\lambda X^\lambda = (N_\rho g^{\mu\nu} - N^\mu g_\rho^\nu) \nabla_\mu \delta g_{\rho\nu}. \quad (2.21)$$

Furthermore, one can decompose the metric

$$g_{\mu\nu} = h_{\mu\nu} + \epsilon N_\mu N_\nu, \quad (2.22)$$

on ∂M (with $N^\mu N_\mu = \epsilon$), then one can see that the terms with 3 N 's cancel and Eq. (2.21) becomes

$$N_\lambda X^\lambda = N_\rho h^{\mu\nu} \nabla_\mu \delta g_{\rho\nu} - N^\mu h_\rho^\nu \nabla_\mu \delta g_{\rho\nu}. \quad (2.23)$$

The first term depends only on the variations $\delta g_{\rho\nu}$ on the boundary, and its tangential derivatives $h^{\mu\nu} \nabla_\mu \delta g_{\rho\nu}$. Therefore, that term is zero if one imposes standard Dirichlet boundary conditions

$$\delta g_{\alpha\beta}|_{\partial M} = 0 \quad (2.24)$$

on the metric at ∂M . on the other hand, the second term, depends on $\delta g_{\rho\nu}$ and its normal derivative $N^\mu \nabla_\mu \delta g_{\rho\nu}$, hence it is non-zero

$$\delta g_{\alpha\beta}|_{\partial M} = 0 \Rightarrow N_\lambda X^\lambda|_{\partial M} = -N^\mu h_\rho^\nu \nabla_\mu \delta g_{\rho\nu}|_{\partial M} = -h_\rho^\nu N^\mu \partial_\mu \delta g_{\rho\nu}|_{\partial M}. \quad (2.25)$$

Therefore with Dirichlet boundary conditions the variation of the Einstein–Hilbert action gives rise to a non-zero boundary term

$$\delta S_{EH} = \int_M \sqrt{-g} d^4x G_{\alpha\beta} \delta g^{\alpha\beta} + \epsilon \int_{\partial M} \sqrt{h} d^3y (-h_\rho^\nu N^\mu \partial_\mu \delta g_{\rho\nu}), \quad (2.26)$$

where one can rewrite the variation more compactly as

$$\delta S_{EH} = \delta S_{EH}^{(bulk)} + \delta S_{EH}^{(bound)}, \quad (2.27)$$

therefore in this case the variation of the Einstein–Hilbert action does not equal the Einstein tensor

$$\frac{\delta}{\delta g^{\alpha\beta}} S_{EH} \neq \sqrt{-g} G_{\alpha\beta}. \quad (2.28)$$

The way to resolve this issue is to add a suitable new boundary term to the gravitational action, in a way that will not change the bulk variation, and its variation will cancel the boundary term above, for Dirichlet boundary conditions. Among several candidates, the preferred boundary term is the Gibbons–Hawking–York boundary term introduced in [278, 279]

$$S_{GHY} = 2\epsilon \int_{\partial M} \sqrt{h} d^3y K, \quad (2.29)$$

where K is the trace of the extrinsic curvature K_{ab} of ∂M

$$\begin{aligned} K &= h^{ab} K_{ab} = h^{\alpha\beta} \nabla_\alpha N_\beta \\ &= h^{\alpha\beta} \left(\partial_\beta N_\alpha - \Gamma^\gamma_{\alpha\beta} N_\gamma \right) \end{aligned} \quad (2.30)$$

where we have used that $\nabla_\beta (N^\alpha N_\alpha) = 0$ implies $N^\alpha \nabla_\beta N_\alpha = 0$. Now we will proceed to the calculation of its variation, and we will show that it cancels against the variation arising from the Einstein–Hilbert action for Dirichlet boundary conditions. The variation of K will be

$$\begin{aligned} \delta K &= -h^{\alpha\beta} \delta \Gamma^\gamma_{\alpha\beta} N_\gamma \\ &= -h^{\alpha\beta} N_\gamma \frac{1}{2} g^{\gamma\sigma} \left(\partial_\beta \delta g_{\sigma\alpha} + \partial_\alpha \delta g_{\sigma\beta} - \partial_\sigma \delta g_{\alpha\beta} \right) \\ &= -\frac{1}{2} h^{\alpha\beta} \left(\partial_\beta \delta g_{\mu\alpha} + \partial_\alpha \delta g_{\mu\beta} - \partial_\mu \delta g_{\alpha\beta} \right) N^\mu \\ &= \frac{1}{2} h^{\alpha\beta} \partial_\mu \delta g_{\alpha\beta} N^\mu, \end{aligned} \quad (2.31)$$

where we have used the fact that the tangential derivatives of $\delta g_{\alpha\beta}$ vanish on $\partial\mathcal{M}$. Substituting the above expression to the variation of the GHY term we obtain

$$\delta S_{GHY} = 2\epsilon \int_{\partial\mathcal{M}} \sqrt{h} d^3y \delta K = \epsilon \oint_{\partial\mathcal{M}} \sqrt{h} d^3y h^{\alpha\beta} \partial_\mu \delta g_{\alpha\beta} N^\mu \quad (2.32)$$

We can see that $\delta S_{GHY} = -\delta S_{EH}^{(bound)}$, hence, the variation of the Gibbons–Hawking–York boundary term cancels the boundary term of the variation of the EH action. Therefore, the GHY term is the established choice, and gives rise to the standard gravitational action in the case of Dirichlet boundary conditions

$$\begin{aligned} S_g &= S_{EH} + S_{GHY} \\ &= \int \sqrt{-g} d^4x R + 2\epsilon \int_{\Sigma} \sqrt{h} d^3y K, \end{aligned} \quad (2.33)$$

with the well-defined variation

$$\delta S_g = \int \sqrt{-g} d^4x G_{\alpha\beta} \delta g^{\alpha\beta} \quad (2.34)$$

for Dirichlet boundary conditions.

2.2.3. Einstein equation in matter

We will present a dimensional analysis of the Einstein Hilbert action as another way to retrieve the coupling constant. Starting from accepting that the coordinates x^μ have dimension of length, then the metric is dimensionless, the Ricci scalar has second derivatives, thus $[R] = [L]^{-2}$, therefore the EH action will behave as $[S] = L^2$. We know that the action should have dimensions of energy $\times$ time, thus the action should have dimensions $[S] = ML^2T^{-1}$. In order for the EH action to be dimensionally consistent it should be multiplied with an appropriate factor $1/2k^2$, which turns out to be $k^2 = \frac{8\pi G}{c^4}$ and is better understood as the coupling strength between geometry and matter. In the case where there is matter, the total action will be the sum of the gravitational plus the matter action

$$S = \frac{1}{2k^2} S_{EH} + S_M, \quad (2.35)$$

where S_M is the action for the matter fields. If we define the energy momentum tensor as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \quad (2.36)$$

then the variation of the total action becomes

$$\delta S = \int d^4x \sqrt{-g} \left(\frac{1}{2k^2} G_{\mu\nu} - \frac{1}{2} T_{\mu\nu} \right) \delta g^{\mu\nu}, \quad (2.37)$$

and the extremization of the action $\delta S = 0$ yields the Einstein equation in matter

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad (2.38)$$

.

2.3. The Schwarzschild Solution and black holes

The unique spherically symmetric vacuum solution to Einstein's field equations of GR is the Schwarzschild metric. It describes the gravitational field outside a spherically symmetric, non-rotating mass M . In spherical coordinates (t, r, θ, ϕ) , the metric is given by

$$ds^2 = - \left(1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2 + r^2 d\Omega^2, \quad (2.39)$$

where $d\Omega^2$ is the metric on a unit two-sphere

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad (2.40)$$

We will remark some important aspects of Eq. (2.39). As $r \rightarrow \infty$, the metric reduces to the Minkowski metric of flat spacetime:

$$ds^2 \rightarrow -c^2 dt^2 + dr^2 + r^2 d\Omega^2. \quad (2.41)$$

The metric coefficients become infinite at $r = 0$ and $r = 2GM/c^2$. While the coordinate singularity at $r = 2GM/c^2$ can be removed by a change of coordinates, the Schwarzschild solution has a physical singularity at $r = 0$. This is revealed by the curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, which diverges as $r \rightarrow 0$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6}, \quad (2.42)$$

This singularity signifies a breakdown of the classical description of spacetime and suggests the need for a quantum theory of gravity [280]. On the other hand, at $r = 2GM/c^2$ the metric exhibits a coordinate singularity, which is not a physical singularity but rather a coordinate artifact [281]. The radius

$$r_s \equiv \frac{2GM}{c^2} \quad (2.43)$$

is defined as the Schwarzschild radius. For $r > r_s$, the metric describes the exterior of a spherically symmetric mass. For $r < r_s$, the metric describes the interior of a black hole, where the roles of t and r are exchanged t becomes spacelike, and r becomes timelike [282].

The event horizon at $r = r_s$ is a null surface that acts as a one-way membrane: nothing, not even light, can escape from inside the horizon. The area of the event horizon is

$$A = 4\pi r_s^2 = \frac{16\pi G^2 M^2}{c^4}. \quad (2.44)$$

The motion of massive particles are described by timelike geodesics. Circular orbits are stable only for $r > 3r_s$. The null geodesics describe the paths of light rays. Photons can orbit at $r = \frac{3r_s}{2}$, known as the photon sphere [283]. The photo of a BH taken in 2019, as it can be seen in Fig. 2.1, exhibits the characteristic photon ring around it.

The Kruskal–Szekeres diagram

The coordinate singularity at $r = r_s$ can be removed by transforming to Kruskal–Szekeres coordinates (T, X) , which cover the entire Schwarzschild spacetime [284]. The

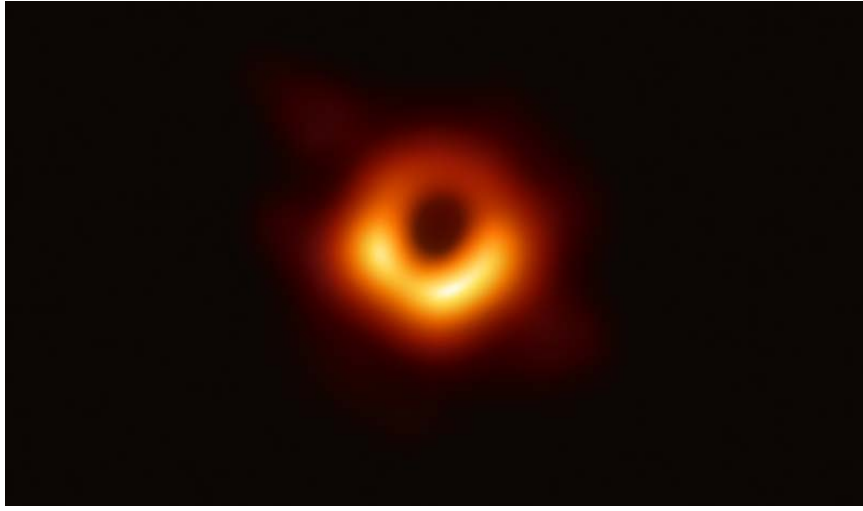


Figure 2.1.: First image of the supermassive black hole in the galaxy M87, obtained by the Event Horizon Telescope (EHT) on April 10, 2019. The image shows the photon ring surrounding the event horizon, formed by gravitationally lensed light from the accretion disk. The bright crescent is due to relativistic Doppler beaming. Credit: EHT Collaboration.

Kruskal coordinates are defined as

$$U = \left(\frac{r}{r_s} - 1 \right)^{1/2} e^{r/2r_s} \sinh \left(\frac{ct}{2r_s} \right), \quad (2.45)$$

$$V = \left(\frac{r}{r_s} - 1 \right)^{1/2} e^{r/2r_s} \cosh \left(\frac{ct}{2r_s} \right), \quad (2.46)$$

for $r > r_s$, and:

$$U = \left(1 - \frac{r}{r_s} \right)^{1/2} e^{r/2r_s} \cosh \left(\frac{ct}{2r_s} \right), \quad (2.47)$$

$$V = \left(1 - \frac{r}{r_s} \right)^{1/2} e^{r/2r_s} \sinh \left(\frac{ct}{2r_s} \right), \quad (2.48)$$

for $r < r_s$. The Schwarzschild metric in Kruskal coordinates becomes:

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/r_s} (-dU^2 + dV^2) + r^2 d\Omega^2, \quad (2.49)$$

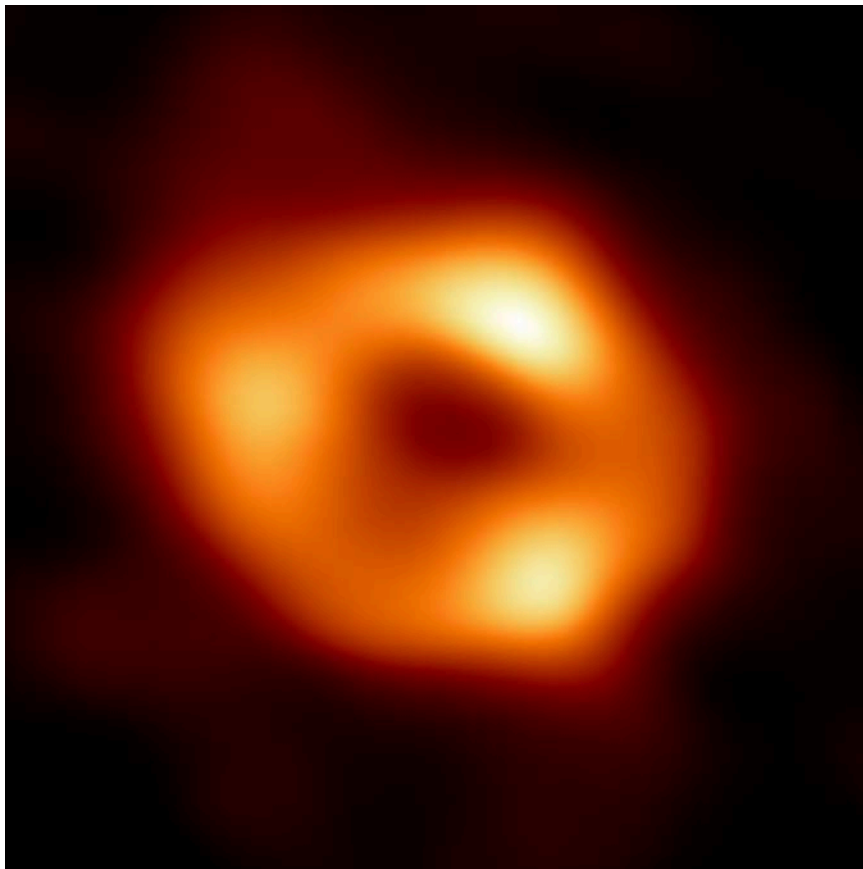


Figure 2.2.: *First image of Sagittarius A*, the supermassive black hole at the center of the Milky Way, captured by the Event Horizon Telescope (EHT) and released on May 12, 2022. Similar to M87*, the image reveals a bright photon ring around the dark event horizon, shaped by gravitational lensing effects. Due to the dynamic nature of Sgr A*, the image represents an average over multiple observations. Credit: EHT Collaboration.*

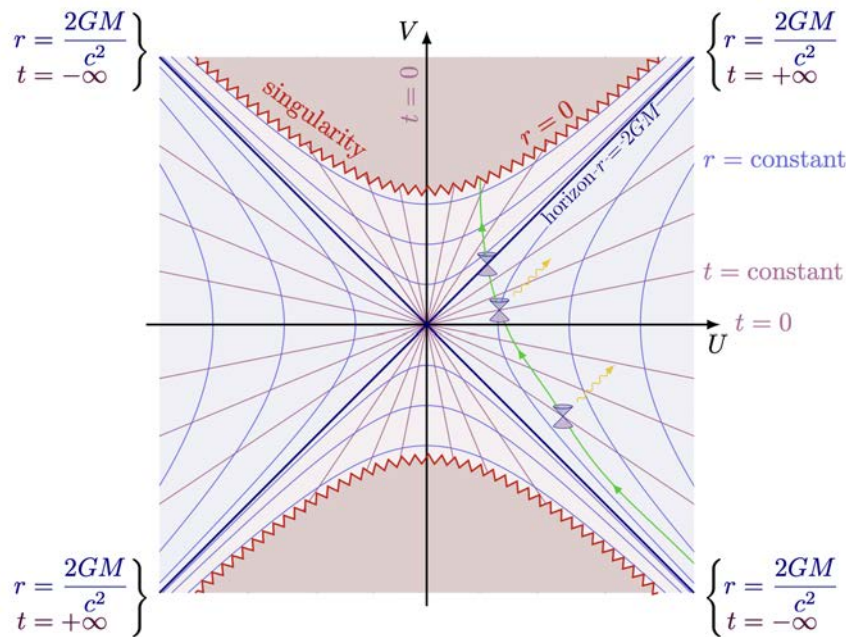


Figure 2.3.: Credits to Izaak Neutelings for making the tikz code publicly available

where r is implicitly defined by:

$$U^2 - V^2 = \left(1 - \frac{r}{r_s}\right) e^{r/r_s}. \quad (2.50)$$

The Kruskal diagram Fig. 2.3 represents successfully the regions of Schwarzschild black hole geometry.

- The event horizon $r = r_s$ corresponds to $U = \pm V$.
- The singularity $r = 0$ corresponds to $U^2 - V^2 = 1$.
- Kruskal coordinates cover all four regions of the maximally extended Schwarzschild spacetime:
 1. **Region I:** Exterior of the black hole ($r > r_s$).
 2. **Region II:** Interior of the black hole ($r < r_s$).
 3. **Region III:** Parallel exterior universe.
 4. **Region IV:** White hole region.

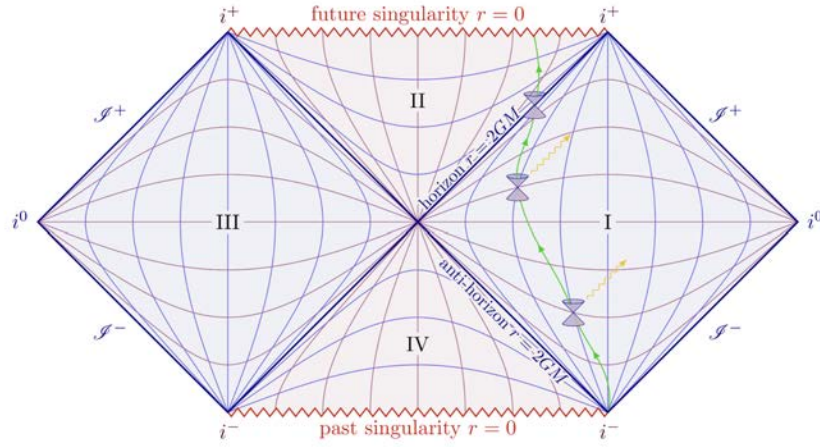


Figure 2.4.: Credits to Izaak Neutelings for making the tikz code publicly available

The Penrose Diagram

The Penrose diagram (or conformal diagram) is a tool to visualize the global structure of spacetime. For the Schwarzschild solution, the Penrose diagram is constructed by compactifying the Kruskal coordinates [285] under the transformation

$$\begin{aligned} U + V &= \tan(u + v) \\ U - V &= \tan(u - v), \end{aligned} \quad (2.51)$$

The resulting diagram Fig. 2.4 is a square, with each point representing a 2-sphere.

- **Event Horizon:** Diagonal lines at 45° , representing $r = r_s$.
- **Singularity:** Hyperbolic boundary at $r = 0$.
- **Asymptotic Infinity:** Boundaries at $U' = \pm \pi/2$ and $V' = \pm \pi/2$, representing $r \rightarrow \infty$.

The Penrose diagram reveals the causal structure of the Schwarzschild spacetime, including the existence of a white hole and a parallel universe. It highlights the global nature of the event horizon and the singularity.

The Schwarzschild solution provides a foundational framework for understanding black holes. The coordinate singularity at $r = r_s$ is resolved by Kruskal coordinates,

revealing the maximally extended spacetime. The Penrose diagram offers a powerful tool for visualizing the global structure and causal relationships in black hole geometry. These concepts are essential for exploring the physics of black holes, including their thermodynamics, quantum properties, and role in cosmology [280,286].

Chapter 3.

Modified theories of gravity

GR has been remarkably successful in describing the solar system, black holes, gravitational waves, weak lensing and has been the gravitational framework for the concordance model of cosmology Λ CDM. Nevertheless, the problems of dark energy, dark matter and the non-renormalization of GR has fueled research for a new theory of gravity that would address these problems and would have GR as a limit. These theoretical attempts have been named as alternative theories of gravity or Modified Theories of Gravity (MTG). One can categorize MTG according to the modification made to the GR framework in order to obtain a modified theory.

- **Higher dimensional theories:** Spacetime is assumed to have higher dimensions than four $D > 4$.
- **Higher order theories:** The Lagrangian contains higher order curvature terms
- **Advanced geometry:** Departure from Riemannian geometry, Levi-Civita connection and non-metricity
- **Extra degrees of freedom:** Adding tensor, vector and scalar components
- **Entropic gravity:** Conceptual shift from GR where entropy is considered the fundamental quantity.

For the purposes of this work, we are going to present a special class of higher order MTG. The Lagrangian formulation of GR provides a framework for modifications since one can assume extra terms in the gravitational Lagrangian

$$S = \int d^4x \sqrt{g} \left(R + c_1 R^2 + c_2 R_{\alpha\beta} R^{\alpha\beta} + c_3 R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + c_4 R \square R + \dots \right). \quad (3.1)$$

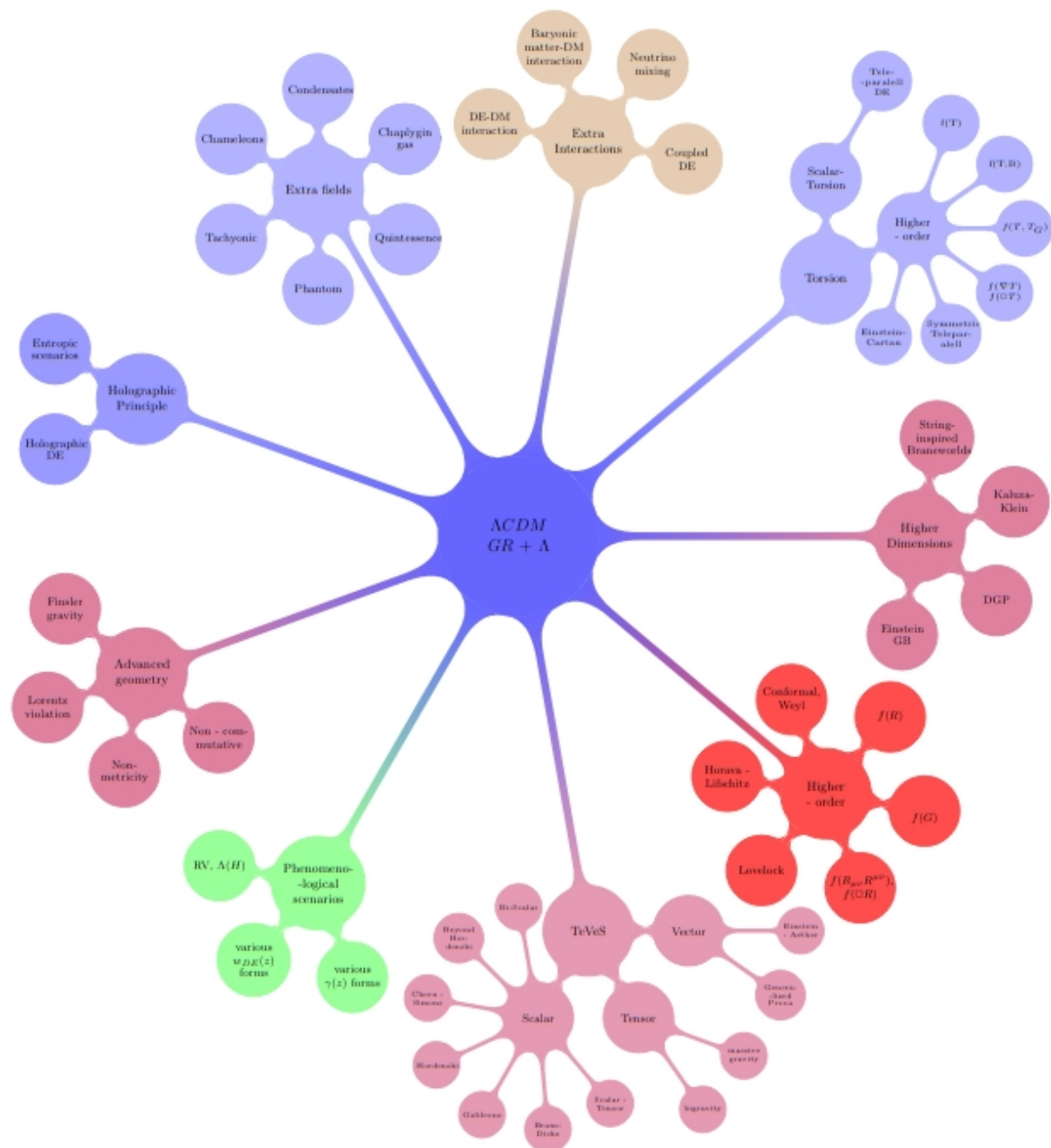


Figure 3.1.: In this diagram a number of important branches of modified gravity theories and corresponding cosmological models are presented, source [287]

The field equations arising from a Lagrangian with higher order terms in general suffers from the appearance of higher than second order derivatives of the metric, that cause ghosts and lead to the Ostrogradski instabilities. Therefore, all candidate tensors $A^{\mu\nu}$ for replacing the Einstein tensor $G_{\mu\nu}$ on the left side of the field equations should contain up to second derivatives of the metric, be covariantly conserved and posses the same symmetries as the energy momentum tensor . Weyl [288] and Cartan [289] showed that if $A^{\mu\nu}$ is to be linear in second derivatives of the metric $g_{\rho\sigma,\tau\chi}$, then $A^{\mu\nu}$ must be a linear combination of the Einstein tensor and a cosmological constant.

3.1. Lovelock gravity

In the same vein, Lovelock asked which set of rank-2 tensors $A^{\mu\nu}$ could satisfy the following three conditions:

- (1) $A^{\mu\nu} = A^{\mu\nu}(g_{\rho\sigma}, g_{\rho\sigma,\tau}, g_{\rho\sigma,\tau\chi})$
- (2) $\nabla_\nu A^{\mu\nu} = 0$
- (3) $A^{\mu\nu} = A^{\nu\mu}$,

but without the linearity condition assumed by Weyl and Cartan, Lovelock found a broader class of solutions that could replace the left-hand side of the Einstein equation. The Lagrangian of the Lovelock gravity theory in D dimensions is defined as [214, 290, 291]

$$L = \sum_m c_m L_m, \quad (3.2)$$

where

$$L_m = \frac{1}{16\pi} \frac{1}{2^m} \delta^{\mu_1\nu_1 \dots \mu_m\nu_m}_{a_1b_1 \dots amb_m} \prod_{i=1}^m R^{\mu_i b_i}_{\mu_i \nu_i}, \quad (3.3)$$

the Lagrangian L_m is the Euler density of a $D = 2m$ dimensional manifold [290]. Thus, the Lovelock action is defined as:

$$S = \int_M d^D x \sqrt{-g} L_m. \quad (3.4)$$

For the study of Lovelock theory it is useful to define the tensor

$$P_{cd(m)}^{ab} \equiv \left(\frac{\partial L}{\partial R_{ab}^{cd}} \right)_{g_{kl}} = \sum_m c_m \frac{m}{2^m} \delta_{c_1 d_1 \dots c_{m-1} d_{m-1}}^{a_1 b_1 \dots a_{m-1} b_{m-1}} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_{m-1} b_{m-1}}^{c_{m-1} d_{m-1}}, \quad (3.5)$$

then the Lagrangian L_m can be expressed as

$$L_m = \frac{1}{m} P_{cd(m)}^{ab} R_{ab}^{cd} \quad (3.6)$$

3.1.1. Full variation of the action

In the general case, when an action $S = \int d^D x \mathcal{L}$ is varied with respect to a dynamical variable, one needs to include the terms from the boundary variations [292], schematically as in [293]

$$\begin{aligned} \delta S &= \int_M d^D x \text{ (equation of motion term) } \delta(\text{dynamical variable}) \\ &+ \int_{\partial M} d^{D-1} x \text{ (total derivative) } + \int_{\partial M} d^{D-1} x \delta(\text{boundary term}) \\ &+ \int_{\partial M} d^{D-1} x \text{ (Conjugate momentum) } \delta(\text{dynamical variable}). \end{aligned} \quad (3.7)$$

By varying the action of the m – th order Lovelock Lagrangian Eqs. (3.3),(3.6) with respect to the inverse metric, as dictated by Eq. (3.7) :

$$\delta S = \delta \int_M d^D x \sqrt{-g} L_m^D = \int_M d^D x \sqrt{-g} A_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial M} d^{D-1} x \delta \mathcal{L}_{\partial M}, \quad (3.8)$$

where $A_{\mu\nu}$ is the Lanczos tensor [213,291,294]:

$$A_\nu^\mu = - \sum_m \frac{c_m}{2m+1} \delta_{\nu a_1 b_1 \dots a_m b_m}^{\mu \rho_1 \sigma_1 \dots \rho_m \sigma_m} \prod_{i=1}^m R_{\rho_i \sigma_i}^{a_i b_i} \quad (3.9)$$

and $\delta \mathcal{L}_{\partial M}$ is the variation on the non-null boundary ∂M [293,295]

$$\delta \mathcal{L}_{\partial M} = \sqrt{h} B[n_c] \quad (3.10)$$

were h is the induced metric, $[n_c]$ the unit normal vector on the boundary ∂M and $B[n_c]$ with the use of Eq. (3.6) is defined as

$$B[n_c] = 2n_c P_a^{bcd} \delta \Gamma_{bd}^a = 2P^{abcd} n_c \nabla_b \delta g_{ad}, \quad (3.11)$$

following [293] we can introduce the induced metric on the boundary ∂M as

$$h_b^a = \delta_b^a - \epsilon n^a n_b, \quad (3.12)$$

where $\epsilon = \pm 1$ for timelike/spacelike boundary. If one projects the components of P^{abcd} onto the boundary, then two useful expressions are introduced

$$B_{(m)}^{med} = 2P_{(m)}^{ancp} n_c h_a^m h_n^e h_p^d \quad (3.13)$$

and

$$C_{(m)}^{dm} = 2P_{(m)}^{ancp} n_c n_n h_a^d h_p^m \quad (3.14)$$

3.1.2. Topological structure of Lovelock theory

In [290] it is explicit analyzed how the Lovelock Lagrangian Eq. (3.3) is the Euler density in D dimensions. As higher order terms m appear in the action series Eq. (3.4) $m \leq D$, while the action terms need not be zero, there is a cut-off in the variation of the action terms in the critical dimension $2m = D$ of topological origin. As the m order Lovelock Lagrangian is the Euler density in $D = 2m$ dimensions, according to the Chern–Gauss–Bonnet theorem, its integral on the manifold is a topological invariant. The Chern–Gauss–Bonnet theorem states that for a compact orientable manifold M with boundary ∂M of even dimension $n = 2m$ [217]

$$\chi(M) = \int_M e(\Omega) + \int_{\partial M} Q, \quad (3.15)$$

where the Euler density or Euler class is

$$e(\Omega) = \frac{1}{(2\pi)^n} Pf(\Omega) \quad (3.16)$$

where we have the Pfaffian $Pf(\Omega)$ of the curvature two-form Ω and Q is an appropriate correction form integrated on the boundary ∂M [296].

The topological structure of the manifold M is described by the Euler characteristic $\chi(M)$. The essence of the theorem is that despite any local deformation of the manifold its total curvature, as expressed by the integral of the GB curvature polynomial, depends only from the topology of the manifold. In that way, for a manifold of fixed topology, χ is considered a topological invariant under smooth variations of the metric [291].

In the modern language of differential forms and in the context of de Rham Cohomology, the Euler character χ is defined as the alternating sum of the Betti numbers of the manifold M

$$\chi(M) = \sum_p (-1)^p B_p \quad (3.17)$$

where the Betti numbers B_p of a manifold are defined as the dimension of the p^{th} de Rham cohomology group

$$B_p = \dim H^p(M), \quad (3.18)$$

the p^{th} de Rham cohomology group $H^p(M)$ is the set of all closed p -forms $Z^p(M)$, modulo, the set of all exact p -forms $B^p(M)$

$$H^p(M) = Z^p(M) / B^p(M). \quad (3.19)$$

The Poincaré Lemma states that a closed form $d\omega = 0$ that is defined on a domain $V \subseteq M$ is also exact $\omega = dn$, if the domain V is contractible to a point. In the light of Poincaré Lemma, de Rhams Cohomology can be seen as a restriction on the global exactness of closed forms. Thus, Betti numbers measure the global inexactness of closed forms as obstructions to contractability to a point originated from holes and disconnections of the domain [297]

3.1.3. The Einstein–Hilbert action as the first term

For $m = 0$ one obtains a constant term, which in the framework of Lovelock gravity could be associated with a cosmological constant Λ , which will be ignored for the moment.

For $m = 1$ and $D = 4$ dimensions in Eqs. (3.3), (3.6), (3.5) one obtains the Einstein

Hilbert Lagrangian

$$L_1 = R, \quad (3.20)$$

and therefore by Eq. (3.4) the Einstein Hilbert action

$$S_{EH} = \int d^4x \sqrt{-g} R, \quad (3.21)$$

similarly, for $m = 1$ from Eq. (3.9) one retrieves the Einstein tensor

$$G_b^a = R_b^a - \frac{1}{2} g_b^a R \quad (3.22)$$

and thus the GR equations of motion in vacuum $G_{ab} = 0$. From Eq. (3.10) and (3.11) for $m = 1$ one obtains the variation of the boundary term [278, 298]

$$\delta \mathcal{L}_{\partial M} = D_b (\sqrt{|h|} \delta u^b) + 2\delta(|h|K) - \sqrt{|h|} (K_{dq} - K h_{dq}) \delta h^{dq}, \quad (3.23)$$

where $K_{\mu\nu}$ is the extrinsic curvature on the boundary ∂M [299] defined as

$$K_{mn} = -\frac{1}{2} \mathcal{L}_{\mathbf{n}} h_{mn} \quad (3.24)$$

where $\mathcal{L}_{\mathbf{n}}$ is the Lie derivative of the induced metric along the normal $\mathbf{n}$ of the boundary ∂M . In the Hamiltonian ADM formalism, the canonical conjugate momentum Π^{ab} coincides with the Gibbons–Hawking–York term [292]

$$\Pi^{ab} = -\sqrt{h} (K^{ab} - h^{ab} K) \quad (3.25)$$

3.1.4. The Gauss–Bonnet action as the second term

For $m = 2$ in Eqs. (3.3), (3.6), (3.5) one obtains the Gauss–Bonnet Lagrangian term

$$L_{(2)} = \mathcal{G}$$

$$\mathcal{G} = R^2 - 2R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \quad (3.26)$$

and for $m = 2$ in Eq. (3.4), one obtains the Einstein Hilbert action S_{EH} plus the Gauss–Bonnet action S_{GB}

$$S = S_{EH} + S_{GB}, \quad (3.27)$$

where the Gauss Bonnet action is

$$S_{GB} = \int d^D x \sqrt{-g} \alpha \mathcal{G} \quad (3.28)$$

and α is a constant parameter with dimensions L^2 which originates from string theory and is related to the string scale l_s [175, 220–222, 300–302]. Similarly, for $m = 2$ in Eq. (3.9) one obtains the Lanczos tensor appropriate expression as the Einstein tensor G_ν^μ plus the tensor H_ν^μ the bulk variation of the GB term

$$A_\nu^\mu = G_\nu^\mu + \alpha H_\nu^\mu = G_\nu^\mu + \alpha (R^{\mu\sigma} R_{\nu\sigma} - 4R^{\rho\sigma} R_{\nu\rho\sigma}^\mu - 4R^{\mu\rho} R_{\nu\rho} + 2RR_\nu^\mu - \frac{1}{2}G_\nu^\mu), \quad (3.29)$$

and for $m = 2$ in Eq. (3.10) as was demonstrated in [293] the full variation of the boundary term

$$\delta \mathcal{L}_{\partial M} = \sqrt{|h|} B[n_c] = \sqrt{|h|} D_b T^b - \delta(\sqrt{|h|} B T_2) + \sqrt{|h|} \Pi_{dq}, \quad (3.30)$$

the first term is the total derivative term $\sqrt{|h|} D_b T^b$ which was derived in [293] as

$$\begin{aligned} \sqrt{|h|} D_b T^b &= B_{(2)}^{qbd} \delta h_{dq} - \epsilon C_q^b{}_{(2)} \delta u^q + (S^{pq} \delta \bar{\gamma}_{pq}^b - S^{ab} \delta \bar{\gamma}_{ab}^c) \\ &\quad - \left[D^b S^{bc} - \frac{1}{2} (D^b S^{cd} - h^{cd} D_a S^{ab}) \right] \end{aligned} \quad (3.31)$$

where by Eq. (3.14)

$$C_{m(2)}^d = 4\epsilon \left(R_m^d - \frac{1}{2} R h_m^d \right) - 4(K K_m^d - K_c^d K_m^c) + 2(K^2 - K_{ab} K^{ab}) h_m^d, \quad (3.32)$$

similarly by Eq. (3.13) one obtains the expression for $B_{(2)}^{qbd}$. Furthermore, the quantity S^{ab} is defined as

$$S^{ab} = 8 \left(K^{ab} - \frac{1}{2} K h^{ab} \right) \quad (3.33)$$

and in the boundary coordinate system the term $\bar{\gamma}$

$$\bar{\gamma}_{ab}^m = \frac{h^{mn}}{2} \left[D_n(h_a^c h_b^d \delta h_{cd}) + D_a(h_b^c h_n^d \delta h_{cd}) + D_b(h_a^c h_n^d \delta h_{cd}) \right] \quad (3.34)$$

the second term $\sqrt{|h|}BT_2$ as was proposed in [295] in agreement with [293]

$$\begin{aligned} \sqrt{|h|}BT_2 = \sqrt{|h|} \left(-4KR + 8Kh^{ab}R_{ab} - 8K^{ac}h^{bd}R_{abcd} + \frac{8^3}{3} \right. \\ \left. - 8\epsilon K K^{ab}K_{ab} + \frac{16}{3}\epsilon K_b^a K_c^b K_a^c \right), \end{aligned} \quad (3.35)$$

the third term is the conjugate momentum Π_{dq} as it was suggested in [303] in agreement with [293]

$$\Pi_{dq} = -2 \left(Q_{dq} - \frac{h_{dq}}{3} Q \right) \quad (3.36)$$

with

$$\begin{aligned} Q_{ab} = -2K^{cd}R_{acbd} - 4R_{ac}K_b^c + 2KR_{ab} + RK_{ab} + \\ K_{ab}(K^{cd}K_{cd} - K^2) + 2KK^{ac}K_b^c - 2K_a^c K_{cd}K_b^d. \end{aligned} \quad (3.37)$$

3.1.5. 4D Einstein Gauss Bonnet gravity

As $D = 4$ is the critical dimension for $m = 2$ the variation of the GB term is expected to vanish in four space time dimensions. However, in higher dimensions $D > 4$, the variation of the GB term doesn't vanish. It's contribution has been extensively studied in heterotic string theory [175, 218, 219], in M theory [221, 222] and in five dimensional Gauss Bonnet Gravity. Recently, Glavan and Lin [304] proposed a way to bypass Lovelocks theory topological cutoff in $D = 4$ dimensions. They expressed the variation of the GB term as

$$\frac{g_{\nu\rho}}{\sqrt{-g}} \frac{\delta S_{GB}}{\delta g_{\mu\rho}} = -2R^{\mu\alpha}{}_{\rho\sigma} + 4R^{\mu\alpha}{}_{\nu\beta} R^\beta{}_\alpha + 4R^\mu{}_\alpha R^\alpha{}_\nu - 2RR^\mu{}_\nu + \frac{1}{2}G\delta_\nu^\mu \quad (3.38)$$

then the trace of Eq. (3.38) yields

$$\frac{g_{\nu\rho}}{\sqrt{-g}} \frac{\delta S_{GB}}{\delta g_{\mu\rho}} = (D-4) \times \mathcal{G}. \quad (3.39)$$

For $D = 4$ both Eq. (3.38) and Eq. (3.39) give zero, regardless of the space time symmetries. Glaval and Lin proposed that under a proper rescaling $a \rightarrow a/(D-4)$ the variation of the GB term in four dimensions need not be zero and contributes to the equation of motion in a nontrivial way, giving rise to an effective cosmological constant and interesting cosmology. Their theory has been named 4D Einstein Gauss Bonnet gravity and has raised significant interest and criticism among the community, for a review see [291].

From the perspective of Poincaré lemma, in the language of differential forms, the GB term, that is the 4D Euler density in the Lovelock theory of Gravity [214, 290, 291, 304, 305], if it is locally exact, then it can be written locally as a total derivative and would contribute to the equations of motion no more than a surface term. What happens when the topology of the manifold doesn't remain stable but changes due to quantum fluctuations? We will try to answer this question in Chapter. 7.

3.2. $f(\mathcal{G})$ gravity

We have seen that the GB term is considered a topological invariant in $4D$, thus the variation of the GB action will have no contribution to the field equations. If the action is an arbitrary function of the GB curvature polynomial, then the metric variation of the action will yield non-zero contribution terms to the equations of motion. This approach is known as $f(\mathcal{G})$ gravity with interesting cosmological behavior [178] and in this section we will derive the equations of motion. The gravitational action for $f(\mathcal{G})$ gravity is given by

$$S_g = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa^2} R + f(\mathcal{G}) \right), \quad (3.40)$$

where $\kappa^2 = 8\pi G$ is the gravitational constant, R is the Ricci scalar, and $\mathcal{G}$ is the GB term in $4D$:

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \quad (3.41)$$

The variation of the action with respect to the metric $g_{\mu\nu}$ will be

$$\delta S_g = \int d^4x \delta \left(\sqrt{-g} \left(\frac{1}{2\kappa^2} R + f(\mathcal{G}) \right) \right), \quad (3.42)$$

now it can be split into two parts

$$\delta S_g = \int d^4x \left[\delta \left(\frac{1}{2\kappa^2} \sqrt{-g} R \right) + \delta \left(\sqrt{-g} f(\mathcal{G}) \right) \right]. \quad (3.43)$$

The first part of the variation, $\delta \left(\frac{1}{2\kappa^2} \sqrt{-g} R \right)$, as we have demonstrated in 2.2.1, directly yields the Einstein tensor $G_{\mu\nu}$

$$\delta \left(\frac{1}{2\kappa^2} \sqrt{-g} R \right) = \frac{1}{2\kappa^2} \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} = \frac{1}{2\kappa^2} \sqrt{-g} G_{\mu\nu} \delta g^{\mu\nu}. \quad (3.44)$$

The second part of the variation yields

$$\delta \left(\sqrt{-g} f(\mathcal{G}) \right) = \delta \left(\sqrt{-g} \right) f(\mathcal{G}) + \sqrt{-g} \delta f(\mathcal{G}), \quad (3.45)$$

where

$$\delta \left(\sqrt{-g} \right) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}, \quad (3.46)$$

hence

$$\delta \left(\sqrt{-g} \right) f(\mathcal{G}) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} f(\mathcal{G}) \delta g^{\mu\nu}. \quad (3.47)$$

We can write the variation $\delta f(\mathcal{G})$ as

$$\delta f(\mathcal{G}) = \frac{df(\mathcal{G})}{d\mathcal{G}} \delta \mathcal{G} = f'(\mathcal{G}) \delta \mathcal{G} = H_{\mathcal{G}} \delta g^{\mu\nu} \quad (3.48)$$

where $f'(\mathcal{G}) = \frac{df}{d\mathcal{G}}$. After a lengthy calculation

$$\begin{aligned} H_{\mathcal{G}} = & 2f'(\mathcal{G}) R R^{\mu\nu} - 4f'(\mathcal{G}) R_{\rho}^{\mu} R^{\nu\rho} + 2f'(\mathcal{G}) R^{\mu\rho\sigma\tau} R_{\rho\sigma\tau}^{\nu} \\ & + 4f'(\mathcal{G}) R^{\mu\rho\sigma\nu} R_{\rho\sigma} - 2 \left(\nabla^{\mu} \nabla^{\nu} f'(\mathcal{G}) \right) R + 2g^{\mu\nu} \left(\nabla^2 f'(\mathcal{G}) \right) R \\ & + 4 \left(\nabla_{\rho} \nabla^{\mu} f'(\mathcal{G}) \right) R^{\nu\rho} + 4 \left(\nabla_{\rho} \nabla^{\nu} f'(\mathcal{G}) \right) R^{\mu\rho} - 4 \left(\nabla^2 f'(\mathcal{G}) \right) R^{\mu\nu} \\ & - 4g^{\mu\nu} \left(\nabla_{\rho} \nabla_{\sigma} f'(\mathcal{G}) \right) R^{\rho\sigma} + 4 \left(\nabla_{\rho} \nabla_{\sigma} f'(\mathcal{G}) \right) R^{\mu\rho\nu\sigma}. \end{aligned} \quad (3.49)$$

Considering the full action with the contribution of matter

$$S = S_g + S_m \quad (3.50)$$

where the energy-momentum tensor for matter fields has been defined in 2.2.1 as

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}. \quad (3.51)$$

Demanding that the extremization of the total action becomes zero and combining the contributions from the Einstein–Hilbert term and the $f(\mathcal{G})$ term, the equations of motion for $f(\mathcal{G})$ gravity will be

$$\begin{aligned} & \frac{1}{2\kappa^2} \left(-R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} R \right) + T^{\mu\nu} + \frac{1}{2} g^{\mu\nu} f(\mathcal{G}) \\ & - 2f'(\mathcal{G}) R R^{\mu\nu} + 4f'(\mathcal{G}) R^\mu_\rho R^{\nu\rho} - 2f'(\mathcal{G}) R^{\mu\rho\sigma\tau} R^\nu_{\rho\sigma\tau} \\ & - 4f'(\mathcal{G}) R^{\mu\rho\sigma\nu} R_{\rho\sigma} + 2 \left(\nabla^\mu \nabla^\nu f'(\mathcal{G}) \right) R - 2g^{\mu\nu} \left(\nabla^2 f'(\mathcal{G}) \right) R \\ & - 4 \left(\nabla_\rho \nabla^\mu f'(\mathcal{G}) \right) R^{\nu\rho} - 4 \left(\nabla_\rho \nabla^\nu f'(\mathcal{G}) \right) R^{\mu\rho} + 4 \left(\nabla^2 f'(\mathcal{G}) \right) R^{\mu\nu} \\ & + 4g^{\mu\nu} \left(\nabla_\rho \nabla_\sigma f'(\mathcal{G}) \right) R^{\rho\sigma} - 4 \left(\nabla_\rho \nabla_\sigma f'(\mathcal{G}) \right) R^{\mu\rho\nu\sigma} = 0. \end{aligned} \quad (3.52)$$

We have derived the equations of motion for $f(\mathcal{G})$ gravity by varying the action with respect to the metric $g_{\mu\nu}$. The Einstein–Hilbert term directly yields the Einstein tensor, while the $f(\mathcal{G})$ term introduces additional contributions involving the Gauss–Bonnet term and its derivatives. These equations generalize the Einstein field equations and are essential for studying modified theories of gravity.

Chapter 4.

Euclidean Quantum Gravity

4.1. The Path Integral in Quantum Field Theory

Path–integral quantization provides a powerful framework for understanding quantum mechanics and quantum field theory. In contrast to canonical quantization, where emphasis is placed on operator algebra, the path–integral approach sums over all possible histories of the system. This formulation, originally developed by Feynman, not only offers deep physical insight but also serves as the backbone for modern perturbative and non-perturbative methods, including the derivation of Feynman rules and the study of gauge theories.

In non-relativistic quantum mechanics, the transition amplitude for a particle to travel from position x' at time t' to position x'' at time t'' is given formally by

$$\langle x'', t'' | x', t' \rangle = \int \mathcal{D}x(t) e^{iS[x(t)]}, \quad (4.1)$$

where $S[x(t)]$ is the classical action and $\mathcal{D}x(t)$ is a formal measure over all possible paths. Although this measure is defined through a limiting procedure, one may illustrate the idea by discretizing time. For a particle in one spatial dimension, the propagator can be expressed as

$$\langle x'', t'' | x', t' \rangle = \lim_{N \rightarrow \infty} \left(\prod_{j=1}^{N-1} \int dx_j \right) \left(\frac{m}{2\pi i \Delta t} \right)^{N/2} \exp \left\{ i \sum_{j=0}^{N-1} \left[\frac{m}{2} \frac{(x_{j+1} - x_j)^2}{\Delta t} - \Delta t V(x_j) \right] \right\}, \quad (4.2)$$

with $\Delta t = (t'' - t')/N$. This expression makes clear that most paths contributing to the integral are continuous yet nowhere differentiable. Despite the formal nature of the measure, such representations have proven extremely useful in developing both semiclassical approximations (via the stationary phase method) and perturbation theory.

The quantum-mechanical path integral generalizes naturally to field theory by replacing the particle trajectory $x(t)$ with field configurations $\phi(x)$. For a real scalar field, the generating functional is written as

$$Z[\phi] = \int \mathcal{D}\phi e^{iS[\phi]}, \quad (4.3)$$

where the action

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi), \quad (4.4)$$

describes the dynamics of the field, and $\mathcal{D}\phi$ denotes integration over all field configurations. Although, in the case of quantum mechanical trajectories the measure $\mathcal{D}x(t)$ is rigorously defined, this is no longer true for $\mathcal{D}\phi$, when one generalizes the procedure to quantum fields. Here, $Z[\phi]$ often represents in-out transition amplitudes or serves as a partition sum for the theory.

A key quantity in QFT is the propagator, which is directly related to the two-point correlation function. In the path-integral formalism, the propagator for a scalar field is defined as

$$G(x, y) = \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle, \quad (4.5)$$

where T denotes time ordering and $|0\rangle$ is the vacuum state. This propagator can be obtained by taking two functional derivatives of the generating functional with respect to an external source $J(x)$. Specifically, defining

$$Z[J] = \int \mathcal{D}\phi \exp \left\{ i \left[S[\phi] + \int d^4x J(x) \phi(x) \right] \right\}, \quad (4.6)$$

the propagator is given by

$$G(x, y) = \frac{1}{Z[0]} \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} \bigg|_{J=0}. \quad (4.7)$$

This relation establishes a direct link between the propagator and the underlying path-integral framework, making it a cornerstone for deriving Feynman rules and calculating scattering amplitudes in perturbation theory.

A significant challenge in the formulation in Eq. (4.3) is the oscillatory nature of the integrand e^{iS} , which can hinder convergence. A common resolution is to apply the Wick rotation, where the time coordinate is rotated to the imaginary dimension via $t \rightarrow -i\tau$. Under this rotation, the action transforms as

$$iS[\phi] \rightarrow -S_E[\phi], \quad (4.8)$$

leading to the Euclidean generating functional

$$Z_E[\phi] = \int \mathcal{D}\phi e^{-S_E[\phi]}. \quad (4.9)$$

The Euclidean formulation offers several advantages. The damping exponential e^{-S_E} replaces the oscillatory e^{iS} , leading to better convergence properties. The field equations derived from S_E are elliptic rather than hyperbolic, simplifying boundary-value problems. There is a direct correspondence with the partition function in statistical mechanics (e.g., $Z = \text{tr} e^{-\beta H}$), which facilitates numerical and lattice studies.

The rationale for employing Wick rotations in quantum field theory is based on the fact that the Euclidean Green functions obtained can be analytically continued to real time while preserving their pole structure [306].

While the path-integral formalism provides a powerful heuristic and computational tool, it is important to acknowledge several subtleties. The definition of the integration measure $\mathcal{D}\phi$ remains formal, and operator ordering ambiguities in quantum theory are reflected in these measure issues. The discretized formulation underpins the continuum limit; however, a rigorous mathematical foundation for the continuum path integral in QFT is still an area of active research. In standard quantum mechanics, the propagator obeys a composition law. While this property is well understood for systems with an external time parameter, its straightforward generalization to field theory (and later to quantum gravity) faces challenges. In summary, despite the formal ambiguities in the definition of the measure, the path-integral method remains a central and unifying pillar of modern theoretical physics.

4.2. The Quantum Gravitational Path Integral

The quantum gravitational path integral, originally introduced by Misner in [307], is formally given by

$$Z[g] = \int \mathcal{D}g_{\mu\nu}(x) e^{iS[g_{\mu\nu}(x)]}, \quad (4.10)$$

where the integration is performed over all metrics $g_{\mu\nu}(x)$ on a four-dimensional manifold M divided by the diffeomorphism group $\text{Diff}(M)$, according to the redundancy due to coordinate invariance. Although one might expect an additional sum over topologies, this extension is highly nontrivial, since four-manifolds remain not fully classifiable [262].

The gravitational path integral is significantly more complex than its quantum field theory progenitor. In particular, due to the absence of an external time parameter, the gravitational path integral behaves more like an energy Green function rather than a conventional propagator. Moreover, the formal definition of the integration measure $\mathcal{D}g_{\mu\nu}(x)$ and the necessity of factoring out the diffeomorphism group pose substantial technical challenges.

To improve convergence properties, a Wick rotation is often performed, transforming the Lorentzian metric into a Euclidean one. However, this procedure introduces several complications in the gravitational context. First, not every Euclidean metric admits an analytic continuation to a Lorentzian signature $(-, +, +, +)$; only metrics with special symmetries possess such a continuation. Second, the inclusion of a sum over distinct topologies is problematic because four-manifolds cannot be exhaustively classified. Third, and most significantly, the Euclidean gravitational action is not bounded from below. In order to demonstrate the unboundness of the gravitational action we will first apply the Wick rotation to the EH action with the GHY boundary term in the presence of a cosmological constant

$$S_E[g] = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} (R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^3x hK, \quad (4.11)$$

which under a conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ transforms as

$$S_E[\tilde{g}] = -\frac{1}{16\pi G} \left[\int_M d^4x \sqrt{\tilde{g}} \left(\Omega^2 R + 6 g^{\mu\nu} \nabla_\mu \Omega \nabla_\nu \Omega - 2\Lambda \Omega^4 \right) + 2 \int_{\partial M} d^3x \sqrt{h} \Omega^2 K \right], \quad (4.12)$$

where R is the Ricci scalar, Λ the cosmological constant, h the induced metric on the boundary ∂M , and K the trace of the extrinsic curvature. For a rapidly varying conformal factor Ω , the action can become arbitrarily negative. This divergence is known as the conformal-factor problem.

There are strong indications that the divergence associated with the conformal mode may be mitigated. In [308] the authors have argued that the divergent contribution from the conformal factor might cancel against a term of opposite sign arising from the integration measure. Similar cancellations have been observed in the context of linearized gravity. Nonetheless, a definitive resolution of the conformal-factor problem remains an open question in quantum gravity research.

In summary, the quantum gravitational path integral extends the path-integral approach to include a sum over geometries on a manifold M , with diffeomorphism invariance properly taken into account. Despite its conceptual appeal, the formalism is burdened by significant technical challenges, including measure ambiguities, difficulties in summing over topologies, and the unboundedness of the Euclidean action due to conformal fluctuations. These issues highlight the inherent complexities of quantum gravity and motivate ongoing efforts toward a consistent and well-defined formulation.

4.3. The Background Field Method

In [309] t'Hooft and Veltman have shown that in order to quantize gravity perturbatively, one employs the background field method, in which the full metric $g_{\mu\nu}$ is split into a fixed classical background $\tilde{g}_{\mu\nu}$ and a quantum fluctuation $h_{\mu\nu}$

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}. \quad (4.13)$$

Here, the quanta of the fluctuation $h_{\mu\nu}$ are interpreted as the gravitons of the theory. Often a constant is factored in front of $h_{\mu\nu}$ to control the perturbative expansion, although in the present treatment this factor is absorbed into the definition of $h_{\mu\nu}$.

The Einstein–Hilbert action (with a cosmological constant Λ) for the full metric is then expanded in powers of $h_{\mu\nu}$. In a schematic notation, one writes

$$S[\tilde{g} + h] = \frac{1}{k^2} \left[S_0 + S_1 + S_2 + \sum_{n \geq 3} S_n \right], \quad (4.14)$$

where S_0 is the classical action for the background metric $\tilde{g}_{\mu\nu}$, S_1 is the term linear in $h_{\mu\nu}$, and S_2 is quadratic in $h_{\mu\nu}$. The higher-order terms S_n (with $n \geq 3$) represent further interactions among the quantum fluctuations.

To derive these terms, one must first express metric-dependent quantities in terms of $h_{\mu\nu}$. For example, the inverse metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \left(\tilde{g}_{\mu\nu} + h_{\mu\nu} \right)^{-1} = \tilde{g}^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\lambda h^{\lambda\nu} + O(h^3). \quad (4.15)$$

Another essential ingredient is the expansion of the square root of the metric determinant. Exploiting the property $\log \det A = \text{tr} \log A$ and expanding logarithms and exponentials, one obtains

$$\sqrt{g} = \sqrt{\tilde{g}} \left[1 + \frac{1}{2} h - \frac{1}{2} h_{\mu\nu} h^{\mu\nu} + \frac{1}{8} (h)^2 + O(h^3) \right], \quad (4.16)$$

where we use the shorthand $h = \tilde{g}^{\mu\nu} h_{\mu\nu}$ and the indices on $h_{\mu\nu}$ are raised and lowered with the background metric $\tilde{g}_{\mu\nu}$.

Inserting these expansions into the Euclideanized Einstein–Hilbert action in the presence of a cosmological constant

$$S_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} (R - 2\Lambda), \quad (4.17)$$

one can organize the result as an expansion in powers of $h_{\mu\nu}$. After a lengthy calculation—employing integration by parts and discarding total derivatives—the action can be written in the form

$$S[\tilde{g} + h] = \frac{1}{k^2} [S_0 + S_1 + S_2 + \cdots], \quad (4.18)$$

with the individual contributions given schematically by

$$S_0 = - \int d^4x \sqrt{\tilde{g}} (\tilde{R} - 2\Lambda), \quad (4.19)$$

$$S_1 = \int d^4x \sqrt{\tilde{g}} h^{\mu\nu} \left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} + \tilde{g}_{\mu\nu} \Lambda \right), \quad (4.20)$$

and

$$S_2 = - \int d^4x \sqrt{\tilde{g}} \left\{ \frac{1}{4} h^{\mu\nu} [\nabla^2 + 2\Lambda] h_{\mu\nu} - \frac{1}{8} [\nabla^2 + 2\Lambda] h + \frac{1}{2} \left(\nabla^\nu h_{\nu\mu} - \frac{1}{2} \nabla_\mu h \right)^2 \right. \\ \left. + \frac{1}{2} h^{\mu\lambda} h^\nu{}_\lambda \tilde{R}_{\mu\nu} + \frac{1}{8} (h^2 - 2 h^{\mu\nu} h_{\mu\nu}) \tilde{R} \right\}. \quad (4.21)$$

For a complete, step-by-step derivation of the expansion of the Einstein–Hilbert action up to quadratic order in $h_{\mu\nu}$, the reader is referred to Appendix A.1.

By applying the principle of least action, demanding $\delta S \rightarrow 0$ as $\delta h \rightarrow 0$, equivalently setting

$$\frac{\delta S_1}{\delta h_{\mu\nu}} = 0, \quad (4.22)$$

leads directly to the classical Einstein field equations for the background metric

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} + \tilde{g}_{\mu\nu} \Lambda = 0. \quad (4.23)$$

Thus, the background field method not only enables the systematic study of quantum fluctuations about a classical spacetime but also ensures that the classical limit of GR is recovered.

4.4. Gauge Fixing and Ghosts: The Faddeev—Popov Method in the Euclidean Background Field Approach

Under the expansion the gravitational path integral becomes

$$Z = \int \mathcal{D}h_{\mu\nu} e^{S[\tilde{g}_{\mu\nu}, h_{\mu\nu}]} \quad (4.24)$$

in order to avoid over-counting, one must consider the addition of gauge fixing and ghost terms. First, they have been calculated by first order by Feynmann [310], then De Witt generalized the procedure to all orders [311] and Faddeev and Popov presented the procedure in a more appealing form [312], thus the procedure became known as the Faddeev–Popov method. In order to present the problem arising in Eq. (4.24) let us start from the fact that GR is invariant under diffeomorphisms. An infinitesimal coordinate transformation,

$$x^\mu \rightarrow x^\mu - \epsilon^\mu(x), \quad (4.25)$$

induces, to linear order, a transformation of the fluctuation

$$\delta h_{\mu\nu} = \tilde{\nabla}_\mu \epsilon_\nu + \tilde{\nabla}_\nu \epsilon_\mu, \quad (4.26)$$

where $\tilde{\nabla}_\mu$ denotes the covariant derivative with respect to the background metric $\tilde{g}_{\mu\nu}$. This gauge redundancy renders the quadratic operator in S_2 non-invertible and gives rise to infinities as the path is integrated over gauge directions. To eliminate the redundancy, we impose the de Donder (harmonic) gauge condition

$$G_\mu[h] \equiv \tilde{\nabla}^\nu h_{\mu\nu} - \frac{1}{2} \tilde{\nabla}_\mu h = 0, \quad (4.27)$$

with $h \equiv \tilde{g}^{\mu\nu} h_{\mu\nu}$. The Faddeev–Popov procedure inserts into the path integral the identity

$$1 = \Delta_{FP}[h] \int D\epsilon \delta(G_\mu[h^\epsilon]), \quad (4.28)$$

where the gauge-transformed fluctuation is

$$h_{\mu\nu}^\epsilon = h_{\mu\nu} - \tilde{\nabla}_\mu \epsilon_\nu - \tilde{\nabla}_\nu \epsilon_\mu, \quad (4.29)$$

and the Faddeev–Popov determinant is defined as

$$\Delta_{FP}[h] = \det \left(\frac{\delta G_\mu[h^\epsilon]}{\delta \epsilon^\nu} \right). \quad (4.30)$$

Expanding $G_\mu[h^\epsilon]$ to linear order in ϵ yields

$$G_\mu[h^\epsilon] = G_\mu[h] - M_\mu^\nu \epsilon_\nu, \quad (4.31)$$

with the Faddeev–Popov operator

$$M_\mu^\nu = \tilde{\nabla}^\rho \tilde{\nabla}_\mu \delta_\rho^\nu + \tilde{\nabla}^2 \delta_\mu^\nu - \tilde{\nabla}_\mu \tilde{\nabla}^\nu. \quad (4.32)$$

Because the integration over the gauge group produces an infinite volume, we factor out this volume, and the path integral becomes

$$Z = \int Dh_{\mu\nu} \Delta_{FP}[h] \delta(G_\mu[h]) e^{-S[\tilde{g}+h]}. \quad (4.33)$$

To exponentiate the determinant, we introduce anticommuting ghost fields c^μ and antighost fields $\bar{c}_\mu$ such that

$$\Delta_{FP}[h] = \det(M_\mu^\nu) = \int Dc D\bar{c} \exp \left\{ - \int d^4x \sqrt{\tilde{g}} \bar{c}_\mu M_\mu^\nu c^\nu \right\}. \quad (4.34)$$

A gauge-fixing term is introduced by replacing the delta function with a Gaussian weight. We choose the gauge-fixing action as

$$S_{\text{gf}} = \frac{1}{2\tilde{\xi}} \int d^4x \sqrt{\tilde{g}} \left(G^\mu[h] \right)^2, \quad (4.35)$$

where $\tilde{\xi}$ is a gauge parameter. One can write it explicitly in terms of $h_{\mu\nu}$

$$S_{\text{gf}} = \frac{1}{2\tilde{\xi}} \int d^4x \sqrt{\tilde{g}} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} \left(\tilde{\nabla}^\gamma h_{\alpha\gamma} - \frac{1}{2} \tilde{\nabla}_\alpha h \right) \left(\tilde{\nabla}^\delta h_{\beta\delta} - \frac{1}{2} \tilde{\nabla}_\beta h \right). \quad (4.36)$$

Thus, the complete gravitational action becomes

$$S_g \rightarrow S_g + S_{\text{gf}} + S_{\text{ghost}}, \quad (4.37)$$

with the ghost action given explicitly by

$$S_{\text{ghost}} = \int d^4x \sqrt{\tilde{g}} \bar{c}_\mu \left(\tilde{\nabla}^2 \delta_\nu^\mu + \tilde{R}_\nu^\mu \right) c^\nu, \quad (4.38)$$

then the Euclidean path integral reads

$$Z = \int Dh_{\mu\nu} Dc D\bar{c} \exp\left\{-\left[S_g + S_{\text{gf}} + S_{\text{ghost}}\right]\right\}. \quad (4.39)$$

This procedure ensures that the integration is restricted to one representative per gauge orbit, thereby removing the nonphysical gauge degrees of freedom and rendering the quadratic operator invertible. The ghost fields, though unphysical, cancel the contributions from the redundant modes.

4.5. The Stationary-Phase (One-Loop) Approximation

If one assumes that the significant expansion terms in the path integral are up to second order then one can write

$$Z = \int \mathcal{D}h_{\mu\nu} e^{-S_0 - S_1 - S'_2}. \quad (4.40)$$

Now if one considers that S_0 is independent of h , $S_1 = 0$ from the equations of motion and S'_2 contains all the contributions from the expansion, the gauge fixing and ghost terms, that are quadratic in h , then one obtains

$$Z = e^{-S_0} \int \mathcal{D}h_{\mu\nu} e^{-S'_2(h_{\mu\nu})}, \quad (4.41)$$

This approach is known as the stationary phase, WKB, or one-loop approximation. Taking the logarithm of the expression we obtain

$$\ln(Z) = -S_0 + \ln\left(\int \mathcal{D}h_{\mu\nu} e^{-S'_2(h_{\mu\nu})}\right). \quad (4.42)$$

In the more general case that matter fields are present, in the form of a scalar field ϕ , then the total action will be $S(g, \phi)$. If one expands the scalar field similarly with the metric, then the total action will expand as $S(g, \phi) = S_0(g, \phi) + S_2(g, \phi) + \mathcal{O}(3)$. Furthermore, if one assumes for simplicity that $\phi_0 = 0$ and the quadratic term splits as $S_2(g, \phi) = S_2(g) + S_2(\phi)$, then the one-loop approximation will yield

$$\ln(Z) = -S_0 + \ln\left(\int \mathcal{D}\phi e^{S_2(\phi)}\right) + \ln\left(\int \mathcal{D}h_{\mu\nu} e^{-S'_2(h_{\mu\nu})}\right). \quad (4.43)$$

Now we are going to present the method implemented by Hawking in [216] to evaluate the matter contribution to the one-loop path integral, which will be used again for the gravitational part. Let us write the $S_2(\phi)$ as

$$S_2(\phi) = \frac{1}{2} \int \sqrt{\tilde{g}} \phi \hat{A} \phi, \quad (4.44)$$

where $\hat{A}$ is a differential operator depended on the background metric $\tilde{g}$, satisfying the eigenvalue equation

$$\hat{A} \phi_n = \lambda_n \phi_n. \quad (4.45)$$

Now, if one express a scalar field as $\phi = \sum y_n \phi_n$ and the measure on the space of ϕ as $\mathcal{D}\phi = \prod \mu dy_n$, where μ is a normalization factor of dimensions of mass, then one can express the matter term, in the case of bosons, as

$$Z_\phi = \left(\det \left(\frac{1}{2} \pi^{-1} \mu^{-2} \hat{A} \right) \right)^{-1/2}, \quad (4.46)$$

while in the case of fermions, the matter term becomes

$$Z_\psi = \det \left(\frac{1}{2} \mu^{-2} \hat{A} \right). \quad (4.47)$$

Thus, the stationary-phase approximation separates the classical background contribution from the quantum corrections, with the latter encoded in the determinants of the quadratic fluctuation operators. These determinants can be systematically evaluated using zeta function regularization (and, equivalently, via the heat kernel method), thereby capturing the effect of quantum fluctuations around the classical background.

4.6. Zeta Function Regularization

To regularize the determinant of an elliptic operator $\hat{A}$ with eigenvalues $\{\lambda_n, \phi_n\}$, one introduces the spectral zeta function defined by

$$\zeta_A(s) = \sum_n \lambda_n^{-s}, \quad (4.48)$$

which converges for sufficiently large $\text{Re}(s)$ (typically $\text{Re}(s) > D/2$ in D dimensions). This function can be analytically continued to a meromorphic function on the complex s -plane with poles only at specific values (for example, at $s = D/2$ and $s = D/2 - 1$ in four dimensions). In particular, $\zeta_A(s)$ is regular at $s = 0$. The regularized determinant of A is then defined via

$$\det A = \exp \left[-\zeta'_A(0) \right], \quad (4.49)$$

which provides a finite value for the otherwise divergent product of eigenvalues. There is a close connection between the zeta function and the heat kernel $K(x, x'; t)$ of the operator A . The heat kernel is defined as the solution of the diffusion equation

$$\left(\frac{\partial}{\partial t} + A_x \right) K(x, x'; t) = 0, \quad K(x, x'; 0) = \delta(x, x'), \quad (4.50)$$

and it admits an asymptotic expansion as $t \rightarrow 0$:

$$K(x, x; t) \sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^{\infty} b_n(x) t^n, \quad (4.51)$$

where the coefficients $b_n(x)$, known as the Seeley—DeWitt coefficients, are local geometric invariants constructed from the metric, curvature, and their covariant derivatives. The spectral zeta function is related to the heat kernel by the Mellin transform

$$\zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \text{Tr} K(t). \quad (4.52)$$

Thus, the pole structure and the finite part of $\zeta_A(s)$ are determined by the short-time asymptotic expansion of the heat kernel. In particular, the first few Seeley—DeWitt coefficients control the ultraviolet divergences of the one-loop effective action.

An important consequence of this regularization procedure is that the regularized determinant, and hence the effective action, is not invariant under conformal transformations of the background metric. For example, if the background metric is rescaled by a constant factor,

$$\tilde{g}'_{\mu\nu} \rightarrow k^2 \tilde{g}_{\mu\nu}, \quad (4.53)$$

the eigenvalues scale as $\lambda_n \rightarrow k^{-2} \lambda_n$. Consequently, the zeta function transforms as

$$\zeta'_A(s) \rightarrow k^{2s} \zeta_A(s), \quad (4.54)$$

so that

$$\zeta'_A(0) \rightarrow \zeta_A(0) + 2 \ln k \zeta_A(0). \quad (4.55)$$

Since $\zeta_A(0)$ is generally nonzero, the determinant transforms as

$$\ln(\det A) \rightarrow -2\zeta_A(0) \ln(k) + \ln(\det A). \quad (4.56)$$

This scaling behavior implies that the effective action acquires an anomalous dependence on the scale k , a phenomenon known as the *conformal anomaly*. The anomaly arises because the regularization procedure introduces a normalization scale (often denoted by μ) with dimensions of mass, and the path-integral measure is not invariant under conformal transformations.

Gibbons Hawking and Perry in [298] implement the Fadeev–Popov method and the zeta function regularization technique, as it was presented in the previous subsection, in order to evaluate the quadratic term in the fluctuations around the background. They obtain the following expression for the partition function

$$\ln(Z) = -I(g_{\mu\nu}) - \frac{1}{2} \ln \left(\det \left(\frac{1}{2} \pi^{-1} \mu^{-2} (-F + G) \right) \right) + \ln \left(\det \left(\frac{1}{2} \pi^{-1} \mu^{-2} \right) \right) \quad (4.57)$$

where μ is the normalization scale, C the operator for the ghosts and $-F + G$ an operator for the expansion and gauge fixing. The determinants can be approximated by ζ functions, where for each eigenvalue function the operator is noted in the subscript.

$$\ln(Z) = -I(g_{\mu\nu}) + \frac{1}{2} \zeta'_F(0) + \frac{1}{2} \zeta'_G(0) - \zeta'_C(0) + \frac{1}{2} \ln(2\pi\mu^2) (\zeta_F(0) + \zeta_G(0) - 2\zeta_C(0)) \quad (4.58)$$

In [313] Hawking calculates a formula for the zeta functions for a space time with a cosmological constant by the implementation of Eqs. (4.51) and (4.52)

$$\zeta_F(0) + \zeta_G(0) - 2\zeta_C(0) = \frac{53}{720\pi^2} \int d^4x \sqrt{g} R_{abcd} R^{abcd} - \frac{87}{120\pi^2} \Lambda^2 V \equiv \gamma \quad (4.59)$$

by applying the Chern–Gauss–Bonnet theorem and taking into account the Einstein equations in vacuum, Hawking calculates the above equation as

$$\gamma = \frac{106\chi}{45} + \frac{73\Lambda^2}{120\pi^2 V}, \quad (4.60)$$

The above calculation was later corrected by Christensen and Duff [232]

$$\gamma = \frac{106}{45}\chi - \frac{87}{120\pi^2}\Lambda^2 V. \quad (4.61)$$

If one rescales the action as $\tilde{g}'_{\mu\nu} = k^2 \tilde{g}_{\mu\nu}$ the eigenvalues of the operators will scale as k^{-2} , therefore one obtains the amplitude in the one-loop approximation as

$$\ln(Z') = \ln(Z) + (1 - k^2)S_0 + \frac{1}{2}\gamma \ln(k). \quad (4.62)$$

In summary, the zeta function regularization method provides a rigorous framework to define the determinant of an operator in Euclidean quantum gravity. By relating the determinant to the spectral zeta function and, through it, to the heat kernel and Seeley–DeWitt coefficients, one can systematically isolate and renormalize the ultraviolet divergences. Moreover, the method naturally explains the emergence of conformal anomalies when the background metric is rescaled.

4.7. Instantons in Euclidean Quantum Gravity

In this section, we present several examples of positive-definite (Euclidean) solutions of the Einstein (and Einstein–Maxwell) equations that serve as instantons in the path-integral formulation of quantum gravity. These instanton solutions, obtained either as analytic continuations of well known Lorentzian metrics or as genuine Euclidean metrics, play a crucial role in describing semiclassical tunneling processes and topology change.

For instanton solutions, we are primarily concerned with complete Euclidean metrics that satisfy the Einstein equation for the background or their Einstein–Maxwell generalizations. In many cases the Euclidean section is obtained by an analytic continuation of a Lorentzian solution (via, e.g., $t \rightarrow -i\tau$), though its global structure and topology may differ substantially. Such Euclidean metrics, even when not admitting a Lorentzian section, can still be interpreted as stationary–phase points in the gravitational path integral.

4.7.1. The Euclidean Schwarzschild solution

A classic example is provided by the Euclidean Schwarzschild solution. By setting $t = -i\tau$ in the Lorentzian Schwarzschild metric, one obtains for $r > 2M$ the Euclidean metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (4.63)$$

Introducing a new radial coordinate near $r = 2M$ shows that the apparent singularity is a coordinate artifact; regularity then requires that τ be identified with period $8\pi M$. Consequently, the boundary at infinity has the topology $S^1 \times S^2$. A similar procedure can be applied to the Reissner–Nordström solution (with appropriate restrictions on the charges to maintain a real electromagnetic field), where the outer horizon serves as an axis of symmetry and the period of τ is determined by the surface gravity.

4.7.2. Reissner–Nordström Instanton

The Reissner–Nordström instanton is the Euclidean version of the Reissner–Nordström black hole, which describes a charged black hole in Euclidean spacetime. The metric for the Reissner–Nordström instanton is given by:

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) d\tau^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \quad (4.64)$$

where M is the mass, Q is the charge, and $d\Omega_2^2$ is the metric on the two-sphere. The Euler characteristic of the Reissner–Nordström instanton is $\chi = 2$ which is consistent with the topology of a disk times a two-sphere, where the disk represents the Euclidean time direction.

4.7.3. The Euclidean Wormhole Instanton

A particularly intriguing instanton solution in Euclidean quantum gravity is the Euclidean wormhole, characterized by the topology $S^1 \times S^3$. In this configuration, the S^1 factor represents the compactified Euclidean time coordinate, while the S^3 factor corresponds to a spatial hypersurface. Such instantons are of interest because they provide saddle-point contributions to the Euclidean path integral and may mediate topology change at the quantum level. A common ansatz for the Euclidean wormhole metric is

$$ds^2 = d\tau^2 + a^2(\tau) d\Omega_3^2, \quad (4.65)$$

where τ is the Euclidean time, $a(\tau)$ is the scale factor, and $d\Omega_3^2$ is the metric on the three-sphere S_3 .

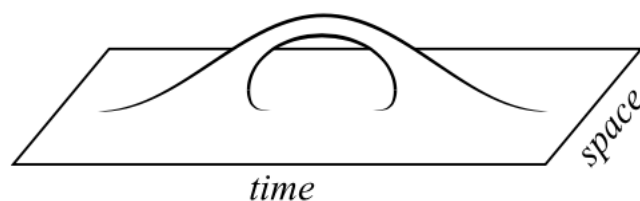


Figure 4.1.: An illustrative sketch of the Euclidean wormhole, source [259].

Inserting this ansatz into the Euclidean Einstein equations (possibly with a cosmological constant or other matter terms) yields an ordinary differential equation for the scale factor $a(\tau)$. The solution typically exhibits a “bounce” behavior, where $a(\tau)$ contracts to a minimum size and then expands, reflecting the wormhole geometry that connects different regions of spacetime or even different universes.

The topological invariants of the Euclidean wormhole are straightforward to compute. Since the Euler characteristic satisfies $\chi(M \times N) = \chi(M) \chi(N)$, and because the

Euler characteristic of both S^1 and S^3 vanishes,

$$\chi(S^1 \times S^3) = \chi(S^1) \chi(S^3) = 0 \times 0 = 0. \quad (4.66)$$

The above topological property is significant when discussing the role of such instantons in the gravitational path integral and their contribution to quantum topology change. Thus, the Euclidean wormhole instanton, serves as a concrete example of a non-singular, positive-definite solution that can contribute to the semiclassical approximation in quantum gravity. Its nontrivial topology may lead to observable effects, such as shifts in coupling constants or modifications of the effective potential, when integrated over in the full path integral.

4.7.4. The Nariai Instanton

A classic example of an instanton in Euclidean quantum gravity is the Nariai solution, which arises as a limiting case of the Schwarzschild—de Sitter geometry when the black hole and cosmological horizons coincide. In this limit, the metric becomes a direct product of two constant-curvature 2-manifolds. We begin by assuming a product-metric ansatz of the form

$$ds^2 = a^2 d\Omega_2^2 + b^2 d\tilde{\Omega}_2^2, \quad (4.67)$$

where $d\Omega_2^2$ and $d\tilde{\Omega}_2^2$ denote the standard metrics on two unit 2-spheres, and a and b are constants representing their respective radii. For an Einstein manifold with a positive cosmological constant Λ , the Euclidean Einstein equations are

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \quad (4.68)$$

On a 2-sphere of radius a , the Ricci tensor is given by

$$R_{ab} = \frac{1}{a^2} g_{ab}, \quad (4.69)$$

so that the scalar curvature is

$$R = \frac{2}{a^2}. \quad (4.70)$$

Similarly, for the second 2–sphere of radius b we have

$$R_{ij} = \frac{1}{b^2} g_{ij} \quad \text{and} \quad R = \frac{2}{b^2}. \quad (4.71)$$

Since the total metric is a direct product, the Einstein equations require that each factor satisfy

$$\frac{1}{a^2} = \Lambda \quad \text{and} \quad \frac{1}{b^2} = \Lambda \quad (4.72)$$

thus, we obtain

$$a = b = \frac{1}{\sqrt{\Lambda}}. \quad (4.73)$$

Substituting these results back into the product–metric ansatz, the Euclidean Nariai metric becomes

$$ds^2 = \frac{1}{\Lambda} \left[d\Omega_2^2 + d\tilde{\Omega}_2^2 \right]. \quad (4.74)$$

A common explicit form for the metric on a 2–sphere is

$$d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (4.75)$$

so that the full metric can be written as

$$ds^2 = \frac{1}{\Lambda} \left[d\theta^2 + \sin^2 \theta d\phi^2 + d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right]. \quad (4.76)$$

The topology of the Nariai instanton is then

$$S^2 \times S^2. \quad (4.77)$$

The Euler characteristic of a product manifold is given by the product of the Euler characteristics of the individual factors. Since

$$\chi(S^2) = 2, \quad (4.78)$$

the Euler characteristic of $S^2 \times S^2$ is

$$\chi(S^2 \times S^2) = \chi(S^2) \chi(S^2) = 2 \times 2 = 4. \quad (4.79)$$

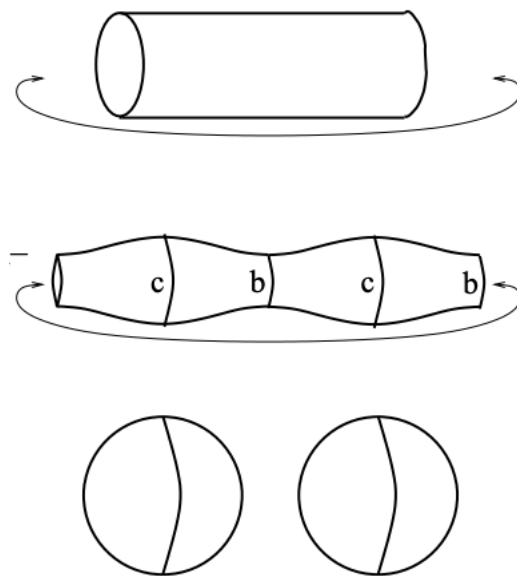


Figure 4.2.: An illustrative sketch of the Nariai instanton, source [314]

In summary, the Nariai instanton is a solution of the Einstein equations with a positive cosmological constant and has topology $S^2 \times S^2$ with Euler characteristic $\chi = 4$. This instanton plays an important role as a stationary-phase point in the Euclidean path integral and has been interpreted either as describing the nucleation of a universe with topology $S^1 \times S^2$ or, alternatively, as representing the creation of a pair of black holes in a cosmological background.

4.7.5. The Taub—NUT Instanton

The Taub–NUT metric represents a class of solutions characterized by an ordinary mass M and a NUT charge N (interpreted as a gravitational “magnetic” charge). In Euclidean signature, the Taub–NUT metric can be written in the form

$$ds^2 = V(r) \left[d\tau + 2N \cos \theta d\phi \right]^2 + \frac{dr^2}{V(r)} + (r^2 + N^2) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.80)$$

with

$$V(r) = \frac{r^2 - 2Mr - N^2}{r^2 + N^2}. \quad (4.81)$$

The metric exhibits a potential Dirac string singularity along the axis ($\theta = 0$ or $\theta = \pi$). This singularity is removed by an appropriate redefinition of the time coordinate and by identifying τ with period $8\pi N$. As a result, the topology of the constant- r surfaces changes from $S^2 \times \mathbb{R}$ to that of S^3 . The Taub—NUT instanton is non-compact, thus surface terms contribute to the calculation of the Euler characteristic which yields $\chi = 1$, reflecting the non-trivial topology induced by the NUT charge.

The instanton solutions discussed, the Euclidean Schwarzschild and Reissner–Nordström, Nariai, Taub–NUT, offer a rich collection of non-singular, positive-definite metrics in Euclidean quantum gravity. These solutions serve as stationary-phase points in the gravitational path integral, facilitating semiclassical approximations and shedding light on possible topology change at the quantum level.

4.8. Hawking's Space Time Foam model

Wheeler [241] suggested that a theory of quantum gravity would describe quantum fluctuations of the metric and topology of space time near the Planck scale. Hawking in [231] constructs a mathematical model for describing the foam structure of space time by defining the partition function for the "volume canonical ensemble" of the field as the amplitude of quantum gravity through the path integral approach:

$$Z[\Lambda] = \sum_n \langle g_n | e^{-I} | g_n \rangle = \int D\tilde{g}_{\mu\nu} e^{-I}, \quad (4.82)$$

for the integral to converge, the space time manifold is Euclideanized by a Wick rotation ($t \rightarrow i\tau$), thus I is the Euclidean action of the gravitational field with a cosmological constant Λ

$$I = -\frac{1}{16\pi} \int d^4x \sqrt{\tilde{g}} (R - 2\Lambda) = -\frac{\Lambda V}{8\pi}, \quad (4.83)$$

and D is a measure over all metrics on all compact manifolds. The partition sum is equivalent to:

$$Z(\Lambda) = \int_0^\infty N(V) e^{-\frac{\Lambda V}{8\pi}} dV, \quad (4.84)$$

then one computes the number of states $N(V)$ by an inverse Laplace transform of $Z(\Lambda)$ employing the saddle point approximation:

$$N(V) = \int_C Z(\Lambda) e^{\frac{\Lambda V}{8\pi}} d\Lambda \approx Z(\Lambda_s) e^{\frac{\Lambda_s V}{8\pi}} \quad (4.85)$$

where the contour of integration C is taken at the imaginary axis to the right and the saddle point Λ_s is given by:

$$\left. \frac{\partial}{\partial \Lambda} \left(\ln(Z) + \frac{\Lambda_s V}{8\pi} \right) \right|_{\Lambda_s} = 0, \quad (4.86)$$

thus the entropy of the system consisting of N microstates will be:

$$S = k_b \ln(N) \quad (4.87)$$

In the one loop WKB approximation $Z[\Lambda]$ is approximated:

$$Z \approx L e^{-I}, \quad (4.88)$$

then the amplitude can be determined by the Zeta functions renormalization technique we presented in the previous subsections. In [231] by dimensional arguments Hawking writes $\Lambda = -f\sqrt{V}$ thus $f^2 = \Lambda^2 V$ where f is a dimensionless parameter, then Λ can be seen as a rescaling of the action by $1/\Lambda$:

$$I = -\frac{\Lambda V}{8\pi} = \frac{1}{\Lambda} \left(-\frac{f^2}{8\pi} \right). \quad (4.89)$$

In that case, Gibbons, Hawking and Perry [315] introduced an appropriate Faddeev–Popov determinant, for a space time that fulfill Einstein field equations in vacuum $R_{\mu\nu} = \Lambda g_{\mu\nu}$, as to obtain an expression for the scaling behavior of gravitational perturbations $L \sim \Lambda^{-\gamma}$. We have seen in Eq. (4.60) that

$$\gamma = \frac{106\chi}{45} + \frac{73\Lambda^2}{120\pi^2 V}. \quad (4.90)$$

Hawking assumes that $\gamma = \alpha\chi$ and interprets α as the extra modes per gravitational instanton. Then, finally, the contribution of Einstein metrics with Euler number χ to

the partition sum is

$$Z[\Lambda, \chi] = \left[\frac{\Lambda}{\Lambda_0} \right]^\gamma e^{\frac{\Lambda V}{8\pi}} \quad (4.91)$$

where Λ_0 depends on μ which is a one-loop normalization regulator. Then by Eq. (4.86) the stationary phase point will be

$$\Lambda_s = \frac{4\pi}{V} \left(-a\chi \pm \left(\sqrt{a^2\chi^2 + \frac{Vd^2\chi}{16/\pi^2}} \right) \right) \quad (4.92)$$

Hawking interprets intuitively the Euler characteristic χ as the number of gravitational instantons or virtual black holes which appear and disappear and pictures the space time foam as a gas of $\chi - 2$ instantons. Subsequently, parameter a is interpreted as the modes per instanton. From Eq. (4.92) he makes the approximation that the density of instantons $c \sim \frac{\chi}{V}$, is about one instanton per Planck volume, and c is a function of Λ_0 . Finally, Hawking builds an argument that in his space time foam model $\Lambda < 0$ must be negative in order to prevent geodesics from converging.

4.9. Modifications of Hawkings model

In [277] Schulz developed a modification of Hawking's space time foam model. He employed the corrected scaling parameter γ from Eq. (4.61) by Christensen and Duff [232]

$$\gamma = \frac{106\chi}{45} - \frac{87\Lambda^2 V}{120\pi^2}, \quad (4.93)$$

and introduced a different renormalization scale, so that the perturbative evaluation converges for a positive $\Lambda > 0$. The modified foam model yields a partition function

$$Z = \left(\frac{\Lambda}{2\pi\mu^2} \right)^{-\frac{1}{2} \left(\frac{106}{45}\chi - \frac{87}{120\pi^2}\Lambda^2 V \right)} e^{\frac{\Lambda V}{8\pi}}, \quad (4.94)$$

and sets the regulator $\mu^2 = H_0^2$ for consistency of the model. Then by the saddle point approximation

$$\Lambda_{s1} = \frac{5\pi}{29} \pm \frac{\sqrt{225\pi^2 V^2 + 24592\pi^2 V \chi}}{87V}, \quad (4.95)$$

when the $\Lambda^2 V$ term is omitted the approximation resembles Hawking's result

$$\Lambda_{s0} = -\frac{424\pi\chi}{45V}. \quad (4.96)$$

Schulz argues that our universe imposes a positive $\Lambda > 0$, therefore by Eq. (4.96) the Euler characteristic must be negative $\chi < 0$. He also interprets χ as the number of spacetime cavities or wormholes, so that EQ. (4.95) becomes

$$|c| = \frac{\chi}{V} = -3\Lambda_{s1} \left(\frac{73\Lambda_{s1}\pi + 60}{848\pi} \right) \quad (4.97)$$

where c is the cavity density. Inserting the observed value of $\Lambda = 10^{-122}$, Schulz claims to get in (s.i) units that $|c| \approx 10^{25}$ cavities were produced in one s per m^3 .

4.10. Space time foam thermodynamics

Further investigations of the relationship between the cosmological constant and Euclidean quantum gravity by Carlip [274], [316] led to an interesting reformulation of Hawking's foam model. The volume canonical ensemble is defined as:

$$Z[\Lambda] = \sum_M \int [dg] e^{-I_E}, \quad (4.98)$$

where the sum is over distinct topological manifolds M . The Euclidean action I_E is

$$I_E = -\frac{1}{16\pi L_p^2} \int_M d^4x \sqrt{g} (R - 2\Lambda), \quad (4.99)$$

then a rescaling of the metric

$$\tilde{g}_{\mu\nu} = \frac{1}{|\Lambda|} g_{\mu\nu} \quad (4.100)$$

will absorb the dependence of Λ into the normalized volume

$$\tilde{u}(M) = \int_M d^4x \sqrt{\tilde{g}}. \quad (4.101)$$

$\tilde{u}(M)$ is a measure of topological complexity, as in some cases is a function of the Euler character χ , and $\tilde{g}$ will be a solution of the Einstein field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with the scalar curvature set to ± 12 . Then the extrema of action in Eq. (4.99) will be

$$\tilde{I}_E = -\frac{9}{8\pi\Lambda L_p^2} \tilde{u}(M). \quad (4.102)$$

With this action into the path integral, the saddle point approximation will give

$$Z[\Lambda] = \sum_M \Delta_M \exp\left(-\frac{9}{8\pi\Lambda L_p^2} \tilde{u}(M)\right), \quad (4.103)$$

by the trace anomaly [232]:

$$\Delta_M \sim \Lambda^{-\frac{\gamma}{2}} \text{ and } \gamma = \frac{106}{45} \chi(M) - \frac{261}{\pi^2} \tilde{u}(M), \quad (4.104)$$

now if one assumes $\rho(\tilde{u}) \sim \Delta_M$ as the density of topologies on M , then the partition function can be written in a more compact form

$$Z[\Lambda] = \sum_{\tilde{u}} \rho(\tilde{u}) \exp\left(-\frac{9}{8\pi|\Lambda|L_p^2} \tilde{u}(M)\right), \quad (4.105)$$

resembling a classical thermal partition function. Based on this analogy, Carlip [317] defines a microcanonical cosmological constant as

$$\Lambda = -\frac{9}{8\pi L_p^2} \left(\frac{\partial \ln \rho(\tilde{u})}{\partial \tilde{u}} \right)^{-1} \Big|_{\tilde{u}_{max}}. \quad (4.106)$$

Carlip comments that Eq. (4.106) can be interpreted as a mechanism that balances the value of Λ , as any tension to increase its value will just lead to the production of more states, thus more complicated space time foam.

The techniques and results presented in this chapter provide a solid foundation for understanding the interplay between quantum fluctuations, gauge invariance, and topology in the gravitational field. This work paves the way for further investigations

into non-perturbative effects and the role of instantons in a comprehensive theory of quantum gravity.

Chapter 5.

Black Hole Thermodynamics

5.1. The Four Laws of Black Hole Mechanics

The study of black hole thermodynamics emerged as a profound intersection of general relativity, quantum mechanics, and statistical physics. The development of this field was pioneered by the groundbreaking work of Jacob Bekenstein and Stephen Hawking, who established a deep connection between black hole mechanics and the laws of thermodynamics. This section outlines the historical development of the four laws of black hole mechanics, tracing their origins and implications.

The Zeroth Law

The zeroth law of black hole mechanics states that the surface gravity κ of a stationary black hole is constant over the event horizon. This law is analogous to the zeroth law of thermodynamics, which asserts that the temperature of a system in thermal equilibrium is uniform. Mathematically, the surface gravity is given by:

$$\kappa = \left. \frac{1}{2} \frac{\partial f(r)}{\partial r} \right|_{r=r_H}, \quad (5.1)$$

where $f(r)$ is the metric function and r_H is the radius of the event horizon [61]. This law was first formulated by Bardeen, Carter, and Hawking in their seminal work on the four laws of black hole mechanics.

The First Law

The first law of black hole mechanics relates the change in mass M , angular momentum J , and charge Q of a black hole to the change in its horizon area A . It is expressed as:

$$dM = \frac{\kappa}{8\pi}dA + \Omega_H dJ + \Phi_H dQ, \quad (5.2)$$

where Ω_H is the angular velocity at the horizon and Φ_H is the electrostatic potential [61]. This law is analogous to the first law of thermodynamics, with the horizon area A playing the role of entropy. Bekenstein first proposed that the area of a black hole is proportional to its entropy, a conjecture later confirmed by Hawking's discovery of black hole radiation [318,319].

The Second Law

The second law of black hole mechanics, also known as the area theorem, states that the total horizon area of a black hole never decreases in classical processes. Mathematically, this is expressed as:

$$\delta A \geq 0. \quad (5.3)$$

This law was derived by Hawking using the theory of general relativity and is analogous to the second law of thermodynamics, which states that the entropy of an isolated system never decreases [320]. Bekenstein's insight that the horizon area corresponds to entropy further solidified this analogy.

The Third Law

The third law of black hole mechanics states that it is impossible to reduce the surface gravity κ to zero through a finite sequence of physical processes. This law is analogous to the third law of thermodynamics, which asserts that the entropy of a system ap-

proaches a constant value (typically zero) as the temperature approaches absolute zero. The third law was first discussed by Israel in the context of black hole physics [321].

The formulation of the four laws of black hole mechanics was a collaborative effort spanning several decades. Bekenstein's proposal that black holes possess entropy was initially met with skepticism but was later vindicated by Hawking's discovery of black hole radiation, which provided a thermodynamic interpretation of black hole mechanics [318, 319]. Subsequent work by Bardeen, Carter, Hawking, and others established a rigorous framework for black hole thermodynamics, bridging the gap between classical and quantum physics. Together, these laws not only encapsulate the elegant interplay between gravitational and thermodynamic principles but also offer deep insights into the nature of spacetime and quantum effects in strong gravitational fields.

5.2. The Bekenstein–Hawking Entropy Law

For space-times with boundaries, a well-defined variational principle is obtained by adding the Gibbons–Hawking boundary term to the Einstein–Hilbert action, as it was presented analytically in 2.2.2. In natural units (with $c = \hbar = 1$), the Euclidean action is written as

$$I_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{\gamma} (K - K_0), \quad (5.4)$$

where R is the Ricci scalar on the manifold M , γ is the determinant of the induced metric on the boundary ∂M , K is the extrinsic curvature of ∂M embedded in M , K_0 is the extrinsic curvature of the boundary embedded in flat space, chosen such that $I_E = 0$ for Minkowski space.

In the case of the Schwarzschild black hole, the Euclidean section is obtained by Wick rotating the time coordinate, $t \rightarrow -i\tau$. Regularity at the event horizon demands that the Euclidean time be periodic with period

$$\beta = 8\pi GM, \quad (5.5)$$

which ensures the absence of a conical singularity at $r = 2GM$.

Since the Schwarzschild solution is a vacuum solution ($R = 0$), the bulk term in Eq. (5.4) vanishes and the action is entirely determined by the boundary contribution:

$$I_E = -\frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{\gamma} (K - K_0). \quad (5.6)$$

A detailed evaluation of this boundary term for the Euclidean Schwarzschild metric (see, e.g., [193]) is provided in Appendix B.1, where it shows that finally one obtains

$$I_E = \frac{\beta M}{2}. \quad (5.7)$$

As it was presented in section 4.5 in the semiclassical approximation, the Euclidean path integral is dominated by the classical solution,

$$Z \simeq e^{-I_E}, \quad (5.8)$$

which is interpreted as the canonical partition function at inverse temperature β . From standard thermodynamics, the free energy F is given by

$$F = -\frac{1}{\beta} \ln Z \simeq \frac{I_E}{\beta}, \quad (5.9)$$

and the expectation value of the energy is

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z. \quad (5.10)$$

For the Schwarzschild black hole, one identifies the ADM mass M as the energy, so consistency demands that

$$\langle E \rangle = M. \quad (5.11)$$

If one substitutes these expressions to $F = \langle E \rangle - TS$, then the entropy S of the system is obtained from the thermodynamic relation

$$S = \beta \langle E \rangle - I_E. \quad (5.12)$$

Substituting $I_E = \frac{\beta M}{2}$ from Eq. (5.7) and using the relation $\beta = 8\pi GM$ from Eq. (5.5), we find

$$S = \beta M - \frac{\beta M}{2} = \frac{\beta M}{2} = 4\pi GM^2. \quad (5.13)$$

Since the event horizon area of a Schwarzschild black hole is

$$A = 4\pi r_S^2 = 4\pi(2GM)^2 = 16\pi G^2 M^2, \quad (5.14)$$

thus from Eq. (5.13) and (5.14) the entropy may be written in the familiar form

$$S = \frac{A}{4G}. \quad (5.15)$$

This derivation rigorously establishes the Bekenstein—Hawking entropy law by analytically continuing the gravitational action, carefully evaluating the boundary contribution, and interpreting the Euclidean path integral as a canonical partition function. The subtraction of the flat-space extrinsic curvature (through K_0) is essential to fix the zero-point of the action, ensuring that Minkowski space has vanishing entropy.

5.3. Wald Black Hole Entropy

Black hole thermodynamics has revealed deep connections between gravity, quantum field theory, and statistical physics. A cornerstone of these developments is the Bekenstein—Hawking entropy, which in General Relativity is proportional to the horizon area. In theories beyond GR, such as higher-curvature gravity, entropy generally acquires corrections. To address this issue, Wald developed a general method based on Noether's theorem, which associates conserved charges to the diffeomorphism invariance of the gravitational Lagrangian [322, 323].

Wald's derivation begins by considering a diffeomorphism-invariant Lagrangian density L that depends on the dynamical fields ϕ (including the metric $g_{\mu\nu}$ and any matter fields) and their derivatives. Under an infinitesimal diffeomorphism generated by a vector field ζ^μ , the dynamical fields vary according to

$$\delta\phi = \mathcal{L}_\zeta\phi, \quad (5.16)$$

where $\mathcal{L}_{\xi}$ denotes the Lie derivative with respect to ξ^μ . The corresponding variation of the Lagrangian can be expressed as

$$\delta L = E_\phi \delta\phi + d\Theta(\phi, \delta\phi), \quad (5.17)$$

where $E_\phi = 0$ are the equations of motion and $\Theta(\phi, \delta\phi)$ is the symplectic potential $(D-1)$ -form. Because L is diffeomorphism invariant, its variation under the diffeomorphism is also given by the Lie derivative

$$\delta L = \mathcal{L}_{\xi} L = d(\xi \cdot L), \quad (5.18)$$

where $\xi \cdot L$ denotes the contraction of ξ^μ with the first index of the D -form L . Equating the two expressions for δL we obtain

$$E_\phi \delta\phi + d\Theta(\phi, \delta\phi) = d(\xi \cdot L). \quad (5.19)$$

Specializing to the case $\delta\phi = \mathcal{L}_{\xi}\phi$, one defines the Noether current $(D-1)$ -form J by

$$J = \Theta(\phi, \mathcal{L}_{\xi}\phi) - \xi \cdot L. \quad (5.20)$$

A standard calculation shows that, when the equations of motion hold (i.e. $E_\phi = 0$), the current is a closed form

$$dJ = 0. \quad (5.21)$$

By Poincaré's lemma, J is also exact, hence a locally defined $(D-2)$ -form Q exists, called the Noether charge, such that

$$J = dQ. \quad (5.22)$$

For a stationary black hole with a Killing horizon generated by ξ^μ , Wald demonstrated that the variation of the Hamiltonian associated with ξ^μ leads to the first law of black hole mechanics, with the entropy being identified as the surface integral of the Noether charge over a bifurcation surface $\mathcal{H}$ of the horizon [322]. In particular, the entropy is given by

$$S_{\text{Wald}} = -2\pi \int_B \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \sqrt{\sigma} d^{D-2}x, \quad (5.23)$$

where B is any $(D - 2)$ -dimensional spacelike cross-section of the horizon, ϵ_{ab} is the binormal to B normalized such that

$$\epsilon_{ab}\epsilon^{ab} = -2, \quad (5.24)$$

and $\sqrt{\sigma} d^{D-2}x$ is the area element on B .

5.4. Wald Black Hole Entropy for the Einstein–Hilbert–Gauss–Bonnet Lagrangian

The expression for S_{Wald} we obtained in the previous subsection is entirely general and applies to any diffeomorphism-invariant theory of gravity. In the case of GR, it reproduces the Bekenstein–Hawking result $S = A/(4G)$. For theories with higher-curvature terms, such as those including the Gauss–Bonnet invariant, additional contributions arise that depend on the intrinsic geometry of the horizon cross-section. In the case of the EH plus the GB term, the Lagrangian will be

$$L = \frac{1}{16\pi G} (R + \alpha \mathcal{G}), \quad (5.25)$$

with

$$\mathcal{G} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2, \quad (5.26)$$

where α is the Gauss–Bonnet coupling constant. In order to evaluate the Wald entropy (5.23), we need to compute the functional derivative

$$\frac{\partial L}{\partial R_{abcd}} = \frac{1}{16\pi G} \left[\frac{\partial R}{\partial R_{abcd}} + \alpha \frac{\partial \mathcal{G}}{\partial R_{abcd}} \right]. \quad (5.27)$$

We will now calculate the contribution from the Einstein–Hilbert Term. Since the Ricci scalar is linear in the Riemann tensor,

$$\frac{\partial R}{\partial R_{abcd}} = g^{ac}g^{bd} - g^{ad}g^{bc}, \quad (5.28)$$

its contribution to the derivative is

$$\left(\frac{\partial L}{\partial R_{abcd}} \right)_{EH} = \frac{1}{16\pi G} \left(g^{ac} g^{bd} - g^{ad} g^{bc} \right). \quad (5.29)$$

When contracted with the binormals on the bifurcation surface and using

$$\epsilon_{ab} \epsilon_{cd} \left(g^{ac} g^{bd} - g^{ad} g^{bc} \right) = -2, \quad (5.30)$$

the Einstein–Hilbert term yields

$$S_{EH} = -2\pi \times \frac{1}{16\pi G} (-2) \int_B \sqrt{\sigma} d^{D-2}x = \frac{1}{4G} \int_B \sqrt{\sigma} d^{D-2}x. \quad (5.31)$$

Now we can will evaluate the Contribution from the Gauss–Bonnet term. In [322] Wald shows that the variation of the Gauss–Bonnet term with respect to the Riemann tensor gives

$$\frac{\partial \mathcal{G}}{\partial R_{abcd}} = 2R^{abcd} - 4 \left(g^{ac} R^{bd} - g^{ad} R^{bc} - g^{bc} R^{ad} + g^{bd} R^{ac} \right) + 2R \left(g^{ac} g^{bd} - g^{ad} g^{bc} \right). \quad (5.32)$$

The above calculation is presented in Appendix B.2. On a stationary black hole horizon the geometry is such that only the intrinsic (i.e., $(D-2)$ —dimensional) curvature of B contributes when the above tensor is contracted with the binormals. One may show (see, e.g., [322] for a detailed discussion) that

$$\epsilon_{ab} \epsilon_{cd} \frac{\partial \mathcal{G}}{\partial R_{abcd}} = 2^{(D-2)} R, \quad (5.33)$$

where $^{(D-2)}R$ is the Ricci scalar of the induced metric on the cross-section B . Thus, the Gauss–Bonnet contribution to the derivative is

$$\left(\frac{\partial L}{\partial R_{abcd}} \right)_{GB} = \frac{\alpha}{16\pi G} 2^{(D-2)} R, \quad (5.34)$$

its contraction with the binormals gives

$$S_{GB} = \frac{\alpha}{4G} \int_B ^{(D-2)}R \sqrt{\sigma} d^{D-2}x. \quad (5.35)$$

Combining the two contributions we obtain the total Wald entropy

$$S_{\text{Wald}} = S_{EH} + S_{GB} = \frac{1}{4G} \int_B \left[1 + 2\alpha^{(D-2)} R \right] \sqrt{\sigma} d^{D-2}x. \quad (5.36)$$

In summary, the Wald entropy formula (5.23) yields, for the Lagrangian (5.25), the result

$$S_{\text{Wald}} = \frac{1}{4G} \int_B \left[1 + 2\alpha^{(D-2)} R \right] \sqrt{\sigma} d^{D-2}x, \quad (5.37)$$

which reduces to the standard Bekenstein–Hawking result in the limit $\alpha \rightarrow 0$ for a spherical horizon B

$$S_{\text{Wald}} = \frac{A}{4G} + 2\alpha \int_B \sqrt{\sigma} d^{D-2}x^{(D-2)} R. \quad (5.38)$$

Therefore, Wald’s Noether charge method not only provides a derivation of black hole entropy from first principles but also offers a systematic procedure to compute entropy corrections in modified theories of gravity. The correction term depends on the intrinsic curvature of the horizon and plays an important role in the thermodynamics of modified gravity theories. The Wald entropy formula provides a universal way to compute black hole entropy in any covariant gravitational theory. The corrections in higher-curvature gravity models, such as Gauss–Bonnet gravity, become significant in the context of black hole thermodynamics and the AdS/CFT correspondence [324, 325]. Recent research has also extended Wald’s approach to Lovelock gravity [290], scalar-tensor theories [326], and non-local gravity models [327]. These studies indicate that black hole entropy is a key probe of fundamental modifications to gravity and quantum corrections to black hole physics.

5.5. Hawking Radiation and the Unruh Effect

The interplay between quantum field theory and curved spacetime leads to surprising phenomena such as Hawking radiation and the Unruh effect. Both effects reveal that what one considers to be the vacuum state is observer-dependent.

5.5.1. Hawking Radiation

Hawking showed that black holes are not completely black but radiate as black bodies with a characteristic temperature [319]. The essential idea is that quantum vacuum fluctuations near the event horizon of a black hole lead to particle creation. In a collapsing geometry, modes that propagate outwards can be related to modes in the asymptotic past via a Bogoliubov transformation. This transformation mixes positive- and negative-frequency modes, implying that an observer at infinity will detect a flux of particles with a black body thermal spectrum.

A key quantity in this derivation is the surface gravity, κ , which measures the gravitational acceleration at the horizon. The Hawking temperature is given by

$$T_H = \frac{\hbar\kappa}{2\pi k_B c}, \quad (5.39)$$

in natural units (where $\hbar = c = k_B = 1$), the expression simplifies to

$$T_H = \frac{\kappa}{2\pi}. \quad (5.40)$$

This result can also be understood via the Euclidean path integral approach. In that derivation, one analytically continues time to imaginary values and finds that the absence of a conical singularity at the horizon forces the Euclidean time to be periodic with period $\beta = 2\pi/\kappa$. Interpreting β as the inverse temperature yields the same result for T_H .

5.5.2. The Unruh Effect

Shortly after Hawking's discovery, it was realized by Davies [328] and Unruh [329] that a uniformly accelerated observer in Minkowski spacetime, of conventional quantum field theory, perceives the usual vacuum as a thermal bath of particles with temperature

$$T_U = \frac{\hbar a}{2\pi k_B c}, \quad (5.41)$$

similarly with Hawking radiation, in natural units it takes the form

$$T_U = \frac{a}{2\pi}. \quad (5.42)$$

An observer with constant proper acceleration a naturally adopts Rindler coordinates, in which the Minkowski metric takes a form that covers only a wedge of spacetime. The derivation involves quantizing a scalar field in Rindler space and comparing the mode decomposition of the field with that in inertial coordinates. The corresponding Bogoliubov transformation shows that the particle content emitted by the Rindler horizon and as seen by the accelerated observer follows a black body distribution with temperature T_U .

5.5.3. Hawking–Unruh Temperature

If one identifies the surface gravity κ as the gravitational acceleration g at the surface of the black hole, then the striking similarity between the two results,

$$T_H = \frac{g}{2\pi}, \quad T_U = \frac{a}{2\pi}, \quad (5.43)$$

indicates that the phenomenon of particle creation is closely tied to the acceleration experienced by the observer, whether due to gravity (in the case of a black hole) or due to proper acceleration in flat spacetime. In this unified view, one may refer to these temperatures collectively as the Hawking–Unruh temperature, emphasizing that the key physical ingredient is the presence of a horizon or an acceleration horizon.

The thermal behavior of quantum fields is not an intrinsic property of the field itself but arises from the observer's kinematical state or the spacetime geometry. These results have profound implications on the merger of quantum field theory and general relativity and sparked intense debates over unresolved questions that are still being actively investigated. This insight forms one of the cornerstones in our quest to understand the interplay between gravitation, thermodynamics, and quantum theory.

5.6. Spacetime Thermodynamics Conjecture

The Spacetime Thermodynamics Conjecture represents a remarkable theoretical framework that intimately connects thermodynamics with the geometric and gravitational aspects of spacetime. Pioneered by Jacobson [190], and significantly developed further by Padmanabhan [330] and Verlinde [331], this conjecture proposes that Einstein's equations can be viewed fundamentally as thermodynamic equations of state. This

perspective implies that gravitational dynamics, horizon entropy, and temperature are not mere analogies but instead profound manifestations of a deeper statistical nature of spacetime.

Jacobson's original derivation [190] is particularly foundational, establishing that Einstein's equations emerge naturally from thermodynamic principles applied locally to spacetime horizons. His starting point is the fundamental thermodynamic relation interpreted in the context of local Rindler horizons,

$$\delta Q = T dS, \quad (5.44)$$

where heat δQ flows across causal boundaries defined by local horizons. The temperature T of these horizons is identified with the Hawking–Unruh temperature . (5.39),(5.41) experienced by an accelerated observer,

$$T = \frac{\hbar \kappa}{2\pi k_B c}, \quad (5.45)$$

where κ is the surface gravity associated with the horizon. The entropy S is taken to be proportional to the horizon area A , following the Bekenstein–Hawking relation,

$$S = \frac{k_B c^3}{\hbar G} \frac{A}{4}. \quad (5.46)$$

Jacobson explicitly demonstrates that requiring the validity of this thermodynamic relation at every local spacetime point imposes stringent conditions on the curvature of spacetime. Specifically, by expressing the heat flow in terms of the energy-momentum tensor T_{ab} ,

$$\delta Q = \int_H T_{ab} \chi^a d\Sigma^b, \quad (5.47)$$

with χ^a being the approximate local boost Killing vector field, one recovers Einstein's field equations,

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}. \quad (5.48)$$

Critically, this derivation hinges on a local equilibrium condition, emphasizing that Einstein's equations may represent classical equations of state, analogous to the macro-

scopic equations describing sound waves in fluids, thus potentially limiting their applicability for canonical quantization.

Jacobson's foundational insight provides the conceptual underpinning that Cai and Kim later leveraged to generalize this thermodynamic interpretation explicitly to cosmological horizons within Friedmann–Robertson–Walker cosmology [194].

Chapter 6.

Λ CDM Cosmology

All physical theories are based on founding axioms. In order to study the Universe, we make the following fundamental assumption

Cosmological principle: On the largest scales, the Universe is spatially homogeneous and isotropic.

Homogeneity states that the Universe seems identical from every point, and isotropy that it looks the same in every direction. One should notice that these properties refer only to space, since the temporal evolution of the Universe remains the main objective of cosmology. Furthermore, at specific time and spatial scales anisotropies become observable and provide valuable insight to the high energy physics at hand, e.g the CMB spectrum anisotropies.

6.1. The FLRW metric

Applying the above principal into the geometry of space and adopting hyperbolic signature $(-, +, +, +)$ according to the relativistic framework one obtains the Friedmann – Lemaitre – Robertson – Walker (FLRW) metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (6.1)$$

where t is the cosmic time, (r, θ, ϕ) are the spherical spatial coordinates, $a(t)$ is the dimensionless scale factor, and k is the curvature

$$k \equiv \begin{cases} 0 & \text{flat} \\ +1 & \text{spherical} \\ -1 & \text{hyperbolic} \end{cases} \quad (6.2)$$

6.2. The Friedmann equations

The requirements of homogeneity and isotropy lead to the assumption that the energy and matter content of the Universe behaves as a *perfect fluid*, therefore the coarse-grained energy momentum tensor will be

$$T^\mu{}_\nu = (\rho + P)U^\mu U_\nu - P \delta^\mu_\nu, \quad (6.3)$$

where ρ and P are the energy density and pressure of the cosmic fluid respectively and U^μ is its four-velocity from an observer. For a comoving observer $U^\mu = (1, 0, 0, 0)$ the energy momentum tensor can be written in matrix form

$$T^\mu{}_\nu = g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix}. \quad (6.4)$$

For describing the temporal dynamics of the cosmic fluid in a relativistic spacetime we will start from the Einstein field equation with a cosmological constant (we have adopted natural units where $c = 1$)

$$G^\mu{}_\nu - g^\mu{}_\nu \Lambda = 16\pi G T^\mu{}_\nu, \quad (6.5)$$

and calculate the metric dependent curvature quantities for the FLRW metric in Eq. (6.1). First the Christoffel symbols, then the Ricci tensor and finally the Ricci scalar. Firstly, we need to calculate all the components of the Ricci tensor, using the FLRW

metric

$$R_{\mu\nu} = \Gamma_{\mu\nu,\beta}^{\beta} - \Gamma_{\mu\beta,\nu}^{\beta} + \Gamma_{\mu\nu}^{\beta} \Gamma_{\beta\lambda}^{\lambda} - \Gamma_{\mu\lambda}^{\beta} \Gamma_{\beta\nu}^{\lambda}. \quad (6.6)$$

and therefore the Christoffel symbols

$$\Gamma_{\lambda\mu}^{\sigma} = \frac{1}{2} g^{\nu\sigma} (g_{\mu\nu,\lambda} + g_{\lambda\nu,\mu} - g_{\lambda\mu,\nu}). \quad (6.7)$$

For the term R_{00} , we have

$$R_{00} = \Gamma_{00,\beta}^{\beta} - \Gamma_{0\beta,0}^{\beta} + \Gamma_{00}^{\beta} \Gamma_{\beta\lambda}^{\lambda} - \Gamma_{0\lambda}^{\beta} \Gamma_{\beta 0}^{\lambda}, \quad (6.8)$$

where

$$\Gamma_{00}^{\beta} = \frac{1}{2} g^{\beta\beta} (g_{0\beta,0} + g_{0\beta,0} - g_{00,\beta}) = \frac{1}{2} g^{\beta\beta} g_{0\beta,0} = 0, \quad (6.9)$$

and

$$\Gamma_{0\beta}^{\beta} = 3 \frac{\dot{a}}{a} \quad \text{and} \quad \Gamma_{0\lambda}^{\beta} \Gamma_{\beta 0}^{\lambda} = 3 \left(\frac{\dot{a}}{a} \right)^2, \quad (6.10)$$

with $\dot{a} = \frac{da}{dt}$. Eq. (6.8) now takes the form

$$R_{00} = -3 \frac{\ddot{a}}{a}. \quad (6.11)$$

To extract the remaining components of the Ricci tensor, we proceed as before and using the following Christoffel symbols

$$\Gamma_{11}^0 = a\dot{a} \frac{1}{1 - kr^2} \quad \Gamma_{11}^1 = \frac{kr}{1 - kr^2} \quad \Gamma_{22}^0 = a\dot{a}r^2 \quad (6.12)$$

$$\Gamma_{33}^0 = a\dot{a}r^2 \sin^2 \theta \quad \Gamma_{01}^1 = \Gamma_{02}^2 = \frac{\dot{a}}{a} \quad \Gamma_{03}^3 = \frac{\dot{a}}{a} \quad (6.13)$$

$$\Gamma_{22}^1 = -r(1 - kr^2) \quad \Gamma_{33}^1 = -r(1 - kr^2) \sin^2 \theta \quad \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{1}{r} \quad (6.14)$$

$$\Gamma_{33}^2 = -\sin \theta \cos \theta \quad \Gamma_{23}^3 = \cot \theta, \quad (6.15)$$

we are leading to

$$R_{00} = -3\frac{\ddot{a}}{a} \quad \text{and} \quad R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \quad (6.16)$$

$$R_{22} = r^2 (a\ddot{a} + 2\dot{a}^2 + 2k) \quad \text{and} \quad R_{33} = r^2 (a\ddot{a} + 2\dot{a}^2 + 2k) \sin^2 \theta. \quad (6.17)$$

According to the symmetries dictated from spatial homogeneity and isotropy the only non-vanishing components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}}{a} \quad (6.18)$$

and

$$R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right]g_{ij}, \quad (6.19)$$

contracting with the metric one can obtain the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$

$$R = -6\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right]. \quad (6.20)$$

Substituting the above expressions one can obtain the non-zero components of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$

$$G^0_0 = 3\left[\left(\frac{\dot{a}}{a}\right) + \frac{k}{a^2}\right], \quad (6.21)$$

and

$$G^i_j = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\delta^i_j. \quad (6.22)$$

The cosmological constant term can be treated as a part of the energy momentum tensor and simultaneously defining the corresponding energy density as

$$T^{(\Lambda)}_{\mu\nu} = \frac{\Lambda}{8\pi G}g_{\mu\nu} \equiv \rho_\Lambda g_{\mu\nu}, \quad (6.23)$$

finally, if one absorbs Eq. (6.23) to (6.4) and then substitutes it along with Eq. (6.21), (6.22) to the Einstein field equation in Eq. (6.5) then the (00) component yields the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \quad (6.24)$$

the spatial component yields the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \quad (6.25)$$

and the covariant conservation of the energy momentum tensor $\nabla_\mu T^\mu_\nu = 0$ yields the continuity equation or the third Friedmann equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0. \quad (6.26)$$

The energy density ρ and pressure P correspond to the total amount from contributions from the different species

$$\begin{aligned} \rho &= \sum_i \rho_i = \rho_\Lambda + \rho_m + \rho_r \\ P &= \sum_i P_i = P_\Lambda + P_m + P_r \end{aligned} \quad (6.27)$$

where ρ_Λ is the dark energy density, ρ_m the (baryonic and dark) matter density and ρ_r the radiation density. The pressure of each specie is related to its energy density according to the equation of state

$$P_i = w_i \rho_i, \quad (6.28)$$

where w_i is the equation of state parameter for each specie. Matter is considered pressureless thus $w_m = 0$, radiation has $w_r = 1/3$ and dark energy has negative pressure hence $w_\Lambda = -1$. By defining the Hubble rate $H \equiv \frac{\dot{a}}{a}$ the Friedmann equations can be written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \quad (6.29)$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3P), \quad (6.30)$$

and

$$\dot{\rho} + 3H\rho(1 + w) = 0. \quad (6.31)$$

Integrating the continuity equation one obtains the temporal evolution of the species density

$$\rho_i = \rho_{i(0)} a^{-3(w+1)}, \quad (6.32)$$

where $\rho_{i(0)}$ is the constant of integration, usually referring to the values at present time ($a_0 = 1$).

Applying the solution for matter where $w_m = 0$ one obtains

$$\rho_m = \rho_{m0} a^{-3}, \quad (6.33)$$

for radiation where $w_r = 1/3$ similarly

$$\rho_r = \rho_{r0} a^{-4}, \quad (6.34)$$

finally the dark energy density ($w_\Lambda = -1$) is constant

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}. \quad (6.35)$$

Substituting to the first Friedmann equation in Eq. (6.30) one obtains

$$H^2 = \frac{8\pi G}{3} (\rho_{m0} a^{-3} + \rho_{r0} a^{-4} + \rho_\Lambda) - k a^{-2}. \quad (6.36)$$

In order to have a flat Universe ($k = 0$), from the first Friedmann equation we can see, that the density must have a specific value

$$\rho = \rho_{\text{crit}} \equiv \frac{3H^2}{8\pi G}, \quad (6.37)$$

which is defined as the critical density, therefore the critical density at the present is

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G}, \quad (6.38)$$

since observations suggest that the observed Universe at the present is spatially flat, we can use $\rho_{\text{crit},0}$ to define the dimensionless density parameter for each specie

$$\Omega_i = \frac{\rho_i}{\rho_{\text{crit},0}}, \quad (6.39)$$

therefore dividing Eq. (6.36) with H_0^2 we define the dimensionless Hubble rate

$$E^2 \equiv \frac{H^2}{H_0^2} = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{k0}a^{-2} + \Omega_{\Lambda0}, \quad (6.40)$$

where $\Omega_{k0} = -k/H_0^2$. We can see that the sum of the dimensionless densities at the present is by definition $\sum_i \Omega_i = 1$. Furthermore, since the observables are acquired and expressed in terms of redshift, it is convenient to express the scale factor in terms of redshift according to

$$a = \frac{1}{1+z}, \quad (6.41)$$

substituting to Eq. (6.40) we acquire the dimensionless Hubble rate with respect to redshift

$$E(z) = \sqrt{\Omega_{m0}(z+1)^3 + \Omega_{r0}(z+1)^4 + \Omega_{k0}(z+1)^2 + \Omega_{\Lambda0}}. \quad (6.42)$$

This parametrization has been useful for expressing cosmological observables, according to latest measurements [46] $\Omega_{\Lambda0} = 0.69$, $\Omega_{m0} = 0.31$, $\Omega_{r0} = 4.9 \cdot 10^{-5}$ and $|\Omega_{k0}| \leq 0.01$.

While in Λ CDM cosmology ρ_{Λ} is constant, this is not the case in many alternative cosmological scenarios where dark energy evolves with respect to time. For confronting these scenarios with observations, a different ansatz proves to be helpful. If one rewrites $\Omega_{\Lambda0}$ as

$$\Omega_{\Lambda0} = \frac{8\pi G}{3E^2 H_0^2} E^2 \rho_{\Lambda} = \frac{8\pi G}{3H^2} \rho_{\Lambda} E^2 = \Omega_{\Lambda} E^2, \quad (6.43)$$

substitutes to Eq. (6.42) and solves per E , then one obtains

$$E(z) = \left(\frac{\Omega_{m0}(z+1)^3 + \Omega_{r0}(z+1)^4 + \Omega_{k0}(z+1)^2}{1 - \Omega_\Lambda(z)} \right)^{1/2}, \quad (6.44)$$

where $\rho_\Lambda(z)$ and thus $\Omega_\Lambda(z)$ can be functions of redshift.

6.3. The evolution of perturbations

To investigate the formation and evolution of cosmic structures, it is necessary to perform a perturbative analysis of important observable quantities. In this subsection, we will present the formalism of cosmological perturbation theory and derive the linearized equations for the evolution of scalar perturbations.

We will consider linear perturbations of the FLRW spacetime, significant only up to first order. Therefore, at linear order the Scalar–Vector–Tensor (SVT) decomposition ensures that the Einstein equations for scalar, vector, and tensor perturbations will remain decoupled, ensuring that they can be analyzed independently. We will implement the Newtonian gauge for our analysis, where ($B = E = 0$), in which the perturbed metric becomes

$$ds^2 = a^2(\eta)(1 + 2\Psi)d\eta^2 - (1 - 2\Phi)\delta_{ij}dx^i dx^j \quad (6.45)$$

where $\eta = dt/a$, is the conformal time, Ψ is the gravitational potential and Φ is a local perturbation of the scale factor a . Lets us know perturb the energy momentum tensor of the cosmic fluid defined in Eq. (6.3)

$$T^\mu{}_\nu = \bar{T}^\mu{}_\nu + \delta T^\mu{}_\nu. \quad (6.46)$$

In a perturbed universe, the energy density ρ , the pressure P and the four-velocity U^μ can be functions of position. Furthermore, the energy momentum tensor may now have a contribution from anisotropic stress, $\Pi^\mu{}_\nu$. The perturbed energy momentum tensor is

$$\delta T^\mu{}_\nu = (\delta\rho + \delta P)U^\mu U_\nu + (\rho + P)(\delta U^\mu U_\nu + U^\mu \delta U_\nu) - \delta P\delta^\mu{}_\nu - \Pi^\mu{}_\nu. \quad (6.47)$$

We will assume that the anisotropic stress is negligible $\Pi^i_j = 0$, and that $\delta T^\mu_\nu = 0$, $\mu \neq \nu$ since we assume that the perturbed fluid remains a perfect fluid. Lets us now consider small perturbations of the Einstein equations

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu}. \quad (6.48)$$

First, we need to calculate the perturbed Christoffel symbols. Using Eq. (2.4) and (6.45) we obtain

$$\Gamma^0_{00} = \mathcal{H} + \Psi', \quad (6.49)$$

$$\Gamma^0_{i0} = \partial_i \Psi, \quad (6.50)$$

$$\Gamma^i_{00} = \delta^{ij} \partial_j \Psi, \quad (6.51)$$

$$\Gamma^i_{j0} = \mathcal{H} - \Phi' \delta^i_j, \quad (6.52)$$

$$\Gamma^0_{ij} = \mathcal{H} \delta_{ij} - \Phi' + 2\mathcal{H}(\Phi + \Psi) \delta_{ij}, \quad (6.53)$$

$$\Gamma^i_{jk} = -2\delta^i_{(j} \partial_{k)} \Phi + \delta_{jk} \delta^{il} \partial_l \Phi \quad (6.54)$$

where $\mathcal{H} = aH$, the Hubble parameter expressed in terms of the conformal time η and the prime notes the derivative with respect to the conformal time. From the conservation of the energy momentum tensor

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha = 0, \quad (6.55)$$

we can obtain the perturbed conservation equations. At this point it will be convenient to define the fractional overdensity

$$\delta \equiv \frac{\delta\rho}{\rho}. \quad (6.56)$$

Substituting the perturbed energy-momentum tensor Eq. (6.47), the fractional overdensity Eq. (6.56) and Christoffel symbols Eq. (6.49)-(6.54) to the $\nu = 0$ part of Eq. (6.55) one derives the perturbed continuity equation

$$\delta' + 3\mathcal{H}(c_s^2 - w)\delta + (1 + w)(\nabla v - 3\Phi') = 0, \quad (6.57)$$

where we have set the speed of sound $c_s^2 = \frac{\delta P}{\delta\rho}$ and the equation of state parameter $w = \frac{P}{\rho}$. Repeating the same procedure for the spatial part $\mu = \nu$ one obtains the

perturbed Euler equation

$$v' + \mathcal{H}(1 - 3w)v = -\frac{c_s^2}{1 + w}\nabla\delta - \nabla\Psi. \quad (6.58)$$

Let us now proceed to the calculation of the perturbed Einstein equations. By plugging the perturbed Christoffel symbols Eq. (6.49)-(6.54) to Eq. (2.6) we can calculate the non-zero components of the perturbed Ricci tensor

$$R_{00} = -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'', \quad (6.59)$$

$$R_{0i} = 2\partial_i(\Phi' + \mathcal{H}\Psi), \quad (6.60)$$

$$R_{ij} = \mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'\delta_{ij} + \partial_i\partial_j(\Phi - \Psi). \quad (6.61)$$

Furthermore, the Ricci scalar is

$$R = g^{00}R_{00} + g^{ij}R_{ij} = \frac{1}{a^2}(1 - 2\Psi)R_{00} - (1 + 2\Phi)\delta^{ij}R_{ij} \quad (6.62)$$

Combining the above equations, the (00) component of the Einstein tensor is

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi', \quad (6.63)$$

while the (0i) is

$$G_{0i} = R_{0i} - \frac{1}{2}g_{0i}R = 2\partial_i(\Phi' + \mathcal{H}\Psi), \quad (6.64)$$

and the (ij) components are given by

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\ &= -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + \nabla^2(\Psi - \Phi) + 2\Phi'' \\ &\quad + 2(2\mathcal{H}' + \mathcal{H}^2)(\Phi + \Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\delta_{ij} + \partial_i\partial_j(\Phi - \Psi). \end{aligned} \quad (6.65)$$

We will now substitute the derived components of the perturbed Einstein tensor Eqs. (6.63),(6.64),(6.65) and of the perturbed energy momentum tensor Eq. (6.47) into

the perturbed Einstein equation Eq. (6.48) and since the anisotropic stress was set to zero, we can assume that $\Phi = \Psi$. The 00 equation yields

$$\nabla^2 \Phi = 4\pi G a^2 \rho \delta + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi), \quad (6.66)$$

while the 0i equation to

$$(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 (\rho + P)v \quad (6.67)$$

similarly the (ij) part of the equation

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta P. \quad (6.68)$$

Considering sub-horizon scales and taking the time derivative of (6.57) and the divergence of (6.58) one obtains

$$\delta'' + (1 + 3w)\mathcal{H}\delta' - c_s^2 \nabla^2 \delta = (1 + w)\nabla^2 \Phi. \quad (6.69)$$

On the sub-horizon scales, time derivatives of Φ can be ignored. Considering the matter specie, $w = 0$ and $c_s^2 \ll 1$ and taking into account (6.66), (6.69) we finally get the matter overdensity evolution equation

$$\delta_m'' + \mathcal{H}\delta_m' - 4\pi G a^2 \rho_m \delta_m \approx 0. \quad (6.70)$$

The above linear equation has been proven useful for studying the distribution of matter in the Universe. In the case of MTG we can get an analogous equation, where the Newtonian G is replaced by a G_{eff} .

6.4. Derivation of Friedmann Equations from Spacetime Thermodynamics

In [194] Cai and Kim begin with an $(n + 1)$ -dimensional Friedmann–Robertson–Walker (FRW) universe, whose metric is given by

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right), \quad (6.71)$$

where $a(t)$ is the scale factor, k denotes the spatial curvature (with $k = +1, 0, -1$ corresponding to closed, flat, and open universes, respectively), and $d\Omega_{n-1}^2$ is the line element on an $(n-1)$ -dimensional unit sphere. A convenient rewriting introduces the areal radius $\tilde{r} = a(t)r$ and the two-dimensional metric $h_{ab} = \text{diag}(-1, a^2/(1-kr^2))$ on the (t, r) subspace

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2. \quad (6.72)$$

The apparent horizon is defined by the condition

$$h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0, \quad (6.73)$$

which leads to the expression for the apparent horizon radius

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \quad (6.74)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter. One can see that for a flat Universe $k = 0$, the apparent horizon coincides with the Hubble horizon $r_H = 1/H$.

Following Cai [194] one may consider the work done by a change in the apparent horizon as the work density at the apparent horizon

$$W = -\frac{1}{2} T^{ab} h_{ab}, \quad (6.75)$$

where T^{ab} is the projection of the energy momentum tensor of a perfect fluid in the direction normal to $(n-1)$ -dimensional sphere. Furthermore, one can define the total energy flow through the apparent horizon by the energy-supply vector

$$\Psi_a = T_a^b \partial_b \tilde{r} + W \partial_a \tilde{r}. \quad (6.76)$$

In [332, 333] it has been shown that the following generalization of the *first law* holds for the total energy

$$\nabla E = A\psi + W\nabla V, \quad (6.77)$$

where A and V are the area and volume of the n dimensional space of radius $\tilde{r}$ and the total energy inside the space is defined by

$$E = \frac{n(n-1)\Omega_n}{16\pi G} \tilde{r}^{n-2} \left(1 - h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}\right). \quad (6.78)$$

The energy supply vector term can be written as

$$A\Psi = \frac{\kappa}{8\pi G} \nabla A + \tilde{r}^{n-2} \nabla \left(\frac{E}{\tilde{r}^{n-2}} \right), \quad (6.79)$$

where the surface gravity is defined as

$$\kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \tilde{r}). \quad (6.80)$$

It can be shown that at the apparent horizon only the first term of the right side of Eq. (6.79) survives. Assuming the standard form of the energy momentum tensor of the Universe content

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}, \quad (6.81)$$

the energy supply vector at the apparent horizon becomes

$$\Psi_A = \left[-\frac{1}{2}(\rho + p)H\tilde{r}, \frac{1}{2}(\rho + p)a \right], \quad (6.82)$$

then by combining Eqs. (6.77), (6.79), (6.80) and (6.82) one can find that the energy crossing the apparent horizon, over an infinitesimal time interval dt , is given by

$$-dE = A\Psi = A(\rho + p)H\tilde{r}_A dt. \quad (6.83)$$

According to thermodynamics the heat flow δQ changes the energy of a system by dE and it is related to the change of entropy by $\delta Q = TdS$. During the time interval dt the heat flow through the apparent horizon is the amount of energy crossing it, hence $\delta Q = -dE$ is the change of energy inside the apparent horizon. Therefore, one can consider the first law of thermodynamics on the apparent horizon in the form

$$-dE = TdS. \quad (6.84)$$

In the case that the entropy is proportional to the horizon area according to Bekenstein–Hawking entropy Eq. (5.15)

$$S = \frac{A}{4G}, \quad (6.85)$$

where the area of the apparent horizon is

$$A = n\Omega_n \tilde{r}_A^{n-1}, \quad (6.86)$$

with Ω_n being the volume of an n –dimensional unit ball. The temperature is defined via the surface gravity κ at the apparent horizon as $T = \kappa/2\pi$ as the Hawking–Unruh temperature Eqs. (5.39),(5.41). For the dynamical horizon, one may show that the temperature is

$$T = \frac{1}{2\pi\tilde{r}_A}, \quad (6.87)$$

where the approximation $\kappa \approx -1/\tilde{r}_A$ is used, which is valid when the horizon is treated as nearly static during an infinitesimal time interval.

Substituting the expressions for T and dS , and noting that the change in the entropy is related to the change in the horizon area, one arrives at a differential equation that relates the time derivative of the Hubble parameter to the matter content

$$\dot{H} - \frac{k}{a^2} = -\frac{8\pi G}{n-1}(\rho + p). \quad (6.88)$$

This equation corresponds to the second Friedmann equation in $(n+1)$ dimensions.

To complete the derivation, we will invoke the continuity (or energy conservation) equation for the perfect fluid

$$\dot{\rho} + nH(\rho + p) = 0. \quad (6.89)$$

Substituting the relation for $H(\rho + p)$ from the energy flux calculation into the differential equation above and integrating, one obtains the first Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{16\pi G}{n(n-1)}\rho. \quad (6.90)$$

It is important to note that during the procedure an integration constant appears that has been absorbed into the fluid density ρ . Thus, the integration constant can be regarded as the cosmological constant Λ .

In summary, the derivation by Cai and Kim shows that by identifying the apparent horizon in a FLRW universe, assigning appropriate thermodynamic quantities (with entropy $S = A/4G$ and temperature $T = 1/(2\pi\tilde{r}_A)$), computing the energy flux across the horizon, and applying the first law of thermodynamics ($-dE = T dS$), one obtains a differential equation relating $\dot{H}$ to ρ and p . Together with the continuity equation, this procedure yields the Friedmann equations, thereby establishing a deep connection between the dynamics of spacetime and thermodynamic laws. This approach supports the view that gravitational dynamics may be emergent from underlying thermodynamic or statistical principles.

Chapter 7.

Dark energy from topology change at the foam level

7.1. Introduction

It is known that the appearance of microscopic objects with distinct topologies and different Euler characteristics, such as instantons and wormholes, at the spacetime-foam level in Euclidean quantum gravity approaches, leads to spacetime topology changes. Such changes, in principle, may affect the field equations that arise through the semiclassical variation procedure of gravitational actions. Although in the case of Einstein–Hilbert action the presence of microscopic wormholes does not lead to any non-trivial result, when the Gauss–Bonnet term is added in the gravitational action, the above effective topological variation procedure induces an effective cosmological constant that depends on the Gauss–Bonnet coupling and the wormhole density. Since the latter in a dynamical spacetime is in general time-dependent, one obtains an effective dark energy sector of topological origin.

In the framework of Euclidean quantum gravity there are solutions such as instantons and wormholes, which exhibit different topology from the background [264]. Wormholes have also been investigated in astrophysics [334–339]. If one assumes that these objects appear at the spacetime-foam level [340], then the spacetime background becomes topologically dynamical. The effects of topology changes have been thoroughly studied in the literature. For instance, in [341] it was shown that topology change in classical Lorentzian spacetime manifolds leads to the development of singularities, and based on this result, Anderson and DeWitt argued that quantum field

theory is inconsistent with such singularities [263]. Additionally, in the framework of quantum gravity, the feasibility of topology change has been supported by several investigations [342–345]. In [346] Sorkin argued that topology change is required in order for quantum gravity to be consistent, while in [264] Gibbons showed that in the context of Euclidean Quantum Gravity the Wick rotation of the time coordinate to the imaginary plane reflects a change of signature and thereby a topology change of the manifold describing a real tunneling geometry. Moreover, there have been recent suggestions that quantum gravity topology change is connected with Perelman’s Ricci flow [347, 348], while in [259] it was argued that a topology change cannot be forbidden due to “censorship” theorems, since it is a well understood feature of string theory in 2D and 10D [265–268]. Finally, in [269, 349] the evolution of the topology was illustrated in loop quantum gravity with topspin network formalism.

Among others, topology changes may in principle affect the field equations that arise through the semiclassical variation procedure of gravitational actions. Although in the case of Einstein–Hilbert action this procedure reproduces the standard field equations, one could investigate whether variation of the Gauss–Bonnet action on a topologically altered spacetime due to the formation of microscopic wormholes could lead to a non-trivial result. Interestingly enough, such an extended analysis induces extra terms in the field equations, which can be interpreted as an effective dark energy sector of topological origin.

This chapter is organized as follows: In section 7.2 we briefly review the topology change in the framework of Euclidean quantum gravity induced by objects with distinct topology, and in section 7.3 we present the effective topological variation procedure. Then in section 7.4 we apply this procedure, and we derive the semiclassical field equations for the cases of Einstein–Hilbert and Gauss–Bonnet actions, while in section 7.5 we show the appearance of an effective dark energy induced by microscopic Gauss–Bonnet wormholes. Finally, section 9.4 is devoted to the conclusions

7.2. Topology change in Euclidean Quantum Gravity

In this section, we discuss the effects of topology change in the framework of Euclidean quantum gravity (EQG). In EQG context, in order for the complex path integral to converge, the time dimension is Wick rotated $t \rightarrow i\tau$, thus the Lorentzian signature $(-+++)$ changes to Euclidean $(++++)$. In the complex path integral there are

saddle point solutions, which correspond to classical solutions, namely instantons, with different topology from the background [298,340] and therefore they mediate topology change [264]. These solutions can represent the creation of a pair of black holes or Euclidean wormholes under a strong field, as in Schwinger process [258,264]. In the sum over history approach, Sorkin [346] has developed a calculus for topology change based on Morse theory, where the transition between two manifolds of distinct topology is being performed by a Morse function provided a cobordism exists between the manifolds [344,350].

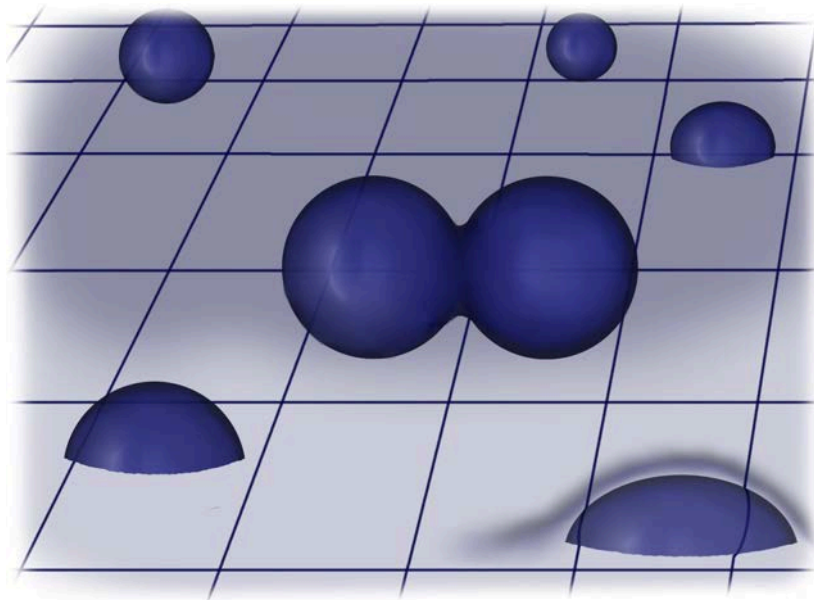


Figure 7.1.: A three-dimensional illustration of the the instanton solutions appearing at the spacetime foam level, credits to Eva Theodoratou for the design.

The topological structure of a manifold M is characterized by topological indices, and one of the most extensively studied is the Euler characteristic $\chi(M)$. In the modern language of differential forms and in the context of de Rham Cohomology, the Euler characteristic χ is defined as the alternating sum of the Betti numbers of the manifold M [351]

$$\chi(M) = \sum_p (-1)^p B_p, \quad (7.1)$$

where the Betti numbers B_p of a manifold are defined as the dimension of the p^{th} de Rham cohomology group [351]

$$B_p = \dim H^p(M). \quad (7.2)$$

In the above expression the p^{th} de Rham cohomology group $H^p(M)$ is the set of all closed p -forms $Z^p(M)$ modulo the set of all exact p -forms $B^p(M)$

$$H^p(M) = Z^p(M) / B^p(M), \quad (7.3)$$

where a closed form satisfies $d\omega = 0$ (where ω is a p -form and d denotes the exterior derivative), and an exact form satisfies $\omega = dn$ (where n is a p -form).

The Poincaré Lemma states that a closed form defined on a domain $V \subseteq M$ is also exact, if the domain V is contractible to a point. In the light of Poincaré Lemma, de Rham Cohomology can be seen as a restriction on the global exactness of closed forms [351]. In summary, Betti numbers measure the global inexactness of closed forms as obstructions to contractability to a point, originated from holes and discontinuities of the domain [352].

In Table 7.1 we present the value of Euler characteristics for different spacetime manifolds.

Spacetime	Euler characteristic χ
Minkowski	0
Extreme Black Holes	0
Self-dual Taub–Newman–Unti–Tamburino	1
Schwarchild and Kerr Black Holes	2
Nariai $S_2 \times S_2$	4
Euclidean Wormhole $S_1 \times S_3$	0

Table 7.1.: Euler characteristics for different spacetime manifolds as it has been calculated in [298, 353–356]. By the product property for product manifolds $\chi(M_1 \times M_2) = \chi(M_1) \cdot \chi(M_2)$, one can easily verify that for a Nariai instanton $\chi_{Na} = 2 \cdot 2 = 4$ and similarly for a Euclidean wormhole $\chi_{EW} = 0 \cdot 3 = 0$.

In order to investigate the topology change, one can decompose a 4D manifold M into a connected sum (symbolized by $\#$) of two 4D manifolds M_1 and M_2 , by gluing them together at the boundaries left by the removal of a four-ball. For connected sums, the Euler characteristic is given by [351] and it has been illustrated in Fig. 7.2

$$\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2. \quad (7.4)$$

Then, following Gibbons [264, 357], the formation of a Euclidean wormhole with

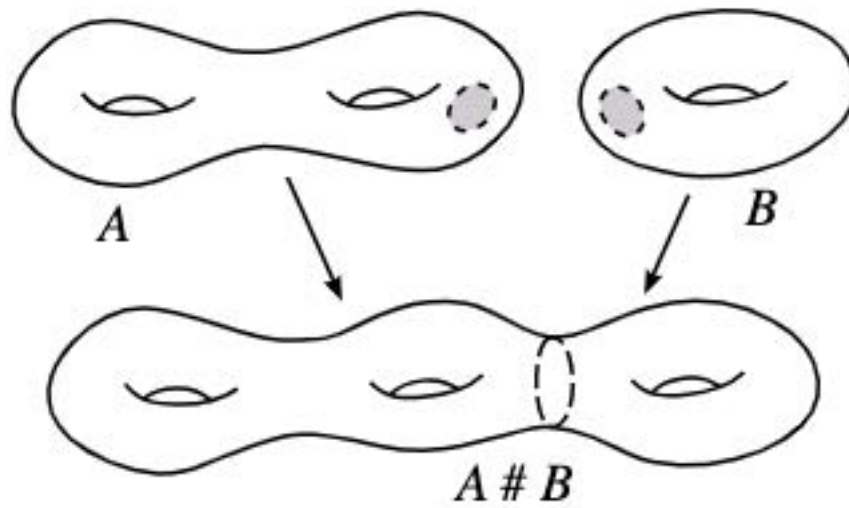


Figure 7.2.: An illustration for the connected sums procedure.

topology $(S_1 \times S_3)$, namely

$$M \rightarrow M \# (S_1 \times S_3), \quad (7.5)$$

decreases χ by 2, thus $\delta\chi = -2$, while the formation of a Nariai instanton with topology $(S_2 \times S_2)$, namely

$$M \rightarrow M \# (S_2 \times S_2), \quad (7.6)$$

increases χ by 2 thus $\delta\chi = 2$.

Since the formation of gravitational instantons or wormholes change the Euler characteristic of the 4D spacetime, one must examine its effect in a systematic way. For a 4D spacetime manifold, equation (7.1) becomes [352]

$$\chi = b_0 - b_1 + b_2 + b_3, \quad (7.7)$$

with b_0 the number of connected components, b_1 the number of one-dimensional holes, b_2 the number of two-dimensional holes, and b_3 the number of three-dimensional holes. Hence, when the Euler characteristic changes, there is a change in the Betti numbers of spacetime [277], which corresponds to a change in the dimension of De Rham Cohomology group and therefore to a change in the proportion of exact to closed forms. The induced non-triviality of De Rham Cohomology group indicates that a closed form exists which is no longer exact. Therefore, by the Poincaré Lemma, the induced inexactness implies that there is an area of the manifold, a wormhole, which is not contractible to a point. In summary, the formation of a wormhole changes the topology and this property will be exploited in the following.

7.3. The effective topological variation procedure

Inspired by Wheeler's conceptualization of spacetime foam [358], where quantum fluctuations of the metric are considered to cause fluctuations of the topology of the spacetime manifold, which was later developed in the context of Euclidean quantum gravity by Hawking, Gibbons, Sorkin and others [340, 359, 360], we are interested in investigating the behavior of the variation of higher-order gravitational actions under the assumption that the variation of the quantum field fluctuations δh causes a variation in the topology of the spacetime manifold $\delta\chi$.

We consider that the process of signature change of the spacetime manifold by the Wick rotation and the Euclidean quantum gravity path integral convergence contour deformation, which yields instanton solutions of different spacetime topology, can be encapsulated into an effective topology change operation (eff_{TC}), as illustrated in Fig. 7.3. Specifically, for manifolds with metric g_i and Euler characteristics χ_i , it can be encapsulated into the variation of the gravitational quantum field δh , namely

$$\begin{aligned} M(g_1, \chi_1) &\xrightarrow{eff_{TC}} M'(g_2, \chi_2), \\ eff_{TC} : \delta h &\longrightarrow \delta\chi. \end{aligned} \quad (7.8)$$

One could argue that EQG topology change is a mathematical artifact, a byproduct of the Wick rotation and of the integral contour deformation. However, there have been many studies supporting that topology change occurs in other quantum gravity approaches too, for instance in string theory [265–268] and loop quantum gravity

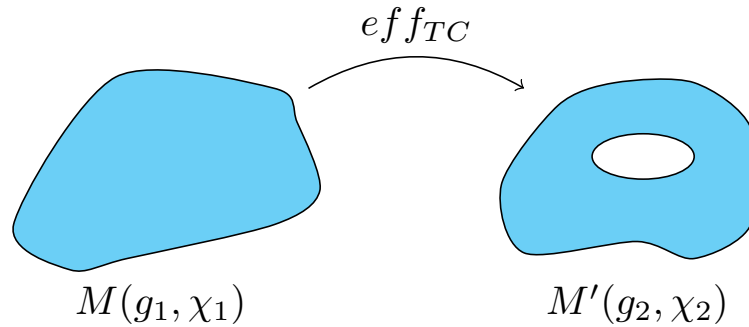


Figure 7.3.: A three-dimensional illustration of the effective topology change from a manifold of Euler characteristic χ_1 to a manifold of Euler characteristic χ_2 .

[269, 349]. Therefore, it seems likely that topology change is a generic feature of quantum gravity [259, 346]. An additional objection could be that processes which change the manifold topology may lead to the development of singularities, in the cases where they give rise to discrete and/or non-differentiable submanifolds, which could make the variational calculus ambiguous. However, in 3D one can introduce novel surgery techniques that successfully treat the singularities that develop in 3D manifolds during their Ricci flow [361], and thus one could in principle follow the same procedure in 4D manifolds. Nevertheless, we should comment here that since 4D topologies remain unclassifiable [262], the path integral approach is still not fully well-defined mathematically, and since topology change can emerge by the path integral approach [264], it is interesting to explore the implications of relation (7.8) as long as it may lead to interesting novel physical results. We will investigate a variational formulation of topology changes in a separate work.

Let us examine the above consideration in more detail. A common technique in many approaches to quantum gravity is to split linearly the full metric g into a background metric $\tilde{g}$ and the quantum fluctuation field h around it [362]

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} + h^{\mu\nu}. \quad (7.9)$$

According to Wheeler's argument [240, 363] and similar more recent ones [364], the quantum fluctuations are scale-dependent as $\delta h \sim \frac{l_p}{l}$, and they become large near the Planck scale, remaining always smaller than one as long as we consider the Planck scale as cut-off. However, from the viewpoint of asymptotic safety, one could argue that interactions could become weak at the Planck scale by an appropriate renormalization-group flow [364]. If one follows the first consideration, fluctuations may induce topology change, nevertheless higher-order terms in the expansion could be non-

negligible. If one follows the second consideration, then classical expansion techniques can be employed, but ambiguities arise on the ability of fluctuations to induce topology change. Since a solid theory of quantum gravity remains far from being complete, in this work we assume that a compromise exists between the two extremes, suggesting that quantum fluctuations can produce topology change at small scales, while being small enough in order for perturbation theory to hold. Consequently, this scaling approach facilitates a reduction in the significance of higher-order terms, thereby allowing the phenomena to be predominantly described by the first-order term.

As mentioned in [365], the quantum fluctuation field $h = g - \tilde{g}$ of the linear split is not a metric and lacks a geometrical meaning, therefore among the other types of split, it is the most suited for describing the fluctuation field that causes topology change. Consequently, we consider $\tilde{g}_{\mu\nu}$ to be the dynamical background metric, and we handle $h_{\mu\nu}$ as an effective “matter” field.

We close this section by commenting on the background independence and the tadpole condition. In particular, in most treatments of quantum gravity there are strong arguments in favor of the independence of the quantum effective action from the background, since physical observables must be background independent and the action must be diffeomorphism invariant. By demanding background independence, one can introduce split symmetry [365], given by all the transformations of the background metric and fluctuation fields that preserve the full metric, namely

$$g(\tilde{g}, h) \rightarrow g(\tilde{g} + \delta\tilde{g}, h + \delta h) = g(\tilde{g}, h). \quad (7.10)$$

In the quantization scheme of [366], background independence is guaranteed by the class of metrics that are self-consistent. Self-consistent metrics $\tilde{g}^{SC}$ are those that allow the effective field equations obtained from the effective action $\Gamma[h_{\mu\nu}, \tilde{g}]$ to admit the solution $h_{\mu\nu} = 0$. Thus, if one incorporates background independence into the extremization condition of the effective action, then one obtains the tadpole condition [366]

$$\left. \frac{\delta}{\delta h_{\mu\nu}} \Gamma[h, \tilde{g}] \right|_{h=0, \tilde{g}=\tilde{g}^{SC}} = 0. \quad (7.11)$$

7.4. Effective cosmological constant of topological origin

We now have all the machinery to perform the variation of gravitational actions in cases where there are topology changes in the underlying spacetime manifold. Firstly, we will apply the procedure in the case of Einstein–Hilbert action, which leads to a trivial result, and then we will apply it in the case of the Gauss–Bonnet modified action, where we will see the surprising result of the appearance of an effective cosmological constant.

7.4.1. Semiclassical approach for the Einstein–Hilbert action

Let us start by presenting the semiclassical approach described above in the case of Einstein–Hilbert action, as it has been demonstrated firstly by ‘t Hooft in [309] and it has been presented analytically in 4.3. By performing a Wick rotation, the spacetime signature becomes $(+, +, +, +)$ and the action will be Euclideanized, i.e.

$$S_{EH} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{g} R, \quad (7.12)$$

with κ^2 the gravitational constant. We expand the Einstein–Hilbert action around the background field according to the metric split (7.9), in orders of the “quantum field”, namely

$$S_{EH} = S_0 + S_1 + S_2 + \sum_{n=3}^{\infty} S_n. \quad (7.13)$$

In order to calculate each term we expand the inverse metric as

$$g^{\mu\nu} = \tilde{g}^{\mu\nu} - h^{\mu\nu} + h_{\lambda}^{\mu} h^{\lambda\nu} + \mathcal{O}(h^3), \quad (7.14)$$

and then by employing the property $\log \det A = \text{tr} \log A$ and performing logarithmic and exponential expansions, we express the determinant of the metric in terms of powers of h as

$$\sqrt{g} = \sqrt{\tilde{g}} \left[1 + \frac{1}{2} \tilde{g}_{\mu\nu} h^{\mu\nu} - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} + \frac{1}{8} (h_{\mu}^{\mu})^2 + \mathcal{O}(h^3) \right], \quad (7.15)$$

where the uppering and lowering of the quantum field indices are performed using the background metric, namely $h = h_\mu^\mu = \tilde{g}^{\mu\nu} h_{\mu\nu}$. Since the Ricci scalar is $R = g^{\mu\nu} R_{\mu\nu}$, by employing (7.14) and (7.15) the Ricci tensor and Ricci scalar can be expanded in the same manner (one can see a more detailed derivation in A.1). After some algebra and neglecting the total derivatives, the first three terms of the Einstein–Hilbert expansion are expressed as

$$\begin{aligned}
S_0 &= -\frac{1}{2\kappa^2} \int d^4x \sqrt{\tilde{g}} \tilde{R} \\
S_1 &= \frac{1}{2\kappa^2} \int d^4x \sqrt{\tilde{g}} \left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) h^{\mu\nu} \\
S_2 &= -\frac{1}{2\kappa^2} \int d^4x \sqrt{\tilde{g}} \left\{ \frac{1}{4} h^{\mu\nu} \nabla^2 h_{\mu\nu} - \frac{1}{8} h \nabla^2 h \right. \\
&\quad + \frac{1}{2} \left(\nabla^\nu h_{\nu\mu} - \frac{1}{2} \nabla_\mu h \right)^2 + \frac{1}{2} h^{\mu\lambda} h^{\nu\sigma} \tilde{R}_{\mu\lambda\nu\sigma} \\
&\quad \left. + \frac{1}{2} \left(h^{\mu\lambda} h_\lambda^\nu - h h^{\mu\nu} \right) \tilde{R}_{\mu\nu} + \frac{1}{8} \left(h^2 - 2h^{\mu\nu} h_{\mu\nu} \right) \tilde{R} \right\}. \tag{7.16}
\end{aligned}$$

In summary, the effective action up to one-loop approximation will be

$$\Gamma = S_{EH} + \Gamma_{1L} + \mathcal{O}(2-loop), \tag{7.17}$$

where the quadratic terms of the quantum field h are absorbed in the one-loop part

$$\Gamma_{1L} = \Gamma_{GF} + \Gamma_{Fgh}, \tag{7.18}$$

with Γ_{GF} and Γ_{Fgh} corresponding to the effective action for the gauge fixing and ghost terms respectively [366]. One can then vary the Einstein–Hilbert action due to quantum fluctuations of the field $h^{\mu\nu} \rightarrow h^{\mu\nu} + \delta h^{\mu\nu}$, i.e. calculate $\delta_h S_{EH}$. Imposing the tadpole condition (7.11) for the effective action (7.17) and taking into account (7.16), one finally retrieves the Einstein equations for the classical background as [366, 367]

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = \kappa^2 T_{\mu\nu}, \tag{7.19}$$

where the stress tensor originates from the one-loop part of the effective action, containing matter as correction terms in the right-hand-side, i.e.

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta}{\delta h^{\mu\nu}} \Gamma_{1L} \big|_{h=0}. \quad (7.20)$$

Actually, this was expected, since the Einstein–Hilbert action term is the Euler density in two-dimensions, and thus its variation due to variations of the quantum field h in 4D will be the standard one [290].

7.4.2. Semiclassical approach for the Gauss–Bonnet action

Let us now perform the above procedure in the case of the Gauss–Bonnet action. The Gauss–Bonnet (GB) curvature polynomial $\mathcal{G}$ is defined as

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad (7.21)$$

and it is known that in four dimensions such a term is a topological invariant. In order to see this in the context of the present manuscript, we recall that the Chern–Gauss–Bonnet theorem [368] states that for the case of a compact orientable manifold M with boundary ∂M of dimension $D = 4$, the Euler characteristic is

$$\chi(M) = \frac{1}{32\pi^2} \int_M d^4x \sqrt{g} \mathcal{G} + \int_{\partial M} Q, \quad (7.22)$$

with Q an appropriate correction form integrated on the boundary ∂M [296]. The essence of the theorem is that despite any local deformation of the manifold, its total curvature, as expressed by the integral of the GB curvature polynomial, depends only on the topology of the manifold. Consequently, for a manifold of fixed topology, χ is considered a topological invariant under smooth variations of the metric [291].

We can now perform the steps of the previous subsections in the case of the Gauss–Bonnet action. Its Euclideanized form is

$$S_{GB} = -\frac{\alpha}{2\kappa^2} \int d^4x \sqrt{g} \left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \right), \quad (7.23)$$

where α is the coupling parameter. The effective action under the one-loop approximation will be

$$\Gamma = S_{GB} + \Gamma_{1L} + \mathcal{O}_{(2-Loop)}. \quad (7.24)$$

Varying the GB action with respect to quantum fluctuations of the field $h^{\mu\nu} \rightarrow h^{\mu\nu} + \delta h^{\mu\nu}$ according to the topological variational procedure and applying the 4D Chern–Gauss–Bonnet theorem (7.22) without a boundary, we obtain

$$\begin{aligned} \delta_h S_{GB} &= \delta_h \left[-\frac{\alpha}{2\kappa^2} \int_M d^4x \sqrt{g} \left(R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right] \\ &= -32\pi^2 \frac{\alpha}{2\kappa^2} \frac{\delta\chi}{\delta h^{\mu\nu}} \delta h^{\mu\nu}. \end{aligned} \quad (7.25)$$

Implementing the substitution $\delta\chi \rightarrow \partial\chi$ and applying the chain rule, we find

$$\begin{aligned} \delta_h S_{GB} &= -16\pi^2 \frac{\alpha}{2\kappa^2} \frac{\partial\chi}{\partial V} \frac{\delta V}{\delta h^{\mu\nu}} \delta h^{\mu\nu} \\ &= -16\pi^2 \frac{\alpha}{2\kappa^2} \frac{\partial\chi}{\partial V} \delta_h \left(\int_M d^4x \sqrt{g} \right) \\ &= -16\pi^2 \frac{\alpha}{2\kappa^2} \frac{\partial\chi}{\partial V} \int_M d^4x \frac{\delta\sqrt{g}}{\delta h^{\mu\nu}} \delta h^{\mu\nu}, \end{aligned} \quad (7.26)$$

with V the manifold volume. If one implements the expansion of the metric determinant (7.15), then the functional derivative into the integral becomes

$$\frac{\delta\sqrt{g}}{\delta h^{\mu\nu}} = \frac{1}{2} \sqrt{\tilde{g}} \tilde{g}_{\mu\nu} + \mathcal{O}(h), \quad (7.27)$$

so after inserting it into (7.26) we finally acquire

$$\delta_h S_{GB} = -16\pi^2 \frac{\alpha}{2\kappa^2} \frac{\partial\chi}{\partial V} \int_M d^4x \sqrt{-\tilde{g}} \tilde{g}_{\mu\nu} \delta h^{\mu\nu}. \quad (7.28)$$

Finally, we make the reasonable approximation that for an infinitesimal integration volume the topology change per volume $\frac{\partial\chi}{\partial V}$ remains constant, and thus it can enter inside the integral, in which case the topological variation of the Gauss–Bonnet term is

expressed as

$$\frac{1}{\sqrt{\tilde{g}}} \frac{\delta S_{GB}}{\delta h^{\mu\nu}} = -16\pi^2 \frac{\alpha}{2\kappa^2} \frac{\partial \chi}{\partial V} \tilde{g}_{\mu\nu} + \mathcal{O}(h). \quad (7.29)$$

Interestingly enough, the variation of the Gauss–Bonnet term on a manifold that has topology changes due to the formation of wormholes is not zero.

7.4.3. Semiclassical approach for the Einstein–Gauss–Bonnet action

Let us now consider the full case of general relativity plus a Gauss–Bonnet correction, namely

$$S_{tot} = S_{EH} + S_{GB}. \quad (7.30)$$

As we analyzed above, although the Einstein–Hilbert term gives the standard classical field equations, the Gauss–Bonnet term leads to a non-trivial semi-classical result. In particular, the effective action under the one-loop approximation will be

$$\Gamma = S_{EH} + S_{GB} + \Gamma_{1L} + \mathcal{O}_{(2-Loop)}, \quad (7.31)$$

calculating the functional derivative of the effective action by employing Eq. (7.16) and Eq. (7.29), we finally obtain

$$\begin{aligned} \frac{1}{\sqrt{\tilde{g}}} \frac{\delta \Gamma}{\delta h^{\mu\nu}} &= \frac{1}{\sqrt{\tilde{g}}} \frac{\delta S_{EH}}{\delta h^{\mu\nu}} + \frac{1}{\sqrt{\tilde{g}}} \frac{\delta S_{GB}}{\delta h^{\mu\nu}} + \frac{1}{\sqrt{\tilde{g}}} \frac{\delta \Gamma_{1L}}{\delta h^{\mu\nu}} \\ &= \frac{1}{2\kappa^2} \left\{ \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} - 16\pi^2 \alpha \frac{\partial \chi}{\partial V} \tilde{g}_{\mu\nu} + \frac{1}{\sqrt{\tilde{g}}} \frac{\delta \Gamma_{1L}}{\delta h^{\mu\nu}} + \mathcal{O}(h) \right\}. \end{aligned} \quad (7.32)$$

Hence, imposing the tadpole condition (7.11) that removes the terms $\mathcal{O}(h)$, we find the semi-classical field equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} + \tilde{g}_{\mu\nu} \Lambda_{eff} = \kappa^2 T_{\mu\nu}, \quad (7.33)$$

where the stress tensor is given by (7.20), and where we have defined

$$\Lambda_{eff} \equiv -16\pi^2 \alpha \frac{\partial \chi}{\partial V}. \quad (7.34)$$

As we observe, we have obtained an effective cosmological constant term of topological origin, induced by the Gauss–Bonnet correction term due to the topology change that microscopic wormholes brought about. This is the main result of the present work.

7.5. Dark energy from microscopic Gauss–Bonnet wormholes

As we showed in the previous sections, the variation of gravitational actions in cases where one has topology changes in the underlying spacetime manifold, may lead to extra terms in the field equations. Although in the case of Einstein–Hilbert action one does not obtain any non-trivial result, incorporating the Gauss–Bonnet correction on such topologically-changed manifolds gives rise to an effective cosmological constant, even in 4D where the GB term is known to have no effect at the classical level. As we discussed, according to the literature, such topological changes can typically arise when objects like instantons or wormholes are formed at the microscopic level.

In particular, the effective cosmological constant is just the term $\frac{\partial \chi}{\partial V}$, which can be interpreted as the density of the non-trivial microscopic objects per four-volume $\rho_{obj} = \frac{N_{obj}}{V}$, since these objects induce the topology change (for instance $\delta \chi = -2$ corresponds to the formation of a Euclidean wormhole, while $\delta \chi = 2$ to the formation of a Nariai instanton). Hence, according to (7.34), the effective cosmological constant equals the density ρ_w of N_w topology changing wormholes per four-volume, i.e.

$$\Lambda_{eff} = -16\pi^2 \alpha \rho_w = -16\pi^2 \alpha \frac{N_w}{V}, \quad (7.35)$$

namely it depends on the GB coupling parameter α and on the wormhole density.

Concerning the value of GB coupling, there is a consensus that, since the GB term appears in the low-energy limit of an effective action [178], α is related to the inverse of the string tension [175, 369, 370] $\alpha \sim (1/\sqrt{a'})$ or equivalently to the square of the string scale $\alpha \sim l_s^2$ [290, 371]. Since the string scale l_s cannot be far from the Planck scale in four dimensions $l_s \sim l_p$ [372], one first estimation could be $\alpha = l_p^2$. In such a case, if we identify the effective cosmological constant Λ_{eff} of (7.35) with the observed cosmological constant $\Lambda_{obs} = 10^{-52} m^{-2}$, we need a microscopic wormhole density of $\rho_w = 10^{16}$ wormholes per cubic meter per second, which is quite reasonable according

to Hawking and Schulz estimations for the spacetime foam [263, 277, 340]. On the other hand, since the upper bound for the wormhole density is one wormhole per Planck volume, namely $\rho_{Mw} = \frac{1}{l_p^4} \sim 10^{140}$, according to (7.35) the upper bound of Λ_{eff} is $\Lambda_M \sim 10^{72} m^{-2}$, or approximately 10^{124} larger than Λ_{obs} .

We stress here that the wormhole density in a dynamical spacetime is not expected to be constant, therefore the obtained effective cosmological constant also acquires a dynamical nature, i.e. it corresponds to an effective dark energy sector.

Note that in different frameworks there have also been approaches where the cosmological constant is driven by spacetime wormholes, but they typically have $\Lambda \rightarrow 0$ at late times. For instance, in [340] Hawking considers space time foam as a gas of instantons of different topology and in the Euclidean quantum gravity one-loop approximation he obtains a negative cosmological constant $\Lambda_s \sim \alpha \frac{\chi}{V}$, which although having been extracted in a totally different framework, it resembles our result (7.34). However, Hawking's calculations are based on the trace anomaly expressed by the invariant GB term, and for that reason χ appears constant (refinements of Hawkings spacetime foam model were presented in [232, 277]). In [246], Coleman proposed a mechanism where wormholes and topological fluctuations of space time induce a distribution of the values of nature's constants, which smear Λ -distribution to peak at zero. In [373, 374] it was claimed that the behavior of the fundamental coupling constants in Coleman's scenario was controlled by the trace anomaly and a similar proposition was the Giddings–Strominger wormhole solution where a wormhole is coupled to an instanton [257]. Additionally, in [258] a semiclassical model of spacetime foam was proposed, in which Casimir-like quantum fluctuations give rise to an arbitrary number of wormholes, as pairs of black-hole and anti-black-hole, which drive the induced cosmological constant to zero as they grow.

Nevertheless, in our approach microscopic wormholes lead to an effective dark energy sector not directly, but due to the topology change they induce on the manifold, which in turn affects the variation of the GB term. That is why it can have an arbitrary dynamical behavior. Additionally, apart from its effects at late-time cosmology, such a dynamical effective sector could play a role in the early universe too, potentially driving inflation. Finally, note that since Nariai instantons correspond to a negative component while Euclidean wormholes to a positive one, one could have richer cosmological behavior as well.

7.6. Conclusions

It is known that the appearance of microscopic objects, such as instantons and wormholes, at the spacetime-foam level in Euclidean quantum gravity approaches, leads to spacetime topology changes, which in principle may affect the field equations that arise through the variational procedure of gravitational actions. Although in the case of Einstein–Hilbert action the presence of microscopic wormholes does not lead to any non-trivial result, when the Gauss–Bonnet term is added in the action the above procedure induces an effective cosmological term that depends on the Gauss–Bonnet coupling and the wormhole density. Since the latter in a dynamical spacetime is in general time-dependent, one results with an effective dark energy sector of topological origin.

In particular, the appearance of objects with distinct topology and thus with different Euler characteristics, leads to a change of the topological character of the spacetime manifold. This process can be encapsulated into an effective approach in which the variation of the quantum fluctuations induces a variation in the Euler characteristic, constituting the effective topological variation procedure. By employing the semiclassical one-loop approach on the linear split of the metric and, additionally, incorporating the background independence through the tadpole condition, we showed that the variation of the Gauss–Bonnet term in the Lagrangian gives rise to a non-trivial term in the field equations. The obtained effective cosmological constant can coincide with the observed value $10^{-52} m^{-2}$ for densities of the order of 10^{16} microscopic wormholes per cubic meter per second, which is quite reasonable according to estimations.

It would be interesting to consider scenarios of time-dependent wormhole density and investigate the behavior of the resulting dynamical dark energy sector, including the confrontation with observational data from Supernovae Type I (SNIa), Baryonic Acoustic Oscillations (BAO), and Cosmic Microwave Background (CMB) observations, as well as with direct Hubble constant measurements through cosmic chronometers (CC). Additionally, one could examine the matter perturbation evolution in such a dynamical scenario. Moreover, one could apply the same considerations at early times and examine the possibility of a successful inflation realization. At the more theoretical level, one could investigate the effective topological variation procedure going beyond the linear expansion level, as well as examine the effect of a topologically dynamical GB term in the trace anomaly behavior, in heterotic strings renormalizability, and in M-theory's β -function.

Chapter 8.

Cosmological implications of Spacetime Foam Topological Dark Energy

We propose a novel cosmological scenario based on microscopic topological anomalies emerging from quantum effects in classical spacetime. In the framework of Euclidean Quantum Gravity, gravitational instantons induce topological transitions at the spacetime foam level. When the latter appear, then the Gauss–Bonnet action term in $4D$ becomes non-trivial and gives rise to an effective cosmological constant proportional to the density of instantons. Using quantum field theory techniques, we determine the evolution of instanton density in a cosmological background and derive the corresponding modified Friedmann equations. We confront this scenario with observational data, including SN Ia, BAO, and cosmic chronometers, and find it to be significantly preferred over the standard Λ CDM model by CC + SN Ia + SH0ES + BAO datasets. The proposed scenario describes a complete mechanism for the origin and dynamics of Dark Energy, while exhibits Dark Energy to Dark Matter conversion. Finally, it offers the possibility of a joint solution of σ_8 and H_0 tensions. Hence, the scenario of topological dark energy may represent a compelling competitor to Λ CDM paradigm.

According to overwhelming observations of various origins, the Universe entered the phase of accelerated expansion in the recent cosmological past [83–88]. The simplest explanation is the introduction of a positive cosmological constant Λ , that corresponds to a constant dark energy density ρ_Λ . Therefore, dark energy must be created as the Universe expands in order to compensate the dilution. A promising candidate for explaining this behavior has been the vacuum, since as spacetime expands more vacuum should be created, nevertheless quantum field theoretical analysis predict a value up to 120 orders of magnitude larger than the observed one [90, 91]. This

notorious discrepancy is known as "the cosmological constant problem". Furthermore, the introduction of Λ results to a cosmic evolution scenario where the present is an extremely special moment of transition from matter to dark energy dominance. This fine-tuning is considered the "coincidence problem". Furthermore, the concordance model, namely Λ CDM, suffers from observational discrepancies such as the H_0 and σ_8 tensions [375, 376]. On the other hand, modified gravity, particularly higher-order curvature terms such as the Gauss–Bonnet (GB) one, can address these issues and are known to arise naturally in string and M-theory [104, 218, 220].

Another attempt to study the quantum behavior of gravity at the Planck scale have led to the concept of spacetime foam [240, 241], where quantum fluctuations give rise to transient topological features such as virtual black holes and wormholes. Spacetime foam has been explored within various approaches to quantum gravity as Loop Quantum Gravity (LQG) [242], string theory [243] and Euclidean Quantum Gravity (EQG) [231, 246].

In the framework of EQG there are solutions such as instantons and wormholes, which exhibit different topology from the background [264]. If one assumes that these objects appear at the spacetime-foam level [313], then spacetime becomes topologically dynamical. Recently, it has been proposed that the behavior of the GB action term at the foam level is not trivial and yields a effective cosmological constant of topological origin which is proportional to the density of gravitational instantons [166, 167].

In this chapter, we intend to explore the cosmological evolution of the density of instantons and confront the theoretical predictions with observations. By implementing standard Quantum Field Theory (QFT) techniques, for the nucleation rate of instantons, we derive the differential equation for the dimensionless dark energy density, $\Omega_{\Lambda_{\text{eff}}}$, of the scenario at hand. Interestingly enough, in the context of likelihood analysis, we find the model to be preferred over Λ CDM by the CC/Pantheon+/SH0ES/BAOs dataset. This chapter is organized as follows: At Sect. 8.1 we construct the novel Spacetime Foam Topological Dark Energy (SFTDE) scenario and we provide the corresponding modified Friedman equations. At Sect. 8.2 we present the methods and datasets that we use in order to check the observational viability of SFTDE scenario, the corresponding results and some implications. At Sect. 8.3 we discuss and comment our findings, while we present other possible implications. Finally, at Sect. 8.4 we outline our conclusions.

8.1. Spacetime Foam Topological Dark Energy

8.1.1. The main concept

Recently in [166, 167] it has been proposed, that DE may originate from topology changes of spacetime foam. Specifically, when the standard Einstein–Hilbert action is extended up to the Gauss–Bonnet term, the appearance of gravitational instantons of alternating topology, as predicted by Euclidean Quantum Gravity (EQG), cause a non-trivial dynamicalization of the GB term. If one splits the metric to the background metric and a quantum fluctuation, i.e. $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$, assumes that the quantum fluctuations encapsulate the EQG procedure of topology change and thus causing a change in the Euler characteristic of spacetime $\delta h \rightarrow \delta\chi$, and moreover employs the semiclassical approximation, then he obtains the Einstein equation for the background

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} + \Lambda_{\text{eff}}\tilde{g}_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (8.1)$$

with an effective cosmological constant Λ_{eff} proportional to the rate of spacetime topology change per four volume

$$\Lambda_{\text{eff}} = -16\pi^2 \alpha \frac{\partial\chi}{\partial V}, \quad (8.2)$$

where α is the GB coupling constant. According to the standard interpretation of EQG, topologically non-trivial gravitational instantons appear at the foam level, causing a change in the topology of spacetime, which can be evaluated by the connected sums formula [264]

$$\chi(M_{\text{init}} \# M_{\text{inst}}) = \chi(M_{\text{init}}) + \chi(M_{\text{inst}}) - 2, \quad (8.3)$$

thus

$$\delta\chi(M) = \chi(M_{\text{inst}}) - 2. \quad (8.4)$$

In Table 8.1 we observe that each instanton species produces a distinct change $\delta\chi_i$ which can be positive or negative. In the case of black holes $\delta\chi = 0$, therefore, the contribution of microscopic black holes in the topology change at the foam level can be neglected. Hence, according to Eq. (8.2), Λ_{eff} can be estimated as the weighted sum density of the non-trivial microscopic objects per four-volume, $n_{\text{obj}(i)} = N_{\text{obj}(i)}/V$ with

Spacetime	Euler characteristic $\delta\chi = w$	
	χ	
FRW	0	
Extreme Black Holes	0	-2
Self-dual Taub–NUT	1	-1
Schwarzschild and Kerr BHs	2	0
Nariai $S_2 \times S_2$	4	2
Euclidean Wormhole $S_1 \times S_3$	0	-2

Table 8.1.: Euler characteristics for different spacetime manifolds as it has been calculated in [298, 353–356]. By the product property for product manifolds $\chi(M_1 \times M_2) = \chi(M_1) \cdot \chi(M_2)$, one can easily verify that for a Nariai instanton $\chi_{Na} = 2 \cdot 2 = 4$ and similarly for a Euclidean wormhole $\chi_{EW} = 0 \cdot 3 = 0$. In the last row $\delta\chi = w$ has been calculated by Eq. (8.4)

weight $w_i = \delta\chi_i$

$$\Lambda_{\text{eff}} = -16\pi^2\alpha \sum_i w_i n_i \quad (8.5)$$

In summary, by bringing together the EQG spacetime topology change at the foam level, with the GB topological action term, we can acquire the Einstein field equation with an effective Λ_{eff} term that is proportional to the density of the topology alternating instantons. Assuming the general non-flat FRW metric at cosmological scales and then following the standard procedure, we can obtain the two Friedmann equations from Eq. (8.1)

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_{\text{DE}}) - \frac{k}{a^2} \quad (8.6)$$

$$3H^2 + 2\dot{H} + \frac{k}{a^2} = -8\pi G(p_r + p_{\text{DE}}), \quad (8.7)$$

where the energy density of the dynamical DE sector is $\rho_{\text{DE}} = \Lambda_{\text{eff}}/8\pi G$, and where we have included curvature, the energy density and pressure of matter and radiation.

Let us mention that other Dark Energy models have been proposed based upon topological aspects, i.e. [272, 377, 378]. In [272, 379] by assuming that the total topology remains constant, compensating topological effects appear, at the foam level [272]

and at the apparent horizon [379] respectively, that compensate the astrophysical BHs topological impact. In contrast, at [377] the cosmological constant is topologically linked to the volume of the boundary of the Universe which is attributed to the volume of BHs. A crucial difference is that the current Topological Dark Energy scenario is based on topology change induced by quantum gravity effects, thus it is dynamic by nature. In [272,377] Λ is static and in [379] the model induces just an extra DE term to the standard Λ . In contrast, the current scenario provides a complete micro-physical explanation for the entire DE sector.

8.1.2. Model formulation

In this work, we aim to formulate the evolution of the effective cosmological constant within a Friedman–Robertson–Walker (FRW) Universe, based on the aforementioned concepts. Since Λ_{eff} is proportional to the density of instantons, which can in principle vary with time, we calculate the evolution of the latter in a cosmological background. The appropriate theoretical framework for describing the nucleation process of gravitational instantons originates from bubble nucleation theory [380]. The latter has been developed to describe the tunneling process from false to true vacuum [381,382] and it has been a part of Quantum Field Theory [383]. More recently, it has been applied to the nucleation process of black holes and gravitational instantons [384,385]. Specifically, the probability per unit volume per unit time for an instanton to occur is given by

$$\Gamma = Ae^{-B}, \quad (8.8)$$

where $B = \Delta I_E$ is the difference in the Euclidean action between the instanton (tunneling) configuration and the initial background configuration [279,381,382,386,387]. The quantity A has been calculated at Ref. [381,382] where the vacuum is represented by a scalar field, nevertheless in the studies of nucleations of black-holes and gravitational instantons [384,385] it has been neglected. Since a tunneling process corresponds to an instanton, the Γ rate can be interpreted as the density of instantons per four volume, therefore we acquire

$$n_i \equiv \Gamma = A_i \exp(-B_i), \quad (8.9)$$

with i running on instanton species. We calculate the exponential factor $B = \Delta I$ in the case where the gravitational action is the Einstein–Hilbert plus the Gauss Bonnet

$$I = I_{EH} + I_{GB} = \frac{1}{16\pi G} \left(\int d^4x \sqrt{g} R + \alpha \int d^4x \sqrt{g} \mathcal{G} \right). \quad (8.10)$$

Hence, the action difference ΔI for each type of instanton consists of a topological and a geometrical contribution $\Delta I = \Delta I_{GB} + \Delta I_{EH}$. Applying the Chern–Gauss–Bonnet theorem,

$$\chi(M) = \frac{1}{32\pi^2} \int d^4x \sqrt{g} \mathcal{G}, \quad (8.11)$$

and taking into consideration that the Euler characteristic of the FRW background is zero ($\chi_{FRW} = 0$), we find that the topological contribution for each instanton specie (i) is given by

$$\Delta I_{GB}(i) = \frac{2\pi\alpha\chi_i}{G}. \quad (8.12)$$

Since gravitational instantons are vacuum solutions, their Ricci scalar is $R_{inst} = 4\Lambda$ [385]. On the other hand, the FRW background metric has a Ricci scalar $R_{back} = 6(2H^2 + \dot{H} + k/a^2)$. In our approach, we consider instantons nucleated at the foam level inside a cosmological background, therefore we assume the same four-volume of integration for the background and the instantons defined from the FRW metric of Lorentzian signature, thus the geometrical part $\Delta I_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} (R_{inst} - R_{back})$ is the same for all instantons. In order to capture the cosmological evolution of n we choose the spatial part of the four volume to be a Hubble sphere of radius $r = 1/H$. Substituting the results of the geometrical and topological contribution to Eq. (8.9) and subsequently to Eq. (8.5), we obtain an integral equation for Λ_{eff} :

$$\Lambda_{eff} = -16\pi^2\alpha \sum_i w_i A_i \exp \left(-\frac{2\pi\alpha\chi_i}{G} \right) \exp \left[\frac{1}{12G} \int_0^t dt \frac{a^3}{H^3} \left(12H^2 + 6\dot{H} + 6\frac{k}{a^2} - 4\Lambda_{eff} \right) \right]. \quad (8.13)$$

The above expression can be transformed into a solvable ansatz by differentiating each side with respect to time. By assuming that A_i is not dependent of the cosmic time, we

obtain the differential equation for the effective cosmological constant as

$$\frac{d\Lambda_{\text{eff}}}{dt} = \frac{1}{12G} \frac{a^3}{H^3} \left(12H^2 + 6\frac{dH}{dt} + 6\frac{k}{a^2} - 4\Lambda_{\text{eff}} \right) \Lambda_{\text{eff}}. \quad (8.14)$$

In the case of dynamic dark energy, a convenient expression for the dimensionless Hubble rate is the following

$$E(z) = \left[\frac{\Omega_{r0}(1+z)^4 + \Omega_{m0}(1+z)^3 + \Omega_{k0}(1+z)^2}{1 - \Omega_{\Lambda_{\text{eff}}}(z)} \right]^{1/2}, \quad (8.15)$$

furthermore, we will change the integration variable as $dt = -dz(1+z)^{-1}H^{-1}$, substitute $\Lambda_{\text{eff}} = 3\Omega_{\Lambda_{\text{eff}}}H^2$, $k/a^2 = -\Omega_{k0}H_0^2(z+1)^2$, $H(z) = H_0E(z)$, in order to obtain the differential equation of the model with respect to the dimensionless quantities. After some simple algebra, we arrive at

$$\begin{aligned} \frac{d\Omega_{\Lambda_{\text{eff}}}(z)}{dz} = & (\Omega_{\Lambda_{\text{eff}}}(z) - 1)\Omega_{\Lambda_{\text{eff}}}(z) \left[4GH_0^2(z+1)^5 f_1(z) \right. \\ & \cdot f_2(z) + (1 - \Omega_{\Lambda_{\text{eff}}}(z)) \cdot (\Omega_{m0}(z+1) - 4(f_1(z) - \Omega_{k0}) \\ & \cdot \Omega_{\Lambda_{\text{eff}}}(z)) \left. \right] \cdot \left[(z+1)f_1(z)(4GH_0^2(z+1)^5 f_1(z) \right. \\ & \left. \left. - (1 - \Omega_{\Lambda_{\text{eff}}}(z))\Omega_{\Lambda_{\text{eff}}}(z)) \right]^{-1} \end{aligned} \quad (8.16)$$

where for convenience we have defined

$$f_1(z) \equiv \Omega_{k0} + (z+1) [\Omega_{m0} + \Omega_{r0}(z+1)] \quad (8.17)$$

$$f_2(z) \equiv 2\Omega_{k0} + (z+1) [3\Omega_{m0} + 4\Omega_{r0}(z+1)]. \quad (8.18)$$

The above equation, (8.16), is a 1st order ODE where the unknown function is $\Omega_{\Lambda_{\text{eff}}}(z)$. At $z=0$, the normalization condition $E(z=0) = 1$ imposes $\Omega_{\Lambda}(z=0) = 1 - \Omega_{m0} - \Omega_{r0} - \Omega_{k0}$. Let us mention that the scenario at hand is derived from first principles, where DE is produced as more gravitational instantons appear quantum mechanically as the Universe expands. In particular, we study two specific models, the “flat SFTDE”, i.e. by setting $\Omega_{k0} = 0$ at the aforementioned equations and “non-flat SFTDE”, where $\Omega_{k0} \neq 0$.

Finally, we are going to derive the DE equation of state parameter. Starting from the continuity equation for the DE species, assuming that the cosmic substance (i.e. matter, radiation etc) are conserved separately, $\dot{\rho}_{\text{DE}} + 3H(1+w)\rho_{\text{DE}} = 0$, transforming time to redshift and substituting $\rho_{\text{DE}} = 8\pi G/3\Omega_{\Lambda_{\text{eff}}}H^2$ one obtains the expression for the dark energy equation of state parameter

$$w_{\text{DE}}(z) = -1 + \frac{(1+z)}{3} \left[\frac{d \ln \Omega_{\Lambda_{\text{eff}}}(z)}{dz} + 2 \frac{d \ln H(z)}{dz} \right]. \quad (8.19)$$

The study of the SFTDE cosmological scenario introduced above, i.e. Eqs. (8.15), (8.16) is the central aim of this work.

8.2. Observational aspects

In this Section, we discuss observational consequences of the proposed topological Dark Energy scenario. Obviously, the complete study of the phenomenology goes well beyond the scope of this work. However, we perform detailed observational fits in order to *compare* SFTDE scenario with the concordance one in terms of its fitting quality. In addition, given the observational constraints, we delve into the physics of the novel SFTDE scenario, and we discuss the implications on the cosmic history.

8.2.1. Data & Methodology

Here we present the methods we use in order (a) to constrain the free parameters of the model at hand, collectively expressed as ϕ^λ , and (b) to assess the fitting quality of the model in comparison with the concordance one. We use late universe data from Supernovae Ia (SNIa) observations, direct measurements of the Hubble function, i.e. Cosmic Chronometers (CC) and data from Baryonic Acoustic Oscillations (BAOs).

Given S independent observational datasets and assuming Gaussian errors, the total likelihood function is defined as:

$$\mathcal{L}_{\text{tot}}(\phi^\lambda) \sim \prod_{p=1}^{\mathcal{P}} \exp[-\chi_p^2(\phi^\psi)], \quad (8.20)$$

where the corresponding expression for χ_{tot}^2 is written as

$$\chi_{\text{tot}}^2 = \sum_{p=1}^{\mathcal{P}} \chi_p^2. \quad (8.21)$$

Note that the statistical χ^2 function must not be confused with the Euler characteristic, χ , defined at Sect. 8.1. The statistical vector has a dimension of k , comprising ν parameters of the model under consideration plus ν_i *intrinsic* parameters from the utilized data sets, yielding $k = \nu + \nu_i$. In the above expression ϕ^μ is the vector containing the free parameters.

Lastly, to obtain the posterior distributions of the model parameters, given the data, we use a Markov Chain Monte Carlo (MCMC) sampler. Instead of the standard Metropolis–Hastings algorithm, to avoid the need of fine-tuning for the hyperparameters, we use an affine-invariant Markov Chain Monte Carlo sampler as implemented within the open-source Python package *emcee* [388, 389], involving 800 “walkers” (chains) and 2500 “states” (steps). Regarding the convergence of the MCMC algorithm, we use the traditional Gelman–Rubin criterion and also the auto-correlation time analysis.

In the assessment of cosmological models based on their predictions with respect to the available data, we employ three widely recognized criteria: the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the Deviance Information Criterion (DevIC), i.e. [390] and references therein.

The AIC criterion addresses the issue of model adequacy from an information theory perspective. Specifically, it serves as an estimator of the Kullback–Leibler information and possesses the property of asymptotic unbiasedness. Under the standard assumption of Gaussian errors, the AIC estimator is expressed as [390]:

$$\text{AIC} = -2 \ln(\mathcal{L}_{\text{max}}) + 2k + \frac{2k(k+1)}{N_{\text{tot}} - k - 1}. \quad (8.22)$$

Here, $\mathcal{L}_{\text{max}}$ represents the maximum likelihood of the considered data set(s), and N_{tot} is the total number of data points. For large N_{tot} , the expression simplifies to $\text{AIC} \simeq -2 \ln(\mathcal{L}_{\text{max}}) + 2k$, which corresponds to the standard form of the AIC criterion. Consequently, it is considered a good practice to utilize the modified AIC criterion in all cases [390].

The BIC serves as an estimator of the Bayesian evidence, and its expression is given by:

$$\text{BIC} = -2 \ln(\mathcal{L}_{\max}) + k \cdot \log(N_{\text{tot}}). \quad (8.23)$$

On the other hand, the DevIC is formulated by incorporating concepts from both Bayesian statistics and information theory [390]. It is expressed as:

$$\text{DevIC} = D(\overline{\phi^\mu}) + 2C_B, \quad (8.24)$$

with $C_B = \overline{D(\phi^\mu)} - D(\overline{\phi^\mu})$ the Bayesian complexity, where the overline represents the mean value. Moreover, $D(\phi^\mu)$ corresponds to the Bayesian Deviation, which, for a general class of distributions (the exponential family), is given by $D(\phi^\mu) = -2 \ln(\mathcal{L}(\phi^\mu))$. This is closely tied to the effective degrees of freedom [390], representing the number of parameters that actually contribute to the fitting.

In the task of ranking a set of competing models based on their fitting quality to observational data, we employ the aforementioned criteria, specifically focusing on the relative difference in Information Criterion (IC) values within the given model set. The difference $\Delta \text{IC}_{\text{model}} = \text{IC}_{\text{model}} - \text{IC}_{\min}$, allows us to assess the fitting quality between the competitive models using the Jeffreys scale [391]. In particular, when $\Delta \text{IC} \leq 2$ the model is statistically compatible with the most favored model by the data, when $2 < \Delta \text{IC} < 6$ implies a moderate tension between the two models, while the case $\Delta \text{IC} \geq 10$ indicates a significant tension.

Supernovae Ia (SNIa)

One of the first objects regarding cosmological applications of standard candles in cosmology is Type Ia supernovae (SNe Ia). We incorporate the full Pantheon+ sample [392], which expands upon the original Pantheon dataset by almost duplicating the sample size, improving the treatment of redshifts and peculiar velocities [393] and greatly increasing the redshift range. The peculiar velocity treatment is particularly crucial for SNIa in the nearby Universe and relevant criticism has been developed regarding the Pantheon sample [394]. A key feature of the Pantheon+ dataset is the inclusion of multiple observations of the same SNe Ia from different surveys, with redundancy properly accounted for through an appropriate covariance matrix. At [395] SH0ES cepheids data from 37 galaxies, along with NGC 4258, LMC and the

Milky Way parallaxes, are used to calibrate the zero point of the SN Ia brightness scale. These calibrated distances to SN Ia hosts define the absolute magnitude $\mathcal{M}$ for SNe Ia, which is then used to anchor the Hubble diagram. The aforementioned high level data product along with the code needed to construct distance moduli can be found at [396]. The dataset contains $N_{\text{SNIa}} = 1702$ data points within the redshift range $0.00122 \lesssim z \lesssim 2.27$.

The chi-square function for this dataset is expressed as:

$$\chi_{\text{SNIa}}^2(\phi^v) = \mathbf{ffi}_{\text{SNIa}}^{-1} \mathbf{C}_{\text{SNIa},\text{cov}}^{-1} \mathbf{ffi}_{\text{SNIa}}^T, \quad (8.25)$$

The vector $\mathbf{ffi}_{\text{SNIa}}^{-1}$ is defined as $\delta\mu_{\text{SNIa}} = \{\mu_1 - \mu_{\text{th}}(z_1, \phi^v), \dots, \mu_N - \mu_{\text{th}}(z_N, \phi^v)\}$. The distance modulus is defined as $\mu_i = \mu_{B,i} - \mathcal{M}$, where $\mu_{B,i}$ is the observed apparent magnitude at maximum in the rest frame for redshift z_i .

The extra free parameter, $\mathcal{M}$, is essential to the analysis, as the observable distance modulus, μ_{obs} , depends on modeling assumptions and crucially, the fiducial cosmology [393]. Additionally, the theoretical form of the distance modulus is given by:

$$\mu_{\text{th}} = 5 \log \left[\frac{d_L(z)}{\text{Mpc}} \right] + 25, \quad (8.26)$$

with the luminosity distance defined as:

$$d_L(z) = c(1+z) \frac{1}{\sqrt{|\Omega_{k0}|}} \Phi \left(\sqrt{|\Omega_{k0}|} \int_0^z \frac{1}{H(\omega, \phi^v)} d\omega \right), \quad (8.27)$$

where

$$\Phi(x) = \begin{cases} \sinh(x), & \Omega_{k0} > 0 \\ x, & \Omega_{k0} = 0 \\ \sin(x), & \Omega_{k0} < 0 \end{cases} \quad (8.28)$$

Note that $\mathcal{M}$ and the normalized Hubble constant h exhibit intrinsic degeneracy within the context of the Pantheon dataset and the inclusion of the SH0ES prior allows the breaking of the latter.

Cosmic Chronometers (CC)

We compile and use a new dataset of $H(z)$ data points, based on [397]. The aforementioned dataset has been built upon two considerations. First, we select data points that have been extracted via the differential age of passive evolving galaxies method e.g. [398, 399] and references therein). This is done in order to minimize the dependency of the data points on the assumed fiducial cosmology. Indeed, the latter method is almost unaffected by cosmological assumptions, except the hypothesis of flat FRW geometry. Secondly, we take into account the fact that authors at Ref. [400] were not able to reproduce the CC data points given by [401]. For the benefit of the reader, we present the dataset at Tab. 8.2, along with the relevant references per data point.

Our analysis incorporates a total of $N_{CC} = 24$ measurements of the Hubble rate, covering the redshift range $0.07 \lesssim z \lesssim 2$.

The corresponding χ_H^2 function is expressed as:

$$\chi_{CC}^2(\phi^\nu) = \mathbf{ffH} \mathbf{C}_{CC,\text{cov}}^{-1} \mathbf{ffH}^T, \quad (8.29)$$

with $\mathbf{ffH} = \{H_1 - H_0 E(z_1, \phi^\nu), \dots, H_N - H_0 E(z_N, \phi^\nu)\}$ where H_i are the observed Hubble rates at redshift z_i ($i = 1, \dots, N$). The covariance matrix has been defined at [410] according to [399] and reads as:

$$\mathbf{C}_{CC,\text{cov}} \equiv \mathbf{C}^{\text{stat}} + \mathbf{C}^{\text{met}} + \mathbf{C}^{\text{young}} + \mathbf{C}^{\text{model}} \quad (8.30)$$

Some words regarding the origin of the aforementioned terms are in place. First, the $\mathbf{C}^{\text{stat}}$ is related with the correlations between the extracted data points due to the employed fitting procedure. Secondly, the $\mathbf{C}^{\text{met}}$ corresponds to the uncertainty for the stellar metallicity estimation. Thirdly, the matrix $\mathbf{C}^{\text{young}}$ is diagonal and represents the contribution to the error budget arising from the assumption that the observed galaxy spectrum is solely dominated by “old” stars. Finally, the term $\mathbf{C}^{\text{model}}$ describes the contribution to the total covariance matrix coming from modeling assumptions, which can be further decomposed as:

$$\mathbf{C}^{\text{model}} = \mathbf{C}^{\text{SFH}} + \mathbf{C}^{\text{IMF}} + \mathbf{C}^{\text{stellar lib.}} + \mathbf{C}^{\text{SPS}} \quad (8.31)$$

No.	z	H [$km/Mpc s^{-1}$]	σ [$km/Mpc s^{-1}$]	Ref.
1	0.070	69.0	19.6	[402]
2	0.120	68.6	26.2	[402]
3	0.1791	75.0	4.0	[403]
4	0.1993	75.0	5.0	[403]
5	0.200	72.9	29.6	[402]
6	0.280	88.8	36.6	[402]
7	0.3519	83.0	14.0	[403]
8	0.3802	83.0	13.5	[404]
9	0.4004	77.0	10.2	[404]
10	0.4247	87.1	11.2	[404]
11	0.4497	92.8	12.9	[404]
12	0.47	89.0	50.0	[405]
13	0.4783	80.9	9.0	[404]
14	0.480	97.0	62.0	[406]
15	0.5929	104.0	13.0	[403]
16	0.6797	92.0	8.0	[403]
17	0.7812	105.0	12.0	[403]
18	0.80	113.1	44.2	[407]
19	0.8754	125.0	17.0	[403]
20	0.880	90.0	40.0	[406]
21	1.037	154.0	20.0	[403]
22	1.26	135.0	65.0	[408]
23	1.363	160.0	33.6	[409]
24	1.965	186.5	50.4	[409]

Table 8.2.: Cosmic Chronometers dataset

z	D	σ_D	$r_{d,\text{fid}}$ (Mpc)	Ref
0.122	539.0 Mpc	17.0 Mpc	147.5	[412]
0.44	1716.40 Mpc	83.00 Mpc	148.6	[413]
0.60	2220.80 Mpc	101.00 Mpc	148.6	[413]
0.73	2516.00 Mpc	86.00 Mpc	148.6	[413]
0.32	6.67	0.13	148.11	[414]
0.57	9.52	0.19	148.11	[414]
1.19	12.6621	0.9876	147.78	[415]
1.50	12.4349	1.0429	147.78	[415]
1.83	13.1305	1.0465	147.78	[415]
2.40	36.0	1.2	-	[416]

Table 8.3.: Baryonic Acoustic Oscillations dataset.

where C^{SFH} has to do with the assumptions regarding the star formation history, the initial mass function C^{IMF} , the stellar library used $C^{\text{stellar lib.}}$ and finally, C^{SPS} has to do with the stellar population synthesis model used.

Baryonic Acoustic Oscillations

Baryon Acoustic Oscillations (BAO) are the result of acoustic waves propagating in the primordial photon-baryon plasma, driven by gravitational instabilities in the early Universe. Prior to recombination, i.e. before redshift $z \sim 1100$), baryons and photons were tightly coupled via Thomson scattering, and overdensities in the matter distribution led to outward-propagating sound waves. When the Universe cooled enough for electrons and protons to form neutral hydrogen, photons decoupled and streamed freely (producing the cosmic microwave background), while the baryon velocity field stalled. This left a residual overdensity at a characteristic co-moving scale—the sound horizon at the drag epoch, r_d , which is imprinted as a preferred separation scale in the late-time matter correlation function and galaxy clustering. BAOs data thus provide a standard ruler that can be used to jointly constrain the Hubble expansion rate $H(z)$ and angular diameter distance $D_A(z)$, making them a powerful probe for DE, see e. g [411].

The relevant χ_{BAO}^2 is given by:

$$\chi_{\text{BAO}}^2 = \mathbf{D} \mathbf{C}_{\text{BAOs, cov}}^{-1} \mathbf{D}^T, \quad (8.32)$$

where $\mathbf{f} \mathbf{D} = \{D(z_1, \phi^\omega) - d_{1, \text{obs}}, \dots, D(z_N, \phi^\omega) - d_{N, \text{obs}}\}$, and $\mathbf{C}_{\text{cov}}^{-1}$ is the corresponding covariance matrix. In this expression, $D(z_i, \phi^\omega)$ is defined as follows:

$$\lambda(z_i, \phi^\omega) = \begin{cases} D_V r_{d, \text{fid}} / r_d, & 1 \leq i \leq 4 \\ D_A / r_d, & 5 \leq i \leq 9 \\ D_M / r_d, & i = 10 \end{cases}, \quad (8.33)$$

where

$$D_M = \frac{D_L(z, \phi^\omega)}{1 + z}, \quad (8.34a)$$

$$D_A = \frac{D_L(z, \phi^\omega)}{(1 + z)^2}, \quad (8.34b)$$

$$D_V = \left[\frac{c D_A^2 z (1 + z)^2}{H(z, \phi^\omega)} \right]^{1/3}. \quad (8.34c)$$

The quantity r_d is the standard ruler, representing a characteristic length scale of over-densities in the distribution of matter. For the concordance Λ CDM model, BAOs originate from sound waves in the early Universe, and r_d is equal to the co-moving sound horizon r_s at the redshift of the baryon drag epoch. However, for other cosmological models, the origin of the standard ruler r_d might differ [417]. The general approach is to treat r_d as a free parameter, rather than calculating it within the Λ CDM context or marginalizing over it [418]. Table 8.3 lists the BAO data and references, with the last column indicating $r_{d, \text{fid}}$, the sound horizon of the fiducial model used to extract the data. Covariances are found in [413], [414], and [415].

Par./Par. combination	Prior
Ω_{m0}	$\mathcal{U}(0.001, 1.0)$
Ω_{k0}	$\mathcal{U}(-1.0, 1.0)$
h	$\mathcal{U}(0.60, 0.90)$
$\mathcal{M}$	$\mathcal{U}(-19.9, -18.0)$
r_d / Mpc	$\mathcal{U}(110, 170)$

Table 8.4.: Priors used in the likelihood analysis.

8.2.2. Data analysis results

Using the data and the methodology described at Sect. 8.2.1, we sample the likelihood function for the topological DE models presented previously. Specifically, we employ the following free parameters/datasets compilations:

- $\phi^\mu = \{\Omega_{m0}, h, \mathcal{M}\}$ and $\mathcal{P} = \{CC, SNIa + SH0ES\}$.
- $\phi^\mu = \{\Omega_{m0}, h, \mathcal{M}, r_d\}$ and $\mathcal{P} = \{CC, SNIa + SH0ES, BAOs\}$.
- $\phi^\mu = \{\Omega_{m0}, h, \mathcal{M}, r_d\}$ and $\mathcal{P} = \{SNIa + SH0ES, BAOs\}$.

We employ flat priors for the free parameters, as presented at Tab. 8.4, with the additional constraint of $\Omega_{m0} + \Omega_{k0} \in (0, 1.0)$. We depict 2D slices of the posterior distributions of the free parameter values for the non-flat and flat SFTDE models at Fig. 8.5, 8.6, respectively, for the case of CC/Pantheon+/SH0ES, CC/Pantheon+/SH0ES/BAOs and CC/Pantheon+/SH0ES/BAOs datasets.

At Tab. 8.5, we summarize the parameter estimation results obtained for the proposed topological dark energy (SFTDE) scenario, both in flat and non-flat spatial geometries, compared against the standard flat Λ CDM cosmological model, along with the 1σ intervals. The latter have been calculated via quantiles on the posterior parameter distributions. Before assessing the physical implications of the results, some consistency checks are in place. We compare the Ω_{m0} values for the concordance Λ CDM presented above with the PLANCK Ω_{m0} value [419] and we report 1σ compatibility within all the dataset compilations considered. Note the appearance of the H_0 tension, while smaller values for the sound horizon at the baryon drag epoch, r_d , also appear. This tendency towards lower r_d values has also been reported elsewhere, whenever the SH0ES data are employed, e.g [420, 421]. For example, authors at [420]

report $r_d = (136.1 \pm 2.7) \text{ Mpc}$, which is consistent with our respective results within $1 - \sigma$. The Hubble constant, for all models considered, stays around $h \sim 0.73$, i.e. it is governed by the SH0ES prior, as expected.

Regarding CC/Pantheon+/SH0ES dataset combination, substantial differences arise in the inferred values of the matter density parameter, Ω_{m0} , with regard to the concordance model. The flat SFTDE model favors a relatively low value of $\Omega_{m0} = 0.246 \pm 0.013$, significantly smaller than the Λ CDM value of $\Omega_{m0} = 0.328^{+0.018}_{-0.017}$, while the non-flat SFTDE model yields an even lower central value of $\Omega_{m0} = 0.207^{+0.089}_{-0.081}$ but with considerably larger uncertainties. The curvature parameter Ω_{k0} in the non-flat SFTDE model is weakly constrained, consistent with an open universe. However, the flat universe ($\Omega_{k0} = 0$) is still allowed as it lies within $2 - \sigma$. The inclusion of BAO data in the second dataset combination (CC/Pantheon+/SH0ES/BAOs) tightens constraints on all parameters, particularly Ω_{m0} and Ω_{k0} , although introducing the sound horizon scale r_d as an extra free parameter (see Sect. 8.2.1). Here, the flat SFTDE model still favors a low matter density ($\Omega_{m0} = 0.234 \pm 0.010$) while predicting $r_d = (135.9 \pm 2.3) \text{ Mpc}$. The non-flat SFTDE model accommodates a mildly open geometry with $\Omega_{k0} = 0.157 \pm 0.082$. Subsequently, the absolute magnitude $\mathcal{M}$ shifts to a lower value in the flat SFTDE model ($\mathcal{M} = -19.409^{+0.061}_{-0.063}$).

Moving on, the exclusion of CC data, i.e. using the compilation Pantheon+/SH0ES/BAOs cause an increment of the Hubble constant for all models. For the non-flat SFTDE model, an increment of about $\sim 10\%$ in both cosmic curvature density and matter energy density ($\Omega_{m0} = 0.216 \pm 0.022$, $\Omega_{k0} = 0.172^{+0.084}_{-0.082}$) appears. In contrast, the matter energy density for both flat SFTDE ($\Omega_{m0} = 0.216 \pm 0.022$) and Λ CDM ($\Omega_{m0} = 0.313 \pm 0.014$) remains almost unchanged. Although for the non-flat SFTDE model the Hubble constant is found to be decreased in comparison with Λ CDM and flat SFTDE. The same is the case for the SNIa calibration parameter, $\mathcal{M}$.

Finally, in Tab. 8.6 we present the values of the Akaike Information Criterion (AIC), the Bayesian Information Criterion (BIC), and the Deviance Information Criterion (DIC) for three cosmological models, i.e flat SFTDE, non-flat SFTDE, and Λ CDM. For each criterion, we also report the corresponding difference $\Delta\text{IC} \equiv \text{IC} - \text{IC}_{\min}$. For all dataset combinations and information criteria, the flat SFTDE model achieves lower ICs values in comparison with Λ CDM, being however statistically indiscernible, as $|IC_{\Lambda\text{CDM}} - IC_{\text{flat SFTDE}}| \ll 2$. Interestingly, for the combined dataset Pantheon+/SH0ES/BAOs, the non-flat SFTDE model yields the minimum AIC and DevIC values, indicating the best overall fit. The non-flat SFTDE model remains competitive or *preferable*, especially

when explicit penalization for extra parameters (as in BIC) is less critical. These results suggest a mild but consistent statistical preference for the non-flat SFTDE scenario over both the flat SFTDE and the standard Λ CDM model. We will discuss this extraordinary result in detail at Sect. 8.3.

Table 8.5.: Parameter estimation results for flat and non-flat versions of the TDE scenario. Results for the concordance model are also included in order to allow for direct comparison.

Model	Ω_{m0}	Ω_{k0}	h	r_d	$\mathcal{M}$	$\chi^2_{\min}$	$\chi_{\min}/dof$
CC/Pantheon+/SH0ES							
flat SFTDE	0.246 ± 0.013	—	0.7343 ± 0.0102	—	-19.25 ± 0.03	1460.32	0.87
non-flat SFTDE	$0.207^{+0.089}_{-0.081}$	$0.050^{+0.105}_{-0.112}$	0.73 ± 0.01	—	-19.25 ± 0.03	1460.06	0.87
Λ CDM	$0.328^{+0.018}_{-0.017}$	—	$0.7343^{+0.0101}_{-0.0103}$	—	-19.25 ± 0.03	1460.33	0.87
CC/Pantheon+/SH0ES/BAOs							
flat SFTDE	0.234 ± 0.010	—	0.7355 ± 0.010	135.9 ± 2.3	$-19.409^{+0.061}_{-0.063}$	1470.78	0.87
non-flat SFTDE	0.196 ± 0.022	0.157 ± 0.082	0.7326 ± 0.010	135.2 ± 2.3	-19.25 ± 0.03	1466.92	0.87
Λ CDM	$0.311^{+0.014}_{-0.013}$	—	$0.73540^{+0.01009}_{-0.00995}$	135.9 ± 2.3	-19.25 ± 0.03	1470.78	0.87
Pantheon+/SH0ES/BAOs							
flat SFTDE	$0.234^{+0.011}_{-0.010}$	—	$0.7369^{+0.0102}_{-0.0100}$	135.5 ± 2.3	-19.25 ± 0.03	1463.40	0.88
non-flat SFTDE	0.216 ± 0.022	$0.172^{+0.084}_{-0.082}$	0.734 ± 0.010	134.6 ± 2.3	-19.24 ± 0.03	1458.98	0.88
Λ CDM	0.313 ± 0.014	—	$0.7368^{+0.0102}_{-0.0100}$	135.5 ± 2.3	-19.25 ± 0.03	1463.40	0.87

8.2.3. Big Bang Nucleosynthesis

A first consistency check regarding the Big Bang Nucleosynthesis (BBN) can be performed via an estimation of the percentage increment of the dilation rate of the universe at BBN era, i.e $z_{BBN} \sim 10^9$, using the best fit values for the free parameter [422]. We calculate the following ratio:

$$C_r = \left| \frac{H_{SFTDE} - H_{\Lambda CDM}}{H_{\Lambda CDM}} \right|^2 \quad (8.35)$$

If $C_r \geq 10\%$, the model cannot describe BBN phase. On the other hand, the inequality $C_r < 10\%$ does not guarantee that the predictions of the model are in good agreement with observations. However, it does motivate further analysis, for example starting with the semianalytic approach of [423] and going to the utilization of data points on primordial abundances via numerical codes such as *alterBBN* [424]. We report that the BBN constraints are satisfied for all SFTDE models considered here (Fig. 8.1). One could also arrive to the same conclusion by observing the tendency of the topological

Model	AIC	Δ AIC	BIC	Δ BIC	DevIC	Δ DIC
CC/Pantheon+/SH0ES						
flat SFTDE	1466.33	0.0	1482.60	0.0	1466.32	0.0
non-flat SFTDE	1468.09	1.76	1489.77	7.17	1467.89	1.57
Λ CDM	1466.34	0.01	1482.61	0.01	1466.35	0.03
CC/Pantheon+/SH0ES/BAOs						
flat SFTDE	1478.81	1.85	1500.52	0.0	1478.77	2.12
non-flat SFTDE	1476.96	0.0	1504.10	3.58	1476.91	0.00
Λ CDM	1478.81	1.85	1500.52	0.0	1478.78	2.13
Pantheon+/SH0ES/BAOs						
flat SFTDE	1471.42	2.60	1493.08	0.01	1471.38	2.38
non-flat SFTDE	1469.02	0.00	1496.08	3.01	1469.00	0.00
Λ CDM	1471.42	2.60	1493.07	0.00	1471.39	2.39

Table 8.6.: The information criteria AIC, BIC, and DevIC for the examined cosmological models, alongside the corresponding differences Δ IC $\equiv$ IC $-$ IC_{min}.

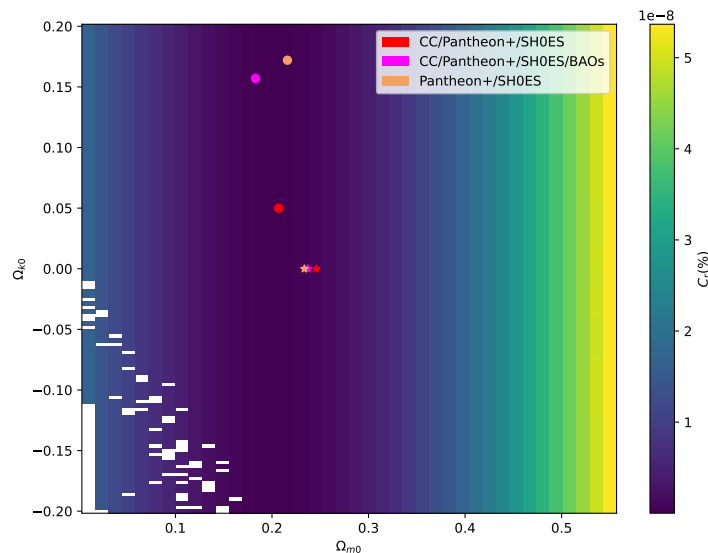


Figure 8.1.: The ratio C_r as a function of the parameters Ω_{m0} and Ω_{k0} . The star (circle) symbol correspond to the flat (non flat) models and each color corresponds to the best fit parameters from different datasets to the best fit values for Pantheon+/CC (Pantheon+/CC/BAOs).

DE model towards Λ CDM at the fish plot, Fig. 8.2, at $z = z_{BBN}$. Note that inclusion of cosmic curvature do not induce any change.

8.2.4. Cosmic evolution of SFTDE

In this subsection we discuss the evolution of matter, radiation, curvature and DE parts of the cosmic substance, depicted at Figs. 8.2, 8.3. In the upper graph of both images we present the flat SFTDE model with the parameters obtained from the best fit to Pantheon+/SH0ES/CC datasets, while in the lower graph we show the non-flat SFTDE model where the parameter values come from the best fit in the Pantheon+/SH0ES/CC/BAO, already presented at Tab. 8.5. We can see that our models thermal history exhibits differences from the Λ CDM one.

While the model's behavior is close to the standard one at the present and recent times, as one looks backwards in the past, can see that $\Omega_{\Lambda_{\text{eff}}}$ becomes greater than Ω_{Λ} and simultaneously Ω_m of the model becomes smaller compared to the standard one. This discrepancy maximizes around $z \approx 10$ and afterwards starts to diminish. The evolution of the radiation component is almost the same with the one in Λ CDM. We can see that at the time of radiation matter equality the evolution of the models Ω_m has almost converged with the standard one, although there is an excess of the models $\Omega_{\Lambda_{\text{eff}}}$ over $\Omega_{\Lambda\text{CDM}}$.

We examine the physical meaning of this discrepancy in light of the DE equation of state parameter's evolution, plotted in Fig. 8.4. We observe that $w_{\text{DE}} = -0.89$ at the present and evolves towards zero as one progresses backwards in time. Interestingly enough, we can see that around $z \approx 10$ and beyond, $w_{\text{DE}} \rightarrow 0$. The latter points out that the DE sector behaves as pressureless DM, i.e. topological DE model exhibits DE to DM conversion. This behavior can be brought to light if one assumes that the model's effective DE sector is a mixture of Λ -like DE, i.e. with $w_{\Lambda} = -1$, and of effective DM (with $w_{\text{DM}_{\text{eff}}} = 0$):

$$\Omega_{\Lambda_{\text{eff}}} = \Omega_{\Lambda} + \Omega_{\text{DM}_{\text{eff}}}. \quad (8.36)$$

Next, we calculate the portion of each component. Since $w_{\text{DM}_{\text{eff}}} = 0$, the pressure of the topological model's DE sector, with $w_{\Lambda_{\text{eff}}}$, will be equivalent to the pressure coming

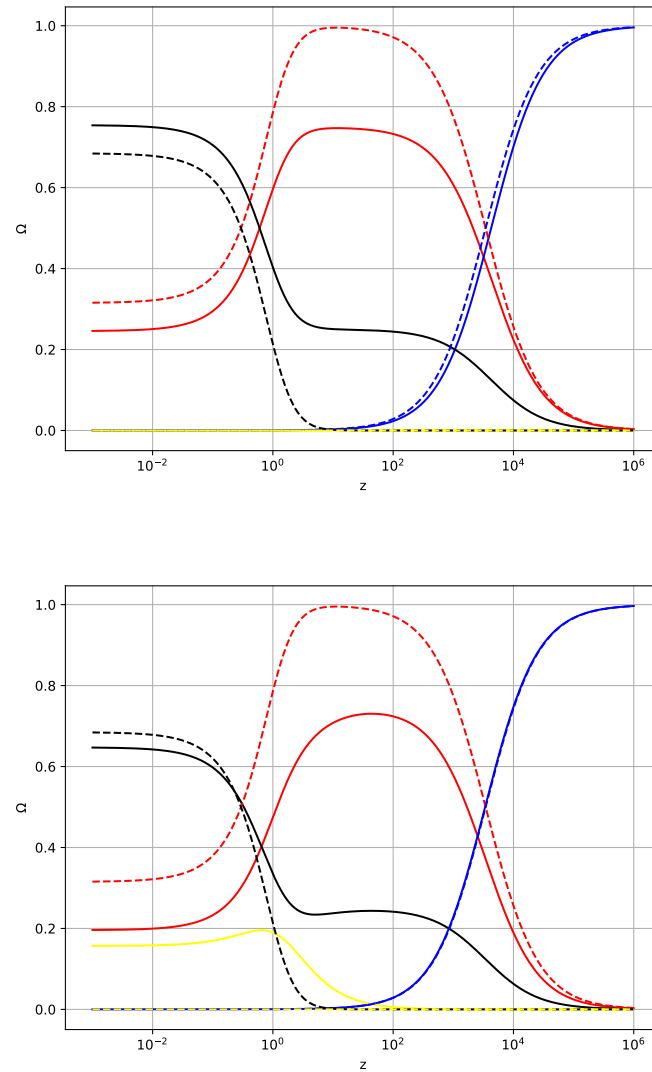


Figure 8.2.: The evolution of matter (red), DE (black), radiation (blue) and curvature (yellow) energy densities as a function of redshift, for the SFTDE model Eq. (8.16) (solid lines), as well as for the Λ CDM paradigm (dashed lines). For the concordance model, we employ the PLANCK values [419] for the free parameters. In the upper graph the parameters for the flat SFTDE model were acquired from the best fit in the CC/Pantheon+/SH0ES dataset $h = 0.7343$, $\Omega_{m0} = 0.246$, $\Omega_{k0} = 0$, while in the lower graph we used the parameters from the best fit in the CC/Pantheon+/SH0ES/BAOs dataset $h = 0.7326$, $\Omega_{m0} = 0.196$, $\Omega_{k0} = 0.157$, assuming a non flat SFTDE model.

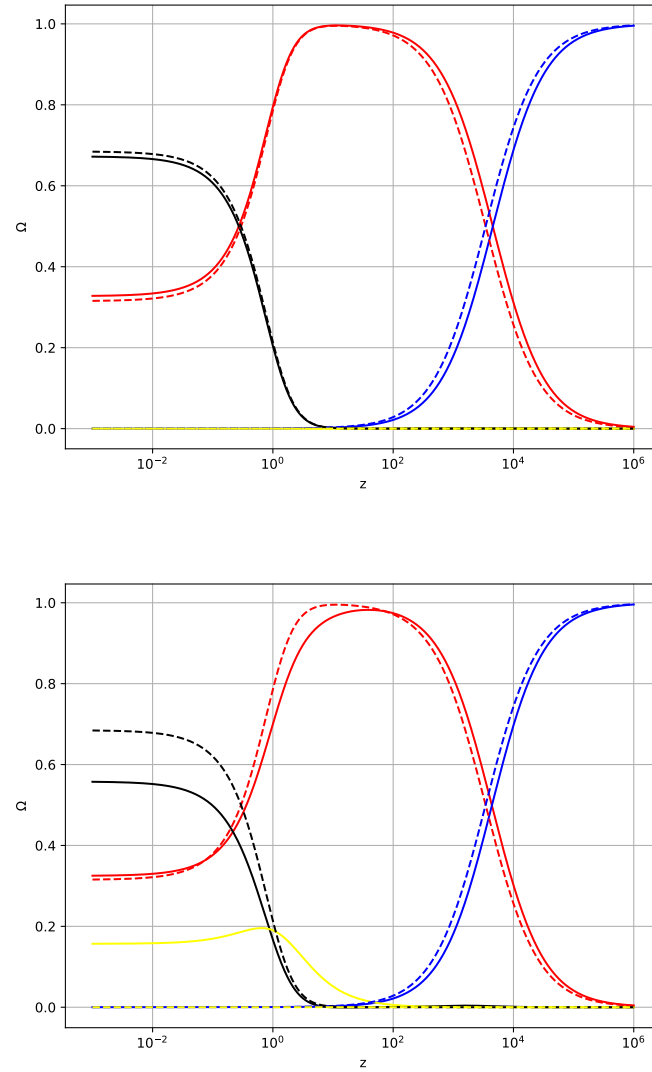


Figure 8.3.: *The evolution of matter (red), DE (black), radiation (blue) and curvature (yellow) energy densities as a function of redshift, for the reconstructed SFTDE model according to Eqs. (8.36)(8.37) (solid lines), as well as for the Λ CDM paradigm (dashed lines) for the PLANCK values [419]. In the upper graph we have implemented the parameters from the best fit in CC/Pantheon+/SH0ES dataset, while in the lower one from the best fit in CC/Pantheon+/SH0ES/BAOs dataset, in accordance with Fig. 8.2.*

from “pure” DE, thus $p_\Lambda = p_{\Lambda_{\text{eff}}}$, after some simple algebra we obtain:

$$\Omega_{\text{DM}_{\text{eff}}} = \left(1 - \frac{w_{\Lambda_{\text{eff}}}}{w_\Lambda}\right) \Omega_{\Lambda_{\text{eff}}}, \quad (8.37)$$

proceeding further, we can evaluate the model’s total DM as $\Omega_{\text{DM}_{\text{tot}}} = \Omega_{\text{DM}} + \Omega_{\text{eff}}$ and the “pure” DE Ω_Λ from Eq. (8.36). For the cosmological parameters from the best fit in the late time Universe datasets, we present the evolution of cosmological species of the reconstructed model in Fig. 8.3. Interestingly, we can see that the reconstructed flat SFTDE model coincides with Λ CDM.

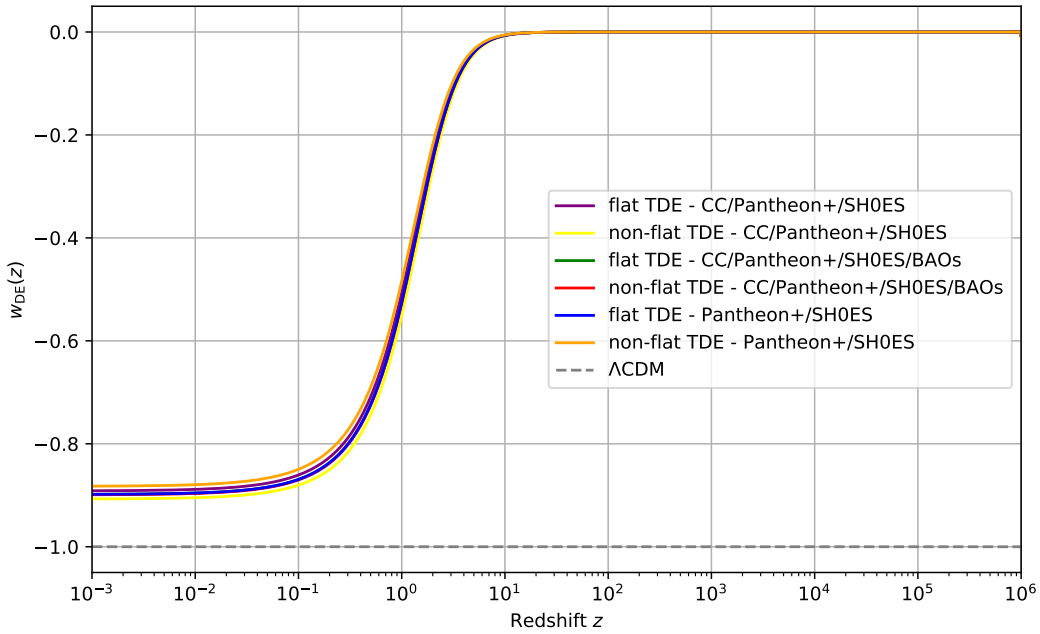


Figure 8.4.: The evolution of the dark-energy equation-of-state parameter w_{DE} , as a function of redshift from Eq. (8.19), for the flat and the non-flat SFTDE model, with parameter values from the best fit in the three datasets as they are presented in Table 8.5 .

8.3. Discussion

The main result of the current work is the fact that the non-flat SFTDE model introduced here, in the case of both CC/Pantheon+/SH0ES/BAOs and Pantheon+/SH0ES/BAOs datasets, *outperforms* the concordance model consistently. Note that $2 < \Delta\text{IC} < 6$ for AIC and DevIC model selection criteria (see lower part of Tab. 8.6), thus non-flat SFTDE model is better, and in moderate tension with Λ CDM. Moreover, the flat

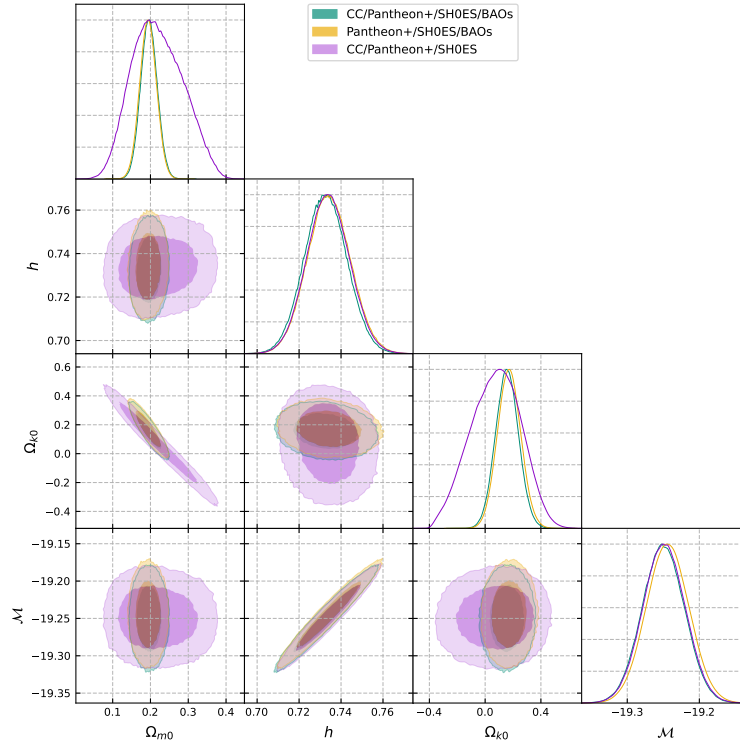


Figure 8.5.: Posterior distributions for parameter pairs regarding the non-flat SFTDE model (8.16), for all dataset compilations. The iso-surfaces correspond to 1σ - 2σ - 3σ areas.

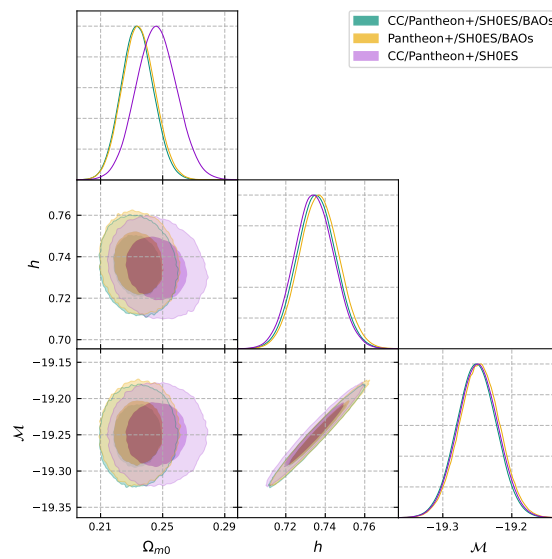


Figure 8.6.: Posterior distributions for parameter pairs regarding the flat SFTDE model (8.16), for all dataset compilations. The iso-surfaces correspond to 1σ - 2σ - 3σ areas.

SFTDE model is slightly better than the concordance model, however remains statistically equivalent with the latter, for all dataset combinations considered here. It is interesting that the non-flat SFTDE model is more favored whenever the CC data are excluded. Indeed, this is expected, as for the derivation of the CC data, a flat universe has been explicitly assumed. Moreover, a preliminary assessment showed that both flat and non-flat SFTDE models are consistent with the standard BBN phase. By observing Fig. 8.1, we note that effects of the cosmic curvature diminish at the BBN era, preserving the thermal history.

Several alternative cosmological models have been proposed in the literature that outperform the concordance model (Λ CDM) in certain cases. One such category includes Running Vacuum Models (RVMs) [422], which are motivated by quantum field theory in curved spacetime, though they are essentially phenomenological parameterizations. Developing a consistent Lagrangian formulation for these models typically requires invoking multiple complex physical mechanisms, as discussed in [425]. Another broad class of models that achieve excellent fits to observational data are Early Dark Energy (EDE) models. Various parameterizations of the dark energy equation of state in these scenarios have been shown to outperform Λ CDM in specific datasets (e.g., [426], [427]). All of these approaches, however, are phenomenological in nature. In contrast, minimally modified gravity theories propose more structural changes to General Relativity by promoting the Hamiltonian constraint to a free function [428]. Similarly, $f(Q)$ cosmologies [429, 430] replace the torsionless, metric-compatible Riemannian geometry with a manifold characterized by non-metricity and introduce a free function of the non-metricity scalar Q into the gravitational action. Although both $f(Q)$ and minimally modified gravity theories are conceptually more fundamental than the phenomenological models mentioned earlier, they generally lack a guiding physical principle or symmetry to constrain the form of the free functions they employ. In contrast, the topological dark energy scenario proposed in this work is of a more fundamental nature. It avoids the use of arbitrary functional forms, requiring no free functions, and thus involves fewer *ad hoc* assumptions than the aforementioned models.

Moving on, another interesting feature of the SFTDE scenario is the fact that it incorporates energy transfer to the dark sector. As we saw, $w_{DE} \rightarrow 0$ at high redshifts, hence the Dark Energy behaves as pressureless matter. Then it evolves towards negative values, and reaches $w = -0.89$ at the present time $z = 0$. When we consider the model's DE density to be consisted of "pure", Λ -like DE ($w_{DE} = -1$) and effective

DM ($w_{DM} = 0$), we saw that the SFTDE scenario produces a significant portion of standard quantity of DM, i.e 25%, in the case of the flat model. Note that the aforementioned fact explains consistently the observed reduction of the matter-energy density for the case of both SFTDE models considered here, i.e Fig. 8.3 and discussion therein. We note that models with interacting dark sector, at least in the context of [431], have been able to solve the σ_8 tension at the CMB level, without worsening the H_0 tension. Thus, it is deemed possible that a solution of σ_8 tension will emerge in the context of SFTDE scenario.

Finally, another point of interest regarding the SFTDE scenario has to do with the increased fitting quality of the non-flat model with regard to the flat one. The latter points out on the importance of cosmic curvature. This result is in line with other relevant works, i.e. [432] for the concordance model. However, in our case, the cosmic energy density of curvature, Ω_{k0} is increased, (see Tab. 8.5) up to approximately $\Omega_{k0} \sim 20\%$, in contrast with $\Omega_{k0} \sim 10\%$ [432]. Further study of the SFTDE scenario using perturbation-related datasets, i.e. LSS and/or CMB should enforce explicit constraints on the relation between $w_i = 0$ and $w_i \neq 0$ components of the Dark sector. This is expected to happen because the contribution of each component to the matter over-densities evolution is governed by its equation of state [433].

Going beyond cosmology, a number of observables associated with spacetime foam have been proposed, particularly in the context of interferometric experiments and photon time delays from astrophysical sources. In particular, laser interferometers like LIGO and the scheduled LISA/eLISA are sensitive to distortions in spacetime and have been suggested as potential tools for detecting spacetime foam-induced noise [247, 248, 434]. In general, spacetime foam might influence the propagation of high-energy photons over vast distances. Observations of gamma-ray bursts (GRBs) and active galactic nuclei (AGNs) have been used to search for energy-dependent time delays that could indicate a foamy spacetime structure [252, 253, 435]. Indeed, tentative evidence for such energy-dependent lags in GRB photon arrivals has been reported by the MAGIC, HESS, and Fermi collaborations, prompting further theoretical investigations [436]. Moreover, in the context of (EQG), spacetime foam could lead to modifications in fundamental constants or induce decoherence effects, that could be possibly observable in astrophysical phenomena [249], providing another potential probe. However, the aforementioned approaches go beyond the scope of the current work and are left for the future.

8.4. Conclusions

This work presents a new Topological Dark Energy (SFTDE) cosmological scenario, which consistently outperforms the standard Λ CDM model across multiple datasets, including Pantheon+, SH0ES, BAO, and Cosmic Chronometers (CC). Model comparison metrics such as AIC and DevIC indicate a moderate but statistically significant preference for the non-flat SFTDE model, especially when CC data (which assume flatness) are excluded. The flat SFTDE version also performs slightly better than Λ CDM, though the difference remains statistically comparable.

A key theoretical strength of the SFTDE scenario lies in its minimal reliance on arbitrary assumptions or extra parameters. Unlike phenomenological alternatives (e.g., Running Vacuum or $f(Q)$ gravity models), SFTDE is fundamentally motivated and grounded in quantum gravity concepts. Importantly, it provides a complete physical mechanism that generates Dark Energy and a fraction of Dark Matter, with energy transfer occurring dynamically within the dark sector. At early times, dark energy behaves like matter ($w \approx 0$), transitions gradually, and asymptotically approaches $w = -0.9$ at late times. This dynamic evolution enables the SFTDE scenario to possibly address key cosmological tensions, including the σ_8 discrepancy and the mismatch in the sound horizon scale r_d between CMB and late-time observations. Notably, the non-flat version performs better than the flat one, pointing to a significant role for cosmic curvature, with inferred values of Ω_{k0} reaching up to 20% -somewhat higher than typical values reported for Λ CDM. In parallel, preliminary tests show that SFTDE scenario is consistent with Big Bang Nucleosynthesis, for both flat and non-flat cases. The aforementioned features make Topological Dark Energy a promising candidate for a more complete description of the Universe. Future work will focus on structure formation and CMB data to further constrain SFTDE scenario. Moreover, the model's connection to Planck-scale physics opens avenues for independent testing via gravitational waves, time-delay measurements, and possible variations in fundamental constants.

Chapter 9.

Topological Dark energy from black hole formation and merging

We apply the gravity-thermodynamics approach in the case of Einstein–Gauss–Bonnet theory, and its corresponding Wald–Gauss–Bonnet entropy, which due to the Chern–Gauss–Bonnet theorem it is related to the Euler characteristic of the Universe topology. However, we consider the realistic scenario where we have the formation and merger of black holes that lead to topology changes, which induce entropy changes in the Universe horizon. We extract the modified Friedmann equations and we obtain an effective dark energy sector of topological origin. We estimate the black-hole formation and merger rates starting from the observed star formation rate per redshift, which is parametrized very efficiently by the Madau–Dickinson form, and finally we result to a dark-energy energy density that depends on the cosmic star formation rate density, on the fraction f_{BH} of stars forming black holes, on the fraction of black holes f_{merge} that eventually merge, on the fraction f_{bin} of massive stars that are in binaries, on the average mass of progenitor stars that will evolve to form black holes $\langle m_{\text{prog}} \rangle$, as well as on the Gauss–Bonnet coupling constant. We investigate in detail the cosmological evolution, obtaining the usual thermal history. Concerning the dark-energy equation-of-state parameter, we show that at intermediate redshifts it exhibits phantom-like and quintessence-like behavior according to the sign of the Gauss–Bonnet coupling, while at early and late times it tends to the cosmological constant value. Finally, we study the effect of the other model parameters, showing that for the whole allowed observationally estimated ranges, the topological dark-energy equation-of-state parameter remains within its observational bounds.

According to extensive observational evidence from various origins, the Universe has recently entered a phase of accelerated expansion [83–88]. In order to explain this behavior, two main directions have been pursued. The first is to retain general relativity as the gravitational framework while introducing new energy components, such as the dark energy sector [102, 103]. The second involves constructing modified and extended theories of gravity by altering the left-hand side of Einstein field equations, adding correction terms to the standard Einstein–Hilbert action [36, 105, 106, 437]. Such modified gravity theories not only address cosmological issues but also offer improved quantum behavior [438], given that general relativity is non-renormalizable [210]. Specifically, incorporating higher-order curvature terms into the Einstein–Hilbert Lagrangian tends to eliminate divergences [218], and among all higher-order terms the Gauss–Bonnet (GB) combination is special since it is topologically invariant in 4 dimensions and thus plays a significant role in heterotic string theory [175, 218–220] and in M-theory [221].

Beyond the aforementioned approaches to modified gravity theories, there exists a well-known conjecture that gravity can be described through the laws of thermodynamics [190–192]. This concept is inspired from black-hole thermodynamics, where a black hole (BH) is assigned a specific temperature and entropy, dependent on its horizon [193]. The “thermodynamics of spacetime” conjecture [190] draws an analogy from BH horizon thermodynamics to the Universe horizon at cosmological scales. In particular, if one applies the first law of thermodynamics in the apparent horizon one can obtain the Friedmann equations [194–196], and vice versa the Friedmann equations can be expressed as the first law of thermodynamics. This procedure has been applied successfully both in the case of general relativity and as well as in modified theories of gravity [197–208], where in the latter case one should use the corresponding modified entropy relation.

In the present work, we are interested in applying the gravity-thermodynamics approach in the case where the gravitational action is extended by the GB term and the corresponding entropy is extended by the Wald–Gauss–Bonnet entropy [322, 323], a term dependent on the BH horizon topology. However, we are interested in considering the realistic scenario where in the Universe we have the formation and merger of black holes, that lead to topology changes which induce entropy changes in the Universe horizon. Hence, applying the gravity-thermodynamics analysis one could extract modified Friedmann equations, with an effective, dark energy sector of topological origin, depending on the black-hole formation and merger rates.

9.1. Modified cosmology from Wald–Gauss–Bonnet entropy

In this section, we present the scenario at hand, extracting the modified Friedmann equations by applying the first law of thermodynamics to the Universe using the Wald–Gauss–Bonnet entropy. We start by considering a homogeneous and isotropic Friedmann–Robertson–Walker (FRW) geometry with metric

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad (9.1)$$

where $a(t)$ is the scale factor, and $k = 0, +1, -1$ corresponds to flat, closed, and open spatial geometry, respectively.

9.1.1. Friedmann equations as the first law of thermodynamics

To extract the Friedmann equations in general relativity from the first law of thermodynamics, we consider an expanding Universe filled with a matter perfect fluid, with energy density ρ_m and pressure p_m . As a boundary, one uses the apparent horizon [194–196]

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \quad (9.2)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The apparent horizon is a marginally trapped surface with vanishing expansion, and it is a causal horizon associated with gravitational entropy and surface gravity [190–192]. The temperature attributed to the apparent horizon is

$$T_h = \frac{1}{2\pi\tilde{r}_A}, \quad (9.3)$$

and the entropy in general relativity is given by the standard Bekenstein–Hawking relation

$$S_h = \frac{A}{4G}, \quad (9.4)$$

where $A = 4\pi\tilde{r}_A^2$ is the area of the apparent horizon. Assuming that the Universe fluid acquires the same temperature with the horizon after equilibrium, the heat flow crossing the horizon during an infinitesimal time interval dt can be found to be [194]

$$\delta Q = -dE = A(\rho_m + p_m)H\tilde{r}_A dt, \quad (9.5)$$

as it has been presented analytically in section 6.4. Since the first law of thermodynamics states that $-dE = TdS$, using the temperature and entropy from (9.3) and (9.4) we find

$$-4\pi G(\rho_m + p_m) = \dot{H} - \frac{k}{a^2}. \quad (9.6)$$

Finally, considering that the matter fluid satisfies the conservation equation

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0, \quad (9.7)$$

integrating (9.6) we obtain

$$\frac{8\pi G\rho_m}{3} = H^2 + \frac{k}{a^2} - \frac{\Lambda}{3}, \quad (9.8)$$

with Λ the integration constant. Equations (9.8) and (9.6) are nothing else but the two Friedmann equations, with the integration constant Λ playing the role of the cosmological constant.

In summary, through the gravity-thermodynamics approach one can extract the Friedmann equations not by varying the gravitational action, but by applying the first law of thermodynamics on the Universe apparent horizon. Hence, this implies that one can obtain modified Friedmann equations without necessarily needing to alter the underlying gravitational theory, but just altering the entropy expression itself.

9.1.2. Wald–Gauss–Bonnet entropy

In the previous subsection, we saw how one can extract the Friedmann equations through the gravity-thermodynamics conjecture. As we mentioned in the Introduction, the same procedure can be applied by changing the entropy relation.

One of the widely studied higher-order theories of gravity is the Gauss–Bonnet one, due to its topological significance [213,214]. The total action in four dimensions consists of the Einstein–Hilbert and the Gauss–Bonnet (GB) term, namely

$$I = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + \tilde{\alpha} \mathcal{G}), \quad (9.9)$$

where R is the Ricci scalar, $\mathcal{G}$ is the GB term, defined as

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}, \quad (9.10)$$

and $\tilde{\alpha}$ is the GB coupling constant. Note that in four dimensions the Gauss–Bonnet term that enters linearly in the action does not lead to any new term in the field equations, since it is a topological invariant, i.e. the above Einstein–Gauss–Bonnet action is just General Relativity. However, this term does lead to new contributions in the entropy expression.

Let us calculate the corrections to the entropy expression. As it was presented in section 5.3, the Wald–Noether charge method [322,323] can be used to derive a horizon entropy S satisfying the first law of thermodynamics for any first-order spacetime perturbation. For any diffeomorphism invariant theory of gravity described by a Lagrangian L , the Wald entropy of a stationary BH with a regular bifurcation surface is given by [323,439]

$$S_{\text{Wald}} = -\frac{1}{8} \int_B \frac{\partial L}{\partial R_{ijkl}} \epsilon_{ij} \epsilon_{kl} \sqrt{\sigma} d^{D-2}x, \quad (9.11)$$

where the integration is over any $(D-2)$ -dimensional spacelike cross-section B of the horizon, and ϵ_{ij} is the binormal on such a cross-section, normalized as $\epsilon_{ij}\epsilon^{ij} = -2$. In section 5.4 we have demonstrated that for the Einstein–Gauss–Bonnet action (9.9), the Wald entropy becomes

$$S_{\text{WGB}} = \frac{1}{4} \int_B \left(1 + 2\tilde{\alpha}^{(D-2)} R\right) \sqrt{\sigma} d^{D-2}x, \quad (9.12)$$

where $^{(D-2)}R$ is the Ricci curvature associated with the $(D-2)$ -dimensional cross-section of the horizon. Applying to $D = 4$ dimensions the expression of the Wald-

Gauss-Bonnet (WGB) entropy becomes

$$S_{\text{WGB}} = \frac{A}{4G} + \frac{\tilde{\alpha}}{2G} \int_h d^2x \sqrt{\sigma} \mathcal{R}, \quad (9.13)$$

where $\mathcal{R}$ is the Ricci scalar of the induced metric on the horizon h .

Now, according to the Chern–Gauss–Bonnet theorem [368], the integral of the Ricci scalar $\mathcal{R}$ over the two-dimensional horizon h equals 4π times the Euler characteristic $\chi(h)$ of the horizon, namely

$$\int_h d^2x \sqrt{\sigma} \mathcal{R} = 4\pi \chi(h). \quad (9.14)$$

The Euler characteristic χ is defined as the alternating sum of the Betti numbers B_p of a manifold M , namely $\chi(M) = \sum_p (-1)^p B_p$ [351], where the Betti numbers are defined as the dimension of the p^{th} de Rham cohomology group $H^p(M)$, i.e. $B_p = \dim H^p(M)$ [166, 167]. Hence, combining (9.13) and (9.14) we find that the entropy for a BH in the presence of the Einstein–Gauss–Bonnet terms is [325]

$$S_{\text{WGB}} = \frac{A}{4G} + \frac{2\pi\tilde{\alpha}}{G} \chi(h). \quad (9.15)$$

Thus, the Wald–Gauss–Bonnet entropy can be interpreted as a sum of an area term (the Bekenstein–Hawking one) and a topological term (the Euler characteristic of the horizon), namely

$$S_{\text{SGB}} = S_{\text{area}} + S_{\text{top}}. \quad (9.16)$$

Nevertheless, the above consideration may lead to a potential problem when we have black hole mergers. As it was discussed in [325, 440], while in the case of general relativity the corresponding Bekenstein–Hawking entropy increases, the Wald entropy of the Einstein–Gauss–Bonnet theory exhibits a decrease of topological origin. In particular, as we saw above, in the case of the GB term the corresponding entropy (9.15) depends on the Euler characteristic of the horizon $\chi(h)$. Since the horizon of a BH is a two-dimensional sphere S^2 , during the merger of two BHs their horizons merge too, as illustrated in Fig. 9.1. Before the merger, the spacetime contains two horizons, i.e.

$$\chi_{\text{in}} = \chi(S^2) + \chi(S^2) = 4, \quad (9.17)$$

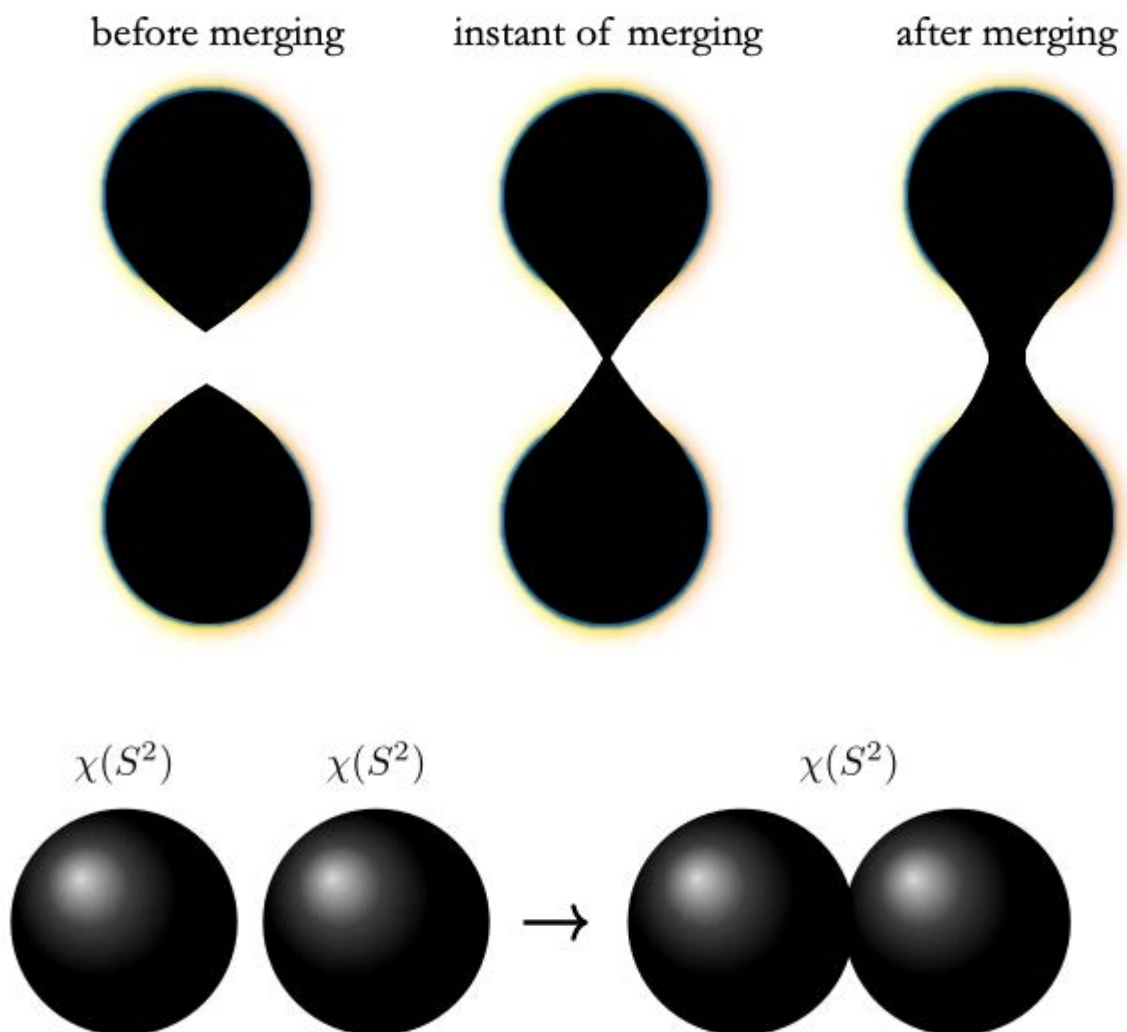


Figure 9.1.: *The topology and the Euler characteristic change of a black hole merger.
Credits to Eva Theodoratou for the design*

while after the merger there is only one horizon, namely

$$\chi_f = \chi(S^2) = 2, \quad (9.18)$$

and therefore during a black hole merger one obtains an Euler characteristic change

$$\delta\chi_h = \chi_f - \chi_{in} = -2. \quad (9.19)$$

Hence, if one considers Einstein–Gauss–Bonnet theory and the extended entropy (9.15) it is straightforward to see that for $\tilde{\alpha} > 0$, the topological transition during the BH merger induces a decrease in the topological part of Wald entropy $\Delta S_{top} \leq 0$, resulting to a violation of the second law (in the same lines, when a BH horizon forms from gravitational collapse an increase of the Wald entropy is induced by the topological transition $\delta\chi = 2$). On the other hand, in the case $\tilde{\alpha} < 0$, there is an increase of Wald entropy during BH mergers and a decrease during a BH formation.

Since the above violation originates from the decrease of the BHs horizon Euler characteristic, and since the decrease is an instantaneous integer jump which cannot be compensated by a continuous procedure [325, 440], one possible way to resolve the second law violation is to assume that a similar process exists that compensates the decrease of the topological part of the entropy. Having in mind the discussion of the gravity-thermodynamics conjecture and that the BH horizon and the apparent horizon are causal horizons, we assume here that the topology of the causally connected boundaries remains constant. In particular, the apparent horizon is the external boundary while the BH horizons are the internal ones, and therefore we can define the total boundary as $\partial M = \mathcal{H} \cup_{i=1}^N h_i$ and then apply the calculation for the Euler characteristics as

$$\chi(\partial M) = \chi(\mathcal{H}) + \sum_{i=1}^N \chi(h_i). \quad (9.20)$$

Since the overall boundary topology remains constant $\delta\chi(\partial M) = 0$ then $\delta\chi(\mathcal{H}) = -\delta \sum_{i=1}^N \chi(h_i)$.

Now, the change in the Euler characteristic of BH horizons at every BH formation from collapse is $\delta\chi(h) = 2$, while at every merger it is $\delta\chi(h) = -2$. Therefore, if δN_{form} BH formations and δN_{merg} BH mergers occur, then the corresponding change

of the Euler characteristic of the apparent horizon will be

$$\delta\chi(\mathcal{H}) = -2 (\delta N_{form} - \delta N_{merg}). \quad (9.21)$$

By demanding that the total topology of the causally connected boundaries remains constant, each time a BH horizon is formed the Euler characteristic of the apparent horizon decreases by 2, while each time two BH horizons merge into one the Euler characteristic of the apparent horizon increases by 2. For two-dimensional surfaces the Euler characteristic is given by [441]

$$\chi = 2 - 2g - b, \quad (9.22)$$

where g is the genus, the number of handles the surface possesses (torus-like holes) and b is the number of boundary components puncture-like holes. An interpretation of (9.21) according to (9.22) could be that each time a BH horizon is formed two compensating puncture disks appear (open up) on the causal horizon, while each time two BH horizons merge two disk punctures disappear (close up) on the causal horizon, as it has been illustrated in Fig. 9.2. We mention here that the connection between the Universe horizon (the largest scale of the theory) and small scales is known to hold according to the holographic principle [314, 442–445], with a famous cosmological application being the holographic dark energy [236, 239, 446–455]. In similar lines, in the following we will obtain a dark energy sector of topological nature.

In summary, a first consequence of the above considerations is that the second law of thermodynamics is satisfied by Hawking area theorem [320]. A second consequence is that the topology of the causal horizon becomes dynamical, thus a new term of topological origin appears when one derives the Friedmann equations in the spacetime-thermodynamics framework for the Wald–Gauss–Bonnet entropy, as we will see in the next subsection.

9.1.3. Modified Friedmann equations form Wald–Gauss–Bonnet entropy

In subsection 9.1.1 we showed how the standard Friedmann equations can be obtained from the first law of thermodynamics in the case of Bekenstein–Hawking entropy that corresponds to general relativity. In the present subsection, we will follow the same

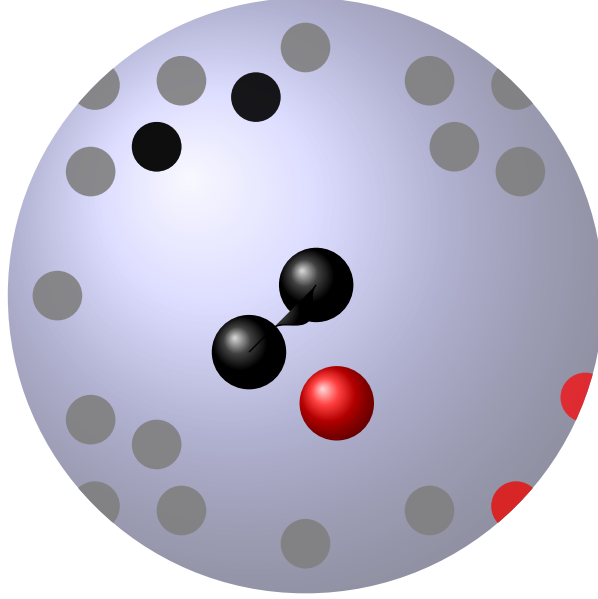


Figure 9.2.: Each time a BH horizon is formed from gravitational collapse (the red sphere), two corresponding puncture holes appear on the horizon (red disks). Similarly, each time two BH horizons merge (black binary), two corresponding puncture holes close up on the apparent horizon (black disks). The grey holes indicate an arbitrary number of disk punctures excess on the horizon, since only a fraction of the BHs merge.

procedure for the case of the Wald–Gauss–Bonnet entropy discussed in subsection 9.1.2.

Differentiating the expression of the Wald–Gauss–Bonnet entropy (9.15), and using (9.2), we find

$$\frac{dS}{dt} = -\frac{2\pi\tilde{r}_A^4}{4G}H\left(\dot{H} - \frac{k}{a^2}\right) + \frac{\pi\tilde{a}}{G}\dot{\chi}(\mathcal{H}). \quad (9.23)$$

Thus, substituting into the first law of thermodynamics $-dE = TdS$ where dE is given by (9.5) and T by (9.3), we obtain

$$-4\pi G(\rho_m + p_m) = \left(\dot{H} - \frac{k}{a^2}\right) - 4\tilde{a}\frac{1}{H}\left(H^2 + \frac{k}{a^2}\right)^2 \dot{\chi}(\mathcal{H}). \quad (9.24)$$

Hence, inserting the conservation equation (9.7) and integrating, we finally acquire

$$H^2 = \frac{8\pi G}{3}\rho_m + \frac{k}{a^2} + \frac{\Lambda}{3} + 4\tilde{a}\int_0^t \left(H^2 + \frac{k}{a^2}\right)^2 \dot{\chi}(H)dt. \quad (9.25)$$

Having in mind the discussion of the previous subsection, if we define the active BH number as the difference between BH formations and BH mergers, namely

$$N = N_{form} - N_{merg}, \quad (9.26)$$

we can express $\dot{\chi}(\mathcal{H})$ according to (9.21). Thus, inserting into (9.25) we finally obtain

$$H^2 = \frac{8\pi G}{3}\rho_m + \frac{k}{a^2} + \frac{\Lambda}{3} - 8\tilde{a} \int_0^t \left(H^2 + \frac{k}{a^2} \right)^2 \frac{dN}{dt} dt. \quad (9.27)$$

Interestingly enough, through the application of the gravity-thermodynamics conjecture in the case of Wald–Gauss–Bonnet entropy, we have obtained modified Friedmann equations depending on the BH formations and mergers and on the GB coupling constant. Note that in the case where the Gauss–Bonnet term is absent one recovers the standard cosmological paradigm, which was expected since in this case the Wald–Gauss–Bonnet entropy (9.15) recovers the standard Bekenstein–Hawking one, and thus the Euler characteristic change does not have any effect on the entropy and hence on the gravity-thermodynamics conjecture. Finally, note that while Einstein–Gauss–Bonnet theory in four dimensions leads to the same field equations with general relativity, since the Gauss–Bonnet term in four dimensions is a topological invariant, its implementation within the gravity-thermodynamics approach does lead to extra terms due to the entropy changes in the horizon caused by the topology changes brought about by the evolution of the Einstein–Gauss–Bonnet black holes.

We can rewrite equations (9.27) and (9.24) as

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_{DE}) \quad (9.28)$$

$$H^2 + \dot{H} = -\frac{4\pi G}{3}(\rho_m + 3p_m + \rho_{DE} + 3p_{DE}) \quad (9.29)$$

by introducing an effective dark energy sector of topological origin, with energy density and pressure defined as

$$\rho_{DE} = \frac{3}{8\pi G} \left(\frac{\Lambda}{3} - 8\tilde{a} \int_0^t \left(H^2 + \frac{k}{a^2} \right)^2 \frac{dN}{dt} dt \right), \quad (9.30)$$

and

$$p_{DE} = -\frac{1}{4\pi G} \left[\frac{\Lambda}{2} + 8\tilde{a} \frac{1}{H} \left(H^2 + \frac{k}{a^2} \right)^2 \frac{dN}{dt} - 12\tilde{a} \int_0^t \left(H^2 + \frac{k}{a^2} \right)^2 \frac{dN}{dt} dt \right]. \quad (9.31)$$

Additionally, we define the effective dark-energy equation-of-state parameter as $w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}}$, leading to

$$w_{DE} = -1 + \frac{2\tilde{a} \left(H^2 + \frac{k}{a^2} \right)^2 \frac{dN}{dt}}{H \left(\frac{\Lambda}{4} - 6\tilde{a} \int_0^t \left(H^2 + \frac{k}{a^2} \right)^2 \frac{dN}{dt} dt \right)}. \quad (9.32)$$

Lastly, from Eqs. (9.28),(9.29), and assuming that the matter sector is conserved, i.e. $\dot{\rho}_m + 3H(\rho_m + p_m) = 0$, we acquire that the effective dark energy sector is conserved too, namely $\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0$.

We close this section by mentioning that in the standard gravity-thermodynamics approach presented in equations (9.2)-(9.8) above, the cosmological constant arises as an integration constant. In the case of the modified Friedmann equations form Wald–Gauss–Bonnet entropy we also have this term, namely in the effective dark energy (9.30) we do have the cosmological constant (arising as integration constant). Thus, the Wald–Gauss–Bonnet extra terms are just corrections on top of the cosmological constant Λ , corrections that aim to improve the behavior comparing to Λ CDM scenario. Lastly, note that in our model the dark matter sector enters in the same way as it does in standard cosmology, namely it is a separate sector that it is added in the model by hand (actually by applying the gravity-thermodynamics conjecture one starts in (9.5) by considering that dark and baryonic matter exist in the Universe), i.e. dark matter is not related to the black holes.

9.2. Cosmic evolution of Wald–Gauss–Bonnet Topological Dark Energy

In the previous section we applied the gravity-thermodynamics conjecture in the case of Wald–Gauss–Bonnet entropy, and we obtained modified Friedmann equations and

an effective dark energy sector of topological origin, dependent on the black hole formation and merger. Therefore, we can now proceed to the investigation of the cosmological implications.

For convenience, we use the redshift z as the independent variable, defined as $1 + z = a_0/a$, and we set the current scale factor to $a_0 = 1$. Additionally, we will focus on dust matter (namely with $p_m = 0$), which implies $\rho_m = \rho_{m0}(1 + z)^3$, with ρ_{m0} the matter energy density at present. As usual, we introduce the dimensionless density parameters

$$\Omega_{DE} = \frac{8\pi G}{3H^2} \rho_{DE}, \quad (9.33)$$

and

$$\Omega_m = \frac{8\pi G}{3H^2} \rho_m. \quad (9.34)$$

Inserting the above into (9.30) we obtain

$$\rho_{DE}(z) = \frac{3}{8\pi G} \left\{ \frac{\Lambda}{3} - 8\tilde{a} \int_{z_i}^z [H^2 + k(1+z)^2]^2 \frac{dN}{dz} dz \right\}, \quad (9.35)$$

while (9.32) gives

$$w_{DE}(z) = -1 - \frac{2\tilde{a} [H^2 + k(1+z)^2]^2 (1+z) \frac{dN}{dz}}{\frac{\Lambda}{4} - 6\tilde{a} \int_{z_i}^z [H^2 + k(1+z)^2]^2 \frac{dN}{dz} dz}. \quad (9.36)$$

Lastly, we can introduce the deceleration parameter $q \equiv -1 - \dot{H}/H^2$, which in terms of redshift becomes

$$q(z) = -1 + \frac{(1+z)}{H(z)} \frac{dH(z)}{dz}. \quad (9.37)$$

As we see, the effective dark energy density depends on the black hole formation and merger rates, namely on $\frac{dN}{dz}$. Thus, in the following subsection, we provide an estimation for its value.

9.2.1. Black hole formation and merger rates

Let us estimate the rate of the number of BHs that form and merge per redshift. In order to achieve that, we need to estimate independently the black-hole formation rate and the black-hole merger rate.

Estimating the black-hole formation rate from the star formation rate

Since black holes typically evolve from massive stars, it is commonly assumed that the formation rate of BHs (BHFR) is proportional to the cosmic star formation rate (SFR) [456–458]. The cosmic star formation rate density is presented in Fig. 9.3, and its best fit form is given by Madau and Dickinson [459] as

$$\psi(z) = 0.015 \frac{(1+z)^{2.7}}{1 + [(1+z)/2.9]^{5.6}} M_{\odot} \text{year}^{-1} \text{Mpc}^{-3}. \quad (9.38)$$

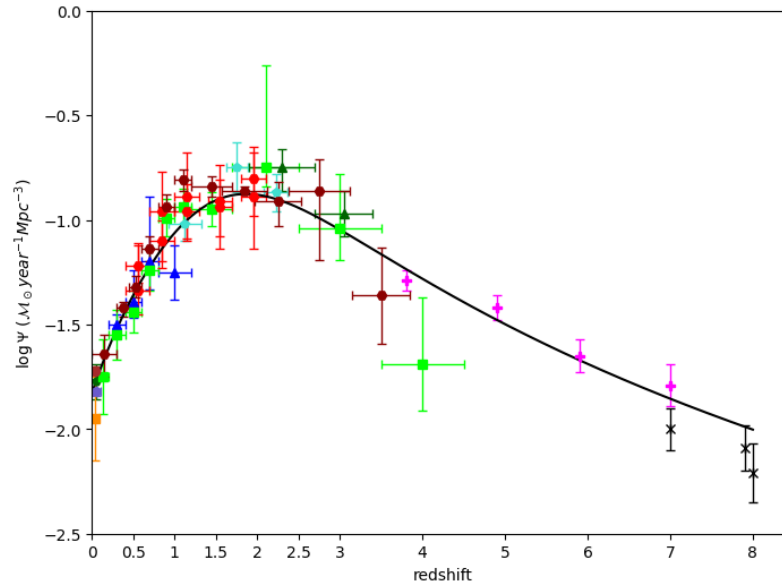


Figure 9.3.: *The history of cosmic star formation rate from [459]. On top of the data points, the black solid curve represents the best fit to the data, namely the Madau–Dickinson form (9.38).*

To estimate the BHFR from the SFR, we will consider the fraction of stellar mass that ends up as BH progenitors. This fraction depends on the initial mass function (IMF)

and the mass range of stars that collapse into BHs. Assuming a Salpeter IMF [460], which gives $\xi(m) \propto m^{-2.35}$, and considering stars with initial masses $m > 25 M_\odot$ as BH progenitors, the fraction f_{BH} of stellar mass forming BH progenitors is

$$f_{\text{BH}} = \frac{\int_{25 M_\odot}^{\infty} m \xi(m) dm}{\int_{0.1 M_\odot}^{\infty} m \xi(m) dm}, \quad (9.39)$$

and it represents the fraction of total stellar mass that goes into stars massive enough to eventually form BHs. According to various studies, f_{BH} ranges from approximately 0.001 to 0.05 [461, 462].

The average mass of progenitor stars that will evolve to form BHs, namely $\langle m_{\text{prog}} \rangle$, is typically in the range of $25 M_\odot$ to $40 M_\odot$ [463, 464]. Thus, the number rate density of BHs formed per unit volume per unit time, $\dot{\rho}_{\text{BH}}(z)$, is given by

$$\dot{\rho}_{\text{BH}}(z) = f_{\text{BH}} \frac{\psi(z)}{\langle m_{\text{prog}} \rangle}. \quad (9.40)$$

Hence, the number rate of BHs that form inside the apparent horizon of volume $\tilde{V}_A = \frac{4\pi}{3} \tilde{r}_A^3$, will be $\dot{N}_{\text{BH}} = \dot{\rho}_{\text{BH}} \tilde{V}_A$, and thus for a flat Universe from (9.2) we have $\tilde{r}_A = 1/H$, which leads to

$$\dot{N}_{\text{BH}}(z) = \frac{4\pi}{3} f_{\text{BH}} \frac{\psi(z)}{\langle m_{\text{prog}} \rangle H^3(z)}. \quad (9.41)$$

Finally, expressing the number rate of BHs formation per redshift instead of time, using $|dt/dz| = 1/H(z)(1+z)$, yields

$$\frac{dN_{\text{BH}}(z)}{dz} = \frac{4\pi}{3} f_{\text{BH}} \frac{\psi(z)}{\langle m_{\text{prog}} \rangle (1+z) H^4(z)}. \quad (9.42)$$

Estimating the black-hole merger rate from the black-hole formation rate

The binary black hole merger rate (BHMR) can also be assumed to be proportional to SFR. Nevertheless, ambiguities arise in this simplified approach as the formation efficiency of compact object is metallicity-dependent and BH formation and binary black hole merging may occur with a significant delay relative to the star formation epoch [465]. We are going to estimate approximately the BHMR from the BHFR by considering as main factors the fraction of massive stars that are in binary systems

and the merger efficiency. Observations suggest that a significant fraction of massive stars are in binaries, which is estimated around $f_{\text{bin}} \approx 0.7$ [466]. Nevertheless, not all binary black holes will merge within the age of the Universe. The merger efficiency f_{merge} accounts for the fraction of binary black holes that will eventually merge within a Hubble time. This efficiency depends on factors such as the initial separation and eccentricity of the binary system, and it is estimated to be between 0.01 and 0.1 [467–469]. Therefore, the black hole merger rate with respect to redshift will be

$$\frac{dN_{\text{BHMR}}(z)}{dz} = f_{\text{bin}} \times f_{\text{merge}} \times \frac{dN_{\text{BH}}(z)}{dz}, \quad (9.43)$$

which using (9.42) yields

$$\frac{dN_{\text{BHMR}}(z)}{dz} = \frac{4\pi}{3} f_{\text{BH}} \times f_{\text{bin}} \times f_{\text{merge}} \times \frac{\psi(z)}{\langle m_{\text{prog}} \rangle H^4(z)(1+z)}. \quad (9.44)$$

9.2.2. Specific examples

Having estimated the black-hole formation and merger rates in (9.42) and (9.44), we can insert in (9.26) and obtain the expression for the rate of the active number of BHs per redshift inside the apparent horizon as

$$\frac{dN(z)}{dz} = C \frac{\psi(z)}{H^4(z)(1+z)}, \quad (9.45)$$

where we have defined the constant

$$C \equiv \frac{4\pi}{3} \frac{(1 - f_{\text{bin}} \times f_{\text{merge}}) f_{\text{BH}}}{\langle m_{\text{prog}} \rangle}. \quad (9.46)$$

Hence, inserting (9.45) into (9.28), (9.35) and (9.36) we acquire

$$H^2 = H_0^2 \Omega_{m0} (1+z)^3 + \frac{\Lambda}{3} - 8\tilde{a}C \int_{z_i}^z \frac{\psi(z)}{(1+z)} dz, \quad (9.47)$$

$$\rho_{DE}(z) = \frac{3}{8\pi G} \left(\frac{\Lambda}{3} - 8\tilde{a}C \int_{z_i}^z \frac{\psi(z)}{(1+z)} dz \right), \quad (9.48)$$

and

$$w_{DE}(z) = -1 - \frac{2\tilde{a}C\psi(z)}{\frac{\Lambda}{4} - 6\tilde{a}C \int_{z_i}^z \frac{\psi(z)}{(1+z)} dz}, \quad (9.49)$$

with H_0 the current value of the Hubble function. As we see, apart from the constants, the only dynamical function that enters the equations is the star formation rate $\psi(z)$, which is given by (9.38). The integral can be evaluated in terms of the hypergeometric function ${}_2F_1(a, b; c; z)$, and gives

$$\int \frac{\psi(z)}{(1+z)} dz = 0.005555 \cdot (1+z)^{2.7} {}_2F_1\left(0.482143, 1.0; 1.48214; -0.00257378 \cdot (1+z)^{5.6}\right). \quad (9.50)$$

Finally, concerning the involved parameters, in Table 9.1 we display the possible values according to the literature.

Table 9.1.: Range of the involved parameters according to the literature.

Parameter	Value	Reference
f_{BH}	0.1% to 5%	[461, 462]
$\langle m_{\text{prog}} \rangle$	25 to 40 $M_{\odot}$	[463, 464]
f_{merge}	1% to 10%	[467–469]
f_{bin}	50% to 80%	[466, 470, 471]

$\tilde{a} > 0$ case

We start our analysis by examining the case where the GB coupling constant $\tilde{a}$ is positive. We elaborate the cosmological equations numerically, imposing $\Omega_{DE}(z=0) \equiv \Omega_{DE0} \approx 0.69$ and $\Omega_m(z=0) \equiv \Omega_{m0} \approx 0.31$ as required by observations [107]. In the upper graph of Fig. 9.4 we present the evolution of the dimensionless density parameters $\Omega_{DE}(z)$ and $\Omega_m(z)$. We observe that we can acquire the usual thermal history of the Universe, the standard sequence of matter and dark energy epochs, while in the asymptotic future ($z \rightarrow -1$) the Universe results to be completely dominated by dark energy. In the middle graph we draw the evolution of the dark-energy equation-of-state parameter $w_{DE}(z)$, where we can see that it lies slightly in the phantom regime for intermediate redshifts, well inside the observational bounds [107], while it

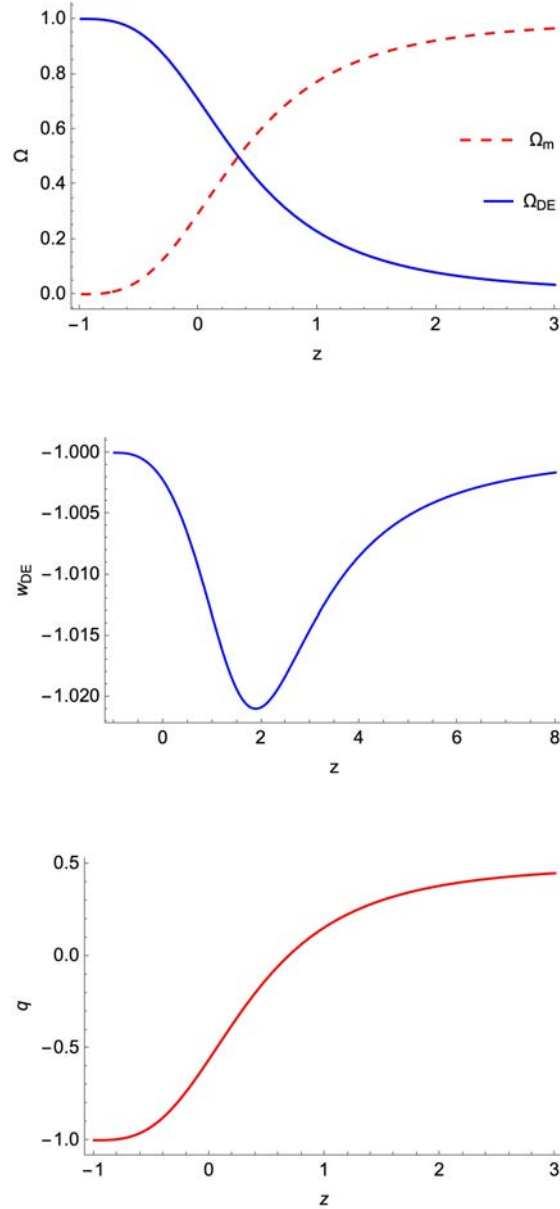


Figure 9.4.: Upper graph: The evolution of the dimensionless dark energy parameter Ω_{DE} (blue-solid) and the corresponding matter density parameter Ω_m (red-dashed) as a function of redshift, for the modified scenario with Wald–Gauss–Bonnet entropy, with $\tilde{\alpha} > 0$. Middle graph: The evolution of the dark energy equation of state parameter w_{DE} . Lower graph: The evolution of the deceleration parameter q . In all graphs we have used the models parameters $\tilde{\alpha} = 10^5$ (in H_0 units), $f_{BH} = 0.025$, $m_{prog} = 30M_{\odot}$, $f_{bin} = 0.65$, $f_{merge} = 0.05$ and we have implemented $\Omega_{DE0} = 0.69$.

approaches $w_{DE} \rightarrow -1$ in the distant past and in the asymptotic future. Finally, in the last graph of Fig. 9.4 we depict the deceleration parameter $q(z)$, where we can see that the transition from deceleration to acceleration occurs at $z_{tr} \approx 0.6$ in agreement with observations.

We proceed by studying the effect of the positive GB coupling constant $\tilde{\alpha}$ on the dark-energy equation-of-state parameter. In Fig. 9.5 we can see that for small values, namely for $0 \leq \tilde{\alpha} \leq 10^3$ in H_0 units, the scenario coincides with Λ CDM paradigm. Nevertheless, as the value of $\tilde{\alpha}$ becomes larger, $w_{DE}(z)$ enters deeper into the phantom regime peaking at $z \approx 2$, while its behavior in the distant past and asymptotic future still resembles Λ CDM one.

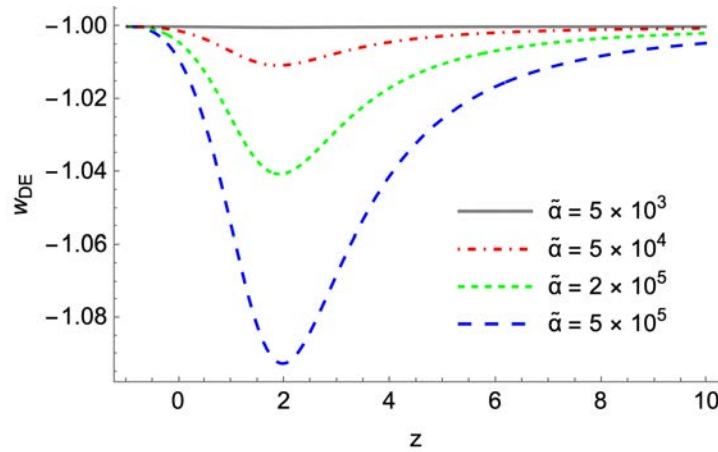


Figure 9.5.: The evolution of the effective dark-energy equation-of-state parameter w_{DE} for various values of positive GB coupling constant $\tilde{\alpha}$ in H_0 units. The other model parameters used in the calculation are $f_{BH} = 0.025$, $m_{prog} = 30M_{\odot}$, $f_{bin} = 0.65$, $f_{merge} = 0.05$, and we have imposed $\Omega_{DE0} = 0.69$. In all cases, the density parameters exhibit similar behaviors with those presented in the upper graph of Fig. 9.4.

In order to examine the effect of the BH formation factor f_{BH} on $w_{DE}(z)$, in Fig. 9.6 we demonstrate its evolution for various values of f_{BH} . We can see that for the smallest value according to the literature, i.e. $f_{BH} = 0.001$, the model tends to Λ CDM behavior. As f_{BH} increases $w_{DE}(z)$ enters deeper into the phantom regime and remains inside observational bounds [107] for the highest estimated value $f_{BH} = 0.001$. Finally, in the distant past and asymptotic future we obtain $w_{DE} \rightarrow -1$, independently of the parameter value.

Finally, we examine the behavior of $w_{DE}(z)$ with respect to the estimated range of values of the BH merging parameter f_{merge} . In Fig. 9.7 one can see that $w_{DE}(z)$ increases slightly as f_{merge} increases, however its effect is minor comparing to the

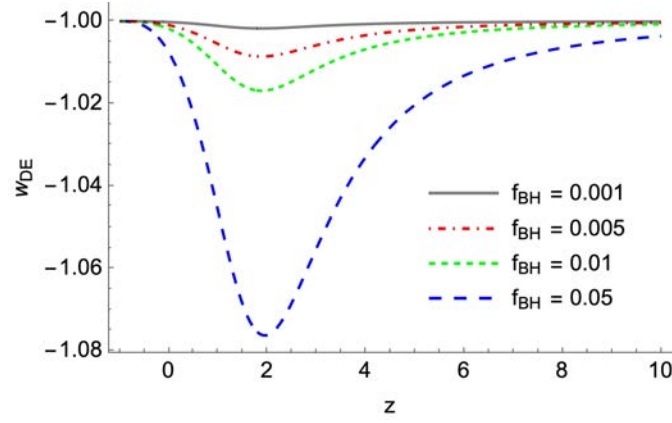


Figure 9.6.: The evolution of the effective dark-energy equation-of-state parameter w_{DE} for various values of the fraction of stars that form black holes f_{BH} . The other model parameters used in the calculation are $\tilde{\alpha} = 2 \times 10^5$ in H_0 units, $m_{prog} = 30M_{\odot}$, $f_{bin} = 0.65$, $f_{merge} = 0.05$ and we have imposed $\Omega_{DE0} = 0.69$. In all cases the density parameters exhibit similar behaviors with those presented in the upper graph of Fig. 9.4.

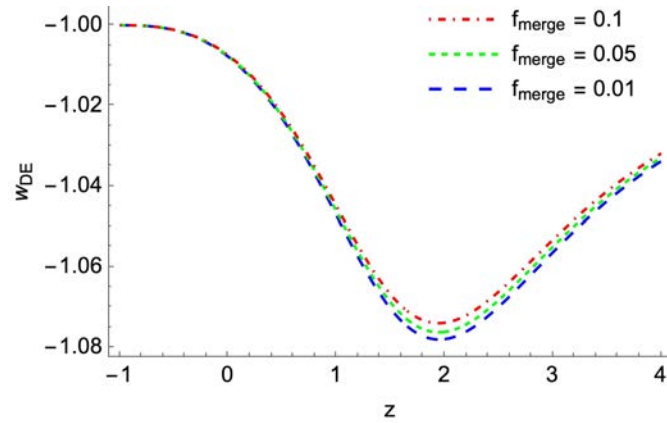


Figure 9.7.: The evolution of the effective dark-energy equation-of-state parameter w_{DE} for various values of the fraction of BHs that eventually merge f_{merge} . The other model parameters used in the calculation are $\tilde{\alpha} = 2 \times 10^5$ in H_0 units, $f_{BH} = 0.025$, $m_{prog} = 30M_{\odot}$, $f_{bin} = 0.65$, and we have imposed $\Omega_{DE0} = 0.69$. In all cases the density parameters exhibit similar behaviors with those presented in the upper graph of Fig. 9.4.

previous parameters. Again, in the distant past and asymptotic future we have $w_{DE} \rightarrow -1$, independently of the parameter value.

$\tilde{\alpha} < 0$ case

We now present the cosmological behavior for negative GB coupling constant $\tilde{\alpha} < 0$. In the upper graph of Fig. 9.8 we present the evolution of $\Omega_{DE}(z)$ and $\Omega_m(z)$, where one can see that the anticipated thermal history of the Universe and the sequence of matter and dark energy epochs are acquired. Furthermore, in the asymptotic future dark energy dominates completely, leading the Universe to a de Sitter phase. In the middle graph we observe that the dark-energy equation-of-state parameter lies in the quintessence regime ($w_{DE} > -1$) for small redshifts, peaking at $z \approx 2$, and in the asymptotic future it approaches $w_{DE} \rightarrow -1$. Finally, in the lower graph we display the evolution of the deceleration parameter, and we can see that the transition from deceleration to acceleration happens at $z_{tr} \approx 0.6$ as required.

In Fig. 9.9 we plot w_{DE} for different values of the negative GB coupling constant. For small absolute values, namely for $-10^3 \leq \tilde{\alpha} \leq 0$ in H_0 units we retrieve Λ CDM behavior, while when the absolute value of the parameter $\tilde{\alpha}$ increases then w_{DE} enters upwards to the quintessence regime. Additionally, in Fig. 9.10 we draw w_{DE} for different values of the fraction f_{BH} of stars that form BHs. As we can see, for the smallest value of the estimated range, i.e. for $f_{BH} = 0.001$, w_{DE} tends to Λ CDM behavior, while as the values of f_{BH} increases, $w_{DE}(z)$ enters higher to the quintessence regime. However, for the whole allowed f_{BH} range showed in Table 9.1, $w_{DE}(z)$ remains always inside the observational bound [107]. Lastly, in Fig. 9.11 we plot $w_{DE}(z)$ for various values of the fraction of BHs that merge. As we observe, $w_{DE}(z)$ decreases slightly as f_{merge} increases, however its effect is minor comparing to the previous parameters.

9.3. Using Wald–Gauss–Bonnet Topological Dark Energy to simultaneously alleviate the H_0 and σ_8 tensions

In this section, we will present how Wald–Gauss–Bonnet topological dark energy can alleviate H_0 and σ_8 tensions simultaneously. As it was demonstrated in the previous

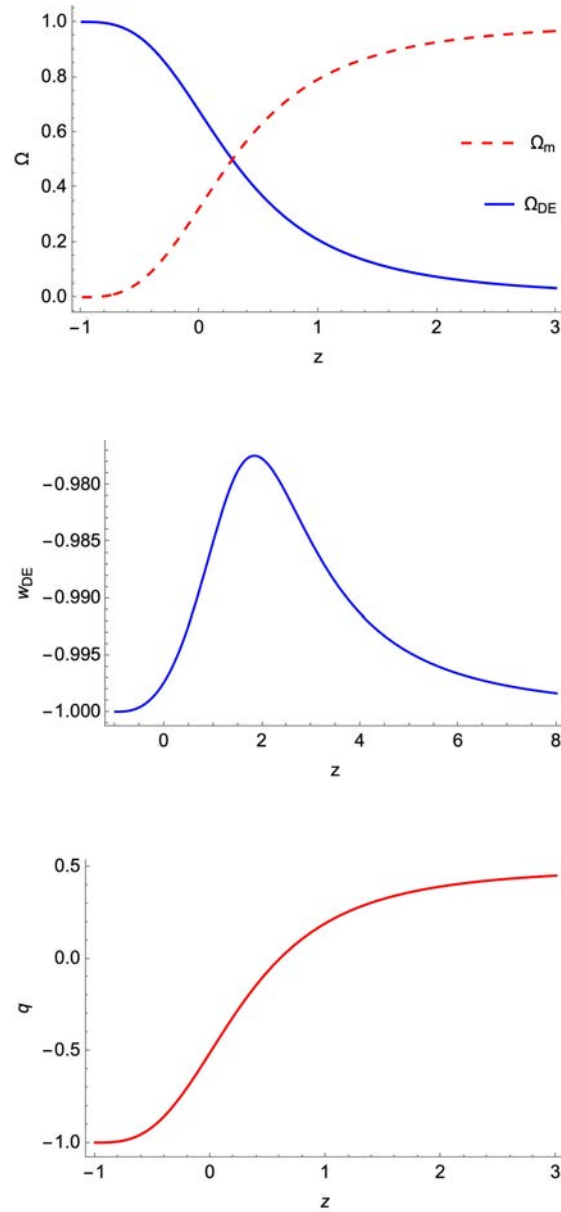


Figure 9.8.: Upper graph: The evolution of the dimensionless dark energy parameter Ω_{DE} (blue-solid) and the corresponding matter density parameter Ω_m (red-dashed) as a function of redshift, for the modified scenario with Wald–Gauss–Bonnet entropy, with $\tilde{\alpha} < 0$. Middle graph: The evolution of the dark energy equation of state parameter w_{DE} . Lower graph: The evolution of the deceleration parameter q . In all graphs we have used the models parameters $\tilde{\alpha} = -10^5$ (in H_0 units), $f_{BH} = 0.025$, $m_{prog} = 30M_{\odot}$, $f_{bin} = 0.65$, $f_{merge} = 0.05$ and we have implemented $\Omega_{DE0} = 0.69$.

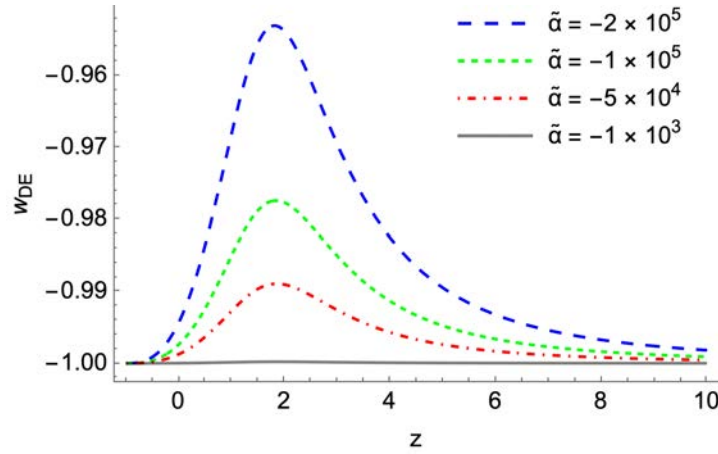


Figure 9.9.: The evolution of the effective dark-energy equation-of-state parameter w_{DE} for various values of negative GB coupling constant $\tilde{\alpha}$ in H_0 units. The other model parameters used in the calculation are $f_{\text{BH}} = 0.025$, $m_{\text{prog}} = 30M_{\odot}$, $f_{\text{bin}} = 0.65$, $f_{\text{merge}} = 0.05$, and we have imposed $\Omega_{\text{DE}0} = 0.69$. In all cases, the density parameters exhibit similar behaviors with those presented in the upper graph of Fig. 9.4.

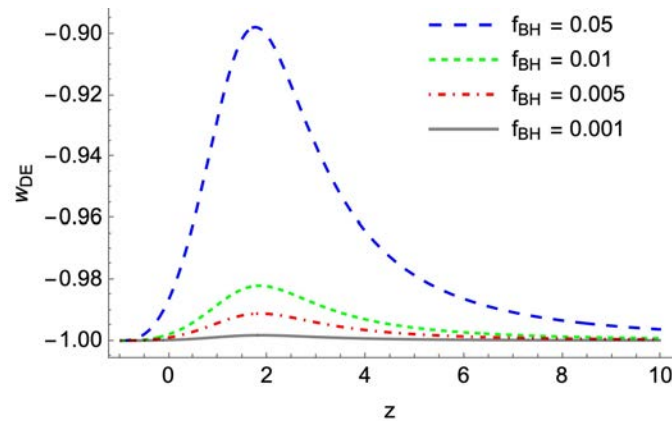


Figure 9.10.: The evolution of the effective dark-energy equation-of-state parameter w_{DE} for various values of the fraction of stars that form black holes f_{BH} . The other model parameters used in the calculation are $\tilde{\alpha} = -2 \times 10^5$ in H_0 units, $m_{\text{prog}} = 30M_{\odot}$, $f_{\text{bin}} = 0.65$, $f_{\text{merge}} = 0.05$, and we have imposed $\Omega_{\text{DE}0} = 0.69$. In all cases the density parameters exhibit similar behaviors with those presented in the upper graph of Fig. 9.4.

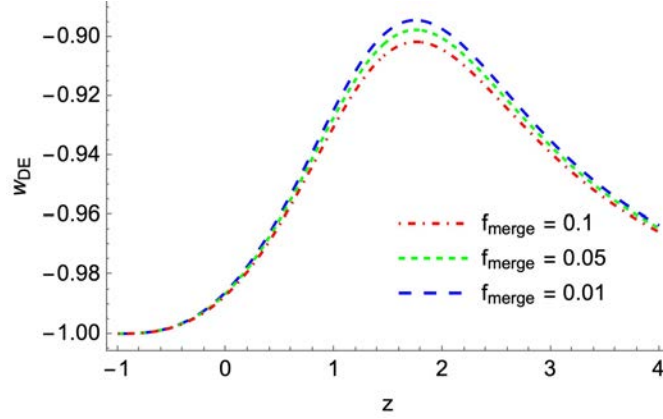


Figure 9.11.: The evolution of the effective dark-energy equation-of-state parameter w_{DE} for various values of the fraction of black holes that eventually merge f_{merge} . The other model parameters used in the calculation are $\tilde{\alpha} = -2 \times 10^5$ in H_0 units, $f_{BH} = 0.025$, $m_{\text{prog}} = 30M_{\odot}$, $f_{\text{bin}} = 0.65$, and we have imposed $\Omega_{DE0} = 0.69$. In all cases, the density parameters exhibit similar behaviors with those presented in the upper graph of Fig. 9.4.

section, Wald–Gauss–Bonnet topological dark energy exhibits phantom behavior at intermediate redshifts for positive coupling constants, which is known to be a sufficient mechanism that can lead to the H_0 tension alleviation. Indeed, we are going to present that for particular regions of the parameter space the scenario at hand does lead to higher H_0 values, and we verify this result performing a confrontation with observational data from Supernovae Type Ia (SNIa) and Cosmic Chronometers (CC) measurements. Additionally, analyzing the growth of structures and the evolution of the matter overdensity we show that the redshift behavior of the $f\sigma_8$ observable obtains smaller values than the ones arising in Λ CDM cosmology, and thus the scenario at hand can alleviate the σ_8 tension too. This is an advantage of Wald–Gauss–Bonnet cosmology, since the simultaneous alleviation of both tensions cannot be easily achieved within other entropic scenarios.

9.3.1. Alleviating the H_0 tension

As we have mentioned above, one efficient mechanism to alleviate the H_0 tension is an equation of state parameter lying in the phantom regime [472–474]. Hence, we will

first explore the parametric space of $\tilde{\alpha}$, as well as of f_{BH} , $\langle m_{prog} \rangle$, f_{bin} , f_{merge} absorbed in the parameter C , in order to fulfill the above requirements.

In Wald–Gauss–Bonnet topological dark energy the Hubble function is given by (9.47), while in the paradigm of Λ CDM it is given by

$$H_{\Lambda CDM}(z) = H_0 \sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda}}, \quad (9.51)$$

where $\Omega_{\Lambda} = 1 - \Omega_{m0}$. Since in our scenario the effective dark energy sector arises from the creation and merger of astrophysical black holes, which are expected to be negligible for $z \geq 20$, we deduce that at the CMB redshift $z_{CMB} \approx 1100$ the model coincides with Λ CDM, namely $H(z) \rightarrow H_{\Lambda CDM}$. Hence, we choose $z_i = z_{CMB}$ as the initial value for the calculation of the integral in Eqs. (9.47),(9.49). Focusing on dust matter we choose values for the model parameters that can give rise to the usual thermal history of the Universe, with the standard sequence of matter and dark energy epochs, as it was done in [379], however to be able to lead to an increased H_0 today.

In Fig. 9.12 we present the normalized $H(z)/(1+z)^3$ as a function of redshift for the Λ CDM scenario as well for the Wald–Gauss–Bonnet topological dark energy, for various values of the coupling constant $\tilde{\alpha}$. As we can observe, the scenario indeed coincides with Λ CDM cosmology at high and intermediate redshifts, while at smaller redshifts it yields higher H_0 values as the coupling constant increases. Specifically, for $\tilde{\alpha} = 4.5 \cdot 10^5$ (in units where $8\pi G = 1$) the value of H_0 approximates $H_0 \approx 74 \text{ km/s/Mpc}$. On the other hand, for values $\tilde{\alpha} \leq 10^4$ the model retrieves Λ CDM behavior, which was expected since for $\tilde{\alpha} \rightarrow 0$ the Friedmann equation (9.47) recovers the standard Λ CDM one.

Another parameter that has a significant effect on H_0 is the fraction f_{BH} of stars that will evolve into BHs. In Fig. 9.13 we depict the normalized $H(z)/(1+z)^3$ for various values of f_{BH} . As we can see, H_0 increases for increasing f_{BH} values. In particular, for $f_{BH} = 0.05$ we obtain $H_0 \approx 74 \text{ km/s/Mpc}$, while for values $f_{BH} \sim 0.001$ the model tends back to Λ CDM.

In order to validate further our results, we confront the scenario with observational data from Supernovae Type Ia (SNIa) and Cosmic Chronometers (CC) measurements, using the publicly available Cobaya code [475].

Concerning the SNIa data, we utilize the binned Pantheon sample [476], which approximates the full dataset with a reduced set $N = 40$ data points within the

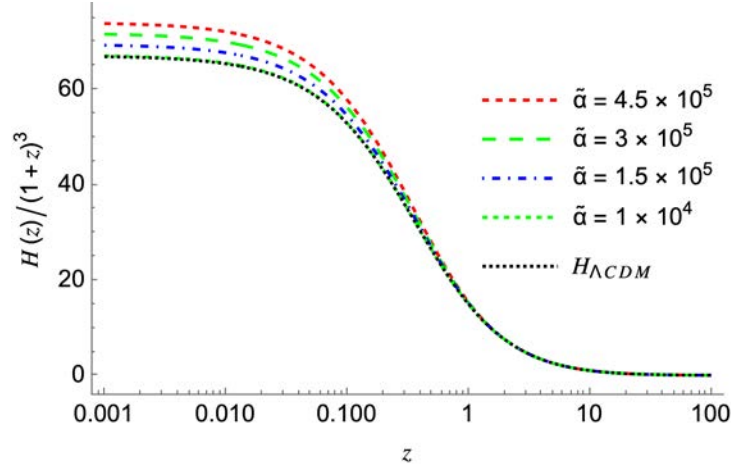


Figure 9.12.: The normalized $H(z)/(1+z)^3$ in units of km/s/Mpc with respect to redshift z , for ΛCDM cosmology (black - dotted) and for Wald–Gauss–Bonnet topological dark energy for $\tilde{\alpha} = 10^4$ (yellow - dashed), $\tilde{\alpha} = 1.5 \cdot 10^5$ (blue - dotted-dashed), $\tilde{\alpha} = 3 \cdot 10^5$ (green - large - dashed) and $\tilde{\alpha} = 4.5 \cdot 10^5$ (red - small-dashing), in units where $8\pi G = 1$. In all cases we have used $f_{\text{BH}} = 0.025$, $m_{\text{prog}} = 30M_{\odot}$, $f_{\text{bin}} = 0.65$, $f_{\text{merge}} = 0.05$ and we have set $\Omega_{m0} = 0.31$.

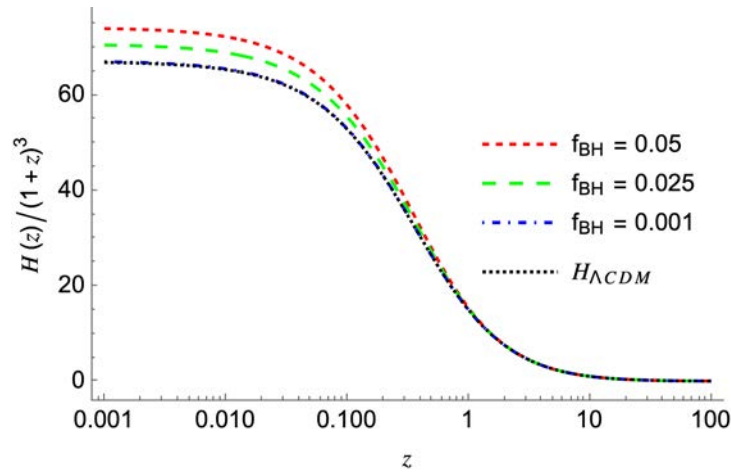


Figure 9.13.: The normalized $H(z)/(1+z)^3$ in units of km/s/Mpc with respect to redshift z , for ΛCDM cosmology (black - dotted) and for Wald–Gauss–Bonnet topological dark energy with $\tilde{\alpha} = 2.3 \times 10^5$ in units where $8\pi G = 1$ and $f_{\text{BH}} = 0.001$ (blue - dotted-dashed), $f_{\text{BH}} = 0.025$ (green - large - dashed) and $f_{\text{BH}} = 0.05$ (red - small-dashing). In all cases we have used $m_{\text{prog}} = 30M_{\odot}$, $f_{\text{bin}} = 0.65$, $f_{\text{merge}} = 0.05$ and we have set $\Omega_{m0} = 0.31$.

redshift range $0.01 \lesssim z \lesssim 1.6$. The chi-square function is given by $\chi_{\text{SNIa}}^2(\phi^\nu) = {}_{\text{SNIa}}^{-1} \mathbf{C}_{\text{SNIa}, \text{cov}}^{-1} {}_{\text{SNIa}}^{-T}$, where ${}_{\text{SNIa}}^{-1} = \{\mu_1 - \mu_{\text{th}}(z_1, \phi^\nu), \dots, \mu_N - \mu_{\text{th}}(z_N, \phi^\nu)\}$, where ϕ^λ is the statistical vector with the free parameters. Moreover, the distance modulus is $\mu_i = \mu_{B,i} - \mathcal{M}$, with $\mu_{B,i}$ the apparent magnitude at maximum brightness in the rest frame of z_i . The parameter $\mathcal{M}$ accounts for the dependence of the observed distance modulus, μ_{obs} , on H_0 , and on the fiducial cosmological model. Lastly, the theoretical distance modulus is $\mu_{\text{th}} = 5 \log \left[\frac{d_L(z)}{\text{Mpc}} \right] + 25$, where $d_L(z) = c(1+z) \int_0^z \frac{dx}{H(x, \phi^\nu)}$ is the luminosity distance assuming a spatially flat Universe.

Concerning the Cosmic Chronometers, we employ the latest compilation of the $H(z)$ dataset, as presented in [477], comprising $N = 36$ measurements data within $0.07 \lesssim z \lesssim 2.36$. In this case, the corresponding χ_H^2 function is expressed as $\chi_H^2(\phi^\nu) = \mathcal{H} \mathbf{C}_{H, \text{cov}}^{-1} \mathcal{H}^T$, with $\mathcal{H} = \{H_1 - H_0 E(z_1, \phi^\nu), \dots, H_N - H_0 E(z_N, \phi^\nu)\}$ and where H_i are the observed Hubble values at z_i ($i = 1, \dots, N$). Finally, r_{drag} is the comoving sound horizon at the drag epoch.

Parameters	1σ Values
H_0	$75.2 \pm 4.7 \text{ km/s/Mpc}$
Ω_{m0}	0.38 ± 0.05
$\tilde{a}$	$(9.4 \pm 7.1) \times 10^4$
Ω_{DE0}	0.62 ± 0.05
r_{drag}	$212.1 \pm 31.2 \text{ Mpc}$

Table 9.2.: *Observational constraints of the cosmological parameters at 1σ confidence level. The coupling $\tilde{a}$ is measured in units where $8\pi G = 1$.*

In Table 9.2 we display the results on the parameters, and in Fig. 9.14 we show the corresponding contour plots for the model parameters. As we can see, $H_0 = 75.2 \pm 4.7 \text{ km/s/Mpc}$ hence, we verify that the Hubble tension can be alleviated. Concerning the coupling parameter $\tilde{a}$, we find that it is constrained at $\tilde{a} = (9.4 \pm 7.1) \times 10^4$.

Let us examine the mechanism that causes the alleviation of the H_0 tension. In Fig. 9.15 we present the evolution of the dark-energy equation of state given by (9.49) for the coupling constant value $\tilde{a} = 4.5 \cdot 10^5$ that can alleviate the tension. As we can see, at intermediate redshifts w_{DE} lies slightly in the phantom regime, which is a known sufficient way to alleviate the tension, as we discussed previously. Note that in

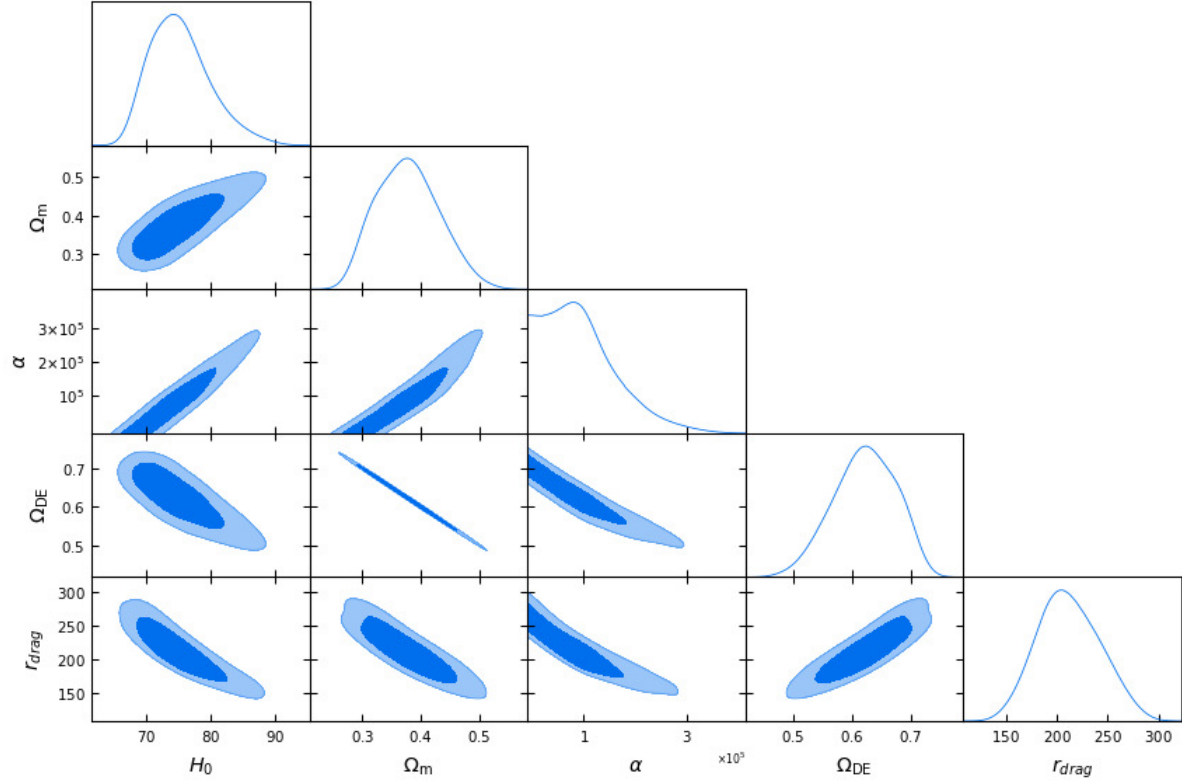


Figure 9.14.: The 1σ and 2σ iso-likelihood contours for Wald–Gauss–Bonnet topological dark energy, for the 2D subsets of the parameter space using Cobaya code [475] (see text).

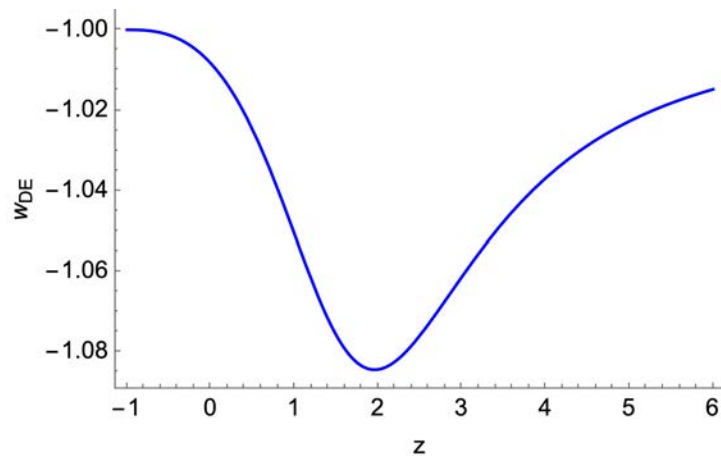


Figure 9.15.: Evolution of the dark-energy equation-of-state parameter w_{DE} with respect to low redshift z , for Wald Gauss Bonnet cosmology with coupling constant value $\tilde{\alpha} = 4.5 \cdot 10^5$ in units where $8\pi G = 1$, and with $f_{\text{BH}} = 0.025$, $m_{\text{prog}} = 30M_{\odot}$, $f_{\text{bin}} = 0.65$ and $f_{\text{merge}} = 0.05$.

the far past, as well as in the far future, w_{DE} acquires the cosmological constant value -1 , as it was discussed in [379].

9.3.2. Alleviating the σ_8 tension

Let us now proceed to the investigation of perturbations. As usual, we introduce the matter overdensity $\delta_m := \delta\rho_m/\rho_m$. As one can see from the Friedmann equation (9.47), in the scenario at hand we do not obtain an effective Newton's constant, i.e. it is still given by G [478]. This was expected since, as we mentioned above, action (9.9) is linear in the Gauss–Bonnet term, and thus it does not have any effect on the field equations, both at the background and perturbative levels. In scenarios where the gravitational constant does not change comparing to general relativity, and assuming that dark energy does not cluster, the differential equation of the overdensity evolution Eq. (6.70) when transformed with respect to the redshift z it becomes [479, 480]

$$\delta_m'' + \left(\frac{H'(z)}{H(z)} - \frac{1}{1+z} \right) \delta_m' - \frac{3 \Omega m_0 H_0^2 (1+z)}{H^2} \delta_m = 0. \quad (9.52)$$

Furthermore, after extracting the solution for $\delta_m(z)$ we calculate the important physical observable

$$f\sigma_8 \equiv f(z)\sigma(z), \quad (9.53)$$

where $f(z) := -\frac{d \ln \delta_m(z)}{d \ln z}$ and $\sigma(z) := \sigma_8 \frac{\delta_m(z)}{\delta_m(0)}$ [472].

Let us now examine the growth of structures and the $f\sigma_8$ observable. We elaborate Eq. (9.52) numerically, with the evolution of $H(z)$ obtained above, for the parameter values that alleviate the H_0 tension, as an input. In Fig. 9.16 we depict the obtained results for $f\sigma_8$ of Wald–Gauss–Bonnet cosmology. For comparison, we also display the corresponding curve for Λ CDM paradigm, as well as the observational data from SDSS-III DR12 and SDSS-IV DR14. As we observe, in Wald–Gauss–Bonnet cosmology we do obtain a lower $f\sigma_8$ values at low redshifts, and thus we can alleviate the tension, while remaining inside the observational bounds.

In summary, we deduce that Wald–Gauss–Bonnet topological dark energy can alleviate the H_0 and σ_8 tensions. We mention that the simultaneous alleviation of both tensions cannot be easily achieved with the same mechanism [472]. Especially in

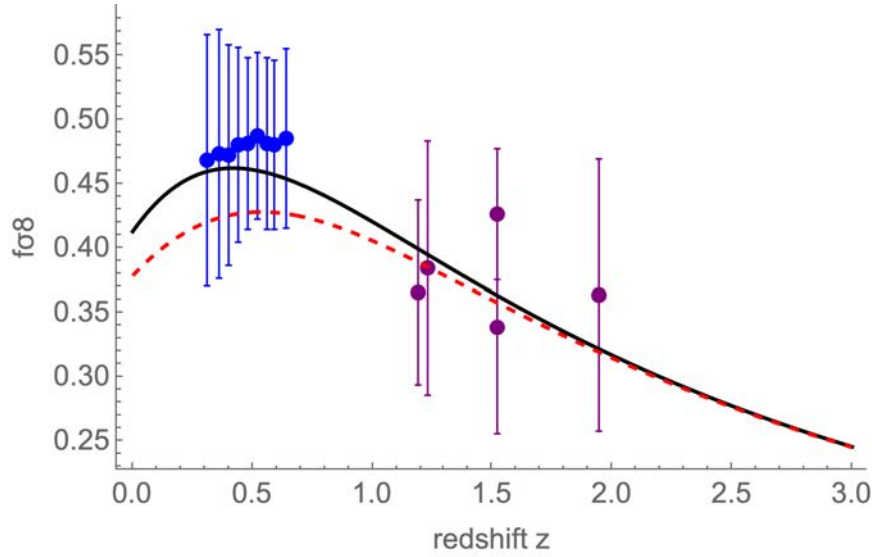


Figure 9.16.: Evolution of $f\sigma_8$ in Wald–Gauss–Bonnet cosmology with $\tilde{\alpha} = 4.5 \cdot 10^5$ in units where $8\pi G = 1$, and with $f_{BH} = 0.025$, $m_{prog} = 30M_{\odot}$, $f_{bin} = 0.65$, $f_{merge} = 0.05$ (red dashed), as well as in Λ CDM paradigm (black solid). Additionally, the blue data points are from Baryonic Acoustic Oscillations (BAO) observations in SDSS-III DR12 [481], while the gray data points at higher redshifts are from SDSS-IV DR14 [481–483].

other modified cosmological scenarios based on extended entropies, such as Barrow cosmology [484,485] and Kaniadakis cosmology [486], the H_0 tension can be alleviated, but the σ_8 one is not improved, only in Tsallis cosmology one may have an alleviation of both tensions [208]. Hence, the aforementioned features act as a significant advantage of Wald–Gauss–Bonnet cosmology.

9.4. Conclusions

In this chapter, we have investigated the Wald–Gauss–Bonnet entropy in the framework of spacetime thermodynamics. The latter is a strong conjecture that connects gravity and thermodynamics, since by applying the black-hole physics in the Universe apparent horizon we can result to the Friedmann equations just starting from the first law of thermodynamics.

We have implemented the above approach in the case of Einstein–Gauss–Bonnet gravity and its corresponding Wald–Gauss–Bonnet entropy, which due to the Chern–Gauss–Bonnet theorem it is related to the Euler characteristic of the Universe topology. Nevertheless, it is known that in the case of the GB extension of general relativity,

during BH merging the second law is violated. In order to remove the violation, we introduced a topological link between the apparent horizon and the BH horizons, a connection that is known to hold according to holographic principle. Hence, through the gravity-thermodynamics approach we extracted modified Friedmann equations, where the new terms depend on the topology changes induced by the black-hole formation and merger. Specifically, we obtained an effective, dark energy sector of topological origin, which evolves in time according to the black-hole formation and merger rates.

In order to investigate the cosmological evolution, we estimated the BH formation rate starting from the star formation rate, and moreover we estimated the black-hole merger rate from the black-hole formation rate. Ultimately, we resulted to a dark-energy energy density that depends only on the cosmic star formation rate density per redshift $\psi(z)$, which is parametrized very efficiently by the Madau–Dickinson form (9.38), with the remaining model parameters being the fraction of stars forming BHs f_{BH} , the fraction of black holes that eventually merge f_{merge} , the fraction of massive stars that are in binaries f_{bin} , and the average mass of progenitor stars that will evolve to form BHs $\langle m_{\text{prog}} \rangle$, as well as the GB coupling constant $\tilde{\alpha}$. The GB coupling is the only completely free parameter, since the other parameters have specific ranges according to the literature.

We investigated in detail the evolution of the dark energy and matter density parameters, of the effective dark-energy equation-of-state parameter, and of the deceleration parameter. As we saw, the Universe evolves according to the usual thermal history, with successive dark energy and matter epochs, and the transition from deceleration to acceleration takes place at $z \approx 0.6$ in agreement with observations, while at asymptotically late times the Universe results in a de Sitter phase completely dominated by dark energy. Concerning the dark-energy equation-of-state parameter, we showed that it exhibits a different behavior according to the sign and value of the GB coupling $\tilde{\alpha}$. For positive values of the GB coupling, the dark energy exhibits phantom-like behavior, while for negative values it exhibits quintessence-like behavior. Interestingly enough, at early and late times w_{DE} tends to the cosmological constant value -1 and the deviation happens only at intermediate redshifts, with a peak at around $z \approx 2$, a behavior that was expected since at early times there are no stars, while at asymptotically late times most black holes will have merged. Furthermore, for small absolute $\tilde{\alpha}$ values, the scenario tends to Λ CDM paradigm. Finally, we investigated the effect of the other model parameters, showing that increasing f_{BH} enhances the

deviations from Λ CDM scenario, while f_{merge} has only a minor effect. Nevertheless, for the whole allowed estimated ranges of the parameters, w_{DE} remains within its observational bounds found by Planck Collaboration.

Subsequently, we have shown how Wald–Gauss–Bonnet topological dark energy can alleviate the H_0 and σ_8 tensions simultaneously. This modified cosmological scenario is obtained by applying the gravity-thermodynamics conjecture to the Universe apparent horizon, employing the Wald–Gauss–Bonnet entropy instead of the standard Bekenstein–Hawking one. Under the assumption that the apparent horizon is topologically linked with the interior BH horizons, we obtained modified Friedmann equations that depend on the BH formation and merging rates, which can be approximated by the star formation rate. Therefore, the modified Friedmann equations depend on the Gauss–Bonnet coupling constant $\tilde{\alpha}$ and on measured astrophysical parameters.

Wald–Gauss–Bonnet topological dark energy has interesting cosmological phenomenology, and it exhibits phantom behavior for positive $\tilde{\alpha}$ and quintessence behavior for negative $\tilde{\alpha}$ [379]. The phantom behavior at intermediate redshifts is a known sufficient mechanism that can lead to the H_0 tension alleviation. Indeed, we have shown that for particular regions of the parameter space, the scenario at hand does lead to higher H_0 values, exactly due to the suitable phantom behavior at intermediate redshifts. For completeness, we have verified this result performing a confrontation with observational data using the Cobaya package.

We proceeded to the investigation of the growth of structures and of the evolution of the matter overdensity. Since in Wald–Gauss–Bonnet cosmology we do not obtain an effective Newton’s constant different than the standard one, under the assumption that the effective topological dark energy does not cluster we were able to calculate the redshift behavior of the observable $f\sigma_8$. Interestingly enough, we saw that we obtained smaller values than the ones arising in Λ CDM cosmology, and thus the scenario at hand can alleviate the σ_8 tension too.

In summary, Wald–Gauss–Bonnet cosmology can relieve the H_0 and σ_8 tensions at the same time. This is an advantage of the scenario at hand, and it places it as a promising candidate for the description of nature.

Chapter 10.

Conclusions – Discussion

In this thesis, we have introduced and studied two new scenarios of dark energy arising from modified theories of gravity. Specifically, we have studied the implication of the Gauss-Bonnet term in the gravitational Lagrangian in $4D$, first at the level of spacetime foam during topological transitions as predicted by Euclidean Quantum Gravity and second in the spacetime thermodynamic conjecture framework in the case of a topologically dynamic apparent Universe horizon.

In chapter 7 we started by asking what happens to the variation of the Gauss-Bonnet term during the appearance of topologically alternating gravitational instantons. Since, the GB term is a topological invariant, during topological transitions at the microscopic scale as predicted by EQG, its variation with respect to the quantum fluctuation should not be trivial. On that ground, we proposed the topological variation procedure, where the variation of the quantum fluctuation causes a change of the Euler characteristic at the microscopic level, thus encapsulating the appearance of topology alternating instantons in the framework of EQG. Incorporating the aforementioned procedure to the semiclassical approximation, in the case of the EH plus the GB action term, we presented the formalism that yields the Einstein equation for the background quantities with an effective cosmological constant term of topological origin. The novel effective term equals the rate of topology change per four volume and can be interpreted as the four volume density of instantons. The formalism developed for the derivation of our result relies on the assumption of the topological variation procedure. This conjecture faces several ambiguities, since topology change involves the change of integer quantities, and we have assumed that this process can be approximated effectively with a classical variational procedure. Nevertheless, the

successful phenomenology of the model could motivate further research towards a rigor mathematical foundation of the aforementioned conjecture.

Proceeding in chapter 8 we developed the phenomenology of the spacetime foam topological dark energy model and confronted it with observations. Applying QFT techniques on the nucleation rate of gravitational instantons we were able to obtain a first order differential equation that describes the evolution of the density of instantons inside an evolving Hubble sphere. Should it be mentioned that although the initial definition of the effective cosmological constant depends on the Gauss–Bonnet coupling parameter, the differential equation of the model depends on the same number of observational parameters as Λ CDM. We confronted the numerical solution of the spacetime foam SFDE model with observational data, including SNIa, Baryonic Acoustic Oscillations, and cosmic chronometers datasets, and find it to outperform the standard Λ CDM model across multiple datasets, including Pantheon+, SH0ES, BAO, and Cosmic Chronometers (CC). Model comparison metrics such as AIC and DevIC indicate a moderate but statistically significant preference for the non-flat SFDE model, especially when CC data (which assume flatness) are excluded. The flat SFDE version also performs slightly better than Λ CDM, though the difference remains statistically comparable.

The models late time thermal history is close to the standard one, however in the past Ω_{DE} becomes greater than Ω_{Λ} and simultaneously Ω_m becomes smaller compared to Λ CDM scenario. This discrepancy maximizes around $z \approx 10$ and afterwards starts to diminish. Furthermore, the evolution of the radiation component is almost the same with the one in Λ CDM scenario. It is apparent that, at the time of radiation-matter equality, the evolution of Ω_m coincides with the standard one, although there is a small excess of Ω_{DE} over Ω_{Λ} . The DE equation of state parameter starts with a value $w = -0.9$ at the present and progresses backwards in time towards $w \rightarrow 0$. The latter behavior implies that the DE sector behaves as pressureless DM, namely the topological DE scenario exhibits the DE to dark matter (DM) conversion. This explains why the excess of DE compensates the lack of DM in comparison with the concordance cosmological model.

In conclusion, the spacetime foam topological DE scenario, consistently outperforms the standard Λ CDM model across multiple datasets. Furthermore, it provides a complete physical mechanism that generates Dark Energy and a fraction of Dark Matter, with energy transfer occurring dynamically within the dark sector. At early times, dark energy behaves like matter ($w \approx 0$), transitions gradually, and asymptotically

approaches $w = -0.9$ at late times. This dynamic evolution enables the SFDE scenario to possibly address key cosmological tensions, including the σ_8 discrepancy and the mismatch in the sound horizon scale r_d between CMB and late-time observations. Notably, the non-flat version performs better than the flat one, pointing to a significant role for cosmic curvature, with inferred values of Ω_{k0} reaching up to 20% -somewhat higher than typical values reported for Λ CDM. In parallel, preliminary tests show that SFDE scenario is consistent with Big Bang Nucleosynthesis, for both flat and non-flat cases. Furthermore, the equation describing the evolution of the cosmological constant results into a specific limit for the initial density at the beginning of time. Therefore, since the origin of spacetime foam topological dark energy resides in the quantum regime, it would be interesting to explore the model's application to the inflationary period. The aforementioned features make Spacetime Foam Topological Dark Energy a promising candidate for a more complete description of the Universe.

In chapter 9 we have explored the implications of the Gauss–Bonnet action term in the framework of the spacetime thermodynamics conjecture. The Einstein–Hilbert action yields the standard Bekenstein–Hawking entropy. Similarly, Wald presented how the Gauss–Bonnet action term corresponds to a new entropy term proportional to the Euler characteristic of the horizon. In that case it has been pointed out that during black hole horizon mergers there would be a decrease of entropy, hence a violation of the second thermodynamical law. Motivated by this ambiguity, we applied the spacetime thermodynamics conjecture formalism for the Wald–Gauss-Bonnet entropy and resulted in modified Friedmann equations containing an extra dark energy term proportional to the rate of topology change of the Universe apparent horizon. In order to resolve the violation of the second law, we made the assumption that the Universe apparent horizon is topologically dynamic in a way that compensates the topological entropy change during black horizon formation and merging. This link relies on the holographic principle. In that way, the rate of topology change of the Universe apparent horizon becomes proportional to the black holes formation and merging rate. We have approximated these rates based on the observational best fit model of star formation rate, and we have obtained an analytical solution for the new dark energy component.

The topological dark energy from black hole formation and merging deviates from Λ CDM only at late times, therefore it yields the same thermal history of the Universe as the concordance model. For different signs of the Gauss–Bonnet coupling parameter, we have obtained quintessence and phantom behavior of the model. The phantom

behavior at intermediate redshifts is a known sufficient mechanism that can lead to the H_0 tension alleviation. Indeed, we have shown that for particular regions of the parameter space, the scenario at hand does lead to higher H_0 values, exactly due to the suitable phantom behavior at intermediate redshifts. For completeness, we have verified this result performing a confrontation with observational data from Supernovae Type Ia (SNIa) and Cosmic Chronometers (CC) measurements, using the publicly available Cobaya code using the Cobaya package. We proceeded to the investigation of the growth of structures and of the evolution of the matter overdensity. Since in Wald-Gauss-Bonnet cosmology we do not obtain an effective Newton's constant different from the standard one, under the assumption that the effective topological dark energy does not cluster we were able to calculate the redshift behavior of the observable $f\sigma_8$. Interestingly enough, we saw that we obtained smaller values than the ones arising in Λ CDM cosmology, and thus the scenario at hand can alleviate the σ_8 tension too.

In summary, Wald-Gauss-Bonnet cosmology can relieve the H_0 and σ_8 tensions at the same time. Furthermore, as we proceed in the multimessenger astronomy era, datasets of increasing accuracy are expected, especially from the space based gravity wave detectors (e.g. LISA). Thus, the model's predictions dependency on black hole formation and merging rate, enhances its falsifiability in the light of the expected measurements. The aforementioned characteristics can be considered advantages of the scenario at hand, and they place it as promising candidate for the description of Nature.

Appendix A.

Background field expansion

*“Le savant n’étudie pas la nature parce que cela est utile;
il l’étudie parce qu’il y prend plaisir,
et il y prend plaisir parce qu’elle est belle.”*
— Henri Poincaré, 1854–1912

A.1. Einstein–Hilbert action expansion under metric perturbation

Under the metric split $g = \tilde{g} + h$, we are going to describe the expansion of the metric depended quantities. We will start with the expansion of the square root of the metric determinant

$$\begin{aligned}\sqrt{g} &= \sqrt{\tilde{g} + h} = \exp \log \sqrt{\tilde{g} + h} \\ &= \exp \left[\log \sqrt{\tilde{g}} + \frac{1}{2} \log \sqrt{1 + \tilde{g}^{-1}h} \right]\end{aligned}\tag{A.1}$$

we will now perform some logarithm and exponential expansions

$$\begin{aligned}
\sqrt{g} &= \sqrt{\tilde{g}} \exp \left[\frac{1}{2} \log \sqrt{1 + \tilde{g}^{-1} h} \right] \\
&= \sqrt{\tilde{g}} \exp \left[\frac{1}{2} \text{tr} \log(1 + \tilde{g}^{-1} h) \right] \\
&= \sqrt{\tilde{g}} \exp \left[\frac{1}{2} \text{tr}(\tilde{g}^{-1} h) - \frac{1}{2} (\tilde{g}^{-1} h)^2 + O(h)^3 \right] \\
&= \sqrt{\tilde{g}} \left(1 + \frac{1}{2} \text{tr}(\tilde{g}^{-1} h) - \frac{1}{4} \text{tr}(\tilde{g}^{-1} h)^2 + \frac{1}{8} \text{tr}^2(\tilde{g}^{-1} h) + O(h)^3 \right) \\
&= \sqrt{\tilde{g}} \left(1 + \frac{1}{2} h^\mu_\mu - h_{\mu\nu} h^{\mu\nu} + \frac{1}{8} (h^\mu_\mu)^2 + O(h)^3 \right)
\end{aligned} \tag{A.2}$$

Now one can proceed to the evaluation of the Levi-Civita connection

$$\begin{aligned}
\Gamma^\rho_{\mu\nu}(\tilde{g} + h) &= \frac{1}{2} g^{\rho\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \\
&= \frac{1}{2} (\tilde{g}^{\rho\sigma} - h^{\rho\sigma} + h^\lambda_\lambda h^{\lambda\sigma}) [(\tilde{g}_{\mu\sigma} + h_{\mu\sigma})_{,\nu} + (\tilde{g}_{\nu\sigma} + h_{\nu\sigma})_{,\mu} - (\tilde{g}_{\mu\nu} + h_{\mu\nu})_{,\sigma}] \\
&= \Gamma^\rho_{\mu\nu}(0) + \Gamma^\rho_{\mu\nu}(1) + \Gamma^\rho_{\mu\nu}(2) + O(h)^3,
\end{aligned} \tag{A.3}$$

where $\Gamma^\rho_{\mu\nu}(0)$ is the Christoffel symbol at 0-th order in h , similarly $\Gamma^\rho_{\mu\nu}(1)$ at 1st order and $\Gamma^\rho_{\mu\nu}(2)$ at 2nd order. Therefore, we can write

$$\Gamma^\rho_{\mu\nu}(0) = \frac{1}{2} \tilde{g}^{\rho\sigma} (\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}) + O(h)^3 \tag{A.4}$$

$$\begin{aligned}
\Gamma^\rho_{\mu\nu}(1) &= \frac{1}{2} \tilde{g}^{\rho\sigma} (h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}) - \frac{1}{2} h^{\rho\sigma} (\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}) \\
&= \frac{1}{2} \tilde{g}^{\rho\sigma} (h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}) - \tilde{g}^{\lambda\sigma} h_{\rho\sigma} \Gamma^\lambda_{\mu\nu}(0).
\end{aligned} \tag{A.5}$$

By using the covariant derivative applied to the fluctuation

$$\nabla_\sigma h_{\mu\nu} = \partial_\sigma h_{\mu\nu} - \Gamma^\lambda_{\sigma\mu} h_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu} h_{\mu\lambda}, \tag{A.6}$$

we can rewrite Eq. (A.5) as

$$\begin{aligned}
\Gamma^\rho_{\mu\nu}(1) &= \frac{1}{2} \tilde{g}^{\rho\sigma} \left[\partial_\nu h_{\mu\sigma} + \partial_\mu h_{\nu\sigma} - \partial_\sigma h_{\mu\nu} - 2h_{\sigma\lambda} \Gamma^\lambda_{\mu\nu} \pm \Gamma^\lambda_{\nu\sigma} h_{\mu\lambda} \pm \Gamma^\lambda_{\mu\sigma} h_{\nu\lambda} \right] \\
&= \frac{1}{2} \tilde{g}^{\rho\sigma} (\tilde{\nabla}_\nu h_{\mu\sigma} + \tilde{\nabla}_\mu h_{\nu\sigma} - \tilde{\nabla}_\sigma h_{\mu\nu}),
\end{aligned} \tag{A.7}$$

where the covariant derivative in the last equation is constructed using the background metric. As one can see, the perturbation at first order in h of the Christoffel symbol, involves only covariant derivatives, hence is a tensor. Similarly, we can evaluate the term

$$\Gamma_{\mu\nu}^\rho(2) = \frac{1}{2}h_\lambda^\rho h^{\lambda\sigma}(\tilde{g}_{\mu\sigma,\nu} + \tilde{g}_{\nu\sigma,\mu} - \tilde{g}_{\mu\nu,\sigma}) - \frac{1}{2}h^{\rho\sigma}(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}) \quad (\text{A.8})$$

$$= -h_\lambda^\rho \Gamma_{\mu\nu}^\lambda(1). \quad (\text{A.9})$$

Once we have expanded the Christoffel symbol, we can evaluate the Riemann tensor, since all the other curvatures can be easily recovered by contraction. In the same fashion, the Riemann tensor can be written as a series of order terms with respect to h

$$R_{\nu\rho\sigma}^\mu(\tilde{g} + h) = R_{\nu\rho\sigma}^\mu(0) + R_{\nu\rho\sigma}^\mu(1) + R_{\nu\rho\sigma}^\mu(2) + O(h)^3, \quad (\text{A.10})$$

where the order 0 in h is given by

$$R_{\nu\rho\sigma}^\mu(0) = \tilde{\nabla}_\rho \Gamma_{\sigma\nu}^\mu(0) - \tilde{\nabla}_\sigma \Gamma_{\rho\nu}^\mu(0), \quad (\text{A.11})$$

and the covariant derivative is written in terms of the background and from this point it will be noted as $\tilde{\nabla}$.

It is known that the variation of the Christoffel symbol is a tensor. Hence, one can write the expansion as

$$R_{\nu\rho\sigma}^\mu(1) = \partial_\rho \Gamma_{\sigma\nu}^\mu(1) - \partial_\sigma \Gamma_{\rho\nu}^\mu(1) + \Gamma_{\rho\lambda}^\mu \Gamma_{\sigma\nu}^\lambda(1) + \Gamma_{\rho\lambda}^\mu(1) \Gamma_{\sigma\nu}^\lambda - \Gamma_{\sigma\lambda}^\mu \Gamma_{\rho\nu}^\lambda(1) - \Gamma_{\sigma\lambda}^\mu(1) \Gamma_{\rho\nu}^\lambda, \quad (\text{A.12})$$

by adding and rearranging $\pm \Gamma_{\rho\sigma}^\lambda \Gamma_{\lambda\nu}^\mu(1)$ the covariant derivatives appear

$$R_{\nu\rho\sigma}^\mu(1) = \tilde{\nabla}_\rho \Gamma_{\sigma\nu}^\mu(1) - \tilde{\nabla}_\sigma \Gamma_{\rho\nu}^\mu(1). \quad (\text{A.13})$$

In Eq .(A.12) we used $\Gamma_{\rho\nu}^\mu := \Gamma_{\rho\nu}^\mu(0)$. By using Eq . (A.7) and (A.13) can be thus expanded as

$$R_{\nu\rho\sigma}^\mu(1) = \frac{1}{2} \left[\tilde{\nabla}_\rho \tilde{\nabla}_\nu h_\sigma^\mu + \tilde{\nabla}_\rho \tilde{\nabla}_\sigma h_\nu^\mu + \tilde{\nabla}_\sigma \tilde{\nabla}_\mu h_{\rho\nu} - \tilde{\nabla}_\sigma \tilde{\nabla}_\nu h_\rho^\mu - \tilde{\nabla}_\sigma \tilde{\nabla}_\rho h_\nu^\mu - \tilde{\nabla}_\rho \tilde{\nabla}_\mu h_{\sigma\nu} \right]. \quad (\text{A.14})$$

The second order perturbation of the Riemann tensor is given by

$$R_{\nu\rho\sigma}^{\mu}(2) = \partial_{\rho}\Gamma_{\sigma\nu}^{\mu}(2) - \partial_{\sigma}\Gamma_{\rho\nu}^{\mu}(2) + \Gamma_{\rho\lambda}^{\mu}\Gamma_{\sigma\nu}^{\lambda}(2) + \Gamma_{\rho\lambda}^{\mu}(2)\Gamma_{\sigma\nu}^{\lambda} \\ - \Gamma_{\sigma\lambda}^{\mu}\Gamma_{\rho\nu}^{\lambda}(2) - \Gamma_{\sigma\lambda}^{\mu}(2)\Gamma_{\rho\nu}^{\lambda} + \Gamma_{\rho\lambda}^{\mu}(1)\Gamma_{\sigma\nu}^{\lambda}(1) - \Gamma_{\sigma\lambda}^{\mu}(1)\Gamma_{\rho\nu}^{\lambda}(1). \quad (\text{A.16})$$

Proceeding in the same way as in the previous case we obtain

$$R_{\nu\rho\sigma}^{\mu}(2) = \tilde{\nabla}_{\rho}\Gamma_{\sigma\nu}^{\mu}(2) - \tilde{\nabla}_{\sigma}\Gamma_{\rho\nu}^{\mu}(2) + \Gamma_{\rho\lambda}^{\mu}(1)\Gamma_{\sigma\nu}^{\lambda}(1) - \Gamma_{\sigma\lambda}^{\mu}(1)\Gamma_{\rho\nu}^{\lambda}(1), \quad (\text{A.15})$$

which can be written in a compact form

$$R_{\nu\rho\sigma}^{\mu}(2) = -h_{\beta}^{\mu}R_{\nu\rho\sigma}^{\beta}(1) - \tilde{g}^{\mu\alpha}\tilde{g}^{\beta\gamma}\Gamma_{\rho\alpha}^{\gamma}(1)\Gamma_{\sigma\nu}^{\beta}(1) - \Gamma_{\sigma\alpha}^{\gamma}(1)\Gamma_{\rho\nu}^{\beta}(1). \quad (\text{A.18})$$

Now we can proceed with the evaluation of the Ricci tensor

$$R_{\nu\sigma} = R_{\nu\sigma}(0) + R_{\nu\sigma}(1) + R_{\nu\sigma}(2) + O(h)^3, \quad (\text{A.16})$$

by contracting the proper indices, thus $R_{\nu\sigma}(0) = R_{\nu\mu\sigma}^{\mu}(0)$ and

$$R_{\nu\sigma}(1) = R_{\nu\mu\sigma}^{\mu}(1) = \frac{1}{2}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}h_{\sigma}^{\mu} + \tilde{\nabla}_{\mu}\tilde{\nabla}_{\sigma}h_{\nu}^{\mu} - \tilde{\nabla}_{\sigma}\tilde{\nabla}_{\nu}h - \tilde{\nabla}^2h_{\sigma\nu}, \quad (\text{A.20})$$

thanks to the simplification of two identical terms, and last

$$R_{\nu\sigma}(2) = R_{\nu\mu\sigma}^{\mu}(2) = -h_{\beta}^{\mu}R_{\nu\mu\sigma}^{\beta}(1) - \tilde{g}^{\mu\alpha}\tilde{g}^{\beta\gamma}\Gamma_{\mu\alpha}^{\gamma}(1)\Gamma_{\sigma\nu}^{\beta}(1) - \Gamma_{\sigma\alpha}^{\gamma}(1)\Gamma_{\mu\nu}^{\beta}(1). \quad (\text{A.17})$$

Similarly the Ricci scalar curvature can be calculated as

$$R = R(0) + R(1) + R(2) = (\tilde{g}_{\nu\sigma} - h_{\nu\sigma} + h_{\nu}^{\lambda}h_{\lambda\sigma})R_{\nu\sigma}, \quad (\text{A.18})$$

where

$$R(0) = \tilde{g}^{\nu\sigma}R_{\nu\sigma}(0), \quad (\text{A.19})$$

$$R(1) = -h_{\nu\sigma}R_{\nu\sigma}(0) + \tilde{g}^{\nu\sigma}R_{\nu\sigma}(1) = -h_{\nu\sigma}R_{\nu\sigma}(0) + \tilde{\nabla}_{\nu\sigma}h_{\nu\sigma} - \tilde{\nabla}^2h, \quad (\text{A.23})$$

and

$$R(2) = -h_{\nu\sigma}R_{\nu\sigma}(1) - \tilde{g}^{\nu\sigma}h_{\beta}^{\mu}R_{\nu\mu\sigma}^{\beta}(1) + R^{\rho\lambda}(0)h_{\rho}^{\alpha}h_{\alpha}^{\lambda} - \tilde{g}^{\nu\sigma}\tilde{g}^{\mu\alpha}\tilde{g}^{\beta\gamma}\Gamma_{\mu\alpha}^{\gamma}(1)\Gamma_{\sigma\nu}^{\beta}(1) - \Gamma_{\sigma\alpha}^{\gamma}(1)\Gamma_{\mu\nu}^{\beta}(1). \quad (\text{A.20})$$

Let us expand all the terms in Eq. (A.20)

$$\begin{aligned} -\tilde{g}^{\nu\sigma}h_{\mu}^{\rho}R_{\nu\rho\sigma}^{\mu} &= -h_{\mu}^{\rho}\tilde{\nabla}_{\rho}\tilde{\nabla}_{\nu}h_{\nu}^{\mu} + h_{\mu}^{\rho}\tilde{\nabla}_{\nu}\tilde{\nabla}_{\mu}h_{\rho\nu} - \tilde{g}^{\nu\sigma}\tilde{g}^{\mu\alpha}\tilde{g}^{\beta\gamma}\Gamma_{\mu\alpha}^{\gamma}(1)\Gamma_{\sigma\nu}^{\beta}(1) - \Gamma_{\sigma\alpha}^{\gamma}(1)\Gamma_{\mu\nu}^{\beta}(1) \\ &= -\frac{1}{4}\tilde{\nabla}_{\mu}h_{\beta}^{\mu}\tilde{\nabla}_{\nu}h^{\beta\nu} + \tilde{\nabla}_{\beta}h\tilde{\nabla}^{\beta}h - 4\tilde{\nabla}_{\mu}h^{\beta\mu}\tilde{\nabla}_{\beta}h - 3\tilde{\nabla}_{\mu}h_{\beta}^{\nu}\tilde{\nabla}^{\mu}h_{\nu}^{\beta} + 2\tilde{\nabla}_{\nu}h_{\beta}^{\mu}\tilde{\nabla}^{\beta}h_{\mu\nu}. \end{aligned} \quad (\text{A.21})$$

If one substitutes the expressions in Eqs. (A.7) and (A.21) to (A.20) then one obtains

$$\begin{aligned} R(2) &= R^{\rho\lambda}(0)h_{\rho}^{\alpha}h_{\alpha}^{\lambda} - h_{\nu\sigma}\tilde{\nabla}_{\mu}\tilde{\nabla}^{\sigma}h_{\nu}^{\mu} - h_{\nu\sigma}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}h^{\sigma\mu} + h_{\nu\sigma}\tilde{\nabla}^2h^{\nu\sigma} + h_{\nu\sigma}\tilde{\nabla}^{\sigma}\tilde{\nabla}_{\nu}h \\ &\quad - \frac{1}{4}\tilde{\nabla}_{\mu}h_{\beta}^{\mu}\tilde{\nabla}_{\nu}h^{\beta\nu} - \frac{1}{4}\tilde{\nabla}_{\beta}h\tilde{\nabla}^{\beta}h + \tilde{\nabla}_{\mu}h^{\beta\mu}\tilde{\nabla}_{\beta}h + \frac{3}{4}\tilde{\nabla}_{\mu}h_{\beta}^{\nu}\tilde{\nabla}^{\mu}h_{\nu}^{\beta} - \frac{1}{2}\tilde{\nabla}_{\nu}h_{\beta}^{\mu}\tilde{\nabla}^{\beta}h_{\mu\nu}. \end{aligned} \quad (\text{A.22})$$

We can collect all the expansion orders of the Ricci scalar in the following expression

$$\begin{aligned} R(\tilde{g} + h) &= R(0) - h_{\nu\sigma}R_{\nu\sigma}(0) + \tilde{\nabla}_{\nu\sigma}h_{\nu\sigma} - \tilde{\nabla}^2h + R^{\rho\lambda}(0)h_{\rho}^{\alpha}h_{\alpha}^{\lambda} \\ &\quad - h_{\nu\sigma}\tilde{\nabla}_{\mu}\tilde{\nabla}^{\sigma}h_{\nu}^{\mu} - h_{\nu\sigma}\tilde{\nabla}_{\mu}\tilde{\nabla}_{\nu}h^{\sigma\mu} + h_{\nu\sigma}\tilde{\nabla}^2h^{\nu\sigma} \\ &\quad + h_{\nu\sigma}\tilde{\nabla}^{\sigma}\tilde{\nabla}_{\nu}h - \frac{1}{4}\tilde{\nabla}_{\mu}h_{\beta}^{\mu}\tilde{\nabla}_{\nu}h^{\beta\nu} - \frac{1}{4}\tilde{\nabla}_{\beta}h\tilde{\nabla}^{\beta}h + \tilde{\nabla}_{\mu}h^{\beta\mu}\tilde{\nabla}_{\beta}h \\ &\quad + \frac{3}{4}\tilde{\nabla}_{\mu}h_{\beta}^{\nu}\tilde{\nabla}^{\mu}h_{\nu}^{\beta} - \frac{1}{2}\tilde{\nabla}_{\nu}h_{\beta}^{\mu}\tilde{\nabla}^{\beta}h_{\mu\nu}. \end{aligned} \quad (\text{A.24})$$

The Einstein-Hilbert action with a cosmological constant is

$$S[\tilde{g} + h] = - \int d^Dx \sqrt{g} (R[\tilde{g} + h] - 2\Lambda) \quad (\text{A.25})$$

if one substitutes the expansion of g from Eq. (A.2) and the expansion of $[\tilde{g} + h]$ from Eq. (A.24) then performing some integration by parts, and making use of the following commutation relationship:

$$[\tilde{\nabla}_{\mu}, \tilde{\nabla}_{\beta}]h_{\gamma}^{\alpha} = R_{\mu\beta}^{\alpha\lambda} + R_{\gamma\beta\mu}^{\lambda}h_{\lambda}^{\alpha}, \quad (\text{A.30})$$

one gets the following orders of the action expansion:

$$S_0 = - \int d^D x \sqrt{\tilde{g}} (\tilde{R} - 2\Lambda), \quad (\text{A.26})$$

$$S_1 = \int d^D x \sqrt{\tilde{g}} \left[h^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda \right) + \text{total derivatives} \right], \quad (\text{A.27})$$

$$S_2 = - \int d^D x \sqrt{\tilde{g}} \left[\frac{1}{4} h^{\mu\nu} (\tilde{\nabla}^2 + 2\Lambda) h_{\mu\nu} - \frac{1}{8} h (\tilde{\nabla}^2 + 2\Lambda) h + \frac{1}{2} \left(\tilde{\nabla}_\nu h^{\nu\mu} - \frac{1}{2} \tilde{\nabla}^\mu h \right)^2 \right. \\ \left. + \frac{1}{2} h^{\mu\lambda} h^{\nu\sigma} R_{\mu\nu\lambda\sigma} + \frac{1}{2} (h^{\mu\lambda} h^\nu{}_\lambda - h h^{\mu\nu}) R_{\mu\nu} + \frac{1}{8} (h^2 - 2 h^{\mu\nu} h_{\mu\nu}) R \right]. \quad (\text{A.28})$$

Appendix B.

Black hole entropy calculations

B.1. Evaluation of the boundary term for the Euclidean Schwarzschild metric

We begin with the Euclidean action for gravity with a boundary,

$$I_E = -\frac{1}{16\pi G} \int_M d^4x \sqrt{g} R - \frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{\gamma} (K - K_0), \quad (\text{B.1})$$

where R is the Ricci scalar on the 4-manifold M , γ is the determinant of the induced metric on the boundary ∂M , K is the extrinsic curvature of ∂M embedded in the spacetime, and K_0 is the extrinsic curvature of the boundary embedded in flat space (chosen so that $I_E = 0$ for Minkowski space).

For the Schwarzschild solution in Euclidean signature, the metric is given by

$$ds^2 = f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (\text{B.2})$$

with

$$f(r) = 1 - \frac{2GM}{r}, \quad (\text{B.3})$$

and the Euclidean time τ is periodic with period

$$\beta = 8\pi GM, \quad (\text{B.4})$$

to avoid a conical singularity at the horizon $r = 2GM$.

Since the Schwarzschild metric is a vacuum solution ($R = 0$), the bulk term in the above action vanishes, and the action reduces solely to the boundary term,

$$I_E = -\frac{1}{8\pi G} \int_{\partial M} d^3x \sqrt{\gamma} (K - K_0). \quad (\text{B.5})$$

Now we will compute the extrinsic curvature. For a hypersurface at constant $r = R$, the induced 3-metric is

$$ds_{\partial M}^2 = f(R) d\tau^2 + R^2 d\Omega^2. \quad (\text{B.6})$$

The outward-pointing unit normal is

$$n^\mu = (0, \sqrt{f(r)}, 0, 0). \quad (\text{B.7})$$

A standard calculation shows that the extrinsic curvature is

$$K = \frac{f'(r)}{2\sqrt{f(r)}} + \frac{2\sqrt{f(r)}}{r}, \quad (\text{B.8})$$

evaluated at $r = R$. For the Schwarzschild metric, since

$$f(r) = 1 - \frac{2GM}{r} \quad \text{and} \quad f'(r) = \frac{2GM}{r^2}, \quad (\text{B.9})$$

we have

$$K = \frac{GM}{R^2 \sqrt{1 - \frac{2GM}{R}}} + \frac{2}{R} \sqrt{1 - \frac{2GM}{R}}. \quad (\text{B.10})$$

We can now proceed to the evaluation of K_0 . For flat space ($M = 0$), we have $f(r) = 1$ and the induced metric on the boundary is

$$ds_{\partial M}^2 = d\tau^2 + R^2 d\Omega^2. \quad (\text{B.11})$$

Then, the extrinsic curvature in flat space is

$$K_0 = \frac{2}{R}. \quad (\text{B.12})$$

Subtracting, we obtain

$$K - K_0 = \frac{GM}{R^2 \sqrt{1 - \frac{2GM}{R}}} + \frac{2}{R} \left(\sqrt{1 - \frac{2GM}{R}} - 1 \right). \quad (\text{B.13})$$

For large R , we expand $\sqrt{1 - \frac{2GM}{R}}$ in powers of GM/R :

$$\sqrt{1 - \frac{2GM}{R}} \approx 1 - \frac{GM}{R} - \frac{(GM)^2}{2R^2} + \dots. \quad (\text{B.14})$$

Thus,

$$\frac{GM}{R^2 \sqrt{1 - \frac{2GM}{R}}} \approx \frac{GM}{R^2} \left(1 + \frac{GM}{R} + \dots \right) = \frac{GM}{R^2} + O(R^{-3}), \quad (\text{B.15})$$

and

$$\frac{2}{R} \left(\sqrt{1 - \frac{2GM}{R}} - 1 \right) \approx \frac{2}{R} \left(-\frac{GM}{R} - \frac{(GM)^2}{2R^2} + \dots \right) = -\frac{2GM}{R^2} + O(R^{-3}). \quad (\text{B.16})$$

Therefore, we have

$$K - K_0 \approx -\frac{GM}{R^2} + O(R^{-3}). \quad (\text{B.17})$$

Now we can finally evaluate the boundary integral. The induced metric on the boundary $r = R$ gives the volume element

$$d^3x \sqrt{\gamma} = \sqrt{f(R)} R^2 \sin \theta d\tau d\theta d\phi. \quad (\text{B.18})$$

Integrating over the angles yields a factor of 4π and over τ the period β , so that

$$\int_{\partial M} d^3x \sqrt{\gamma} = \beta 4\pi R^2 \sqrt{1 - \frac{2GM}{R}}. \quad (\text{B.19})$$

Substituting the above result and the asymptotic form of $K - K_0$ into the boundary action gives

$$I_E = -\frac{1}{8\pi G} \beta 4\pi R^2 \sqrt{1 - \frac{2GM}{R}} \left(-\frac{GM}{R^2} \right). \quad (\text{B.20})$$

Canceling R^2 and noting that for large R

$$\sqrt{1 - \frac{2GM}{R}} \rightarrow 1, \quad (\text{B.21})$$

we obtain

$$I_E \approx \frac{\beta GM}{2G} = \frac{\beta M}{2}. \quad (\text{B.22})$$

B.2. Variation of the Gauss–Bonnet term with respect to the Riemann tensor

In this appendix, we derive the variation of the Gauss–Bonnet density with respect to the Riemann tensor, as needed for the computation of the Wald entropy in higher-curvature gravity theories.

The Gauss–Bonnet density is defined by

$$\mathcal{G} = R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2, \quad (\text{B.23})$$

where the Ricci tensor and scalar are given by

$$R_{ab} = g^{cd} R_{acbd}, \quad (\text{B.24})$$

$$R = g^{ab} R_{ab} = g^{ac} g^{bd} R_{abcd}. \quad (\text{B.25})$$

We now compute the variation of each term in $\mathcal{G}$ with respect to R_{abcd} , assuming that the metric is held fixed.

Variation of $R_{abcd}R^{abcd}$. Since this term is quadratic in the Riemann tensor, its variation is

$$\delta \left(R_{abcd} R^{abcd} \right) = 2 R^{abcd} \delta R_{abcd}, \quad (\text{B.26})$$

thus, we obtain

$$\frac{\partial \left(R_{efgh} R^{efgh} \right)}{\partial R_{abcd}} = 2 R^{abcd}. \quad (\text{B.27})$$

Variation of $-4 R_{ab}R^{ab}$. Using the definition

$$R_{ab} = g^{cd} R_{acbd}, \quad (\text{B.28})$$

its variation is

$$\delta R_{ab} = g^{cd} \delta R_{acbd}. \quad (\text{B.29})$$

Hence,

$$\delta \left(R_{ab} R^{ab} \right) = 2 R^{ab} \delta R_{ab} = 2 R^{ab} g^{cd} \delta R_{acbd}. \quad (\text{B.30})$$

Taking into account the antisymmetry properties of R_{abcd} (i.e. $R_{acbd} = -R_{adb c}$, etc.), a careful analysis shows that the variation can be written as

$$\delta \left(R_{ab} R^{ab} \right) = 4 \left(g^{ac} R^{bd} - g^{ad} R^{bc} - g^{bc} R^{ad} + g^{bd} R^{ac} \right) \delta R_{abcd}. \quad (\text{B.31})$$

Thus, the contribution from this term is

$$\frac{\partial \left(-4 R_{ef} R^{ef} \right)}{\partial R_{abcd}} = -4 \left(g^{ac} R^{bd} - g^{ad} R^{bc} - g^{bc} R^{ad} + g^{bd} R^{ac} \right). \quad (\text{B.32})$$

Variation of R^2 . Since

$$R = g^{ac} g^{bd} R_{abcd}, \quad (\text{B.33})$$

its variation is given by

$$\delta R = g^{ac} g^{bd} \delta R_{abcd}. \quad (\text{B.34})$$

Thus,

$$\delta (R^2) = 2R \delta R = 2R g^{ac} g^{bd} \delta R_{abcd}. \quad (\text{B.35})$$

However, to account for the antisymmetry of δR_{abcd} , the correct expression is

$$\frac{\partial R^2}{\partial R_{abcd}} = 2R (g^{ac} g^{bd} - g^{ad} g^{bc}). \quad (\text{B.36})$$

Combining the three variations, we obtain

$$\frac{\partial \mathcal{G}}{\partial R_{abcd}} = 2 R^{abcd} - 4 (g^{ac} R^{bd} - g^{ad} R^{bc} - g^{bc} R^{ad} + g^{bd} R^{ac}) + 2R (g^{ac} g^{bd} - g^{ad} g^{bc}). \quad (\text{B.37})$$

Appendix C.

Numerical methods

C.1. Some numerical aspects on SFTDE treatment

Regarding the numerical methods for the analysis of the SFTDE scenario in Chapter. 8, in order to perform likelihood analysis given a model where $\dim \{\phi^\lambda\} > 2$, i.e. there exist more than 2 free parameters, numerous likelihood calculations are required, even in the context of an MCMC sampler. The latter fact means that the likelihood analysis gets computationally expensive, thus one needs to use binned datasets and other approximations. For example, where the Hubble rate is not known analytically, which is the case for the model constructed in this work, analytical approximation for the solution of (8.16) was the traditional path to follow. Note that the errors of the analytical approximation not necessarily converge uniformly for arbitrary values of the free parameters and z . Thus, they could potentially disguise new physics or produce fake imprints [487]. Moreover, even for the concordance model, where analytical form exists for the Hubble rate, the luminosity distance calculation demands an integration per data point and per likelihood calculation. Although the “brute force” mindset, i.e. using large computational resources, could serve adequately in some cases, the optimal utilization of the available datasets requires a more efficient approach. In this work, we use interpolation methods to optimize both the Hubble rate and the luminosity distance calculations. In particular, we use a cubic spline, as implemented within *scipy* version 1.14.1 [488]. This implementation offers a callable method that calculates the integral of the fitted function analytically, which significantly reduces the computational burden of the analysis. We tested different interpolator available by *scipy*, such as linear, pchip and akima and we report that, in our case, the cubic spline is the most accurate. Accuracy has been tested in comparison with the full

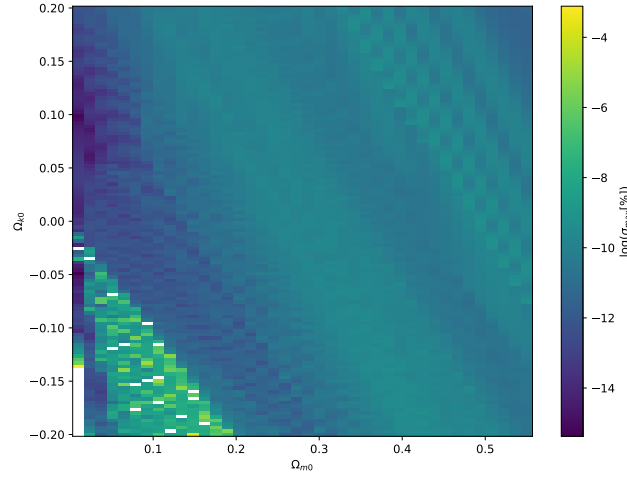


Figure C.1.: The logarithm of the mean percentage error of the approximation scheme on $\Omega_{\Lambda\text{eff}}$ as a function of the free parameters Ω_{m0}, Ω_{k0} .

numerical solution as follows. Numerical solution of (8.16) is performed, given a set of free parameters ($Y = \{\Omega_{m0}, \Omega_{k0}\}$), at $N_{\text{num.}} = 1000$ points within $z \in [0, 2.5]$. Then, we use $N_{\text{interp}} = 51$ points to train the interpolator. The evaluation is performed with the remaining 949 points. Finally, we assess the impact of the free parameters (Ω_{m0}, Ω_{k0}) on the mean percentage error, defined as

$$\sigma(Y) = \frac{1}{N_{\text{num.}}} \sum_{i=1}^{N_{\text{num.}}} \frac{|\Omega_{\Lambda\text{eff, num.}}(z_i; Y) - \Omega_{\Lambda\text{eff, interp.}}(z_i; Y)|}{\Omega_{\Lambda\text{eff, num.}}(z_i; Y)} \quad (\text{C.1})$$

We report that the max percentage error is $\sigma_{\text{max}} \sim 10^{-4}$ for all Y_i and z_i considered. The effect of different parameter sets on the mean percentage error is presented at Fig. C.1. The variation of the approximation error with regard to the free parameters lies at the range $\log(\sigma[\%]) \in [-14, -4]$. The latter range at least 2 orders of magnitude less than statistical uncertainties of $1 - \Omega_{m0}$ for an arbitrary cosmology, e.g. [397, 419, 430] and references therein.

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Colophon

This thesis was made in $\text{\LaTeX}2_\epsilon$ using the “hepthesis” class.