

UNIVERSITY OF THESSALY  
FACULTY OF SCIENCE  
DEPARTMENT OF PHYSICS

**Quantum Field Theory:  
Conformal Field Theories  
and the Conformal Bootstrap**

BACHELOR'S THESIS

**SOKRATIS VARELOGIANNIS**

**Advisor:** Dr Andreas Stergiou (*King's College London*)

**Co-advisor:** Dr Dionysios Vavougios (*University of Thessaly*)

Lamia, Greece, 2023

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*A thesis submitted in fulfilment of the requirements for the degree of  
Bachelor of Science in Physics*



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Lamia, Greece, 2023

*I dedicate this thesis to my family for their constant support,  
to my mother and my father, Dimitra and Dimitris*

# Abbreviations

|                |                                                                    |
|----------------|--------------------------------------------------------------------|
| <b>LHS</b>     | <b>L</b> eft <b>H</b> and <b>S</b> ide                             |
| <b>RHS</b>     | <b>R</b> ight <b>H</b> and <b>S</b> ide                            |
| <b>IR</b>      | <b>I</b> nfra- <b>R</b> ed                                         |
| <b>UV</b>      | <b>U</b> ltra- <b>V</b> iolet                                      |
| <b>CFT</b>     | <b>C</b> onformal <b>F</b> ield <b>T</b> heory                     |
| <b>QFT</b>     | <b>Q</b> uantum <b>F</b> ield <b>T</b> heory                       |
| <b>RG</b>      | <b>R</b> enormalization <b>G</b> roup                              |
| <b>SCT</b>     | <b>S</b> pecial <b>C</b> onformal <b>T</b> ransformation           |
| <b>SOC/SOM</b> | <b>S</b> tate <b>O</b> perator <b>C</b> orrespondence/ <b>M</b> ap |
| <b>OPE</b>     | <b>O</b> perator <b>P</b> roduct <b>E</b> xpansion                 |
| <b>CB</b>      | <b>C</b> onformal <b>B</b> locks                                   |
| <b>MM</b>      | <b>M</b> inimal <b>M</b> odels                                     |
| <b>EFM</b>     | <b>E</b> xtremal <b>F</b> unctional <b>M</b> ethod                 |

# Symbols

|                      |                                                                                                                                                                         |
|----------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\mathcal{O}_i(x_i)$ | Local operator inserted at $x_i$                                                                                                                                        |
| $f_{ijk}$            | Coefficient of the 3-pt function $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle /$ OPE coefficient $\mathcal{O}_i \mathcal{O}_j \sim f_{ijk} \mathcal{O}_k$ |
| $x_{ij}$             | the difference $x_i - x_j$                                                                                                                                              |
| $d/D$                | the number of spacetime dimensions                                                                                                                                      |
| $l$                  | the spin label for a traceless symmetric representation of $SO(D)$                                                                                                      |

|               |                                                                           |
|---------------|---------------------------------------------------------------------------|
| $\Delta$      | The full, quantum-corrected scaling dimension of an operator              |
| $\phi$        | A scalar operator, usually used as the one with lowest non-zero dimension |
| $\varepsilon$ | A scalar operator, usually used as the second lowest non-zero dimension   |
| $\Lambda$     | A general cutoff, or a vector in the context of linear functionals        |

The physical constants  $c$ ,  $k_B$ ,  $\hbar$  are set to 1.

The metric signature is Euclidean or  $(- + + \cdots +)$ , when applicable.

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## **Abstract**

The aim of this Bachelor thesis is the review of features of Quantum Field Theory and especially the study of characteristic theories that remain invariant under the Group of Conformal Transformations, the Conformal Field Theories.

The main purpose is to understand their structure, operation and application in areas of Physics such as Statistical Physics, Condensed Matter Physics and High Energy Physics.

Through the theory's characteristic of providing fascinating universal physics of scale invariant critical points, a special focus is attributed to the study of the physical foundations of conformal symmetry, conformal kinematics, radial quantization and the OPE in  $d \geq 3$ .

An extensive view of the keynote aspects of the Conformal Bootstrap is been presented within the framework of this thesis.

**Keywords:** Conformal Field Theories, Fixed Points, Critical Phenomena, OPE, Conformal Bootstrap

# Chapter 1

## Motivation

### 1.1 Introduction

Modern understanding of Physics and its methods of description of physical systems have concluded to two main ideas of nature that seem to exhibit universal application to all aspects of nature.

The first concept, has been widely accepted and used from all generations of modern physicists, and is the Symmetries of a system.

Systems frequently present properties of Symmetry and consequently provide the existence of conserved quantities, such as momentum, energy, . . . , or (the more exotic) supersymmetry. This frameworks permits the definition and behaviour prediction of natural systems. . . but only if it is combined with the second concept.

The second major advancement in our understanding of nature is based in the idea of Scale dependence of systems. Nature seems to behave in a different manner when the scale of the system evolves. This characteristic led to new insight of physical theories and gave rise, among many ideas, to the new concept of the Renormalization Group.

Further research, mainly provided by the work in Renormalization Group by L. Kadanoff, M. Fisher and K. Wilson, allowed the connection of the behavior of a plethora of systems varying from particle and statistical physics to condensed physics, cosmology and string theory.

With this new framework of understanding nature with the evolution of natural theories in accordance with the Renormalization Group predictions, allows us to perceive a new view of the Renormalization Group (RG) flows as trajectories in the space of theories.

The ends of these trajectories provide special points of high physical interest, they are commonly referred as the *Fixed Points*.

## 1.2 Fixed Points

Fixed points are essential since they allow the perception of Quantum Field Theories as the interpolation between Fixed Points while they systematically appear in, with a first look, different theories while their advantage and source of research curiosity is their characteristic of exhibiting the same natural behavior regardless the system studied.

The classification of Fixed Points is an issue of high priority for theoretical physics since, as seen in [1], they are milestones in predicting the large scale behavior of systems where the action of the Renormalization Group practically suppresses the high momentum modes of the fields leading to Wilson's Effective Action which directly affects the theory's couplings strengths.

These changes in couplings are analogous to the dimension of the interaction they parameterize. The terminology used for the classification of these operators has as its the main criterion the reflection of their contri-

bution importance in the Infrared (IR) regime as:

- High contribution in the IR is provided by the *Relevant Operators* whose characteristic is their dimension,  $\dim[\mathcal{O}] < 4$
- Stable contribution through all scales by the *Marginal Operators* with  $\dim[\mathcal{O}] = 4$
- Insignificant contribution in the IR by the *Irrelevant Operators* with  $\dim[\mathcal{O}] > 4$

For a theory characterized by fundamental fields with global symmetry, after the action of the Renormalization Group, we notice that only a specific set of interaction survives the procedure. This set leads to the shared same behavior for systems that initially seem to be described by different combinations of fields but eventually tend to be inseparable.

This physical characteristic was highly recognizable in a plethora of systems and lead to the form of the concept of *Universality*.

### 1.3 Cases of Universality

*Universality*, in a non-strict definition, may be interpreted as the property that determines that all physical systems with only a difference of some irrelevant operators do present the same fixed points at long distances.

Furthermore, Universality seems to be valid since it provides us a logic connection for the plethora of experimental observations of a variety of -at first sight- different systems to behave with amazingly the same way.

Between the phenomena explained by Universality, is the resemblance of the behavior of ferromagnets with the boiling water! The behavior is

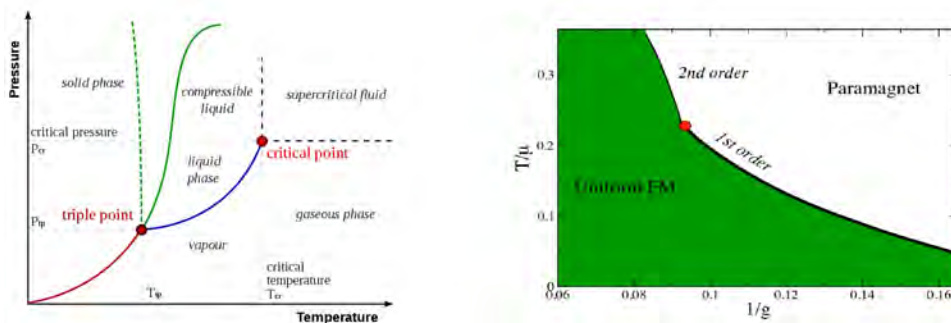


Figure 1.1: Phase diagram of two ensembles exhibiting common behaviour. Boiling water (left) and of the uniform ferromagnet (right), including quantum fluctuations. Below the tricritical point (shown as circle), quantum fluctuations drive the transition first order. (Source: [2] )

the same when the first ensemble being at its critical temperature and the second ensemble being at the tri-critical point which in fact is the same fixed point.

## 1.4 Information provided by Critical Phenomena

Critical phenomena, in the case of water and all systems with at the same Universality class, can be induced by the traditional-common used practises of the study of their phase transitions diagrams.

In a usually seen system, as the  $H_2O$ , a common form of phase diagram does diagrammatically indicates the dependence of the state to macroscopic characteristics of the system such as Temperature and Pressure:

In these diagrams it is common for phases to be distinguished by lines corresponding to first order transitions. Lines separating solid and liquid phase seem not to have a specifying ending point. On the other hand, the

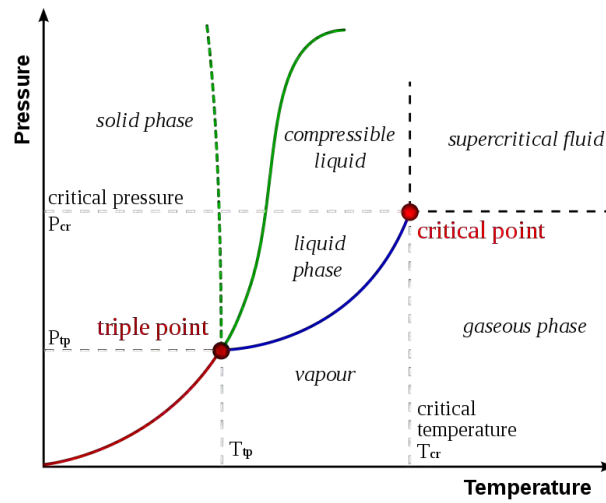


Figure 1.2: *Diagram of Phase transitions of water*

line between liquid and gas phase seems to continue up to a point, the *critical point*.

From the Critical point and beyond there is no further way to distinguish the phase into only one phase (liquid or gas).

Critical points are widely seen in a plethora of natural systems and they commonly appear with the same behaviour for all systems near the regions surrounding the critical point, this leads to the conclusion that they obey the concept of *universality*.

This common behaviour is seen by the validity of the same formulas for calculating macroscopic quantities of the systems. Characteristically, for this system the Heat Capacity is defined as:

$$c \sim \frac{1}{|T - T_c|^\alpha} \quad (1.1)$$

where for  $H_2O$  we have an experimental value for  $a = 0.11008$  [3].

This common detail of the system allows us, to consider nature being

organised within two main physical concepts, Symmetries and the Scale dependence.

A great insight of the impact of symmetries in natural systems was provided by L. Landau stating that different phases of matter are characterised by symmetry changes. So for a breaking of e.g. a translation symmetry, it is implied a destruction of the lattice structures leading to a phase transition from solid to liquid (for our example system of  $H_2O$ ).

Equivalently, the impact of scale dependence in the behaviour of a theory is highly presented through the work of K. Wilson and L. Kadanoff stating that small scale changes do affect large scale behaviours but they do not impact the largest scales, a principle widely referred as the basis of the Renormalization Group which will be mentioned in this introductory chapter.

## 1.5 The Ising model

The above ideas may be detected in the Ising model. This model is based on an lattice in  $d$ -dimensions with  $N$ -sites. Each lattice site admits one degree of freedom called spin (a term with no association to the quantum mechanical homonym term), labelled in the range  $i = 1, \dots, N$ . This discrete variable has values

$$s_i = \pm 1 \quad (1.2)$$

Furthermore, this model admits a magnetic field  $B$  which is coupled with a sum of spins and a coupling strenght  $J$  providing the Energy of the system as:

$$E = -B \sum_i s_i - J \sum_{\langle ij \rangle} s_i s_j \quad (1.3)$$

For aligned spins and  $J$  being positive we provide the case of ferromagnetism and for negative values of  $J$  we have the anti-ferromagnetism.

Adding a Temperature term we notice that the magnetic field  $B$  tends to align spins and  $J$  tends to align spins. In this statistical ensemble we have to investigate the behaviour of the Free Energy instead of only the energy, since the Free Energy as a function of Energy  $E$  and Entropy  $S$  is:

$$F = E - TS \quad (1.4)$$

Anticipating a minimum value for the Free Energy it is a requirement of achieving the maximum of the Temperature-Entropy term. This case coincides with the complete randomness of all spins pointing orientation.

Our statistical ensemble accepts a full description by the Partition function:

$$Z = \sum_{\{s_i\}} e^{-\beta E[s_i]} \quad (1.5)$$

with  $\beta \equiv 1/T$ . Analogously we define the magnetisation  $m$  of the system:

$$m = \frac{1}{N} \left\langle \sum S_i \right\rangle = \frac{1}{N\beta} \frac{\partial \log Z}{\partial B} \quad (1.6)$$

This model is completely solvable only for  $d = 1$  or  $d = 2$  dimensions.

The introduction of the Partition function means that this quantity is no furthermore dependant on the system's spins but is dependent on the macroscopic properties of the system, the coupling  $J$  and the magnetisation  $m$  (where  $-1 \leq m \leq 1$ ).

Strong intuition for the validity of Ising model is gained by considering an independent derivation of the Partition function.

It is achieved our by considering the Partition function to obtain a term

of Temperature  $T$  and a term of the Effective Free Energy  $F(m)$ :

$$Z = \sum_m e^{-\beta F(m)} \rightarrow \int_{-1}^1 dm e^{-\beta F(m)} \quad (1.7)$$

## 1.6 The Mean Field Approximation

The Mean Field Approximation provides a different summing procedure. Instead of summing in terms of spins the summation is now over magnetisation leading to the Energy expression:

$$E = B \sum_i m - J \sum_i m^2 \quad (1.8)$$

Defining the Energy per spin:

$$\frac{E}{N} = -Bm - \frac{1}{2}Jqm^2 \quad (1.9)$$

with  $q$  being the number of the nearest neighbours. The Effective Free Energy per unit spin:

$$f(m) = \frac{F(m)}{N} = -Bm - \frac{1}{2}Jqm^2 - T[\log 2 - \frac{1}{2}(1+m)\log(1+m) - \frac{1}{2}(1-m)\log(1-m)] \quad (1.10)$$

At equilibrium:

$$\frac{\partial f}{\partial m} = 0 \Rightarrow m = \tanh(\beta B_{\text{eff}}) \quad (1.11)$$

Magnetization in equilibrium:

$$B_{\text{eff}} = B + Jqm \quad (1.12)$$

## 1.7 The Landau Theory

A search for an expression for  $f(m)$  in the Mean Field Approximation leads to the use of the expansion of  $f(m)$

$$f(m) = -Bm + \frac{1}{2}(T - Jq)m^2 + \frac{1}{12}Tm^4 + \dots \quad (1.13)$$

For the first term being zero, the second order term  $\frac{1}{2}(T - Jq)$  produces of hyperbolic graph with one minimum for  $T > Jq$  and a graph with two hyperbolas with two minima each one for  $T < Jq$ :

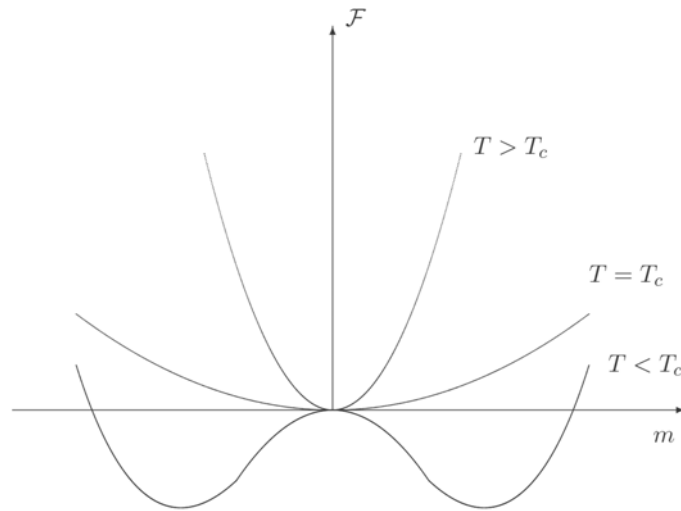


Figure 1.3: *Diagram of values of  $\mathcal{F}$  and  $m$*

The product  $Jq$  corresponds to the critical Temperature  $T_c \equiv Jq$ . There exist two states for the magnetisation, for  $m = 1$  and  $m = -1$  and for  $T > T_c$  there is one state:

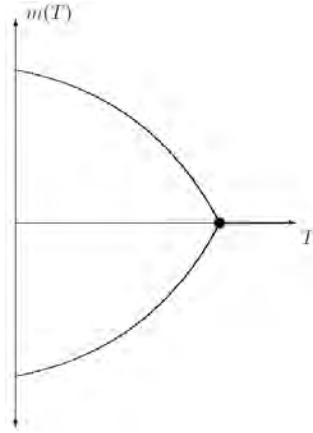


Figure 1.4: Representation of the system's states with dependence on  $m$  and  $T$

## 1.8 The Landau-Ginzburg Theory

Indeed Landau's theory can accept a new approach provided through the work of Ginzburg and Landau [3] by considering the lattice of spins to be organised to a large number of spin clusters containing the edges of the lattice which act as subsystems of the ensemble. In this scheme the magnetisation becomes a field with dependence to the position of each cluster as  $m(\vec{x})$  leads to an integral expression of the Partition Function:

$$Z = \int Dm(\vec{x}) e^{-\beta F[m(\vec{x})]} \quad (1.14)$$

This approach is characterised by the locality of the interaction where spin interact with their nearest neighbours so  $F[m(\vec{x})]$  is expressed as:

$$F[m(\vec{x})] = \int d^d \vec{x} f[m(\vec{x})] \quad (1.15)$$

where the quantity  $f[m(\vec{x})]$  is dependent only on local terms.

The continuum of the Partition Function induces the existence of sym-

metries of the ensemble. We require a translational symmetry and rotational invariance of the system.

Furthermore, the system shall obey  $\mathbb{Z}_2$  symmetry so for  $m \rightarrow -m$  and  $B \rightarrow -B$  occurs the same theory. Analyticity is required for the consistency of the theory and fluctuations are distributed smoothly respecting the condition:  $\nabla m \gg \nabla^2 m \gg \nabla^3 m$ . The Free Energy with dependence on expansion terms is:

$$F[m(\vec{x})] = \int d^d x \left( \frac{1}{2} \gamma(T) (\nabla m)^2 + \frac{1}{2} \alpha_2(T) m^2 + \frac{1}{4} \alpha_4(T) m^4 + \dots \right) \quad (1.16)$$

Recognising the magnetisation as a field that has been derived we continue by following the most familiar symbolism for a field as  $m(\vec{x}) \rightarrow \phi(\vec{x})$ . Further insight is provided through the connected two-point Correlation function. For a Free Energy expression up to second order terms with an extra source term:

$$F[\phi(\vec{x})] = \int d^d x \left( \frac{1}{2} \gamma (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 - B \phi \right) \quad (1.17)$$

there is the formation of the 2-pt Correlation function:

$$\langle \phi(\vec{x}) \phi(\vec{y}) \rangle = \frac{1}{\beta^2} \frac{\delta^2 \log z}{\delta B(\vec{x}) \delta B(\vec{y})} \Big|_{B=0} = \frac{1}{\beta} G(\vec{x} - \vec{y}) \quad (1.18)$$

where  $G(\vec{x} - \vec{y})$  is the Green's function.

By the translational invariance of the system and with setting  $\vec{y} = \vec{0}$  alongside the introduction of a new term  $\xi = \frac{\sqrt{\gamma}}{\mu}$  leads to the Green's function:

$$G(\vec{x}) = G(r = |\vec{x}|) = \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^4} \frac{e^{-i\vec{k}\vec{x}}}{k^2 + 1/\xi^2} \quad (1.19)$$

Subsequently a re-express of terms provides:

$$\frac{1}{k^2 + \frac{1}{\xi^2}} = \int_0^\infty dt e^{-t(k^2 + 1/\xi^2)} \quad (1.20)$$

The Green's function is:

$$G(\bar{x}) = G(r = |\vec{x}|) = \frac{1}{\gamma} \int \frac{d^d k}{(2\pi)^2} \frac{e^{-\vec{k}\bar{x}}}{k^2 + \frac{1}{\xi^2}} \quad (1.21)$$

Trick:  $\frac{1}{k^2 + \frac{1}{\xi^2}} = \int_0^\infty dt e^{-t(k^2 + 1/\xi^2)}.$

$$G(x) = \frac{1}{r(4\pi)^{1/2}} \int_0^\infty dt t^{-d/2} e^{-r^2/4t - t/\xi^2}$$

$$= \frac{2^{d/2}}{(r\xi)^{\frac{1}{2}(d-2)}} K_{\frac{1}{2}(d-2)}(r/\xi)$$

$$= \begin{cases} 2^{d-2} \Gamma\left(\frac{d}{2} - 1\right) \frac{1}{r^{d-2}}, & r \ll \xi \\ 2^{(d-1)/2} \sqrt{\pi} \frac{e^{-r/\xi}}{\xi^{\frac{d-3}{2}} r^{(d-1)/2}}, & r \gg \xi \end{cases} \quad (1.22)$$

with  $K$  expressing the modified Bessel function of the second kind.

The Correlation length  $\xi$  provides the limit in where interactions change their form and the system's behaviour:

$$\xi \sim \frac{1}{|T - T_c|^{\frac{1}{2}}} \quad (1.23)$$

Near a critical point the correlation length diverges and the form  $\sim \frac{1}{r^{d-2}}, r \ll \xi$  is the case of correlation lengths occurring in Conformal Field Theories.

Away from the critical point, the correlation length  $\xi$  is expressed in terms of the critical exponent  $\nu$  and the reduced temperature  $t$ :

$$\xi \sim t^{-\nu} \quad \text{where} \quad t = \frac{|T - T_c|}{T_c} \quad (1.24)$$

the reduced temperature does also depend from the system's scale transformations, since  $\xi \rightarrow \xi/\zeta$ , the corresponding scaling dimension is  $\Delta_\xi =$

–1. This leads to a scaling for  $t$  with the form  $t \rightarrow \zeta^{\Delta_t} t$  and the scaling dimension of  $t$  is:

$$\Delta_t = \frac{1}{\nu} \quad (1.25)$$

Another characteristic of the Free Energy  $F$  is its scaling in terms of critical exponentials, since:

$$f(t) \sim t^{d\nu} \quad (1.26)$$

and the heat capacity is analogous to the hyperscaling relation  $\alpha = 2 - d\nu$ , where:

$$c\partial^2 f \partial t^2 \sim t^{d\nu-2} \sim t^{-\alpha} \quad (1.27)$$

Magnetization is associated with the  $\beta$  critical exponent:

$$m \sim t^\beta \quad (1.28)$$

With the corresponding scaling operators to be connected as:

$$\Delta_m = \beta\Delta_t \Rightarrow \beta = \nu\Delta_\phi = \frac{(d-2+\eta)\nu}{2} \quad (1.29)$$

The existence of a magnetic field  $B$ , allows the introduction of the  $\gamma$  critical exponent, seen in the expression of the susceptibility of the system:

$$\chi = \left. \frac{\partial m}{\partial B} \right|_T \sim t^{-\gamma} \quad (1.30)$$

and the relation of the scaling dimensions of eq. 1.30 is:

$$\Delta_m - \Delta_B = -\frac{\gamma}{\nu} \Rightarrow \gamma = \nu(2 - \eta) \quad (1.31)$$

Last but not least, a new critical exponent may be seen in terms of the

magnetization  $m$  and the magnetic field  $B$  when  $t = 0$ :

$$m \sim B^{1/\delta} \Rightarrow \delta = \frac{\Delta_B}{\Delta_\phi} = \frac{d+2-\eta}{d-2+\eta} \quad (1.32)$$

As it is summed up in [3], the values of the critical exponents are:

|            | $\alpha$        | $\beta$       | $\gamma$      | $\delta$ | $\eta$        | $\nu$         |
|------------|-----------------|---------------|---------------|----------|---------------|---------------|
| Mean Field | $\frac{4-d}{2}$ | $\frac{1}{2}$ | 1             | 3        | 0             | $\frac{1}{2}$ |
| $d = 2$    | 0               | $\frac{1}{8}$ | $\frac{7}{4}$ | 15       | $\frac{1}{4}$ | 1             |
| $d = 3$    | 0.1101          | 0.3264        | 1.2371        | 4.7898   | 0.0363        | 0.6300        |

### 1.8.1 The Renormalization Group

In order to provide a logic derivation of the expansion terms used in equation 1.16 an investigation of the connection of our Landau-Ginzburg theory with the theory of the Renormalization Group is necessary.

An elimination of the appearing  $\gamma$  factor is achieved by the rescaling of the field  $\phi$ :

$$F[\phi] = \int d^d x \left( \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4!} g \phi^4 + \dots \right) \quad (1.33)$$

Furthermore, an important insight to the resulting theory is obtained by concerning the scale of the described physical system. Characteristically for an effective field theory valid for up to length scale  $L$  a decomposition of the momentum modes of the ensemble is possible. Considering only the modes within our area of interest:

$$\phi_{\vec{k}} = \int d^d x e^{-i\vec{k}\vec{x}} \phi(\vec{x}) \quad (1.34)$$

for  $|\vec{k}| < \frac{1}{L} \sim \frac{1}{\alpha}$ . where for  $|\vec{k}| > \Lambda$  the modes are zero:

$$\phi_\kappa = 0 \quad (1.35)$$

An intermediate cutoff  $\Lambda' = \frac{\Lambda}{\zeta} < \Lambda$  with  $\zeta > 1$  raises the question of the Renormalization Group for the agreement between a theory calculated in  $\Lambda$  with a theory calculated in a smaller cutoff  $\Lambda'$ . Theory presents dependence on parameter  $\zeta$  consequently leading to the presence of RG flows.

## 1.9 Renormalization Group properties

The focus of the following paragraphs is to provide a short derivation of the basic framework of Renormalization Group. For simplicity, the  $\phi^4$  theory is used, while further insight to the notions introduced here may be obtained by [2].

### 1.9.1 Dimensional analysis

The underlying dimensions of the theory's space-time become critical in the study of Renormalizability outcoming properties.

The production of this result initially begins by considering an action:

$$S = \int d^d x \mathcal{L} \quad (1.36)$$

of which dimensions, in a natural unit description, are equal to length dimension  $L$  raised to the negative power of the space-time dimension:

$$[\mathcal{L}] = L^{-d} \quad (1.37)$$

equivalently, a momentum dimension expression is available as:

$$[\mathcal{L}] = \Lambda^d \quad (1.38)$$

A four-derivative is equal to a negative one power, symbolically denoted:

$$[\partial_\mu] = L^{-1} \quad (1.39)$$

The  $\partial^\mu \phi \partial_\mu \phi$  kinetic energy term determines the dimensions of the field  $\phi$  as:

$$[\phi] = L^{1-d/2} \quad (1.40)$$

Considering a  $\phi^4$  theory interaction term weighted by a  $\lambda$  factor of dimension  $[\lambda] = L^{-\delta}$ :

$$\delta = 4 - d \quad (1.41)$$

Hereafter, only in the special case of four dimensions, the  $\phi^4$  interaction theory admits a dimensionless  $\lambda$  coupling constant.

However it is possible to gain insight to the theory by re-expressing  $D$  in the form:

$$D = d - \left( \frac{d}{2} - 1 \right) E - n\delta \quad (1.42)$$

where it is induced that the only option for a renormalizable theory is to admit a non negative mass dimension  $\delta \geq 0$ , directly allowing for non diverging higher order diagrams.

### 1.9.2 Renormalized Perturbation Theory

The scheme of Renormalization provides the method of connecting the bare results of an interacting theory with the direct measurable quantities

of mass  $m$ , coupling constant  $\lambda$  and field-strength  $\phi$ .

Considering a complete propagator that provides the equivalence between bare variable, such as  $m_B$  and measurable quantities:

$$G^{(2)}(p) = \frac{i}{p^2 - m_B^2 - \Sigma(p)} \quad (1.43)$$

Under this scheme a shift of the pole of the  $G^{(2)}(p)$  propagator moves it from  $m$  to  $\tilde{m}$  where it is defined as following:

$$\tilde{m}^2 - m_B^2 - \Sigma(\tilde{m}) = 0 \quad (1.44)$$

An expansion in the neighborhood of  $\tilde{m}$  provides:

$$G^{(2)}(p) = \frac{iZ}{p^2 - \tilde{m}^2} + \text{terms regular at } p^2 = \tilde{m}^2 \quad (1.45)$$

$Z$  is the residuum of the pole and simultaneously it can be seen as a probability amplitude of a vacuum state  $\Omega$  and  $\lambda_0$  being an exact one-particle state  $Z = |\langle \lambda_0 | \phi(0) | \Omega \rangle|^2$ .

A unitary normalisation of the probability amplitude for the renormalized field  $\phi_r$  is:

$$\phi = Z^{1/2} \phi_r \quad (1.46)$$

Since the new Lagrangian density for the system obtains the form:

$$\mathcal{L} = \frac{1}{2} Z (\partial_\mu \phi_r)^2 - \frac{1}{2} m_B^2 Z \phi_r^2 - \frac{\lambda_B}{4!} Z^2 \phi_r^4 \quad (1.47)$$

Applying the substitutions:

$$\delta_Z = Z - 1, \quad \delta_m = m_B^2 Z - m^2, \quad \delta_\lambda = \lambda_B Z^2 - \lambda \quad (1.48)$$

the theory is able to predict the physically measurable mass and coupling constant.

With the application of the above technique, the theory's Lagrangian density is:

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_r)^2 - \frac{1}{2} m^2 \phi_r^2 - \frac{\lambda}{4!} \phi_r^4 + \frac{1}{2} \delta_Z (\partial_\mu \phi_r)^2 - \frac{1}{2} \delta_m \phi_r^2 - \frac{\delta_\lambda}{4!} \phi_r^4 \quad (1.49)$$

After these renormalization conditions we are keen on producing new diagrammatic expression, characteristically the renormalized full propagator is:

$$\text{---} \bigcirc \text{---} = \frac{i}{p^2 - m^2} + (\text{terms regular at } p^2 = m^2) \quad (1.50)$$

which omits that the pole of the propagator sets the physical mass and that the residuum of the pole is fixed. The four point vertex function follows with the form:

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = G^{(4)}(p_1, p_2, p_3, p_4) G^{(2)^{-1}}(p_1) G^{(2)^{-1}}(p_2) G^{(2)^{-1}}(p_3) G^{(2)^{-1}}(p_4) \quad (1.51)$$

With the assistance of the Mandelstam variables  $s = (p_1 + p_2)^2$ ,  $t = (p_1 + p_3)^2$  and  $u = (p_1 + p_4)^2$ , the second condition comes as:

$$\text{---} \bigcirc \text{---} = -i\lambda \quad \text{at } s = 4m^2, t = u = 0 \quad (1.52)$$

The modified scheme of the Feynman perturbative diagrams is composed:

$$\longrightarrow = \frac{i}{p^2 - m^2} \quad (1.53)$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \bullet \begin{array}{c} \diagdown \\ \diagup \end{array} = -i\lambda \quad (1.54)$$

$$\text{---} \bigcirc \text{---} = i(p^2 \delta_Z - \delta_m) \quad (1.55)$$

$$\begin{array}{c} \diagup \\ \diagdown \end{array} \bigcirc \begin{array}{c} \diagdown \\ \diagup \end{array} = -i\delta_\lambda \quad (1.56)$$

### 1.9.3 The Callan-Symanzik equation

A new equation able to describe the flow of the coupling constant within the context of a dimensionless coupling theory, such as the renormalized  $\phi^4$  interaction theory, is more than a necessary need. Initiating from the unrenormalized bare Green functions:

$$\langle \Omega | T(\phi(x_1) \phi(x_2) \cdots \phi(x_n)) | \Omega \rangle \quad (1.57)$$

we step to the renormalized Green function by:

$$\langle \Omega | T(\phi_r(x_1) \phi_r(x_2) \cdots \phi_r(x_n)) | \Omega \rangle = Z^{-n/2} \langle \Omega | T(\phi(x_1) \phi(x_2) \cdots \phi(x_n)) | \Omega \rangle \quad (1.58)$$

where a shift in the coupling constant and in the rescaling factor has been of the form:

$$\begin{aligned} M &\rightarrow M + \delta M \\ \lambda &\rightarrow \lambda + \delta \lambda \\ Z &\rightarrow Z + \delta Z \end{aligned} \tag{1.59}$$

leads to:

$$\phi_r \rightarrow (1 + \delta\eta)\phi_r \tag{1.60}$$

with  $\delta\eta = \delta Z/Z$ . For the  $n$  field components of the Green function follows:

$$G^{(n)} \rightarrow (1 + \delta\eta)^n G^{(n)} \approx (1 + n\delta\eta)G^{(n)} \tag{1.61}$$

The shift in the Green functions provides a relation of differentials:

$$dG^{(n)} = \frac{\partial G^{(n)}}{\partial M} \delta M + \frac{\partial G^{(n)}}{\partial \lambda} \delta \lambda = n\delta\eta G^{(n)} \tag{1.62}$$

where two new high impact dimensionless parameters are defined as:

$$\beta = \frac{M}{\delta M} \delta \lambda \quad \text{and} \quad \gamma = -\frac{M}{\delta M} \delta \eta \tag{1.63}$$

The Callan-Symanzik equation arises:

$$\left[ M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + n\gamma(\lambda) \right] G^{(n)}(\{x_i\}; M, \lambda) = 0 \tag{1.64}$$

with dimensionless parameters being only a function of  $\lambda$ .

Counterterms are then determined so as to satisfy the renormalization condition:

$$\begin{aligned} \delta_\lambda &= (-i\lambda)^2 \cdot 3V(-M^2) \\ &= \frac{3\lambda^2}{2(4\pi)^2} \left[ \frac{1}{2-d/2} - \log(M^2) + \text{finite} \right] \end{aligned} \tag{1.65}$$

An application of the Callan-Symanzik equation to the leading terms of the renormalized four point function provides the  $\beta$ -function:

$$\left[ M \frac{\partial}{\partial M} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 4\gamma(\lambda) \right] G^{(4)}(p_1, p_2, p_3, p_4) = 0$$

$$\frac{3i\lambda^2}{16\pi^2} + \beta(\lambda)(-i) + 4\gamma(\lambda)(-i\lambda) = 0 \quad (1.66)$$

For  $\gamma(\lambda)$  being at least of order  $\lambda^2$ , this equation is only valid if:

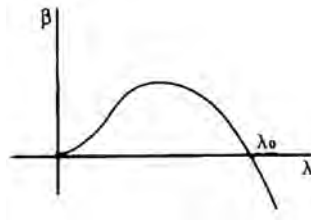
$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \quad (1.67)$$

#### 1.9.4 $\beta$ -function and triviality of $\phi^4$ theory

A variable momentum scale  $M$  provides the behaviour of the  $\beta$ -function with  $\lambda$  as its variable:

$$\beta(\lambda) = M \frac{\partial \lambda}{\partial M} \quad (1.68)$$

Characteristically, for a system obeying a  $\beta$ -function of the form:



the  $\beta$ -function is zero at two different points, at 0 and at  $\lambda_0$ .

For  $\lambda < \lambda_0$ :

$$M \frac{\partial \lambda}{\partial M} > 0 \quad (1.69)$$

with the variable  $\lambda$  approaching  $\lambda_0$  with increasing values of momentum

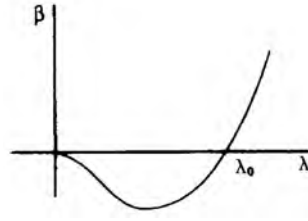
and for  $\lambda > \lambda_0$

$$M \frac{\partial \lambda}{\partial M} < 0 \quad (1.70)$$

where  $\lambda$  decreases towards  $\lambda_0$  with increasing momentum.

Subsequently,  $\lambda_0$  is an UltraViolet stable fixed point. Where on the other hand, the InfraRed fixed point is situated at  $\lambda = 0$ .

In the case of different curvature of the graph, such as:



there exist two fixed points where in the intermediate region,  $0 \leq \lambda \leq \lambda_0$ , an increasing coupling occurs for decreasing momentum determines an InfraRed stable fixed point  $\lambda_0$ .

For increasing values of momentum the coupling  $\lambda$  tends to zero and for large momentum values the coupling constant vanishes providing the Asymptotic Freedom behaviour.

In the special case of the  $\phi^4$  theory, the beta function in leading terms is provided by:

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \quad (1.71)$$

an equation whose further study provides that there is no existence of an UltraViolet fixed point fact usually referred as the triviality of the  $\phi^4$  theory.

## 1.10 The epsilon expansion

Working in the same universality class we may obtain another method: the epsilon expansion. Based on a Lagrangian of a scalar field  $\phi$  in an

Euclidean setup with an interaction term and a mass term:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{4!} \lambda \phi^4 + \frac{1}{2} m^2 \phi^2 \quad (1.72)$$

The method is to consider the free theory (the first term) of the Lagrangian, where we search for the dimension of  $\phi$ , so as this term to be four dimensional. This choice enables the use of the Lagrangian in order to produce the action functional that describes the theory.

The dimension of  $\phi$  as a function of dimensions has the form:

$$\Delta_\phi = \frac{d-2}{2} \quad (1.73)$$

Equivalently, the dimension of  $\phi^4$ :

$$\Delta_{\phi^4} = 4\left(\frac{d-2}{2}\right) = 2(d-2) \quad (1.74)$$

For  $d = 2$  or  $3$  the operator  $\phi^4$  becomes a very strong relevant operator. The addition of this term in the Lagrangian initiates a Renormalization Group flow that does not accept a perturbative analysis.

The  $\varepsilon$ -expansion method was introduced by K. Wilson by setting  $d = 4 - \varepsilon$  where  $\varepsilon$  is a small quantity,  $\varepsilon \ll 1$ , transforming the Scaling Dimensions of the operators as:

$$\Delta_\phi = \frac{d-2}{2} = 1 - \frac{\varepsilon}{2} \quad (1.75)$$

$$\Delta_{\phi^4} = 2(d-2) = 4 - 2\varepsilon \quad (1.76)$$

The introduction of the  $\varepsilon$ -expansion provides the study of the Lagrangian

through perturbation methods. Furthermore the strongly relevant operator  $\phi^2$  vanishes for a theory with zero mass.

The Renormalization Group flow is then described by the Beta function, as also being produced in eq. (1.71):

$$\beta = -\varepsilon\lambda + \frac{3\lambda^2}{16\pi^2} + \dots \quad (1.77)$$

Searching for the critical points of the beta function:

$$\beta = 0 \Rightarrow \lambda_* = 0 \text{ or } \lambda_* = \frac{16\pi^2}{3}\varepsilon \quad (1.78)$$

These solutions for  $\varepsilon \rightarrow 1$  do produce results that are in the same universality class of the critical points (such as of  $H_2O$ ) discussed in previous sections.

## 1.11 Further motivation

Expanding our scope, now we may have obtained a new way of studying natural theories and obtaining, hopefully, fascinating key characteristics of theories. Characteristically studying two models, with scalar and fermion fields, like the Gross-Neveu-Yukawa and the Nambu-Jona-Lasinio-Yukawa model with Lagrangians of the form:

$$\mathcal{L}_{\text{GNY}} = \frac{1}{2} (\partial_\mu \phi)^2 + i\bar{\psi} \not{\partial} \psi + g_1 \phi \bar{\psi} \psi + \frac{g_2}{4!} \phi^4 \quad (1.79)$$

$$\mathcal{L}_{\text{NJLY}} = \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \bar{\psi}_j \not{\partial} \psi^j + g_1 \bar{\psi}_j (\phi_1 + i\gamma_5 \phi_2) \psi^j + \frac{1}{24} g_2 (\phi \bar{\phi})^2 \quad (1.80)$$

Searching for the RG flow of this theory, as seen in the paper of Fei,

Klebanov and G. Tarnopolsky [5], through the use of epsilon-expansion technique, we obtain the beta function and consequently lead to the description of the RG trajectories. The beta functions of GNY model (in  $4 - \varepsilon$  dimensions):

$$\begin{aligned}\beta_{g_2} &= -\varepsilon g_2 + \frac{1}{(4\pi)^2} (3g_2 + 8g_1^2 g_2 - 48g_1^4) \\ \beta_{g_1} &= -\frac{\varepsilon}{2} g_1 + \frac{5}{(4\pi)^2} g_1^3\end{aligned}\tag{1.81}$$

While for the NJLY model the beta functions calculated in two loops:

$$\begin{aligned}\beta_1 &= -\frac{\varepsilon}{2} g_1 + \frac{1}{(4\pi)^2} \left( \frac{N}{2} + 2 \right) g_1^3 + \frac{1}{(4\pi)^4} \left( -\frac{8}{3} g_1^3 g_2 + \frac{1}{9} g_1 g_2^2 + (7 - 3N) g_1^5 \right) \\ \beta_2 &= -\varepsilon g_2 + \frac{1}{(4\pi)^2} \left( \frac{10}{3} g_2^2 + 2N g_2 g_1^2 - 12N g_1^4 \right) - \frac{1}{(4\pi)^4} \left( \frac{20}{3} g_2^3 - 96N g_1^6 - 2N g_2 g_1^4 + \frac{10}{3} N g_1^2 g_2^2 \right)\end{aligned}\tag{1.82}$$

These calculations lead us to a "chart" of the behaviour of the two referenced theories:

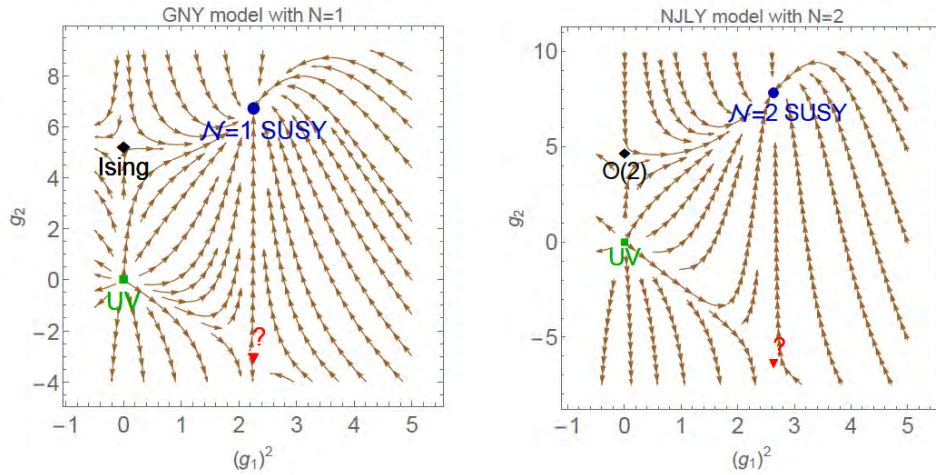


Figure 1.5: RG flow and fixed point structure for the GNY model with  $N = 1$  (one Majorana fermion in  $d = 3$ ) and the NJLY model with  $N = 2$  (one Dirac fermion in  $d = 3$ ), obtained from the one-loop  $\beta$ -functions in  $d = 4 - \varepsilon$ . Source: [5]

The attractive IR fixed points have emergent supersymmetry with 2 and 4 supercharges respectively. The red triangles denote unstable fixed points with negative quartic potential which can be seen in the one-loop analysis in  $d = 4 - \varepsilon$  their fate in  $d = 3$  is unclear.

This framework of study of the two theories allows us to discover that they obtain an emergent supersymmetry property since it is seen that there exists a symmetry exchanging fermions and bosons.

Nevertheless, the above perturbative methods indeed provide an immense amount of understanding for these physical systems but there also exist non-perturbative methods that unveil properties of the Fixed/Critical Points.

Fixed Points seem to obey Conformal Invariance, a property which allows them to exhibit scale invariant behavior. This characteristic does arise very interesting implications to theoretical physics while it is an active field of nowadays cutting edge research.

The development of this special kind of Quantum Field Theories, named Conformal Field Theories alongside with a special arising method of non-perturbative calculations, the Conformal Bootstrap, will be the main subject of this thesis.

## Chapter 2

# Foundations of Conformal Symmetry

The appearing plethora of fixed points allows a further expansion of thoughts about the nature of the Quantum Field Theories. The scheme of the Renormalization Group leads to a consideration of QFTs as RG flows from the Ultraviolet (UV) to Infrared (IR) regime.

$$\left. \begin{array}{c} \text{CFT}_{UV} \\ \downarrow \\ \text{CFT}_{IR} \end{array} \right\} \text{QFT}$$

The initial (UV) and final (IR) fixed points obey conformal invariance. What and how conformal invariance works will be the topic explored in this chapter.

## 2.1 Conformal Transformations

The idea of providing a scale independent physical system, geometrically, can be seen as the property of performing an arbitrary change of system's scale while this action does not alter its shape and maintains constant angles before and after the process. This is widely referred as a *Conformal Transformation*.

Conformal transformations are diffeomorphisms  $x^\mu \rightarrow x'^\mu$  which act on CFTs in Lorentzian or flat Euclidean space(time).

In this setup, the infinitesimal length element is defined in the  $\mathbb{R}^d$  space with a flat metric  $\eta_{\mu\nu}$  with  $diag(-1, 1, \dots, 1)$  and signature  $(p, q)$  as:

$$dx^2 = \eta_{\mu\nu} dx^\mu dx^\nu \quad (2.1)$$

The scale variation of the system is described by the introduction of the scale factor  $\Omega(x)$ :

$$dx'^2 = \Omega(x)^2 dx^2 \quad (2.2)$$

The variable  $\Omega$  establishes the *subgroup of Conformal transformations* and leads our diffeomorphism to look like an action of a composition of a rotation  $\Lambda_\nu^\mu(x)$  and a dilatation  $\Omega(x) \geq 0$  on the system. Consequently:

$$\eta_{\mu\nu} dx^\mu dx^\nu = \Omega(x) \eta_{\alpha\beta} dx^\alpha dx^\beta \Rightarrow \eta_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta} = \Omega(x) \eta_{\alpha\beta} \quad (2.3)$$

The Jacobian is:

$$\frac{\partial x^\mu}{\partial x^\nu} = \Omega(x) \Lambda_\nu^\mu(x), \quad \eta_{\rho\sigma} \Lambda_\mu^\rho(x) \Lambda_\nu^\sigma(x) = \eta_{\mu\nu} \quad (2.4)$$

While the reader of this chapter may find  $\Omega$  rescaling factor some-

what absurd, practically its introduction does not reduce our previous QFT working scheme since for  $\Omega(x) = 1$  we obtain the Lorentz transformations.

A necessary step in working with this theory, is to obtain the form of the transformations and their generators, so *Conformal transformations* may be derived through different but eventually same resulting methods, in our case, we search them by adding an infinitesimal term  $v^\mu(x)$  to system's coordinates:

$$x'^\mu = x^\mu + v^\mu(x) \quad (2.5)$$

While the rescaling (or dilatation) factor is formed as:

$$\Omega(x) = 1 + \sigma(x) \quad (2.6)$$

The infinitesimal length element is next to acquire a new form:

$$dx'^2 = dx^\mu dx^\nu (\eta_{\mu\nu} + 2\partial_\mu v_\nu + O(v^2)) \quad (2.7)$$

Equating the two expressions for the infinitesimal length elements, we obtain the differential equation:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = 2\sigma\eta_{\mu\nu} \quad (2.8)$$

Equation [2.8](#) admits high importance and is widely referred as the *Conformal Killing equation*.

Aiming at an explicit expression of the  $\sigma$  operator we search for the Trace of [2.8](#):

$$\text{Tr}[2\partial_\mu v^\mu = 2d\sigma] \Rightarrow \sigma = \frac{1}{d}\partial_\mu v^\mu \quad (2.9)$$

Correspondingly, the *Conformal Killing equation* accepts its dimensional-

dependent form:

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \frac{2}{d} \eta_{\mu\nu} \partial_\rho v^\rho \quad (2.10)$$

Having established an expression of the *Conformal Killing equation* we proceed on the search of solutions. This is quite not a trivial procedure with a re-expression of the derivative term  $\partial_\rho \partial_\nu v_\mu$  being a prerequisite:

$$\begin{aligned} \partial_\rho \partial_\nu v_\mu &= \frac{1}{2} (\partial_\rho (\partial_\mu v_\nu + \partial_\nu v_\mu) + \partial_\nu (\partial_\mu v_\rho + \partial_\rho v_\mu) - \partial_\mu (\partial_\rho v_\nu + \partial_\nu v_\rho)) \\ \partial_\rho \partial_\nu v_\mu &= \partial_\rho \sigma \eta_{\mu\nu} + \partial_\nu \sigma \eta_{\mu\rho} - \partial_\mu \sigma \eta_{\nu\rho} \end{aligned} \quad (2.11)$$

Proceeding by acting with  $\partial^\mu$  operator:

$$d \partial_\rho \partial_\nu \sigma = 2 \partial_\nu \partial_\rho \sigma - \eta_{\nu\rho} \partial^2 \sigma \quad (2.12)$$

the solution is provided by the search of the Trace of both parts of the equation:

$$\stackrel{\text{trace}}{\Rightarrow} d \partial^2 \sigma = 2 \partial^2 \sigma - d \partial^2 \sigma \Rightarrow (d-1) \partial^2 \sigma = 0 \quad (2.13)$$

Consequently, there are two possible cases either  $d = 1$  or  $\partial^2 \sigma = 0$ .

Introducing these results to eq. 2.8 we deduce the condition:

$$(d-2)(d-1) \partial_\rho \partial_\nu \sigma = 0 \quad (2.14)$$

For  $d \neq 2$  or  $1$ , *Conformal Transformations* are obtained for  $\partial_\rho \partial_\nu \sigma = 0$ , leading to the conclusion that  $\sigma$  can be at most linear at  $x$ .

Finally, the most general solution for  $\sigma$  admits the form:

$$\stackrel{d \neq 1,2}{\Rightarrow} \sigma(x) = k - 2b^\mu x_\mu \quad (2.15)$$

Replacing the  $\sigma(x)$  result to eq. 2.13 which has been a second order differential equation for  $v_\mu$  and  $b_\rho$  a constant vector we obtain the most general expression:

$$v_\mu(x) = a_\mu - \omega_{\mu\nu}x^\nu + \kappa x_\mu + b_\mu x^2 - 2x_\mu b_\nu x^\nu \quad (2.16)$$

with  $\omega_{\mu\nu} = -\omega^{\mu\nu}$  being antisymmetric.

## 2.2 Identifying Conformal Transformations

In the previous section we derived the most general solution of Conformal Killing equation, the solution 2.16 can be further understood if seen term per term:

$$\alpha_\mu \rightarrow \text{translations with dim } d \quad (2.17)$$

$$\omega_{\mu\nu}x^\nu \rightarrow \text{rotations with dim } (1/2)d(d-1) \quad (2.18)$$

$$\kappa x_\mu \rightarrow \text{scale transformations with dim } 1 \quad (2.19)$$

$$b_\mu x^2 - 2x_\mu b_\nu x^\nu \rightarrow \text{special conformal transformations with dim } d \quad (2.20)$$

The formed Group is a Lie Group of dimension  $(\frac{1}{2})(d+1)(d+2)$  which is of high physics importance since it is the largest non-trivial subgroup of diffeomorphisms of  $\mathbb{R}^d$ .

## 2.3 Conformal Algebra

Having acknowledged the Conformal Group via the use of infinitesimal terms, equivalently we can proceed in the definition of the corresponding Conformal Algebra.

For this we use the following operators:  $\mathcal{P}_\mu$  and  $M_{\mu\nu}$  for the Poincaré generators,  $D$  for the Dilatation Operator and  $K_\mu$  expressing the operators of Special Conformal Transformations.

These generators produce six non-zero commutators:

$$[M_{\mu\nu}, \mathcal{P}_\rho] = i (\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \quad (2.21)$$

$$[M_{\mu\nu}, K_\rho] = 2 (\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \quad (2.22)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i (\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho}) \quad (2.23)$$

$$[D, \mathcal{P}_\mu] = -i P_\mu \quad (2.24)$$

$$[D, K_\mu] = i K_\mu \quad (2.25)$$

$$[K_\mu, P_\nu] = 2i (\eta_{\mu\nu} D - M_{\mu\nu}) \quad (2.26)$$

In the case of Euclidean space the Conformal Algebra is isomorphic to  $SO(d+1, 1)$  algebra proved by the map:

$$\begin{aligned} \mathcal{J}_{d+1\mu} &= (P_\mu - K_\mu) / 2, & \mathcal{J}_{d+2\mu} &= (P_\mu + K_\mu) / 2, \\ \mathcal{J}_{\mu\nu} &= M_{\mu\nu}, & \mathcal{J}_{d+1d+2} &= D \end{aligned}$$

satisfying the  $SO(d+1, 1)$  commutation relations

$$[\mathcal{J}_{AB}, \mathcal{J}_{CD}] = \eta_{BC} \mathcal{J}_{AD} - \eta_{AC} \mathcal{J}_{BD} + \eta_{BD} \mathcal{J}_{CA} - \eta_{AD} \mathcal{J}_{CB},$$

with  $\eta_{AB}$  being the Lorentzian metric on  $\mathbb{R}^{d+1,1}$ .

## 2.4 The Finite form of the transformations

The Conformal Transformation also maintain the property of being expressed as finite forms. For this we will introduce a new parameter  $t$  as:

$$\frac{dx^\mu}{dt} = v^\mu(x) \quad (2.27)$$

For scale transformations we obtain:

$$\dot{x}^\mu = kx^\mu \Rightarrow x^\mu(t) = e^{\kappa t}x^\mu(0) \Rightarrow x'^\mu = e^\kappa x^\mu \quad (2.28)$$

while for the case of Special Conformal Transformations (SCT):

$$\dot{x}^\mu = b^\mu x^2 - 2x^\mu b_\nu x^\nu \quad (2.29)$$

The following change of variables provides an expression for  $x'^\mu$ :

$$y^\mu = \frac{x^\mu}{x^2} \text{ then } \dot{y}^\mu = b^\mu \quad (2.30)$$

$$\frac{x^\mu}{x^2}(t) = \frac{x^\mu}{x^2}(0) + b^\mu t \quad (2.31)$$

$$\Rightarrow \frac{x^\mu}{x^2}(t) = \frac{x^\mu}{x^2} + b^\mu \quad (2.32)$$

Eventually for  $x'^\mu$  the formula is shaped as:

$$x'^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2x^\mu b_\mu + b^2 x^2} \quad (2.33)$$

A geometrical interpretation of our actions can be sensed that a *Translation* is like having "fixed"  $\infty$  and "moved" 0, meaning that an object can move further than 0 while *Special Conformal Transformations* act as oppo-

site and its like having "fixed" 0 and "moved"  $\infty$ . So SCTs can be seen as *Translations* in the *inverse* space.

## 2.5 Generators of Conformal Transformations

*Generators of Conformal Transformations* are useful on pointing out how a transformation on spacetime is translated into a transformation on a field of spacetime.

In QFT the *Generator* for a scalar field  $\phi$  is traditionally produced via a Taylor expansion of the field:

$$\begin{aligned}\phi(x + \varepsilon) &= \phi(x) + \varepsilon^\mu \partial_\mu \phi(x) + \dots \\ &= \phi(x) + \varepsilon^\mu i \mathcal{P}_\mu \phi(x) + \dots\end{aligned}\tag{2.34}$$

where  $\mathcal{P}_\mu \equiv -i\partial_\mu$  is the Generator of Translations.

It is characteristic that the *Generator* and the *Differential Representation* of the *Generator* are different since their commutator is:

$$[\mathcal{P}_\mu, \phi] = i\partial_\mu \phi = -\mathcal{P}_\mu \phi\tag{2.35}$$

The appearing minus sign is important as:

$$[A, [B, \phi]] = [A, -B^D \phi] = -B^D [A, \phi] = B^D A^D \phi\tag{2.36}$$

a minus sign is also necessary in the relation formula of both pictures:

$$[[A, B], \phi] = -[A^D, B^D] \phi\tag{2.37}$$

Equivalently, all the *Generators* of the *Transformations* are formed:

$$[\mathcal{P}_\mu, \mathcal{O}_I(x)] = i\partial_\mu \mathcal{O}_I(x) \quad (2.38)$$

$$[D, \mathcal{O}_I(x)] = -i(x \cdot \partial + \Delta)\mathcal{O}_I(x) \quad (2.39)$$

$$[M_{\mu\nu}, \mathcal{O}_I(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu) \mathcal{O}_I(x) - \mathcal{O}_J(x) (S_{\mu\nu})_I^J. \quad (2.40)$$

$$[K_\mu, \mathcal{O}_I(x)] = i(x^2 \partial_\mu - 2x_\mu x \partial - 2\Delta x_\mu) \mathcal{O}_I(x) + 2\mathcal{O}_J(x) (s_{\mu\nu})_I^J x^\nu \quad (2.41)$$

In the following table, [\[6\]](#), we summarize the results of a CFT's *Transformations* and *Generators* along with the corresponding scaling factor  $\Omega$ :

| <i>Transformation</i> | <i>Infinitesimal</i>                               | <i>Finite</i>                                          |
|-----------------------|----------------------------------------------------|--------------------------------------------------------|
| <i>Translation</i>    | $x'^\mu = x^\mu + v^\mu$                           | $x'^\mu = x^\mu + V^\mu$                               |
| <i>Rotation</i>       | $x'^\mu = (\delta_\nu^\mu + \omega_\nu^\mu) x^\nu$ | $x'^\mu = \Lambda_\nu^\mu x^\nu$                       |
| <i>Dilatation</i>     | $x'^\mu = (1 + c)x^\mu$                            | $x'^\mu$                                               |
| <i>SCT</i>            | $x'^\mu = x^\mu + 2x^\mu(x \cdot b) - x^2 b^\mu$   | $x'^\mu = \frac{x^\mu + x^2 b^\mu}{1 + 2xb + b^2 x^2}$ |

| <i>Generator</i>                                           | $\Omega(x)$                         |
|------------------------------------------------------------|-------------------------------------|
| $P_\mu = -i\partial_\mu$                                   | 1                                   |
| $L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$  | 1                                   |
| $D = -ix^\mu \partial_\mu$                                 | $\lambda^2$                         |
| $K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$ | $(1 - 2(b \cdot x) + b^2 x^2)^{-2}$ |

## 2.6 Primaries and Descendants

In this section the aim is to introduce and gain insight in the methods used in CFT for describing fields. In CFT it is common to replace the QFT term

"fields" with the term "Operators", which are denoted as  $\mathcal{O}_I(x)$ .

A special category of Operators in the CFT which respects the property:

$$\mathcal{O}_I(x) \rightarrow \mathcal{O}'_I(x') = \Omega_L(x)^{-\Delta} \mathcal{R}_I^J(x) \mathcal{O}_J(x) \quad (2.42)$$

defines a specific category of them, the *Primary Operators*.  $\Delta$  denotes the scaling dimension of the operator  $\mathcal{O}$  and  $\mathcal{R}_I^J(x)$  is the Rotation matrix in the appropriate representation in accordance with what  $(I, J)$ -indices implies for the system.

In the case of  $\mathcal{O}(0) \equiv \mathcal{O}_{\Delta,r}^i(0)$  forming a finite-dimensional irreducible representation  $r$  of the rotation group. This group is determined by its scaling dimension which is the dilatation eigenvalue  $\Delta$ , consequently the commutators accept the form:

$$\begin{aligned} [D, \mathcal{O}_{\Delta,r}^i(0)] &= \Delta \mathcal{O}_{\Delta,r}^i(0), \\ [M_{\mu\nu}, \mathcal{O}_{\Delta,r}^i(0)] &= (R_{\mu\nu})^i \mathcal{O}_{\Delta,r}^j(0). \end{aligned} \quad (2.43)$$

The action of a derivative in a *Primary Operator* does not produce a Primary operator but rather leads to a new category of operators, the *Descendants*.

Descendants are important since they hold the very useful property of allowing the derivative of a correlation function behave as:

$$\langle \partial_\mu (\mathcal{O}_1(x) \mathcal{O}_2(0)) \rangle = \partial_\mu \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \quad (2.44)$$

## 2.7 Correlation functions of fields

Our scope now is expanded on the impact of conformal symmetry in quantities that characterize a physical system. Such example is the well-known Correlation function of fields. Correlation function is the main output information of a Quantum Field Theory. In terms of operators, it is the diagonal matrix element of a product of local operators corresponding to the vacuum operator. It is a generalized function of position of local operators [23].

Enhancing more on the correlation function in the case of two scalar fields  $\phi_1$  and  $\phi_2$  we would notice that a Lorentzian theory leads to 2-point correlation function with a  $x_1^\mu - x_2^\mu$  spacetime difference dependence for translations and a  $(x_1^\mu - x_2^\mu)^2$  dependence for rotations.

Overall, this 2-point correlation function is expressed:

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = cf(x_{12}^2), \quad x_{12}^\mu = x_1^\mu - x_2^\mu \quad (2.45)$$

An action of the Dilatation operator  $D$  to this 2-point correlation function provides:

$$D \langle \phi_1(x_1) \phi_2(x_2) \rangle = -i(x_1 \partial_{x_1} + \Delta_1 + x_2 \partial_{x_2} + \Delta_2) cf(x_{12}^2) \quad (2.46)$$

A requirement for this correlation function in a theory respecting scale invariance, the 2-point function should be invariant under Dilatations and so eq. 2.46 is equal to zero:

$$D \langle \phi_1(x_1) \phi_2(x_2) \rangle = -i(x_1 \partial_{x_1} + \Delta_1 + x_2 \partial_{x_2} + \Delta_2) cf(x_{12}^2) = 0 \quad (2.47)$$

An evaluation of the action of the derivatives in the above expression:

$$f'(\underbrace{2x_1x_{12} - 2x_2x_{12}}_{2x_{12}^2}) + (\Delta_1 + \Delta_2) f = 0 \quad (2.48)$$

provides that the constraint on the two-point function of two operators with different scaling dimensions is:

$$\frac{f'}{f} = -\frac{\Delta_1 + \Delta_2}{x_{12}^2} \Rightarrow f(x_{12}^2) = \frac{1}{(x_{12}^2)^{\frac{1}{2}(\Delta_1 + \Delta_2)}} \quad (2.49)$$

Equivalently, when the Generator of the Special Conformal Transformation acts to the two point correlation function:

$$K_\mu^{(D)} \langle \phi_1(x_1) \phi_2(x_2) \rangle = 0 \quad (2.50)$$

a restrictive condition is induced:

$$\Rightarrow c(\Delta_1 - \Delta_2) x_{12}^\mu = 0 \quad (2.51)$$

The resulting outcome is that CFT's Correlation functions are consistent with SCT if they only accept the same scaling dimension:

$$\Delta_1 = \Delta_2 \text{ or } c = 0 \quad (2.52)$$

Two points functions, with scalar operators, in CFT are connected with the formula:

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = c \delta_{\Delta_1 \Delta_2} \frac{1}{(x_{12}^2)^{\Delta_1}} \quad (2.53)$$

Considering four point functions with points  $x_1, x_2, x_3, x_4$ , we can create non trivial conformal invariant combinations by introducing the quantities

$u$  and  $v$  can be introduced:

$$u = \frac{x_{12}^2 x_{24}^2}{x_{13}^2 x_{24}^2} \quad , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \quad (2.54)$$

With these new variables, the four point function is expressed in a condensed form:

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u, v) \quad (2.55)$$

Since the combinations of points may become random, when we swap the points in eq. 2.55, per example for the interchange  $2 \leftrightarrow 4$ , the new expression:

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} f(\tilde{u}, \tilde{v}) \quad (2.56)$$

has the function  $f(\tilde{u}, \tilde{v})$  dependent on the conformal invariant cross-ratios calculated with the interchanged indices:

$$\tilde{u} = v, \quad \tilde{v} = u \quad (2.57)$$

For no loss of generality, it is required for both eq. 2.55 and eq. 2.56 to be equal:

$$\frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u, v) = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} f(\tilde{u}, \tilde{v}) \quad (2.58)$$

After a multiplication by  $x_{14}^{2\Delta} x_{24}^{2\Delta} f(u, v)$  this condition is emerged:

$$\left(\frac{v}{u}\right)^\Delta f(u, v) = f(\tilde{u}, \tilde{v}) \quad (2.59)$$

So far this procedure has been keen to produce results for the scalar operators we used. Expanding further this procedure, to include the 2-point function of operators in the traceless symmetric irreducible representation

of  $SO(d)$  with spin  $\ell$ . For the scaling dimensions  $\Delta$  being the same, a non zero result is produced in the form:

$$\langle \mathcal{O}^{\mu_1 \dots \mu_\ell}(x_1) \mathcal{O}_{\nu_1 \dots \nu_\ell}(x_2) \rangle = C^{(\ell)} \left( \frac{I_{\nu_1}^{(\mu_1)}(x_{12}) \dots I_{\nu_\ell}^{(\mu_\ell)}(x_{12})}{x_{12}^{2\Delta}} - \text{traces} \right),$$

$$I_\nu^\mu \equiv \delta_\nu^\mu - 2 \frac{x^\mu x_\nu}{x^2}$$
(2.60)

The object  $I_\nu^\mu(x)$  is an orthogonal matrix and it's associate inversion is given by:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Omega(x) I_\nu^\mu(x)$$
(2.61)

The 3-point function of two operators with a spin  $\ell$  operator is:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_{\Delta, \ell}^{\mu_1 \dots \mu_\ell}(x_3) \rangle = \frac{\lambda_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_{\Delta, \ell}} (Z^{\mu_1} \dots Z^{\mu_\ell} - \text{traces})}{x_{12}^{\Delta_1 + \Delta_2 - \Delta + \ell} x_{23}^{\Delta_2 + \Delta - \Delta_1 - \ell} x_{31}^{\Delta + \Delta_1 - \Delta_2 - \ell}},$$

$$Z^\mu \equiv \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$$
(2.62)

The antisymmetric behaviour of  $Z^\mu$  in the interchange  $1 \leftrightarrow 2$  means that the 3-point function of two identical scalar fields is vanishing for odd  $\ell$ 's and exists only for even  $\ell$ 's.

Non identical scalar fields provide a four point function of the form:

$$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \mathcal{O}_k(x_3) \mathcal{O}_l(x_4) \rangle = \left( \frac{x_{24}}{x_{14}} \right)^{\Delta_{ij}} \left( \frac{x_{14}}{x_{13}} \right)^{\Delta_{kl}} \frac{f_{ijkl}(u, v)}{x_{12}^{\Delta_i + \Delta_j} x_{34}^{\Delta_k + \Delta_l}}$$
(2.63)

with  $\Delta_{ij} = \Delta_i - \Delta_j$ .

With the correlator being invariant under interchanges the corresponding *Crossing equation* is:

$$v^{\frac{(\Delta_j + \Delta_k)}{2}} f_{ijkl}(u, v) = u^{\frac{(\Delta_i + \Delta_j)}{2}} f_{kjil}(v, u)$$
(2.64)

## Chapter 3

# Radial Quantization

One of the basic milestones of CFTs and especially of the Conformal Bootstrap method is the definition of the core structure of this Quantum Field Theory and the discovery of the properties between the Operators living there.

### 3.1 Radial Quantization

Under this notion, the construction of the underlying geometric structure of a Quantum Field Theory can consist out of a  $d$ -dimensional spacetime foliation with  $(d - 1)$ -surfaces, equipped with a time parameter  $t$ . The evolution of  $t$  characterizes each surface.

In a plethora of Poincaré invariant Conformal Field Theories the Generator of time translations is selected to be the zeroth component of four-momentum, the Hamiltonian.

In this scheme, states are defined on each leaf of foliation by an assignment of a Hilbert space equipped with an inner product and an evolution operator of the form  $U = e^{iHt}$ . In CFTs a special foliation of spheres  $S^{d-1}$

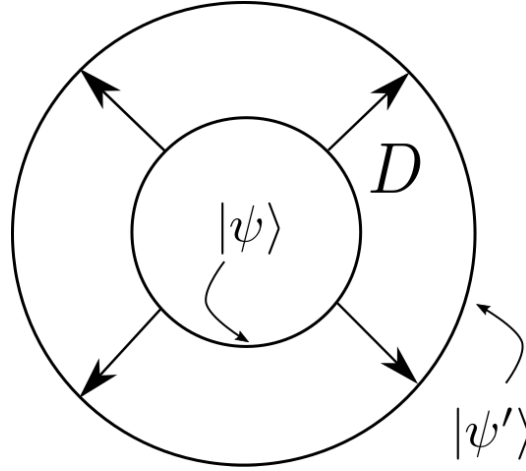


Figure 3.1: *Graphic representation of sphere foliation* [Source:[\[7\]](#)]

of different radii is selected.

The Dilatation operator produces the evolution of the system, while the radial quantization states are eigenvalues of the Dilatation operator.

For an operator  $\mathcal{O}_I(x)$ :

$$[D, \mathcal{O}_I(x)] = -i(x\partial + \Delta) \mathcal{O}_I(x) \quad (3.1)$$

Having denoted the states as  $|\Delta\rangle$ , for  $x = 0$ :

$$\begin{aligned} D|\Delta\rangle &= D\mathcal{O}_\Delta|0\rangle = [D_1, \mathcal{O}_\Delta]|0\rangle + \mathcal{O}_\Delta D|0\rangle \\ &= -\Delta\mathcal{O}_\Delta|0\rangle \\ &= -i\Delta|\Delta\rangle \end{aligned} \quad (3.2)$$

States are generated on the sphere by inserting operators inside the sphere while a no-insert on the sphere produces the vacuum state  $|0\rangle$ .

An insertion of the operator  $\mathcal{O}_\Delta(0)$  leads to the state  $|\Delta\rangle$ , while for  $\mathcal{O}_\Delta(x \neq 0)$ :

$$\mathcal{O}_\Delta(x)|0\rangle = e^{iPx}\mathcal{O}_\Delta(0)e^{-iPx}|0\rangle \quad (3.3)$$

Furthermore, the introduction of an exponential series expansion can give the a series expression:

$$\mathcal{O}_\Delta(x)|0\rangle = e^{iPx}|\Delta\rangle = \sum_n \frac{1}{n!} (iPx)^n |\Delta\rangle \quad (3.4)$$

It is particularly usefull, to explore the action of only one power of momentum of  $\Delta$  acting on the Dilatation Operator:

$$\begin{aligned} D(P_\mu|\Delta\rangle) &= [D, P_\mu]|\Delta\rangle + P_\mu D|\Delta\rangle \\ &= -iP_\mu|\Delta\rangle - i\Delta P_\mu|\Delta\rangle \\ &= -i(\Delta + 1)(P_\mu|\Delta\rangle) \end{aligned} \quad (3.5)$$

The resulting pattern immediately allows to recognise that an action with momentum raises the scaling dimension of the state  $\Delta$ , with the Operator  $\mathcal{O}_\Delta(x)$  to insert states of increasing  $\Delta$  to the foliation:

$$|\Delta\rangle \xrightarrow{P} |\Delta + 1\rangle \xrightarrow{P} |\Delta + 2\rangle \xrightarrow{P} \dots \quad (3.6)$$

Equivalently, the reduction to states with lower scaling dimension occurs with the Special Conformal Transformations operator  $K$ :

$$|\Delta\rangle \xleftarrow{K} |\Delta + 1\rangle \xleftarrow{K} |\Delta + 2\rangle \xleftarrow{K} \dots \quad (3.7)$$

Furthermore, states are produced from the operators, while by reverse, if states exist, then the definition of operators defines the correlation functions. This scheme is referred as the *State-Operator Correspondence*.

## 3.2 Unitarity in Radial Quantization

In an Euclidean spacetime a map, the Weyl rescaling,  $\mathbb{R}$  to  $\mathbb{R} \times S^{d-1}$  does associate time quantization in Euclidean setup with radial quantization:

$$\begin{aligned} ds_{\mathbb{R}^d}^2 &= dr^2 + r^2 ds_{S^{d-1}}^2 \\ &= r^2 \left( \frac{dr^2}{r^2} + ds_{S^{d-1}}^2 \right) \\ &= e^{2\tau} (d\tau^2 + ds_{S^{d-1}}^2), \quad r \equiv e^\tau \end{aligned} \tag{3.8}$$

The variable  $\tau$  range from  $-\infty$  to  $+\infty$  permits the interpretation of  $\tau$  as the time direction in radial quantization. Dilatations do affect the  $\tau$  variable:

$$r \rightarrow \lambda r \Rightarrow \tau \rightarrow \tau + \log \lambda \tag{3.9}$$

Eq. [3.9](#) provides a framework of understanding the character of conjugation in radial quantization by exhibiting the conformal transformation between the plane and the cylinder.

Working in polar coordinates, the action of rescaling of  $r$  in [3.9](#) corresponds to time translation for the  $\tau$  coordinates which behave like time. Furthermore, it is now noticed that rescalings of  $r$  provide an Euclidean theory with standard time on the cylinder. Conjugation in Euclidean theories actually means a change in the sign of time and so the operators obey:

$$\mathcal{O}_{\text{cyl}}(\tau, \vec{n})^\dagger = \mathcal{O}_{\text{cyl}}(-\tau, \vec{n}) \tag{3.10}$$

Correspondingly, a use the definition of cylinder operators which dictates that:

$$\mathcal{O}_{\text{cyl}}(\tau, \vec{n}) = e^{\Delta\tau} \mathcal{O}_{\text{flat}}(x = e^\tau \vec{n}) \tag{3.11}$$

allows a view back to the plane in radial quantization where the conjugation acts as:

$$\mathcal{O}_{\text{flat}}^\dagger(x) = e^{-2\Delta t} \mathcal{O}_{\text{flat}}(x = e^{-\tau} \hat{n}) = \frac{1}{x^{2\Delta}} \mathcal{O}_{\text{flat}}\left(\frac{x^\mu}{x^2}\right) \quad (3.12)$$

Equation [3.12](#) has an immediate effect on the Special Conformal Transformations operator  $K$ . The action of an Inversion  $I$  on SCT operator  $K$  yields that:

$$P^\dagger = IPI = K \quad (3.13)$$

This arising property [3.13](#) presents high interest and importance for Unitarity Bounds, which is the subject of the following section.

### 3.3 Unitarity Bounds

A plethora of QFTs with physical meaning maintain the property of being unitary. Unitarity provides the preservation of probability in quantum mechanics and dictates positive state norms:  $\langle \phi | \phi \rangle \geq 0$ .

Searching the impact and mainly the constraints arising from this property with no loss of generality, we notice that at point  $x = 0$  it is true that:

$$(P_\mu |\mathcal{O}_I\rangle)^\dagger P_\nu |\mathcal{O}_J\rangle \succeq 0 \quad (3.14)$$

with the operator  $\mathcal{O}_J$  being a primary. The introduction of the commutators of  $K_\mu$  and  $P_\nu$  operators provides as set of matrix inequalities:

$$\begin{aligned} \langle \mathcal{O}_I | [K_\mu, P_\nu] | \mathcal{O}_J \rangle &\succeq 0 \\ \langle \mathcal{O}_I | D\delta_{\mu\nu} - M_{\mu\nu} | \mathcal{O}_J \rangle &\succeq 0 \end{aligned} \quad (3.15)$$

An inclusion of a spin term  $s_{\mu\nu}$  in the operators leads to a bit more constraining set of inequalities:

$$\begin{aligned}\langle \mathcal{O}_I | 2\Delta\delta_{\mu\nu} - 2(s_{\mu\nu}) | \mathcal{O}_J \rangle &\succeq 0 \\ (\Delta\delta_{\mu\nu} - s_{\mu\nu}) \langle \mathcal{O}_I | \mathcal{O}_J \rangle &\succeq 0\end{aligned}\tag{3.16}$$

Unitarity is valid after requiring that the parenthesis term of eq. 3.16 is positive. The resulting constraint for  $\Delta$  is:

$$\Delta \geq \text{max eigenvalue of } s_{\mu\nu}\tag{3.17}$$

The case of the operator  $\mathcal{O}_I$  being a scalar leads to  $\Delta \geq 0$ . Additionally, for this case a stronger condition of requiring the norm not being negative is acquired:

$$||P^2|\mathcal{O}\rangle|| \geq 0\tag{3.18}$$

Hereafter, the scaling dimension of the scalar constraints to:

$$\Delta \geq \frac{d-2}{2}\tag{3.19}$$

A spin- $l$  operator will shape the condition for  $\Delta$  as:

$$\Delta \geq l + d - 2\tag{3.20}$$

The concept of using properties of Quantum Field Theories in order to raise constraints is the basic concept of the Conformal Bootstrap. A scheme which will be investigated with detail in the following chapters.

# Chapter 4

## OPE and the Conformal Blocks

Operator Product Expansion in Conformal Field Theory is an appearing convergent expansion. This characteristic allows the use of OPE in  $n$ -point correlation functions to algorithmically reduce them to  $(n - 1)$ -point functions, a procedure closely related to Conformal Blocks which are a topic of this chapter.

### 4.1 Operator Product Expansion (OPE)

A common feature often provided in Quantum Field Theories regards the expression of bilocal product of operators in the limit where the operators being very close to each other. This product in the Operator Product Expansion (OPE) is expressed as a new series of operators located at the midpoint.

A form of two bilocal operators  $\mathcal{O}(x)$  and  $\mathcal{O}(y)$  in this scheme will be analogous to the sum of coefficients  $c_k$  with dependence on the distance

between the two points and the operator  $\mathcal{O}_k(y)$ :

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \sim \sum_k c_k(x-y)\mathcal{O}_k(y) \quad (4.1)$$

Conformal Field Theories share the same feature with more powerful properties mainly due to OPE's connection with the Radial Quantization. OPEs are derived through the framework of Radial Quantization by the consideration of two operators inside a sphere. These operators generate a state  $|\psi\rangle$  on the surface of the sphere. The state  $|\psi\rangle$  is expanded on a sum of coefficients on the basis  $\Delta_n$ :

$$|\psi\rangle = \sum_n c_n |\Delta_n\rangle, \quad c_n = c_n(x) \quad (4.2)$$

The principle of state-operator correspondence, declares that the states  $|\Delta_n\rangle$  will be in one-to-one correspondence with operators that are either primaries or descendants of primaries:

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\mathcal{O} \text{ primaries}} C_{\mathcal{O}}(x, \partial_y) \mathcal{O}(y) \Big|_{y=0} |0\rangle \quad (4.3)$$

where  $C_{\mathcal{O}}(x, \partial_y)$  is interpreted as a power series in  $\partial_y$ .

Generally in QFTs the use of OPEs is restricted only to the asymptotic small distance limit. In CFTs the OPEs may be seen as an expansion in a Hilbert space leading to the conclusion that OPEs in CFT is not just asymptotic but it provides a convergent expansion at finite point separation.

On the following terms of the series:

$$\phi_1(x)\phi_2(0) \sim \frac{\text{const.}}{|x|^k} [\mathcal{O}(0) + \dots] + \text{contributions of other primaries} \quad (4.4)$$

The  $k$ -factor is fixed by scaling, while an action with the operator  $D$  on the LHS produces:

$$\begin{aligned} [D, \phi_1(x)\phi_2(0)] &= -i (\Delta_1 + x \cdot \partial + \Delta_2) \phi_1(x)\phi_2(0) \\ &= -i (\Delta_1 + \Delta_2 - k) \frac{c}{|x|^k} \mathcal{O}(0) + \dots \end{aligned} \quad (4.5)$$

The action of the Dilatation operator  $D$  on the RHS results to:

$$D \frac{\text{const.}}{|x|^k} [\mathcal{O}(0) + \dots] = \frac{\text{const.}}{|x|^k} [i\Delta_{\mathcal{O}} \mathcal{O}(0) + \dots] \quad (4.6)$$

while the requirement that the LHS coincides with the RHS, induces condition that:

$$\Delta_1 + \Delta_2 - k = \Delta_{\mathcal{O}} \Rightarrow k = \Delta_1 + \Delta_2 - \Delta_{\mathcal{O}} \quad (4.7)$$

Equivalently, the descendant terms have the form:

$$\phi_1(x)\phi_2(0) = \frac{\text{const.}}{|x|^k} [\mathcal{O}(0) + cx^\mu \partial_\mu \mathcal{O}(0) + \dots] + \text{contributions of other primaries} \quad (4.8)$$

where action of  $K_\mu$  operation on both sides of eq.4.8 provides on the LHS:

$$\begin{aligned} K_\mu \phi_1(x)\phi_2(0) &= (\sim x_\mu + \sim x^2 \partial_\mu) \phi_1(x)\phi_2(0) \\ &\sim \frac{\text{const.}}{|x|^k} x_\mu [\mathcal{O}(0) + \dots] \end{aligned} \quad (4.9)$$

with  $K_\mu \phi_2(0) = 0$ . Since  $\phi_2(0)$  is a primary operator substituted by the OPE in the second line. the action on the RHS is:

$$\begin{aligned} K_\mu \frac{\text{const.}}{|x|^k} [\mathcal{O}(0) + x^\nu \partial_\nu \mathcal{O}(0) + \dots] |0\rangle \\ = \frac{\text{const.}}{|x|^k} [K_\mu \mathcal{O}(0) + cK_\mu x^\nu \partial_\nu \mathcal{O}(0) + \dots] |0\rangle \\ \sim \frac{\text{const.}}{|x|^k} [c\Delta x^\mu \mathcal{O}(0) + \dots] \end{aligned} \quad (4.10)$$

it is noticed that the action with  $K_\mu$  induces a reduction of the dimension by one, so  $K_\mu \mathcal{O}(0) = 0$  and  $K_\mu P_\nu \mathcal{O}(0) \sim \delta_{\mu\nu} \Delta \mathcal{O}(0)$ .

Through Conformal invariance the function  $C_{\mathcal{O}}(x, \partial_y)$  is fixed up to a numerical prefactor  $\lambda_{\mathcal{O}}$ :

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}} C_{\mathcal{O}}(x, \partial_y) \mathcal{O}(y) \Big|_{y=0} |0\rangle, \quad \text{with } \lambda_{\mathcal{O}} \text{ a free parameter.} \quad (4.11)$$

Furthermore, a distinct method of computing the function  $C_{\mathcal{O}}(x, \partial_y)$  is to use OPE to reduce  $n$ -point functions to  $(n - 1)$ -point functions. An introduction of the OPE inside the correlator of the scalar 3pt function provides:

$$\langle \phi_1(x) \phi_2(0) \Phi(z) \rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}} C_{\mathcal{O}}(x, \partial_y) \langle \mathcal{O}(y)|_{y=0} \Phi(z) \rangle \quad (4.12)$$

The 2pt function is diagonal resulting to the summation there being be a term for  $\mathcal{O} = \Phi$ :

$$\langle \phi_1(x) \phi_2(0) \Phi(z) \rangle = \lambda_{\Phi} C_{\Phi}(x, \partial_y) \langle \Phi(y)|_{y=0} \Phi(z) \rangle \quad (4.13)$$

For the 2pt and 3pt functions being inserted on the above equation the following expressions are generated:

$$\langle \phi_1(x_1) \phi_2(x_2) \Phi(x_3) \rangle = \frac{\lambda_{\Phi}}{(x_{12}) \cdots (x_{13}) \cdots (x_{23}) \cdots} \quad (4.14)$$

$$\langle \Phi(y) \Phi(z) \rangle = \frac{1}{|y - z|^{2\Delta_{\Phi}}} \quad (4.15)$$

An expansion on both sides around  $x = 0$  and a fix of the coefficients terms provides the function  $C_{\Phi}(x, \partial_y)$ . Matching of the 3pt function to the OPE

for  $\Delta_1 = \Delta_2 = \delta$  and  $\Delta_\Phi = \Delta$ :

$$C_\Phi(x, \partial_y) = \frac{1}{|x|^{2\delta-\Delta}} \left[ 1 + \frac{1}{2} x^\mu \partial_\mu + \alpha x^\mu x^\nu \partial_\mu \partial_\nu + \beta x^2 \partial^2 + \dots \right] \quad (4.16)$$

it is accomplished to find the constraints of the coefficients  $\alpha$  and  $\beta$  in terms of the Scaling Dimension:

$$\alpha = \frac{\Delta + 2}{8(\Delta + 1)} \quad \text{and} \quad \beta = -\frac{\Delta}{16 \left( \Delta - \frac{D}{2} + 1 \right) (\Delta + 1)} \quad (4.17)$$

## 4.2 Conformal Blocks

Two and three point functions are determined by Conformal Symmetry up to some undetermined constants. Simultaneously, their kinematical structure is fully fixed.

Presenting this scheme, we will implement these ideas to the four-point function with four identical scalar operators  $\phi$  at points  $x_i$ :

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle \quad (4.18)$$

Making a selection of the points  $x_1$  with  $x_2$  and  $x_3$  with  $x_4$  to be close to each other, we see that each of these two pairs is found encircled within a sphere separate from each other. OPE is then applied to the pairs of operators:

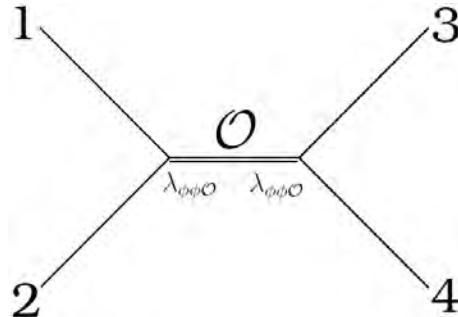
$$\begin{aligned} & \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \\ & = \sum \sum \lambda_{\phi\phi\mathcal{O}} C_{\mathcal{O}}(x_{12}, \partial_2) \lambda_{\phi\phi\mathcal{O}'} C_{\mathcal{O}'}(x_{34}, \partial_4) \langle \mathcal{O}_{\text{prim}}(x_2) \mathcal{O}'_{\text{prim}}(x_4) \rangle \end{aligned} \quad (4.19)$$

The  $\langle \mathcal{O}_{\text{prim}}(x_2) \mathcal{O}'_{\text{prim}}(x_4) \rangle$  operators are not scalars and do accept spin indices. The two-point function in a Conformal Field Theory is diagonal

allowing for the reduction of the sum over primes:

$$\begin{aligned} & \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \\ & = \sum_{\mathcal{O}'} \lambda_{\phi\phi\mathcal{O}} C_{\mathcal{O}}(x_{12}, \partial_2) \lambda_{\phi\phi\mathcal{O}'} C_{\mathcal{O}'}(x_{34}, \partial_4) \langle \mathcal{O}_{\text{prim}}(x_2) \mathcal{O}_{\text{prim}}(x_4) \rangle \end{aligned} \quad (4.20)$$

Diagrammatically eq. 4.20 is represented as:


(4.21)

Resuming the contributions of all descendants occurring in eq. 4.20 then the Conformal Block decomposition of the four-point function leads to an expression of eq. 4.20 in the form:

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{(x_{12}^2 x_{34}^2)^{\Delta_\phi}} \sum_{\mathcal{O}_{\text{prim}}} \lambda_{\phi\phi\mathcal{O}}^2 g_{\Delta_{\mathcal{O}}, \lambda_{\mathcal{O}}}(u, v) \quad (4.22)$$

where  $u, v$  are conformal invariant cross ratios:

$$u = \frac{x_{12}^2 x_{24}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \quad (4.23)$$

For a *Generalized Free Field theory*, (the theory where the connected  $n$ -point functions  $G_n(x_1, \dots, x_n)$  vanish for  $n > 2$ ), the primaries seen in the

above equation are equal to:

$$\sum_{\mathcal{O}_{prim}} \lambda_{\phi\phi\mathcal{O}}^2 g_{\Delta\mathcal{O},l\mathcal{O}}(u,v) = 1 + u^{\Delta_\phi} + \left(\frac{u}{v}\right)^{\Delta_\phi} \quad (4.24)$$

The consideration of a two-point function of spin- $l$  primaries provides:

$$\langle \mathcal{O}^{\mu_1 \dots \mu_l}(x) \mathcal{O}_{\nu_1 \dots \nu_l}(0) \rangle = \frac{I_{(\nu_1}^{(\mu_1} \dots I_{\nu_l)}^{\mu_l)} - \text{traces}}{(x^2)^{\Delta_\mathcal{O}}} \quad (4.25)$$

with  $\mathcal{O}^{\mu_1 \dots \mu_l}$  being the traceless symmetric representations of the Lorentz Group and tensor  $I_\nu^\mu$ :

$$I_\nu^\mu = \delta_\nu^\mu - \frac{2x^\mu x_\nu}{x^2} \quad (4.26)$$

Correspondingly, by replacing eq. [4.26](#) to eq. [4.20](#) the *Conformal Block* is obtained:

$$g_{\Delta,l}(u,v) = x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi} \frac{C_{\mu_1 \dots \mu_l}(x_{12}, \partial_l) C_{\nu_1 \dots \nu_l}(x_{34}, \partial_4) I^{\mu_1 \dots \mu_l \nu_1 \dots \nu_l}(x_{24})}{\lambda_{\phi\phi\mathcal{O}}^2 x_{24}^{2\Delta}} \quad (4.27)$$

Enforcing the limit for  $x_1 \rightarrow x_2$  and  $x_3 \rightarrow x_4$ :

$$C_{\mu_1, \dots, \mu_l}(x, \partial) \xrightarrow{x \rightarrow 0} \frac{x_{\mu_1} \dots x_{\mu_l}}{x^{2\Delta_\phi - \Delta + l}} \quad (4.28)$$

Proceeding on performing the action of contractions of the indices results to:

$$g_{\Delta,l}(u,v) \underset{\substack{u \rightarrow 0 \\ \nu \rightarrow 1}}{\sim} (-2)^{-l} u^{\frac{\Delta-l}{2}} (1-u)^l \quad (4.29)$$

The Casimirs of the Conformal Group are a combination of the generators of the this group. They are obtained by noticing that theory if expressed in

Euclidean signature, as:

$$SO(d, 2) \xrightarrow{\text{Euclidean}} SO(d + 1, 1) \quad (4.30)$$

is seen to be isomorphic to Lorentz Group in  $(d + 2)$  dimension and becomes the basis of the *embedding space formalism* for CFTs, a formalism presented in detail in [20].

Merging all Generators of the Conformal Group, such as translations, rotations, dilatations and SCTs, into one matrix. the Generator of Lorentz transformations in  $(d + 2)$  dimensional space is created:

$$[M_{AB}] = \begin{pmatrix} M_{\mu\nu} & -\frac{1}{2}(P_\mu + K_\mu) & \frac{1}{2}(P_\mu - K_\mu) \\ \frac{1}{2}(P_\nu + K_\nu) & 0 & -D \\ -\frac{1}{2}(P_\nu - K_\nu) & D & 0 \end{pmatrix} \quad (4.31)$$

The Generator  $[M_{AB}]$  satisfies the Lorentz algebra:

$$[M_{AD}, M_{CD}] = i(\eta_{AC}M_{BD} + \dots) \quad (4.32)$$

So it is established that Casimirs are the squares of these generators:

$$\begin{aligned} C &= \frac{1}{2}M_{AB}M^{AB} \\ &= \frac{1}{2}M_{\mu\nu}M^{\mu\nu} + P_\mu K^\mu + D(D - d) \end{aligned} \quad (4.33)$$

Furthermore since the conformal block is a function of  $x$  the differential representation is produced:

$$Dg_{\Delta,l}(u, v) = c_{\Delta,l}g_{\Delta,l}(u, v) \quad (4.34)$$

where  $c_{\Delta,l}$  is an eigenvalue and its equal to:

$$c_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2) \quad (4.35)$$

Consequently, the differential operator  $D$  obtains the form:

$$D = (1 - u - v)\partial_v v\partial_u + u\partial_u (2u\partial_u - d) + (1 - u - v)(u\partial_u + u\partial_v)(u\partial_u + v\partial_v) \quad (4.36)$$

A change of variables in this point makes the results be expressed in a more elegant manner:

$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z}) \quad (4.37)$$

resulting to the new form of the operator  $D$ :

$$D = 2(z^2(1 - z)\partial_z^2 - z^2\partial_z) + 2(\bar{z}^2(1 - \bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}) + 2(d - 2)\frac{z\bar{z}}{z - \bar{z}}((1 - z)\partial_z - (1 - \bar{z})\partial_{\bar{z}}) \quad (4.38)$$

In odd dimension this form can not be solved but for even dimensions we may provide two solutions. For  $d = 2$ :

$$g_{\Delta,l}(z, \bar{z}) = k_{\Delta+l}(z)k_{\Delta-l}(\bar{z}) + k_{\Delta-l}(z)k_{\Delta+l}(\bar{z}) \quad (4.39)$$

with the hypergeometric function  $F_1$  introduced:

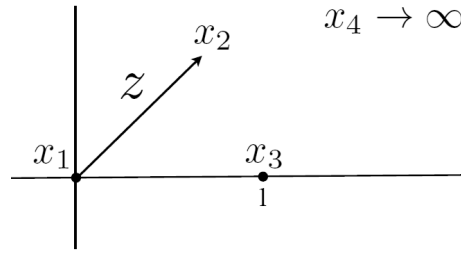
$$k_{\beta}(x) = x^{\beta/2} {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}; \beta \mid x\right) \quad (4.40)$$

. For the four dimensional case,  $d = 4$ :

$$g_{\Delta,l} = \frac{z\bar{z}}{z - \bar{z}} (k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - k_{\Delta-l-2}(z)k_{\Delta+l}(\bar{z})) \quad (4.41)$$

The change of variables  $u, v \rightarrow z, \bar{z}$  establishes a scheme where the action of Special Conformal Transformations takes a point and moves it to infinity, here  $x_4 \rightarrow \infty$ . Performing translations the point is send to zero,  $x_1 = 0$ .

Performing rotations there is a fix  $x_3 = (1, 0, \vec{0})$  where the  $\vec{0}$  is a  $(d-2)$  dimensional vector. Furthermore, by rotations a map  $x_2$  to the same plane occurs and diagrammatically is expressed as:



(4.42)

The point  $x_2 = (a, b, \vec{0})$  and the two complex variables are:

$$\begin{aligned} z &= a + ib \\ \bar{z} &= a - ib \end{aligned} \quad (4.43)$$

Equivalently, introducing the Radial Coordinates and by setting all points to the same plane:

$$(4.44)$$

We obtain the coordinates being  $x_4 = 1$ ,  $x_3 = 1$ ,  $x_2 = \rho$  and  $x_1 = -\rho$ .

Furthermore:

$$\begin{aligned} \rho &= re^{i\theta} \\ \rho &= \frac{z}{(1 + \sqrt{1 - z})^2} \end{aligned} \quad (4.45)$$

And:

$$z = \frac{4\rho}{(1 + \rho)^2} \quad (4.46)$$

This motion shows that the OPE is convergent whenever  $|\rho| < 1$  which enables the extraction of meaningful information for operators of low enough dimensions.

# Chapter 5

## Conformal Bootstrap

Having acquired the basic characteristics of Conformal Field Theory we are in position to search if new methods can be applied in order to solve and retrieve useful information from them.

Conformal Bootstrap has been such a method as it uses CFT data, Conformal Symmetry and consistency conditions in order to extract properties of these systems. Its ability to use the CFT itself and not some (UltraViolet) Lagrangians means that this method is non-perturbative. This characteristic is an immense advantage which allows us to use the Conformal Bootstrap in problems where previously we only could treat with approximate series methods, such as the epsilon expansion, or in some cases we could not handle at all.

### 5.1 Conformal Bootstrap and the Crossing equation

In this section we are based to [9], [14], [15], [16], [22], [24], [26], [27]. The basic concept of this method is to consider the crossing equation for the

four-point functions  $\langle \phi \phi \phi \phi \rangle$  of scalar operators  $\phi$  in the form:

$$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}}^2 (v^{\Delta_\phi} g_{\Delta,l}(u, v) - u^{\Delta_\phi} g_{\Delta,l}(v, u)) = 0 \quad (5.1)$$

The summation occurs over the primaries and it is infinite.

For each operator there is a corresponding conformal block which sums the contribution of descendants, while the parenthesis terms appearing in eq. [5.1](#) constitute the *Convolved Conformal Block*:

$$F_{\Delta,l}(u, v) = v^{\Delta_\phi} g_{\Delta,l}(u, v) - u^{\Delta_\phi} g_{\Delta,l}(v, u) \quad (5.2)$$

Considering  $F_{\Delta,l}(u, v)$  functions in a geometrical manner, we can interpret them as infinite dimensional vectors in the vector space of functions  $u$  and  $v$ . We have imposed the condition of the sum of infinite vectors to be equal to zero, this allows us to introduce a functional  $\alpha$  within this infinite dimensional space.

This step is useful since it permits us to set bounds on the scaling dimensions  $\Delta$ .

For this process we can use a series of steps, an *algorithm*, as seen in [\[24\]](#), with the scope of providing the *bound of scaling operator dimensions*:

1. Set the normalization for  $\alpha$ :

$$\alpha[F_{0,0}^{\Delta_\phi}(u, v)] = 1 \quad (5.3)$$

2. Make an assumption that the scaling dimensions  $\Delta_\ell$  of all spin  $\ell$  operators in  $\phi \times \phi$  except  $\mathcal{O}_{0,0}$  obey lower bounds  $\Delta_\ell \geq \Delta_\ell^B$ .
3. Initiate a search for non-negative linear functional  $\alpha$  whose action on

$F_{\Delta,l}$  satisfies:

$$\alpha[F_{\Delta,\ell}^{\Delta_\phi}] \geq 0 \quad (5.4)$$

since we would find a positive quantity to be equal to zero by finding such  $\alpha$  we would be lead to a contradiction with eq. 5.1 so as the assumption  $\Delta \geq \Delta_\ell^B$  being wrong.

This algorithm can run by differentiating our choice of parameters such as setting the lower bounds  $\Delta_\ell^B$  to their unitary values or by selecting different options for  $\Delta_{\ell^G}^B$  and  $\Delta_\phi$ .

## 5.2 Truncating Crossing

A necessary requirement for this algorithm to be used as a program, is to solve the arising infinities in order to become well defined. This is achieved by imposing four conditions to the crossed blocks  $F_{\Delta,\ell}^{\Delta_\phi}(u, v)$ :

1. The spin  $\ell$  value shall be any even positive number
2. The scaling dimensions of the operators,  $\Delta$ , shall be both a real number and larger than the unitarity bounds.
3. The conformal blocks are calculated with the use of an infinite series expansion in  $r$  (radial coordinates).
4. The action of the functional  $\alpha$  occurs over the infinite dimensional vector space of  $u$  and  $v$

Implementing our conditions, for the third, we introduce two cutoff values  $\ell_{max}$  and  $r_{max}$ , so as making possible a finite expansion in  $r$ . While for the fourth, we interpret the infinite dimensional space of  $u, v$  as a finite

dimensional space defined by the coefficients of the Taylor expansion of  $F_{\Delta,\ell}^{\Delta\phi}(z, \bar{z})$  up to a certain order  $\Lambda$  around the crossing symmetric point  $z = \bar{z} = 1/2$ .

Near the value  $r_c = 3 - 2\sqrt{2}$  we notice that the series is highly convergent, a fact which justifies the introduction of the  $r_{max}$  cutoff. On the other hand the value of the  $\ell_{max}$  cutoff tends to be based on a trial-error principle.

Consequently, the *crossing equation* obtains the form:

$$\sum_{\Delta} \sum_{\ell \leq \ell_{max}} \lambda_{\phi\phi\mathcal{O}_{\Delta,\ell}}^2 \partial_z^m \partial_{\bar{z}}^n F_{\Delta,\ell}^{\Delta\phi,r_{max}}(z, \bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}} = 0 \quad \text{for } m+n \leq \Lambda \quad (5.5)$$

in this equation we notice the action of the functionals  $\alpha_\mu\nu$  on the elements of this new finite vector space  $V_\Lambda$ . This space is a result of the derivatives appearing in the crossing equation.

$F_{\Delta,\ell}^{\Delta\phi}(z, \bar{z})$  is symmetric under an interchange  $z \leftrightarrow \bar{z}$  so it allows a restriction to odd  $(m+n)$ . Being odd under crossing  $(z, \bar{z}) \leftrightarrow (1-z, 1-\bar{z})$  allows the restriction  $m$ , leading to a space  $V_\Lambda$  with dimension:

$$|V_\Lambda| = \frac{1}{2} \left\lfloor \frac{\Lambda+1}{2} \right\rfloor \left( \left\lfloor \frac{\Lambda+1}{2} \right\rfloor + 1 \right) \quad (5.6)$$

Finite issue is resolving the infinity appearing in the continuum of dimensions  $\Delta$ . By using an approximation  $\partial_z^m \partial_{\bar{z}}^n F_{\Delta,\ell}^{\Delta\phi,r_{max}}(r_c, 1)$  as a polynomial in  $\Delta$  for each value of  $\ell$ , then the truncated crossing equation [5.6](#) obtains a form with positivity constraints on a finite number of polynomial.

Hereafter, there are many programming options with semidefinite programming being the main topic covered in more detail here.

### 5.3 The algorithm in 2-dim

A characteristic example of the abilities of this algorithm can be presented within the framework of the two-dimensional Ising model. This frequently used model is in fact equivalent to a two dimensional Conformal Field Theory.

It consists out of a real scalar primary operator  $\phi$  of dimension  $\Delta_\phi = \frac{1}{8}$ . Following [12], the algorithm initiates with an assumption of a finite 2D subspace of the vector space of functions of  $u, v$  as its is defined by values of conformal blocks in their crossing symmetric form.

For every set of conformal data  $(\Delta, \ell)$  appearing in the  $\phi \times \phi$  OPE a pair of  $(u, v)$  is introduced:

$$(u, v) = \left(\frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{2}, \frac{1}{3}\right) \quad (5.7)$$

and

$$(u, v) = \left(\frac{1}{2}, \frac{3}{5}\right), \left(\frac{1}{3}, \frac{1}{4}\right). \quad (5.8)$$

The normalization via the Identity operator, can be implemented with the function  $H_{\Delta, \ell}^{\Delta_\phi}$ :

$$H_{\Delta, \ell}^{\Delta_\phi} = \frac{F_{\Delta, \ell}^{\Delta_\phi}}{u^{\Delta_\phi} - v^{\Delta_\phi}} \quad (5.9)$$

Acquiring the two-dimensional space, it follows a projection of the crossing equation to the space via the linear map provided by the functional:

$$\vec{v}(F) = \left( H\left(\frac{1}{2}, \frac{3}{5}\right) - H\left(\frac{1}{2}, \frac{1}{3}\right), H\left(\frac{1}{2}, \frac{3}{5}\right) - H\left(\frac{1}{3}, \frac{1}{4}\right) \right) \in \mathbb{R}^2 \quad (5.10)$$

$$u = z\bar{z}, v = (1 - z)(1 - \bar{z})$$

Linearity properties of the system determine the sum of the vectors

$\vec{v}(F_{\Delta,\ell}^{\Delta_\phi})$  to be zero:

$$\sum_{\Delta,\ell} p_{\Delta,\ell} \vec{v}(F_{\Delta,\ell}^{\Delta_\phi}) = 0 \quad (5.11)$$

Having acquired the above condition a plot of all values of pairs  $\Delta, \ell$  can be presented in a plot, following [12] calculations:

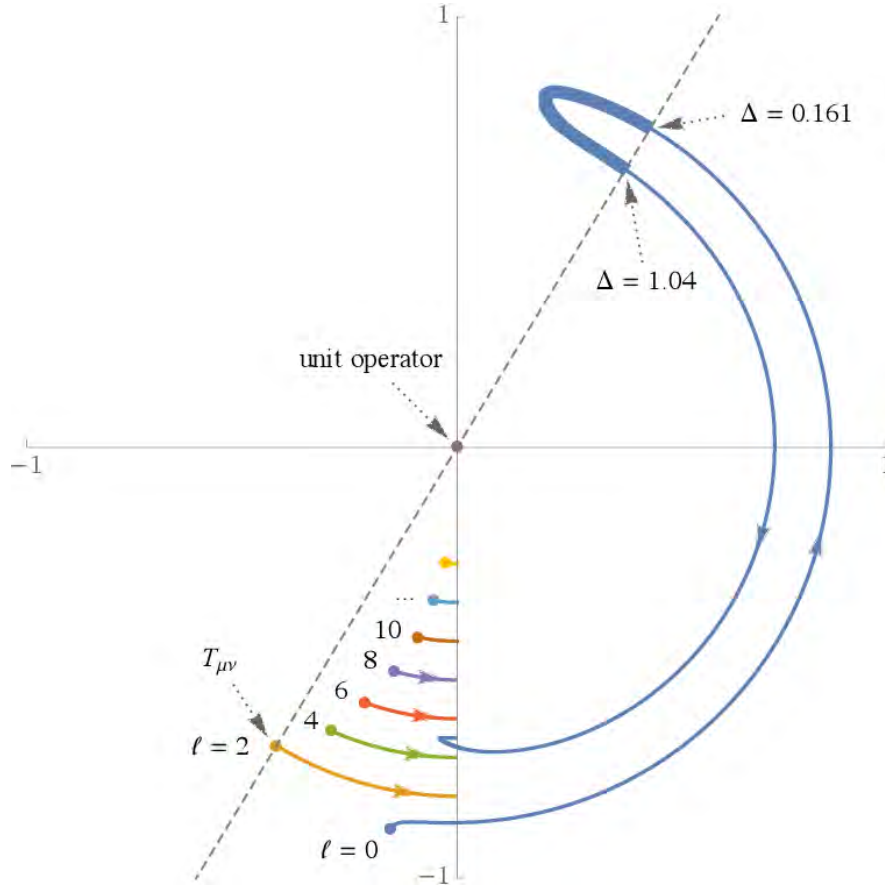


Figure 5.1: Representation of two dimensional vectors for  $\Delta_\phi = \frac{1}{8}$ . Separation in half-planes is noted via the dashed line. [Source [12]]

The method of creating the plot is to initiate from a fixed spin and gradually increase the value of the dimension  $\Delta$  from 0 to infinity.

Each plotted vector is denoted as a point whose coordinates correspond to vector's components. In our case, different shaded curves represents

different spin values for scalar  $\ell = 0$  (blue),  $\ell = 2$  (orange), etc.

Full filling the requirement of the vectors to sum up to zero allows the extraction of the conclusion that the  $\ell = 0$  (scalar) vector shall exist on the above dashed line and especially to be located with dimension  $\Delta$  values:

$$\Delta \in [0.161, 1.035] \quad (5.12)$$

We directly conclude that a two dimensional unitary Conformal Field Theory can not exist with only one scalar for dimension  $\Delta_\phi = \frac{1}{8}$ . Simultaneously, we are in position to claim that in this physical setup there must exist a different scalar operator, say  $\epsilon$ , with scaling dimension  $\Delta_\epsilon \in [0.161, 1.035]$

**Claim:** For a 2D CFT with a scalar of dimension  $\Delta_\phi = \frac{1}{8}$  there exists another scalar operator,  $\epsilon$ , with dimension  $\Delta_\epsilon \in [0.161, 1.035]$ .

This result yields the importance of this algorithm since the  $\epsilon$  operator can be identified as the energy operator of our 2-dim unitary Ising model. In this model the real scalar spin operator  $\sigma$  has dimension  $\Delta_\sigma = \frac{1}{8}$  while the energy operator  $\epsilon$  has  $\Delta_\epsilon = 1$  coinciding with plot's upper bound  $\Delta_{scalar} \leq 1.04$  with (only) a four percent variation from the actual price!

## 5.4 Mixed correlators

Even though the use of four-point correlator functions of identical scalar operators does produce interesting results, an expansion of this setup is necessary for producing operators that enhance the study of more complex systems.

### 5.4.1 Mixed four point scalar correlators

This new scheme is introduced via the use of different real scalar functions  $\phi_i(x)$ , where  $i$  denotes different scalars that follow different irreducible representations  $\mathcal{R}_i$  of the global symmetry group  $G$ .

A fix of these terms is obtained by the conformal symmetry and especially in the form:

$$\langle \phi_i(x_1) \phi_j(x_2) \phi_k(x_3) \phi_l(x_4) \rangle = g_{ijkl}(u, v) \frac{\left[ \frac{x_{24}}{x_{14}} \right]^{\Delta_{ij}} \left[ \frac{x_{14}}{x_{13}} \right]^{\Delta_{kl}}}{x_{12}^{\Delta_i + \Delta_j} x_{34}^{\Delta_k + \Delta_l}} \quad (5.13)$$

where  $\Delta_{ij} \equiv \Delta_i - \Delta_j$ . Using the channels  $1 \leftrightarrow 2-3 \leftrightarrow 4$  setup in the Operator Product Expansion we produce an expansion of  $g_{ijkl}(u, v)$  term:

$$g_{ijkl}(u, v) = \sum_{\Delta, \ell, R} \lambda_{\phi_i \phi_j \mathcal{O}_{\Delta, \ell, R}} \lambda_{\phi_k \phi_l \mathcal{O}_{\Delta, \ell, R}} T_{ij, kl}^R g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}(u, v), \quad (5.14)$$

Notable is the introduction of tensor structure terms  $T_{ij, kl}^R$  corresponding to irreducible representations set by  $\mathcal{R}_i \times \mathcal{R}_j$  and  $\mathcal{R}_k \times \mathcal{R}_l$ . It is anticipated that interchanges between OPE channels do not alter the system and thereafter they yield:

$$g_{ijkl}(u, v) = \sum_{\Delta, \ell, R} \lambda_{\phi_i \phi_j \mathcal{O}_{\Delta, \ell, R}} \lambda_{\phi_k \phi_l \mathcal{O}_{\Delta, \ell, R}} T_{ij, kl}^R g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}(u, v), \quad (5.15)$$

The Conformal Casimir equation dictates new restraints for the Conformal

Blocks:

$$\begin{aligned}\mathcal{D}_{\Delta_{ij},\Delta_{kl}}g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v) &= c_{\Delta,\ell}g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u,v), \\ \mathcal{D}_{\Delta_{ij},\Delta_{kl}} &\equiv (1-u-v)\partial_v \left( v\partial_v + \frac{\Delta_{kl}-\Delta_{ij}}{2} \right) + u\partial_u (2u\partial_u - d) \\ &\quad - (1+u-v) \left( u\partial_u + v\partial_v - \frac{\Delta_{ij}}{2} \right) \left( u\partial_u + v\partial_v + \frac{\Delta_{kl}}{2} \right),\end{aligned}\tag{5.16}$$

We notice an additional dependency on the external operator's scaling dimensions. It is a characteristic with high computational value since it permits to compute Conformal Blocks separately for every value of  $\Delta_{ij}$  and  $\Delta_{kl}$ . This allows for the Casimir equation to be exactly solvable in even dimensions. Whereas for the often seen cases of  $d = 2$  or  $d = 4$  the Correlator Blocks to be generalized as:

$$k_\beta(x) \rightarrow k_\beta^{\Delta_{ij},\Delta_{kl}}(x) = x^{\frac{\beta}{2}} {}_2F_1 \left( \frac{\beta - \Delta_{ij}}{2}, \frac{\beta + \Delta_{kl}}{2}, \beta, x \right) \tag{5.17}$$

On the other hand, working in odd-dimensions requires to the use of Zamolodchikov's recursion relation and so-after an expansion to  $r, \eta$  terms.

$$\tilde{h}_\ell^{\Delta_{ij},\Delta_{kl}}(r,\eta) = \frac{4^\Delta \ell! C_\ell^{\frac{d-2}{2}}(\eta)}{(-2)^\ell \left(\frac{d-2}{2}\right)_\ell (1-r^2)^{\frac{d-2}{2}} (1+r^2+2r\eta)^{\frac{1}{2}(1+\Delta_{ij}-\Delta_{kl})} (1+r^2-2r\eta)^{\frac{1}{2}(1-\Delta_{ij}+\Delta_{kl})}} \tag{5.18}$$

Along with the coefficients  $c_{I,m}$ :

$$\begin{aligned}
c_{m,1} &= -\frac{m(-2)^m}{(m!)^2} \left( \frac{\Delta_{ij} + 1 - m}{2} \right)_m \left( \frac{\Delta_{kl} + 1 - m}{2} \right)_m, \\
c_{m,2} &= -\frac{m\ell!(d/2 + \ell)_{-m}(d/2 + \ell - 1)_{-m}}{(-2)^m(m!)^2(\ell - m)!(d - 2 + \ell)_{-m}} \left( \frac{\Delta_{ij} + 1 - m}{2} \right)_m \left( \frac{\Delta_{kl} + 1 - m}{2} \right)_m, \\
c_{m,3} &= \frac{-m(-1)^{m/2}((d - m)/2 - 1)_m}{2((m/2)!)^2((d - m)/2 + \ell - 1)_m((d - m)/2 + \ell)_m} \left( \frac{\Delta_{ij} - (d + m)/2 - \ell + 2}{2} \right)_{\frac{m}{2}} \\
&\times \left( \frac{\Delta_{ij} + (d - m)/2 + \ell}{2} \right)_{\frac{m}{2}} \left( \frac{\Delta_{kl} - (d + m)/2 - \ell + 2}{2} \right)_{\frac{m}{2}} \left( \frac{\Delta_{kl} + (d - m)/2 + \ell}{2} \right)_{\frac{m}{2}}.
\end{aligned} \tag{5.19}$$

Leading to results, summarized in the table:

| $I$ | $m \in \mathbb{B}_I$ | $\Delta_{I,m}$     | $\ell_{I,m}$ |
|-----|----------------------|--------------------|--------------|
| 1   | $1, 2, 3, \dots$     | $1 - \ell - m$     | $\ell + m$   |
| 2   | $1, 2, \dots, \ell$  | $\ell + d - 1 - m$ | $\ell - m$   |
| 3   | $2, 4, 6, \dots$     | $(d - m)/2$        | $\ell$       |

(5.20)

Table: Values of the poles  $\Delta_{I,m}$  in  $g_{\Delta,\ell}$ , shifts of the residue conformal blocks  $g_{\Delta_{I,m}+m,\ell_{I,m}}^{\Delta_{ij},\Delta_{kl}}$ , and range  $\mathbb{B}_I$  of possible values of  $m$  for each  $I = 1, 2, 3$ .

## 5.5 Crossing equations of mixed correlators

Consequently the interchange of OPE channel between different scalar operators,  $(1, i) \leftrightarrow (3, k)$ , shapes the Crossing equation as:

$$\begin{aligned}
&\sum_{\Delta,\ell,R} \left[ \lambda_{ij\mathcal{O}_{\Delta,\ell,R}} \lambda_{kl\mathcal{O}_{\Delta,\ell,R}} T_{ij,kl}^R v^{\frac{\Delta_j + \Delta_k}{2}} g_{\Delta,\ell}^{\Delta_{ij},\Delta_{kl}}(u, v) \right] \\
&= \sum_{\Delta,\ell,R'} \left[ \lambda_{kj\mathcal{O}_{\Delta,\ell,R'}} \lambda_{il\mathcal{O}_{\Delta,\ell,R'}} T_{kj,il}^{R'} u^{\frac{\Delta_i + \Delta_j}{2}} g_{\Delta,\ell}^{\Delta_{kj},\Delta_{il}}(v, u) \right]
\end{aligned} \tag{5.21}$$

Here the irreducible representations are a consequence of the tensor

product  $\mathcal{R}_i \times \mathcal{R}_j \cap \mathcal{R}_k \times \mathcal{R}_l$  and are denoted as  $R$ .

A useful re-expression of the Crossing equation is:

$$\sum_{\Delta, \ell, R} \left[ \lambda_{ij\mathcal{O}_{\Delta, \ell, R}} \lambda_{kl\mathcal{O}_{\Delta, \ell, R}} T_{ij, kl}^R F_{\mp, \Delta, \ell}^{ij, kl}(u, v) \right] \pm \sum_{\Delta, \ell, R'} \left[ \lambda_{kj\mathcal{O}_{\Delta, \ell, R'}} \lambda_{il\mathcal{O}_{\Delta, \ell, R'}} T_{kj, il}^{R'} F_{\mp, \Delta, \ell}^{kj, il}(u, v) \right] = 0 \quad (5.22)$$

allowing us to define, the symmetric and antisymmetric under  $u \leftrightarrow v$  interchanges, Crossing equations  $F_{\pm, \Delta, \ell}^{ij, kl}(u, v)$  :

$$F_{\pm, \Delta, \ell}^{ij, kl}(u, v) \equiv v^{\frac{\Delta_k + \Delta_j}{2}} g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}(u, v) \pm u^{\frac{\Delta_k + \Delta_j}{2}} g_{\Delta, \ell}^{\Delta_{ij}, \Delta_{kl}}(v, u) \quad (5.23)$$

## 5.6 Constraints on matrices

In this section our main scope is to locate which is the basic difference that changes our algorithmic approach when we meet mixed scalar crossing equations of different scalars. This is highly seen by examining the OPE coefficients,  $\lambda_{kj\mathcal{O}_{\Delta, \ell, R'}} \lambda_{il\mathcal{O}_{\Delta, \ell, R'}}$ , terms which are no longer squares and not always positive as we have assumed in our previous algorithm.

Semi-definite programming allows the use of constraints of positive semi-definite symmetric matrices of a polynomial, hereafter engaging our search towards providing a catalogue of matrix constraints valid for all four-point scalar  $\phi_w(x)$  functions, with  $w = i, j, k, l$ . In this occasion, the vectors of OPE coefficients will be involved in a multiplication of symmetric matrices of polynomials.

Setting as our goal to solve systems that appear to have global use in physics, we focus on creating restraints for a CFT characterized by a  $\mathbb{Z}_2$  symmetry. We expect such theory to accept its lowest dimension  $\mathbb{Z}_2$  even operator being the  $\epsilon$  and its lowest dimension  $\mathbb{Z}_2$  operator being the  $\sigma$ .

This setup accepts a consideration of combinations between the four non-zero  $\epsilon$  and  $\sigma$  correlators:

$$\langle \sigma \sigma \sigma \sigma \rangle, \langle \epsilon \epsilon \sigma \sigma \rangle, \langle \epsilon \epsilon \epsilon \epsilon \rangle \quad (5.24)$$

A set of functions is then generated by the application of Crossing equations:

$$\begin{aligned} 0 &= \sum_{\Delta, \ell^+} \lambda_{\sigma\sigma\mathcal{O}_{\Delta, \ell, +}}^2 F_{-, \Delta, \ell}^{\sigma\sigma, \sigma\sigma}(u, v), \\ 0 &= \sum_{\Delta, \ell^+} \lambda_{\epsilon\epsilon\mathcal{O}_{\Delta, \ell, +}}^2 F_{-, \Delta, \ell}^{\epsilon\epsilon, \epsilon\epsilon}(u, v), \\ 0 &= \sum_{\Delta, \ell} \lambda_{\sigma\epsilon\mathcal{O}_{\Delta, \ell, -}}^2 F_{-, \Delta, \ell}^{\sigma\epsilon, \sigma\epsilon}(u, v), \\ 0 &= \sum_{\Delta, \ell^+} \lambda_{\sigma\sigma\mathcal{O}_{\Delta, \ell, +}} \lambda_{\epsilon\epsilon\mathcal{O}_{\Delta, \ell, +}} F_{\mp, \Delta, \ell}^{\sigma\sigma, \epsilon\epsilon}(u, v) \pm \sum_{\Delta, \ell} (-1)^\ell \lambda_{\sigma\epsilon\mathcal{O}_{\Delta, \ell, -}}^2 F_{\mp, \Delta, \ell}^{\epsilon\sigma, \sigma\epsilon}(u, v), \end{aligned} \quad (5.25)$$

In our notation,  $\ell^\pm$  expresses the existence of only even or odd spins. Simultaneously, we have used that  $\lambda_{\epsilon\sigma\mathcal{O}_{\Delta, \ell, -}} = (-1)^\ell \lambda_{\sigma\epsilon\mathcal{O}_{\Delta, \ell, -}}$ .

These constraints can also be seen more geometrically by re-arranging them as a vector:

$$\sum_{\Delta, \ell} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}_{\Delta, \ell, +}} & \lambda_{\epsilon\epsilon\mathcal{O}_{\Delta, \ell, +}} \end{pmatrix} \vec{V}_{\Delta, \ell, +} \begin{pmatrix} \lambda_{\sigma\sigma\mathcal{O}_{\Delta, \ell, +}} \\ \lambda_{\epsilon\epsilon\mathcal{O}_{\Delta, \ell, +}} \end{pmatrix} + \sum_{\Delta, \ell} \lambda_{\sigma\epsilon\mathcal{O}_{\Delta, \ell, -}}^2 \vec{V}_{\Delta, \ell, -} = \vec{0} \quad (5.26)$$

we have introduced two geometric elements,  $\vec{V}_{-, \Delta, \ell}$  to provide the ket of scalar constraints:

$$\vec{V}_{-,\Delta,\ell} = \begin{pmatrix} 0 \\ 0 \\ F_{-,\Delta,\ell}^{\sigma\epsilon,\sigma\epsilon} \\ (-1)^\ell F_{-,\Delta,\Delta,\ell}^{\epsilon\sigma,\Delta\epsilon} \\ -(-1)^\ell F_{+,\Delta,\ell}^{\epsilon\sigma,\sigma\epsilon} \end{pmatrix} \quad (5.27)$$

and  $\vec{V}_{+,\Delta,\ell}$ , with  $2 \times 2$  symmetric elements, to include the matrix constraints:

$$\vec{V}_{+,\Delta,\ell} = \begin{pmatrix} \begin{pmatrix} F_{-,\Delta,\ell}^{\sigma\sigma,\sigma\sigma} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & F_{-,\Delta,\ell}^{\epsilon\epsilon\epsilon\epsilon} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} F_{-,\Delta,\ell}^{\sigma\sigma,\epsilon\epsilon} \\ \frac{1}{2} F_{-,\Delta,\ell}^{\sigma\sigma,\epsilon\epsilon} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} F_{+,\Delta,\ell}^{\sigma\sigma,\epsilon\epsilon} \\ \frac{1}{2} F_{+,\Delta,\ell}^{\sigma\sigma,\epsilon\epsilon} & 0 \end{pmatrix} \end{pmatrix} \quad (5.28)$$

Further examining the matrix [5.28](#) we notice the existence of the identity operator through the matrix constraint  $\vec{V}_{+,0,0}$  yielding the constraint:

$$\vec{V}_{0,0,+}^{\text{scalar}} \equiv \begin{pmatrix} 1 & 1 \end{pmatrix} \vec{V}_{0,0,+} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (5.29)$$

this new quantity can be now be used as the norm of the Algorithm, namely the quantity which is an operator characterizing an OPE coefficient which will be maximized within the program after it is raised to its square.

## 5.7 Constraints and Gaps

A necessary tool to move closer to predictions of real systems is to assume the existence of gaps. In the model of the  $\mathbb{Z}_2$ -symmetric CFT that we are producing our considerations, we can identify this model as the Ising model. This allows us to use the fact that the Ising model accepts two relevant operators  $\epsilon$  and  $\sigma$ . So, we introduce a new constraint on the OPE coefficient  $\lambda_{\sigma\sigma\epsilon}$  taking advantage of its even spin and we require it to remain invariant under rearrangement of its indices and so after receiving the form:

$$\begin{aligned} \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} & \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} \vec{V}_{\Delta_\epsilon,0,+} + \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} + \lambda_{\sigma\epsilon\sigma}^2 \vec{V}_{\Delta_\sigma,0,-} \\ = \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} & \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix} \left[ \vec{V}_{\Delta_\epsilon,0,+} + \vec{V}_{\Delta_\sigma,0,-} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \lambda_{\sigma\sigma\epsilon} \\ \lambda_{\epsilon\epsilon\epsilon} \end{pmatrix}. \end{aligned} \quad (5.30)$$

We further fix eq. 5.30 so as for the  $\epsilon$  and  $\sigma$  operators to be the only to accept their characteristic scaling dimensions  $\Delta_\epsilon$  and  $\Delta_\sigma$ . This is achieved by replacing the  $2 \times 2$  matrix constraint in eq. 5.30 with a new form of scalar constraint:

$$(\lambda_{\sigma\sigma\epsilon}^2 + \lambda_{\epsilon\epsilon\epsilon}^2) \begin{pmatrix} \cos \theta & \sin \theta \end{pmatrix} \left[ \vec{V}_{\Delta_\epsilon,0,+} + \vec{V}_{\Delta_\sigma,0,-} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \cos \theta & \sin \theta \end{pmatrix} \quad (5.31)$$

the variable is found within the range  $\theta = \tan^{-1}(\lambda_{\epsilon\epsilon\epsilon}/\lambda_{\sigma\sigma\epsilon}) \in [0, \pi)$ . Last step, is to collect all the pairs of points  $(\Delta_\sigma, \Delta_\epsilon, \theta)_{\text{allowed}}$  provided by the bootstrap process.

Numerical predictions of the scaling dimensions of the 3d Ising model, using the technique that we described, were first reported in 2014 by F.

Kos, D. Poland and D. Simmons-Duffin [14]. Allowed and disallowed regions in the plot of  $\Delta_\epsilon$  and  $\Delta_\sigma$  were produced and seen as a graph:

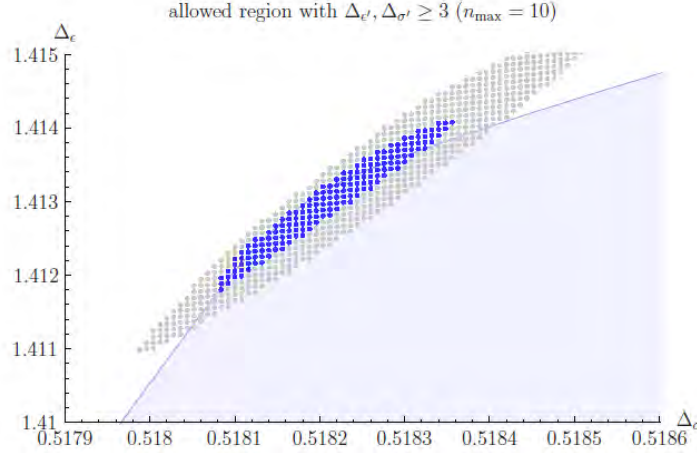


Figure 5.2: Allowed and disallowed  $(\Delta_\sigma, \Delta_\epsilon)$  points in a  $\mathbb{Z}_2$ -symmetric  $\text{CFT}_3$  with only one relevant  $\mathbb{Z}_2$ -odd and  $\mathbb{Z}_2$ -even scalar, using the constraints of crossing symmetry and unitarity for  $\langle\sigma\sigma\sigma\sigma\rangle, \langle\sigma\sigma\epsilon\epsilon\rangle, \langle\epsilon\epsilon\epsilon\epsilon\rangle$  at  $n_{\max} = 10$  (275 components),  $\nu_{\max} = 14$ . The light grey points are ruled out, while the dark blue points are allowed. The light blue shaded region shows the region allowed by crossing symmetry and unitarity of the single correlator  $\langle\sigma\sigma\sigma\sigma\rangle$  at  $n_{\max} = 14$ , computed in [22]. The final allowed region is the intersection of this shaded region with the region indicated by the dark blue points. Source: [14]

This progress in the use of numerical conformal bootstrap algorithms lead to an increased interest for further calculations. New calculations showed that this method provides values with staggering accuracy. Characteristically, calculating and presenting in the following table, the scaling dimensions for the Critical Ising model with three different techniques, the epsilon expansion, Monte Carlo simulations and Bootstrap we notice the precision that our non perturbative method provides!

| Method                | $\Delta_\sigma$ | $\Delta_\epsilon$ | (5.32) |
|-----------------------|-----------------|-------------------|--------|
| $\epsilon$ -expansion | 0.5182(25)      | 1.4139(62)        |        |
| Monte Carlo           | 0.51813(5)      | 1.41275(25)       |        |
| Bootstrap             | 0.5181489(10)   | 1.412625(10)      |        |

Table: Values of critical dimensions of the 3d Ising model. Values obtained by [27], [28] and [14].

# Chapter 6

## Epilogue

The presentation of the main aspects of the Conformal Field Theories along with the key concepts of the method of Conformal Bootstrap leads to the notice of the importance and enhanced impact of this special category of Quantum Field Theories.

The physical ideas that rule nature, symmetries and scale, are observed in every aspect subfield of physics. As we have noticed, common behaviours seen in condensed matter ensembles, in statistical physics, cosmology, particle physics and string theory can all be sum up to the framework of Conformal Field Theories.

Our ability to classify QFTs with the use of CFTs and the recent development of the non-perturbative Conformal Bootstrap method is currently leading the search for further insight to physics theories.

As seen in [26], the Bootstrap method has been already used for testing a variety of models. A citation of some of our era's interesting research results, includes, in  $d = 3$  dimensions, the  $O(N)$  models, CFTs with fermion operators such as the Gross-Neveu-Yukawa models, the  $\mathcal{QED}_3$  theories and the Critical Ising model.

In  $d = 4$ , Conformal Bootstrap has been used to understand the hierarchy problem of the Standard Model, to study the Banks-Zaks fixed points of asymptotically free non-abelian gauge theories.

Furthermore, this mechanism has been used in search of characteristics of Superconformal Theories either with 4d  $\mathcal{N} = 4$  supersymmetry or 3d  $\mathcal{N} = 2$  supersymmetry while also many non-unitary theories have been explored by the Conformal Bootstrap method.

As a conclusion of this Bachelor Thesis, we may state that Conformal Field Theories have already been established as one of main key concepts of our era's Theoretical Physics. Their characteristic of providing solutions and enhanced insight to physical theories through symmetry and scale considerations does place them in the peak of nowadays physics research, with the technique of Conformal Bootstrap being one of the most promising tools for future research.

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